Principal Component Analysis

Jenn-Jier James Lien (連 震 杰)
Professor

Computer Science and Information Engineering
National Cheng Kung University

(O) (06) 2757575 ext. 62540
jjlien@csie.ncku.edu.tw
http://robotics.csie.ncku.edu.tw

Major Issues

1. **Fundamental of machine learning:**
   - PCA, PPCA to Factor analysis (including noise) by Markov Model.
2. **PCA:**
   - Dimensionality reduce => Domain knowledge
   - Reconstruction
   - Low frequency components (PCA, Gaussian) +
     high frequency components (ICA, non-Gaussian) + noise
3. **Covariance matrix Vs. correlation matrix**
4. **Covariance matrix => PCA - SVD: $C=UWV^T$**
   - Gaussian Model
   - Affine transform:
     - translation (mean shift),
     - rotation (eigenvectors) and
     - scaling (eigenvalue)
5. **Camera 2D image plane Vs. Object surface/plane rotation (2D)**
   - **Optical axis Vs. Normal direction of 2D plane/surface at object**

Why PCA

- **256x256-Pixel Image => 256x256 Matrix => 256x256-dimensional Vector $(\ldots)_{256x256}$**
  - One sample/image at high-dimensional space/coordinate

- **Optimization => Vector or Matrix, NOT equations**
  - $Ax = b$
    - Pseudo Inverse
    - SSD $E = \mathrm{Sum}\,(Ax-b)^2$
  - $Ax = 0$
    - Closed Form Solution => SVD => PCA/LDA

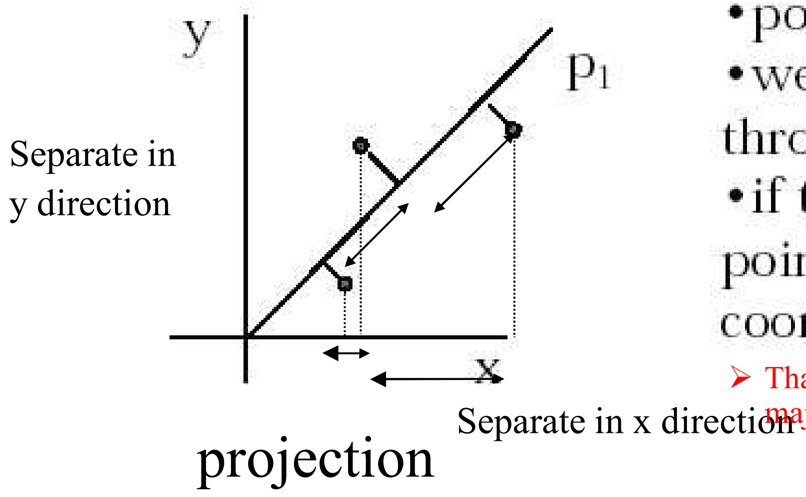
- **Property of PCA: One is to reduce dimension**
  - Computational time
  - Storage

- **Domain Knowledge**
  - PCA, LDA, LLE….
- **Similarity Measure**

- **Bayes' theory: P(F|I) = P(I|F) * (P|F) / P(I)**
- **MAP: Face detection F'=arg max P(F|I);**

- **N >> W*H**
- **N<< or < W*H**
  - Supervised Learning Vs. Unsupervised learning
  - MCMC (Markov Chain Monte Carlo), particle filters, kalman filter

- We are given:
  - a set of M objects" (images) ex.
  - each is represented, initially, by a set of N^2 features (128^2) or pixels
  - This data is organized as an (very large!) N^2xM matrix
- Let's look at a small example



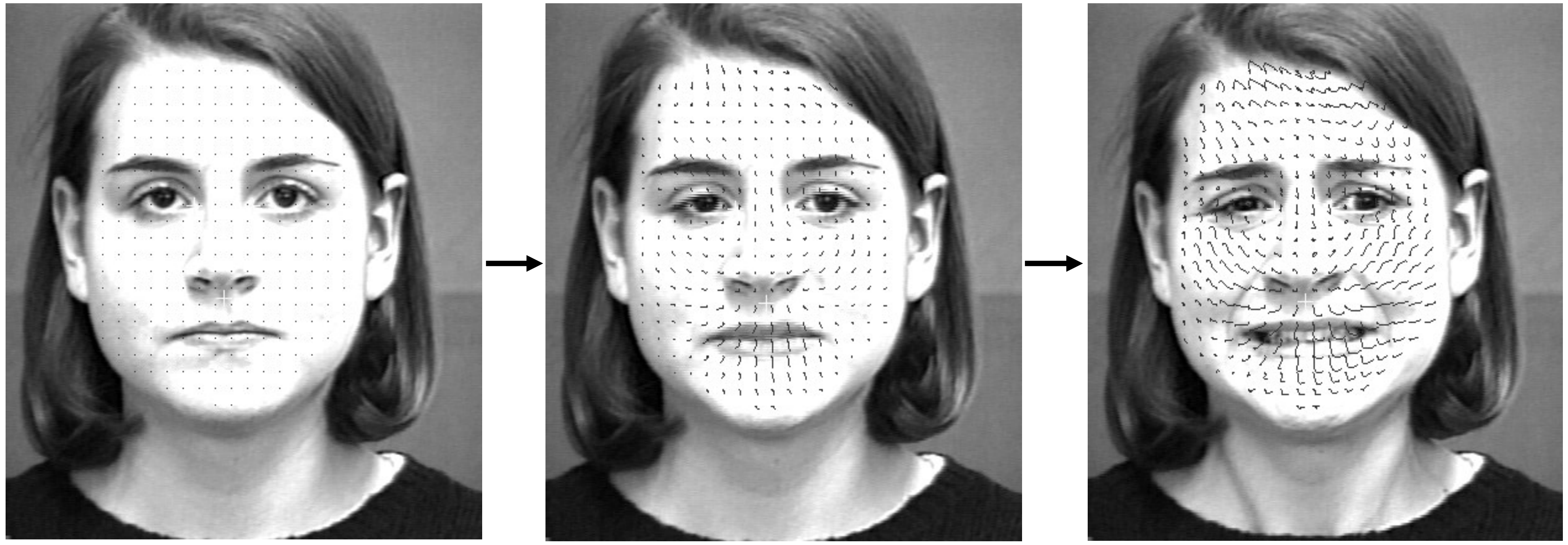
- points are 2-D points
- we find the axis that most closely passes through these points
- if the axis passed exactly through these points, then we would need only one coordinate to represent each point.

➢ That is, sum of all shortest vertical distance of samples or points to the major axis

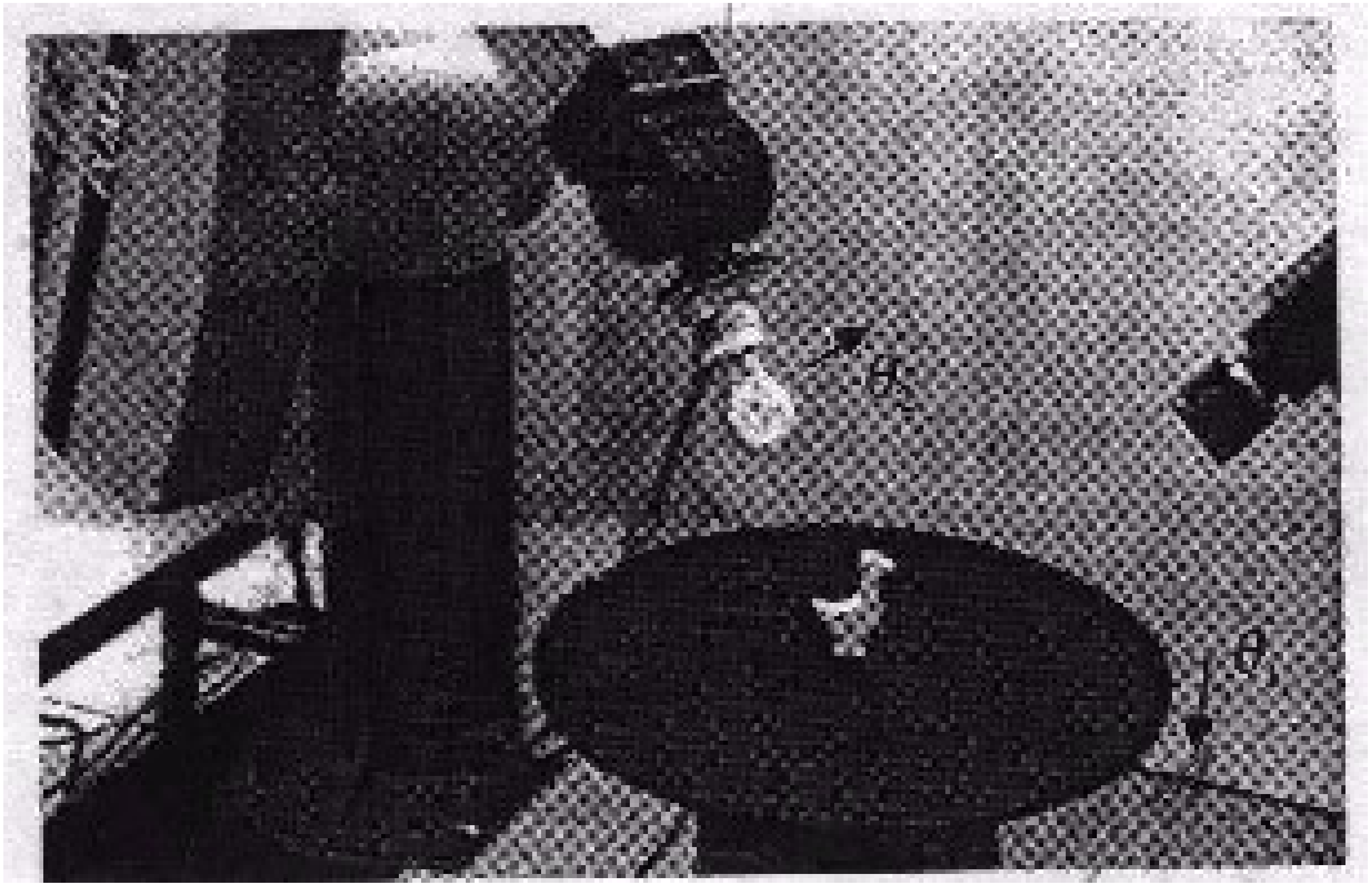
PCA Applications

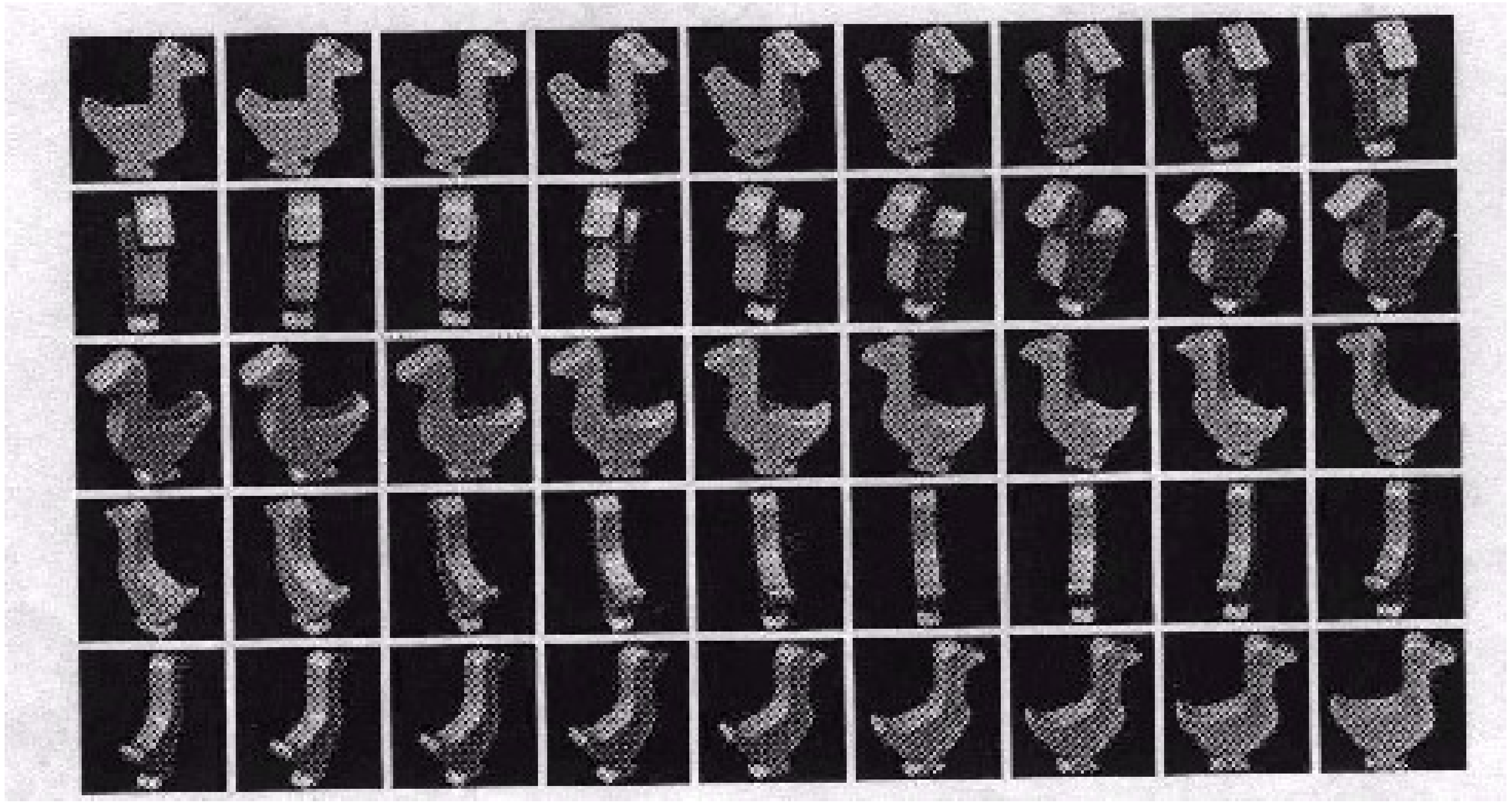
- Face Recognition: Input/output gray values

- Facial Expression Recognition: Input/output dense flows (x-axis and y-axis components)

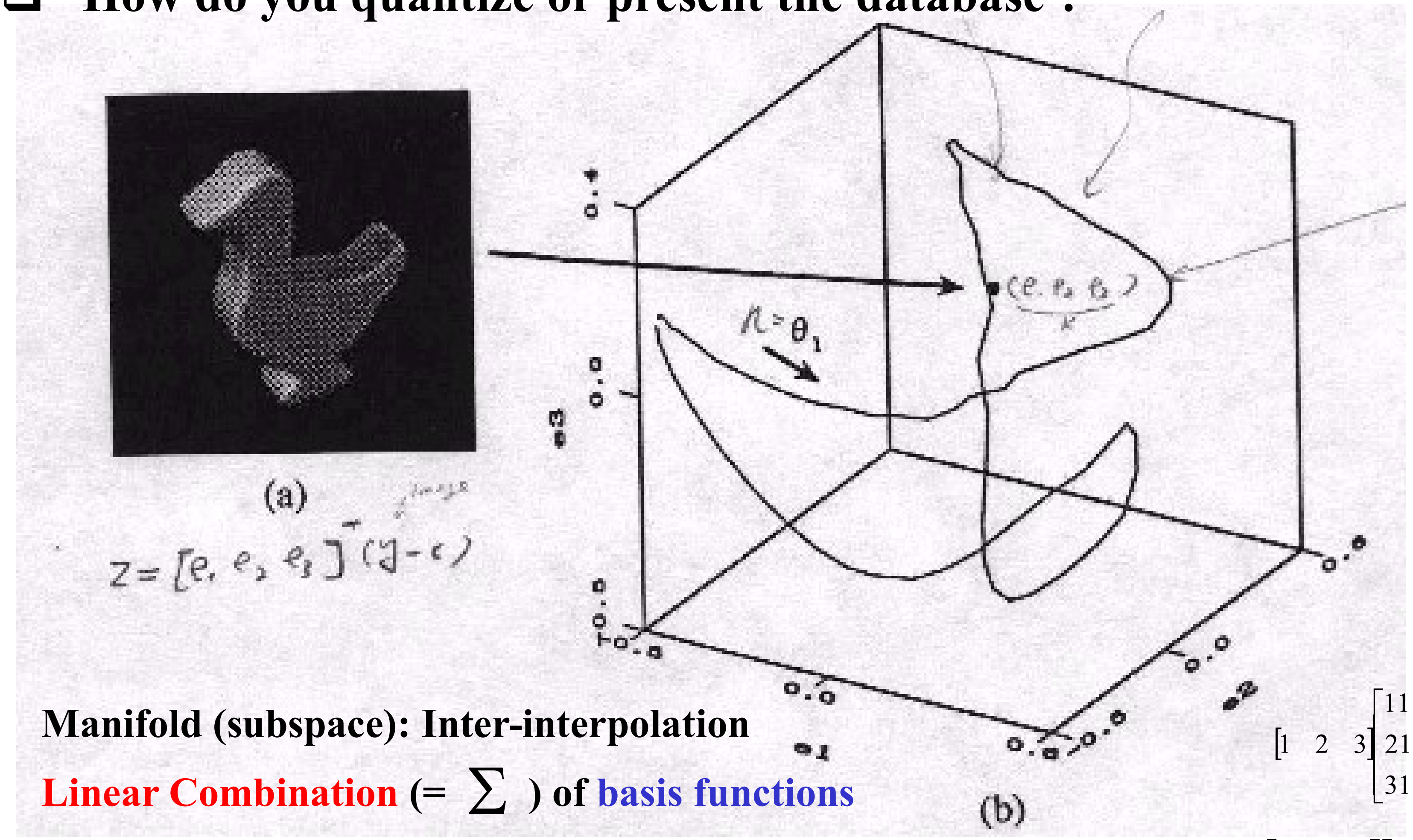


•Object Recognition





❑ How do you quantize or present the database ?



Manifold (subspace): Inter-interpolation

Linear Combination (= $\sum$) of basis functions

$$\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}11 & 12 & 13\\ 21 & & \\ 31 & & 33\end{bmatrix}$$

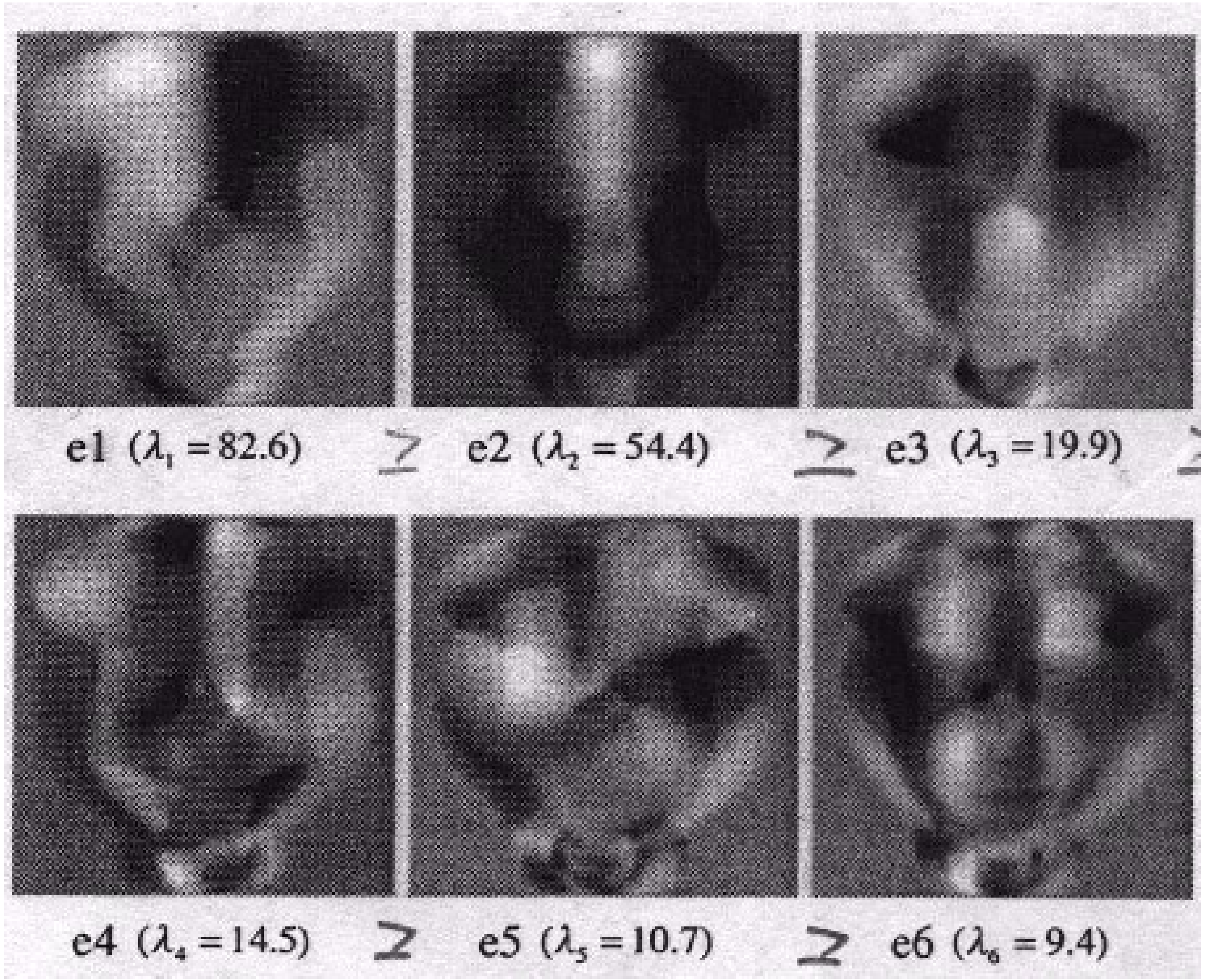
(=>Homogenous Matrix or Matrix * [1 1 … 1]^T):

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}\begin{bmatrix}11 & 12 & 13\\ 21 & & \\ 31 & & 33\end{bmatrix}$$

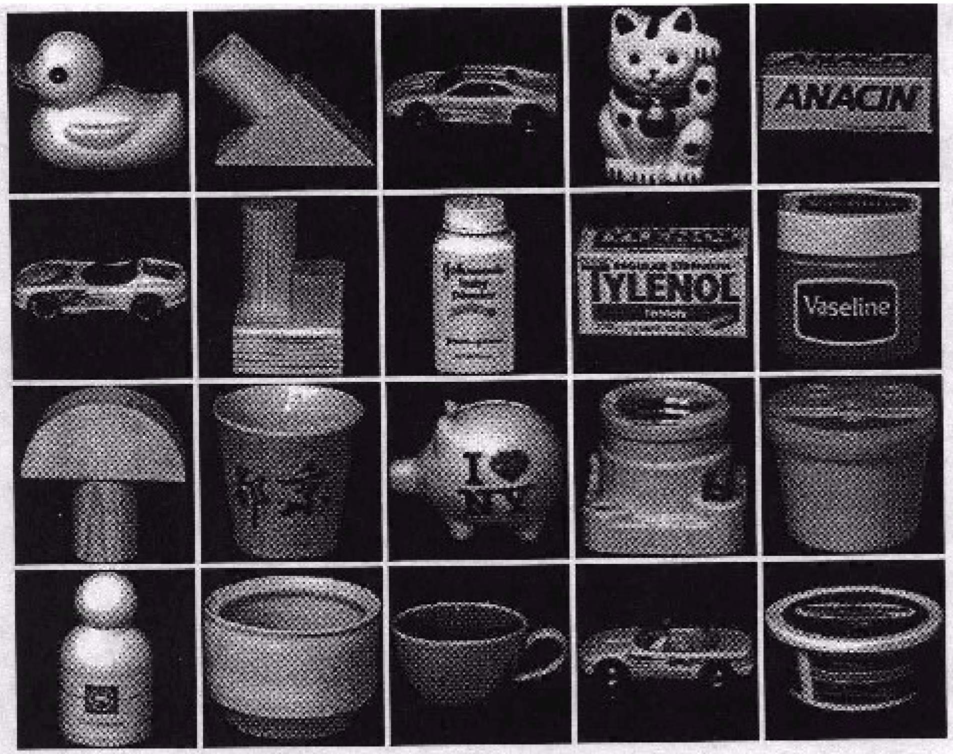
1*v1+1*v2+…+1*v_M=landa1*e1+...+landaM'*eM'+…+landaM*eM

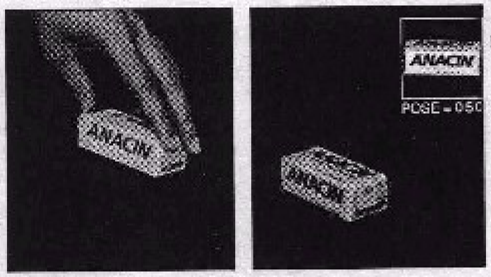
given any vi = w1*e1+w2*e2+…+wM'*eM'+…+w_M*e_M

Independent Vs. Orthogonal

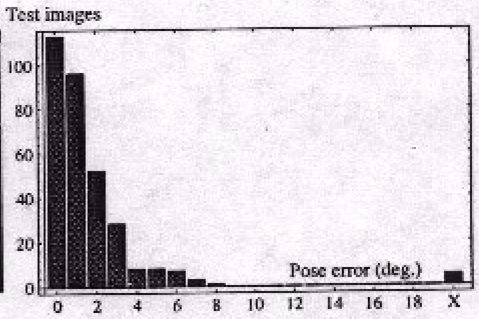

e1 ($\lambda_1 = 82.6$) $\geq$ e2 ($\lambda_2 = 54.4$) $\geq$ e3 ($\lambda_3 = 19.9$) $\geq$

e4 ($\lambda_4 = 14.5$) $\geq$ e5 ($\lambda_5 = 10.7$) $\geq$ e6 ($\lambda_6 = 9.4$)

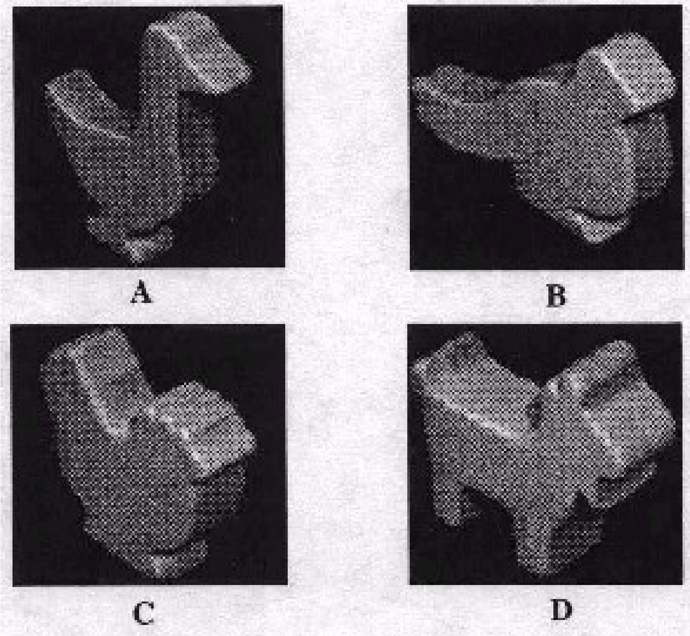

(a) Object set

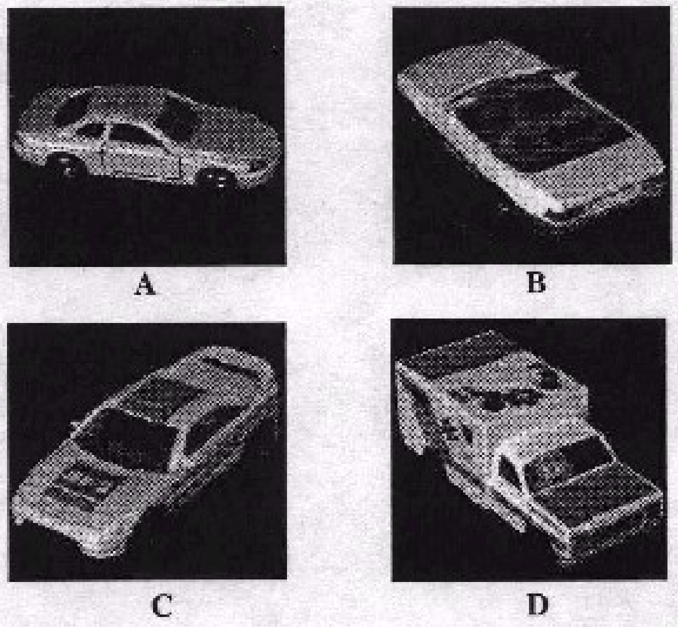


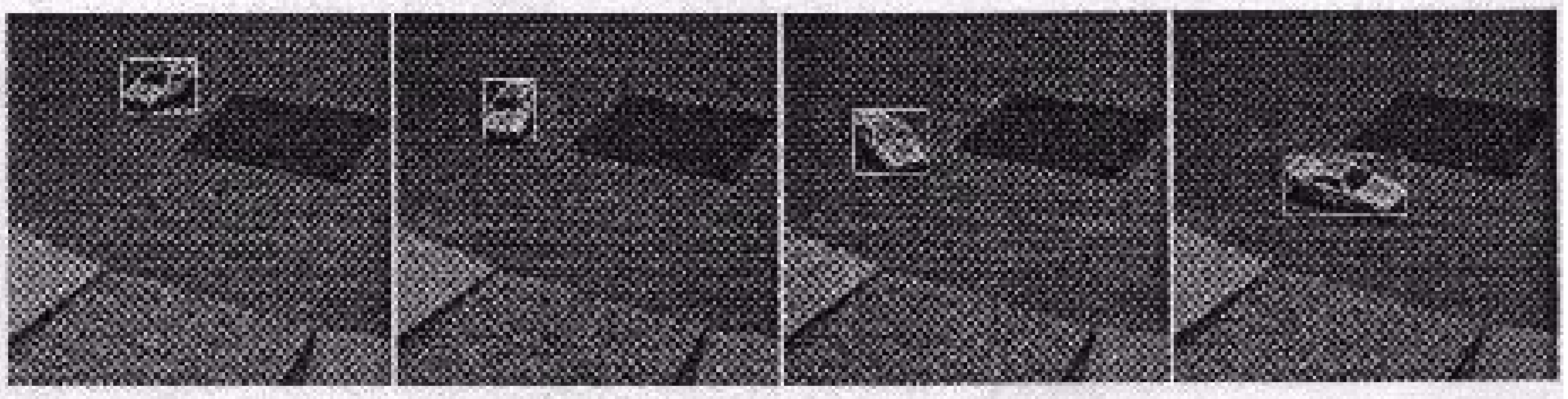
(b) Real-time recognition



(c) Pose estimation accuracy

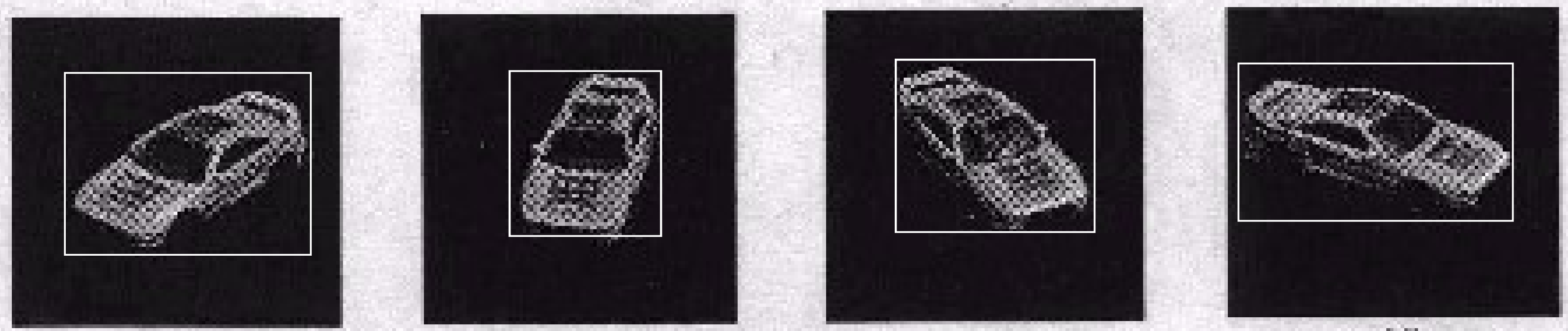

(a) Object Set 1


(b) Object Set 2



(1) (2) (3) (4)

(a) Automatic segmentation of the moving object.



(1) (2) (3) (4)

(b) Learning sample with closest pose.

PCA

1. Covariance Matrix A Vs. Correlation Matrix A'

2. $A=(A'-m)=UWV^T$

- Gaussian distribution N(m, $Lamda^2$),
- as affine transform (translation by m, rotation by U and V, scaling by U and W)

3. Affine transform –

   m: as the translation terms

   U: Eigenvector (orthonormal vector) matrix as the rotation matrix and scaling.

   W: Eigenvalues ($Lamda^2$) as the variance/scaling terms at denomination for normalization.

   V: Rotation matrix

Domain Knowledge

- **Input Data => PCA (Reduce the dimension => Input) =>**
  - => Similarity Measure (=> Machine Learning)
  - => Detection or Recognition Result

- **Domain Knowledge**

- **Find the best space => Manifold**
  - Lower-dimensional space
    - PCA, LDA
  - Higher-dimensional space
    - SVM, Kernel function

From Bayes' Theorem/Rule to a Posterior Probability

Machine learning

$$P(F \mid I) = \frac{P(I \mid F)P(F)}{P(I)}$$

$$= \frac{P(I \mid F)P(F)}{P(I \mid F)P(F) + P(I \mid \sim F)P(\sim F)}$$

$$P(A \cap B) = P(A \mid B)P(B)$$

$$= P(B \mid A)P(A)$$

Preprocssing

- **Geometric Normalization: Affine (Vs. Perspective)**
  - Translation
  - Rotation
  - Scaling
- **Illumination Normalization (Gray Value (0~255) Histogram) /Histogram Equalization**
  - Translation
  - Rotation
  - Scaling

$$(a-m_a)/lo\_a = (b-m_b)/lo\_b$$

- **Similarity Measure (Gaussian Model ?): Absolute Relation => Relative Relation**
  - Translation (shift): Zeroing
  - Rotation: ? (eigenvector ? Scale-Invariant Feature Transform (SIFT), Histogram Of Gradient (HOG) ?), Rotation invariance
  - Scaling: Normalization to the same unit (NT$ vs US$), Scaling invariance

- **Geometric Normalization: Affine (Vs. Perspective) => PCA**
  - Translation
  - Rotation
  - Scaling

- **3D model =>**
  - Intrinsic + Extrinsic
  - 3D-3D affine transfromation
  - Homogenous Matrix
  - Equations
  - 3D-2D perspective projection
  - Decay
  - 2D-2D Affine transformation

Similarity Measure

- **Similarity Measure: Detection / Recognition**
  - +/- Euclidean Distance => Absolute distance | |
  - +/- Gaussian Model => Relative distance => Btw 0.0 ~ 1.0 => Probability
  - +/- Gibbs Distribution/Boltzmann Distribution (kernel)
    P(x)=1/Z Exp(-U/kT)
  - +/- Sum of Squared Difference (SSD)
  - */ / Template Matching
  - ….
  - Norm ½ = $L^{1/2}$ = Euclidean Distance ?
  - Norm 1 = L^1 = SAD, Median Filter ?
  - Norm 2 = L^2 = SSD, Mean Filter (MSD) ? (covariance matrix?),
- **PCA is for dimension reduce, not for similarity measure**

Optimization

- **Affine, Perspective**
- **Ax = b**
  - Pseudo Inverse
  - SSD min E = min || Ax – b||^2
    - b: ground truth = observation
    - Ax=b', estimation or reconstruction
      - x unknown, A and b known
      - A, b, x all unknown => EM => Posterior Prob.
  - Lagrange
- **Ax = 0**
  - SVD of A

Local optimization (0 moment): 1 and 2
Global optimization: 3

Solution/Optimization of Homogenous Matrix

1. **Close Form Solution:**
   $Ax = 0$ => A^tA = Covariance Matrix => PCA => Smallest not-zero eigenvalue => eigenvector
2. **Pseudo Inverse:**
   $Ax = b$, $x=?$
3. **Sum of Squared Difference: (max liklihood – exponential term)**
   min $E = \sum [Ax - b]^2$

   3.1 $Ax = b'$: estimation value. b: ground truth, $E = \sum [b - b']^2$
   - a. Initial value estimation => Pseudo Inverse
   - b. L-M (non-linear approach)
     - b.1 First order Taylor series expansion
     - **b.2** 2nd order Taylor series expansion (sensitive to noise)

   3.2 $Ax = b'$: estimation value. b'': estimation value, $E = \sum [b'' - b']^2$
   - **a. EM (Expected-Maximization), initial b = average value**

Machine learning for prediction

4. **Lagrange Approach (outlier)**
   min $E = \sum [Ax - b]^2 + \lambda \ (x^2+y^2)^2$

- ❑ **Euclidean Distance**
- ❑ **Gaussian Model (Probability)**
  - ➢ m+-Lo (66%?), m+-2Lo (95%?), m+-3Lo (99%)
- ❑ **Gibb Distribution**
- ❑ **Gaussian Mixture Model (GMM) => Weighted GMM**
  - ➢ EM

- ❑ **Probability from Real Histogram (no model)**
  - ➢ Domain Transform => Histogram => GMM

- **Template Matching: Correlation Coefficient**
  - One image per person for few persons
- **PCA => PCA (Gaussian) + ICA (Non-Gaussian)**
  - One image per person for many persons
  - Consider relation between persons
- **LDA**
  - One image set per person for many persons
  - Consider relation within the same person and between different persons

- **LLE: Locally Linear Embedding**
- **SVM: Linear and Non-Linear Discriminant**

  - Subspace => Kernel Function to High Dimension

- ❑ **PCA: Between only**
  - ➢ One subject has one image
- ❑ **LDA: Within + Between**
  - ➢ One subject has several images

- **Similarity Measure:**
  - min SSD => min $(x-y)^2 = x^2 - 2xy + y^2$
    - » Not between 0.0~1.0 => Gaussian Model: exp(….), btw 0.0~1.0
  - max xy
    - » Not between 0.0~1.0 => Normailzed Correlation Coefficient, btw 0.0~1.0

- ❑ **Weakness: Fix, not flexible**

- ❑ **Sum-Square-Difference: Optical Flow (Matching)**

- ❑ **X1*X2: Correlation Matrix**

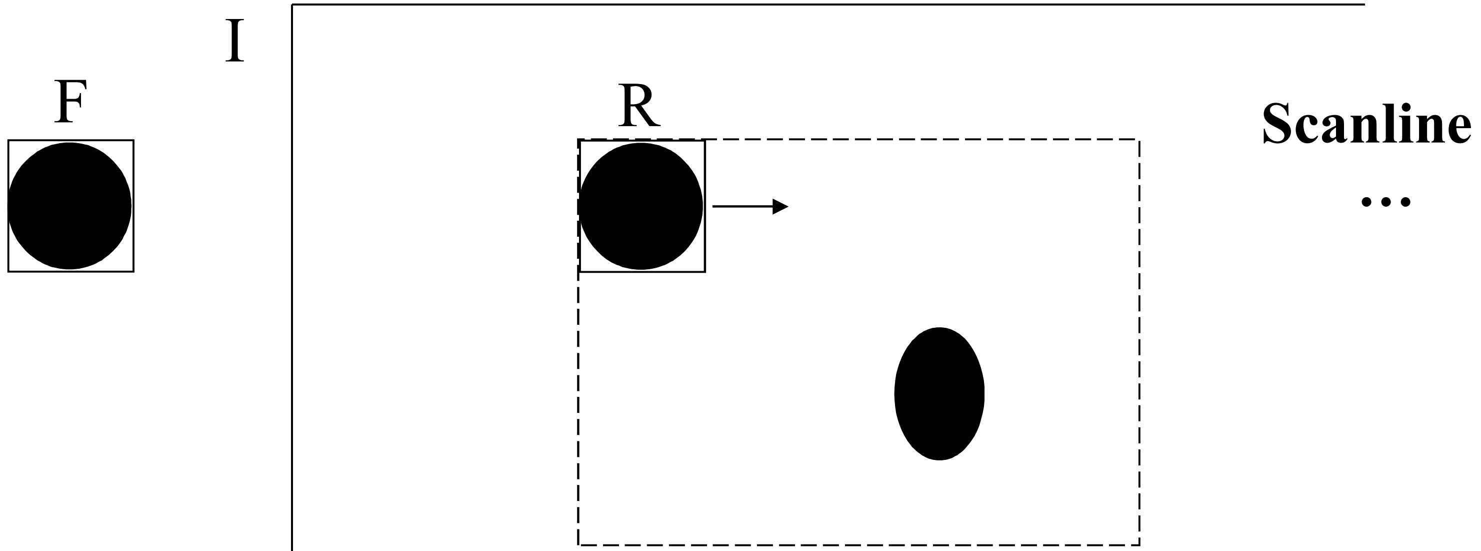
- ❑ **X1*X2 / (Lo_X1*Lo_X2): Correlation Matrix**
  - ➢ Normalization

- ❑ **(X1-mX1)*(X2-mX2): Covariance Matrix**
  - ➢ Shift

- ❑ **(X1-mX1)*(X2-mX2)/(Lo_X1*Lo_X2): Covariance Matrix**
  - ➢ Shift + Normalization

- **Strong correlation/covariance:**
  - Diagonal components of correlation/covariance matrix has brightest gray values

Template Matching: Correlation Coefficient



$$\rho(x) = \left| \frac{\sum_{r\in R}[F(r)-\overline{F}][I(x+r)-\overline{I}(x)]}{\sqrt{\sum_{r\in R}[F(r)-\overline{F}]^2}\sqrt{\sum_{r\in R}[I(x+r)-\overline{I}(x)]^2}} \right|$$

$(-1.0 \le \rho \le 1.0)$

$0.0 \le \rho \le 1.0$

Why minus $\overline{I}(x)$ and minus $\overline{F}$?

Gaussian Distribution Normalization: Shift

Why divided standard deviation ? $\frac{\#}{\sigma} = \frac{@}{1}$

NT$ Vs. US$

Principal Component Analysis

- **Principal Component Analysis = Hotelling Transform = Karhunen-Loeve (K-L) Transform (or Expansion)**
- **Principle components = Eigenvectors**
- **Properties: (Covariance matrix)**
  - Compression (not regular compression, it reduces the dimensions)
  - Correlation (self and cross correlations – Significant variance)

- **Compression:**
  - DCT, Wavelets… (ex. MPEG, H.264)
    - » Spatial domain
    - » Temporal domain
  - PCA
- **Decomposition/Factorization (SVD) Vs. 3D Reconstruction**

Computation of PCA

1) **Preprocessing: Normalization**
   - Geometric normalization (ex. Affine transformation) (Template)
   - Illumination normalization: ex. Histogram equalization => Purpose ?

2) **Let face images in the training set be** $\Gamma_1, \Gamma_2, \dots, \Gamma_M$
   - Represent the images as column vectors, e.g., an image Γ_i of size NxN becomes an N^2x1 column vector $\updownarrow$ or row vector $[\ ,\ ,\ ,\dots,\ ]^T$

3) **The average/mean face of the set is:** $$\Psi = \frac{1}{M}\sum_{n=1}^{M}\Gamma_n$$

   (an N^2x1 column vector)

4) **Each face differing from the average one is the vector:**

   (i.e., centralized image vector unbiased $\Phi_i = \Gamma_i - \Psi$

   i.e., deviations/variances from the mean face => variance matrix

   i.e., absolute difference => relative difference)

   (an N^2x1 column vector)

- **Variance matrix => SVD => SVD again means ?**
- **Covariance matrix => SVD => SVD again, means ?**
  - Correlation matrix
  - Inner Product => Orthogonal or not ?
    - Self Correlation
    - Cross Correlation

Feel enhancing ?

Γ_i Ψ $\Phi_i = \Gamma_i - \Psi$

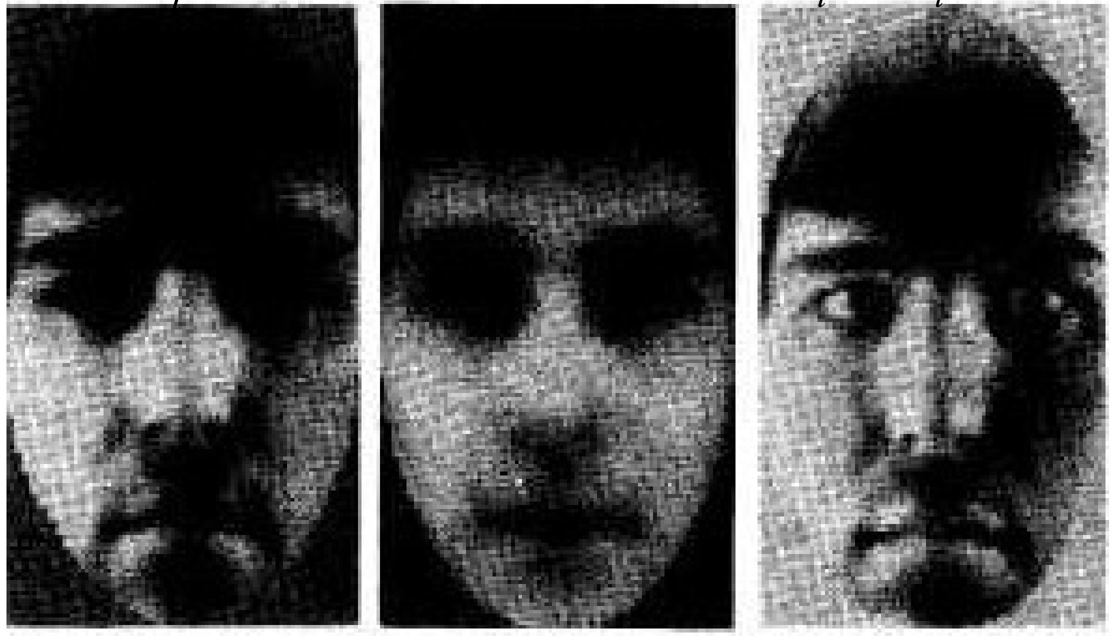
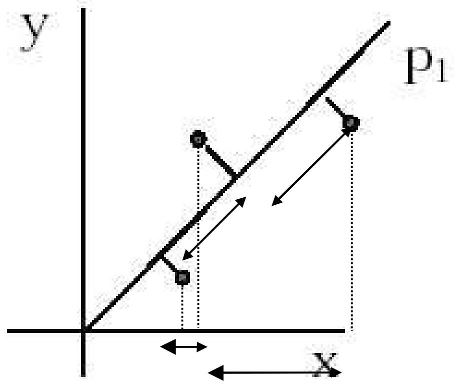


Plate 1. From left to right: sample face, average taken over extended ensemble, caricature of sample face.

- We are given:
  - a set of M objects" (images) ex.
  - each is represented, initially, by a set of N^2 features (128^2) or pixels
  - This data is organized as an (very large!) N^2xM matrix
- Let's look at a small example



projection

- points are 2-D points
- we find the axis that most closely passes through these points
- if the axis passed exactly through these points, then we would need only one coordinate to represent each point.

$$\Phi_i = \Gamma_i - \Psi \qquad \Psi = \frac{1}{M}\sum_{n=1}^{M}\Gamma_n$$

- PCA seeks the axis which the cloud of points are closest to
  - this is mathematically identical to find the axis on which the variance of the point projections is greatest (that is, on which the projections are most spread out). spreading out to be easily separated.)

    Vs. LDA (within+btw)
  - for high dimensional objects, like pictures, it is unlikely that there will be a single axis that passes close to all of the objects.
    - So, in this case, after we find the best axis (u_1), we then find the next best one (orthogonal to the first - u_2), and then the third best (u_3), etc.

      weight
    - Images are then represented by their projections on these axes: $w_i = \Phi_i \bullet u_i$. This is exactly analogous to the Fourier transform, with the u_i replacing the sinusoids.

Basis function

Why orthogonal ? => Liner combination

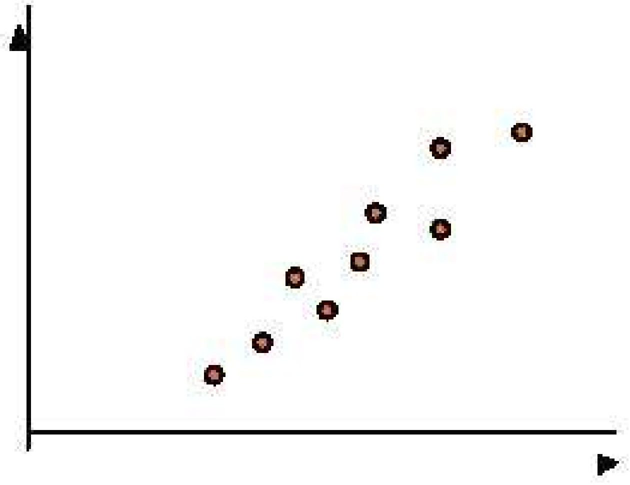
Eigenvectors, orthonormal vector

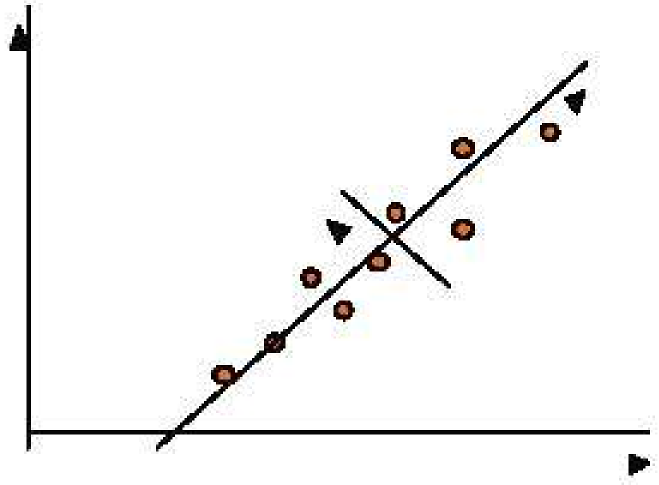
- Find an orthogonal set of basis vectors for the feature space, subject to the requirement that the new features have zero correlation with each other.

eigenvectors

- Dimensional reduction.

❑ **Affine**





.

- ❑ **Orthogonal is independent**
  - ➢ x+y: Linear combination or orthogonal … can be separated
  - ➢ x*y: correlation cannot be separated, => log x + log y
- ❑ **Independent does not need to be orthogonal (ICA)**
  - ➢ Why independent? => P(A,B) = P(A) P(B)

- ❑ **Orthonormal = orthogonal + unit vector (normalization) = eigenvectors**

- ❑ **PCA: Physical meaning, why eigenvector, eigenvalue**
- ❑ **Gaussian Model: Mean, variance**

- ❑ **SVD => Factorization**

From Covariance Matrix to Eigenvectors and Eigenvalues (see DIP book)

- The principle components are the eigenvectors of the covariance matrix
- Covariance matrix:

Pixel over all M training images

(Linear Combination => Matrix)

$$C = \frac{1}{M}\sum_{n=1}^{M}\Phi_n\Phi_n^T = AA^T$$

Note : this is an outer product

$$C = \frac{1}{M}\sum_{n=1}^{M}(\Gamma_n - \Psi)(\Gamma_n - \Psi)^T = \frac{1}{M}\left(\sum_{n=1}^{M}\Gamma_n\Gamma_n^T\right) - \Psi\Psi^T = C' - D = E[\Gamma^2] - E[\Gamma]^2$$

Covariance matrix

Correlation matrix

$$\Psi = \frac{1}{M}\sum_{n=1}^{M}\Gamma_n$$

Correlation matrix

$$A = [\Phi_1, \Phi_2, \ldots, \Phi_M]$$

A is a N^2xM matrix, so C is an N^2xN^2 huge symmetric matrix ! Wow !!

$= \boldsymbol{W} >= \mathbf{0}$

By the way,

Square term

Singular ?

C is positive semi-definite: $uCu^T \geq 0 \quad for\ \ all\ \ u \neq 0$

Non-invertible

So the eigenvalues of a positive semi-definite matrix are real and non-negative.

ex. Covariance matrix

Square term **C, W: Covariance matrix**

?

1. **Ax=b, x=A^{-1}b, but if A is non-invertible,**
   - Pseudo inverse
2. **SSD**

- **U I V^T or U I U^T**

1*v1+1*v2+…+1*v_M=landa1*e1+landa2*e2+…+landa_M*e_M

vi = w1*e1+w2*e2+w3*e3+…+w_M'*e_M'+…+ w_M*e_M

- Eigenvalues are numbers λ such that $Cu = \lambda u$ for some vectors $u \neq 0$.
  (A u = Lamda v)
- Eigenvectors are the vectors u such that $Cu = \lambda u$. They are orthogonal to each other.
- For general matrices, the eigenvalues are complex numbers.
- The eigenvalues of a positive semi-definite matrix are real and non-negative.

  In reality, for example, by using book - numerical recipes, negative value happens - error.

$CU=WU$ $\quad$ $C=U^TWU$

$U^TCU=W$ $\quad$ $landa^2$

N^2-d eigenface

$$u = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N^2} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1N^2} \\ e_{21} & e_{22} & \cdots & e_{2N^2} \\ \vdots & \vdots & \cdots & \vdots \\ e_{N^21} & e_{N^22} & \cdots & e_{N^2N^2} \end{bmatrix} \quad 0$$

Actually only M objects not N^2

- The eigenvectors are the principal components.
- Each eigenvector can be determined up to a scaling factor, so usually they are normalized to unit length:

$$u = \frac{u'}{\|u'\|}$$

vector = direction (unit vector) + magnitude (scale)

- When stacked in a matrix, the eigenvectors form the Karhunen-Loeve transform matrix u with the property $C = UWV^T$ where W is a diagonal matrix, whose diagonal entries are the eigen-values of C. U orthonormal.

$$u_l^T u_k = e_l^T e_k = \delta_{lk} = \begin{cases} 1, & if \quad l=k \\ 0, & otherwise \end{cases}$$

Orthonormal ?

Correlation/Covariance Matrix

- $C = AA^T = UWU^T$, W => Lamda^2
- $A = UWV^T$, W => Lamda =?

$$u = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ \hdashline e_{N^2} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1N^2} \\ e_{21} & e_{22} & \cdots & e_{2N^2} \\ \vdots & \vdots & \cdots & \vdots \\ \hdashline e_{N^2 1} & e_{N^2 2} & \cdots & e_{N^2 N^2} \end{bmatrix}$$

0 Actually only M objects not N^2

Let $\lambda = \begin{bmatrix} 0 \\ \lambda_i \\ \vdots \\ 0 \end{bmatrix} = u(\Gamma_i - \Psi)$

Average variance

$$\Psi_\lambda = \frac{1}{N^2}\sum_{i=1}^{N^2} u(\Gamma_i - \Psi) = u\left(\frac{1}{N^2}\sum_{i=1}^{N^2}\Gamma_i - \frac{1}{N^2}N^2\Psi\right) \overset{??}{=} 0$$

Actually only M objects not N^2

$$C_\lambda = \frac{1}{N^2}\sum_{i=1}^{N^2}(\lambda_i - \Psi_\lambda)(\lambda_i - \Psi_\lambda)^T = \frac{1}{N^2}\sum_{i=1}^{N^2}\lambda_i\lambda_i^T \boxed{-\Psi_\lambda\Psi_\lambda^T} = E[\lambda^2] - (E[\lambda])^2$$

$$= \frac{1}{N^2}\sum_{i=1}^{N^2}\lambda_i\lambda_i^T = \frac{1}{N^2}\sum_{i=1}^{N^2}u(\Gamma_i - \Psi)\cdot[u(\Gamma_i - \Psi)]^T$$

$$= \frac{1}{N^2}\sum_{i=1}^{N^2}u(\Gamma_i - \Psi)(\Gamma_i - \Psi)^T u^T = u\left(\frac{1}{N^2}\sum_{i=1}^{N^2}(\Gamma_i - \Psi)(\Gamma_i - \Psi)^T\right)u^T$$

$$= uCu^T (= W) \geq 0$$

( )²>=0

Square term

DIP book

$$C = uC_\lambda u^T = UWV^T$$

$$C_\lambda = \begin{bmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_{N^2}^2 \end{bmatrix}$$

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M \geq \underbrace{\lambda_{M+1} \geq \ldots \geq \lambda_{N^2}}_{\mathbf{=0}}$$

Solving for Eigenvectors and Eigenvalues: Method I:

- Solve for the roots of the characteristic polynomial:

$Cu = \lambda u$ $\quad\quad$ $Cu - \lambda u = 0$

$(C - \lambda I)u = 0$ $\quad\quad$ $u \neq 0$

By Cramer's theorem, this has non - trivial solution

iff $\det(C - \lambda I) = 0$

If **C** is 2x2 :

$$\begin{bmatrix} C_{11} - \lambda & C_{12} \\ C_{21} & C_{22} - \lambda \end{bmatrix} = 0$$

$$(c_{11} - \lambda)(c_{22} - \lambda) - c_{12}c_{21} = 0$$

$$\lambda^2 - (c_{11} + c_{22})\lambda + c_{11}c_{22} - c_{12}c_{21} = 0$$

An Example

- Consider 4 points in 2D: (0,1), (2,2), (4,6), (6,7)
- Mean values: E[x1]=3, E[x2]=4
- Centered data: (-3,-3),(-1,-2), (1,2), (3,3)
- Covariance matrix C :

1=x, 2=y

$C_{11} = (9+1+1+9)/4 = 5$ $\quad$ $C_{12} = (9+2+2+9)/4 = 5.5$

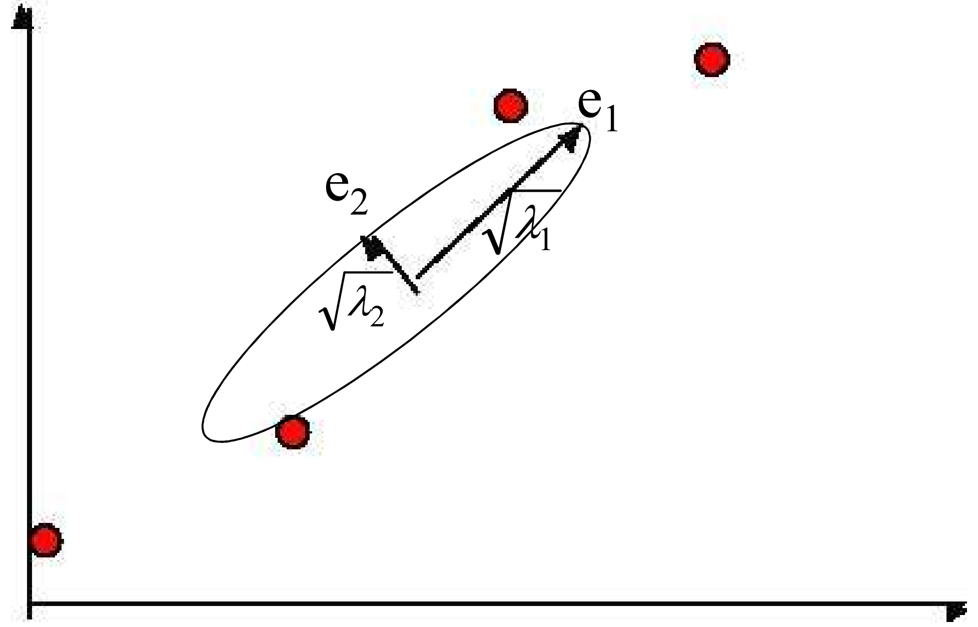
$C_{21} = (9+2+2+9)/4 = 5.5$ $\quad$ $C_{22} = (9+4+4+9)/4 = 6.5$

- Characteristic polynomial:

$$\lambda^2 - \underbrace{(5+6.5)}_{11.5}\lambda + \underbrace{5(6.5) - 5.5(5.5)}_{2.25} = 0$$

- Eigenvalues: $\lambda_{1,2} = (11.5 +/- \sqrt{123.25})/2 = (11.3, 0.2)$
- Eigenvectors: (0.75, 0.66), (-0.66,0.75)

Put in Lamda1 and Lamda2



- ❑ **Uncertainty**
- ❑ **Hessian Matrix**

Method II: Numerical Computation: Singular Value Decomposition (SVD)

- A fundamental problem in science and engineering
- Singular Value Decomposition (SVD) is now the preferred method to do eigen-analysis.
- That is, there is an easier way to do PCA in using Singular Value Decomposition:

Book 'Numerical Recipes'

- $[U\ \mathbf{W}\ V^T] = \text{svd}(C)$
- U is the eigenvector arrranged in descending eigenvalues. **W** contains a diagonal matrix with descending eigenvalues.

$$U = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ \hdashline \\ e_{N^2} \end{bmatrix} = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1N^2} \\ e_{21} & e_{22} & \cdots & e_{2N^2} \\ \vdots & \vdots & \cdots & \vdots \\ \hdashline \\ e_{N^2 1} & e_{N^2 2} & \cdots & e_{N^2 N^2} \end{bmatrix} \; 0$$

$$W = \begin{bmatrix} \lambda_1 & & \multicolumn{1}{c:}{} & 0 \\ & \lambda_2 & \multicolumn{1}{c:}{} & \\ & & \multicolumn{1}{c:}{\ddots} & \\ 0 & & \multicolumn{1}{c:}{} & \lambda_{N^2} \end{bmatrix}$$

$$\qquad\qquad 0$$

Solving for Eigenvectors and Eigenvalues: Method for Huge Covariance Matrix

$$C = \frac{1}{M}\sum_{n=1}^{M}\Phi_n\Phi_n^T = AA^T$$

$\Phi, u : N^2 x1$ *column vector* $\Phi_i = \Gamma_i - \Psi$

$A : N^2 xM$ *matrix* $A = [\Phi_1, \Phi_2, \ldots, \Phi_M]$

$A^T : M \; x \; N^2$ *matrix*

$$Cu = \lambda u$$

$C = AA^T, \lambda : N^2 x \; N^2$ matrix pixel base

$A^T A, v : M \; x \; M$ matrix image base

Image base $\underline{A^T} Av_i = \mu_i v_i$

Pixel base $AA^T \underline{Av_i} = \mu_i \underline{Av_i}$

M' < M << N²,

ex. M'=7, M=16, N=128

Image base $Eigenfaces \; \boxed{u_l} = \sum_{k=1}^{M} \boxed{v_{lk}}\Phi_k = Av_l$ *where* $l = 1,2,\ldots,M$

Physical Meaning:
Phi_1 take/enhance nose by * v_1 +…+
Phi_M take/enhance eye by * v_M
= U_1

For each ul like extracting eye, it consists of vl over all unbiased faces

Eigenvalue/ weight $\omega_k = u_k^T(\Gamma - \Psi) \;\; for \; k = 1,\ldots,M',\ldots,M$

Dimensional Reduction

- Given

$$C = \begin{bmatrix} 59.14 & -1.55 & 4.56 \\ -1.55 & 0.09 & -0.11 \\ 4.56 & -0.11 & 27.02 \end{bmatrix} \qquad \det(C - \lambda I) = 0$$

The resulting eigenvectors and eigenvalues are

$$\mathrm{u}_1 = \begin{bmatrix} -0.99 \\ 0.03 \\ -0.14 \end{bmatrix}, \lambda_1 = 59.82 \quad \mathrm{u}_2 = \begin{bmatrix} 0.14 \\ -0.00 \\ -0.99 \end{bmatrix}, \lambda_2 = 26.39 \quad \mathrm{u}_3 = \begin{bmatrix} 0.03 \\ 1.00 \\ -0.00 \end{bmatrix}, \lambda_3 = 0.04$$

Note $\frac{\lambda_3}{\sum \lambda_i} < 0.001,$ what is the implication?

- We can throw u_3 away, and keep w=[u_1 u_2].
- You may find the different objects/textures forming nice clusters in this 2D space.
- But there is an easier way to do PCA in using Singular Value Decomposition:

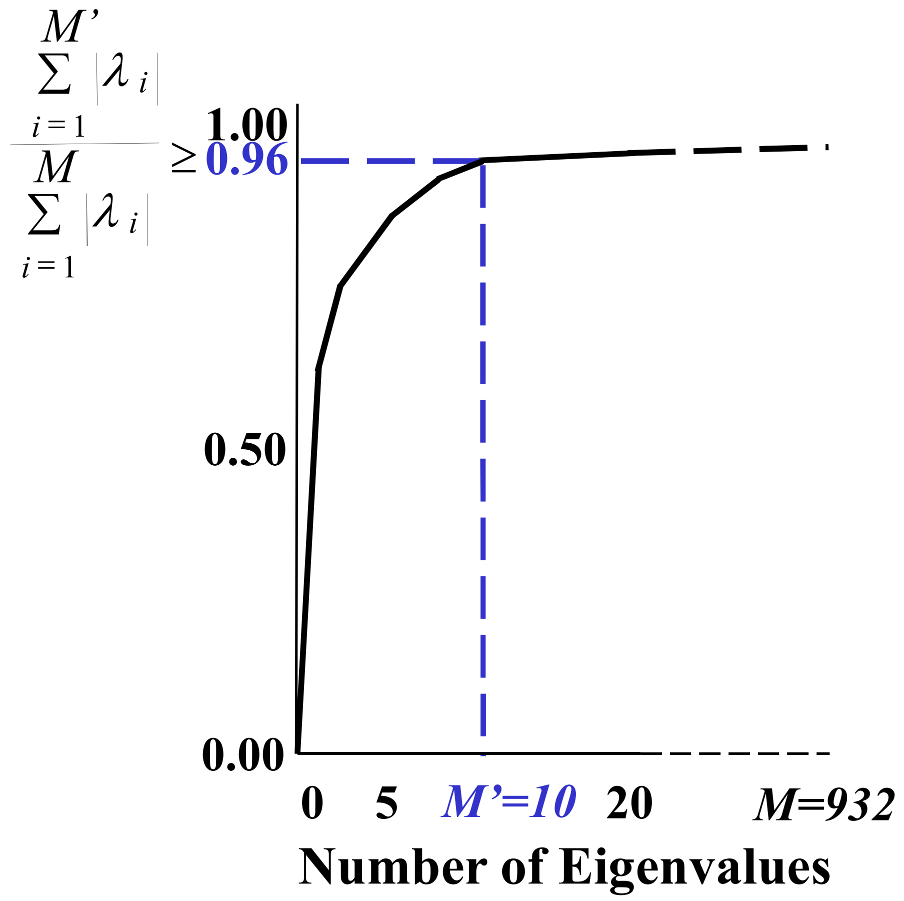
Book 'Numerical Recipes'

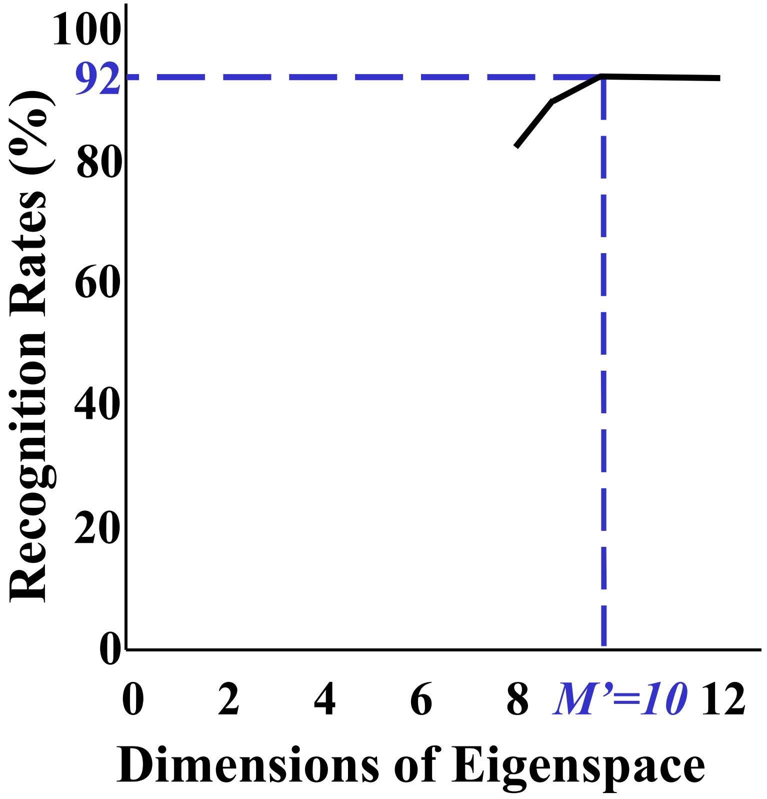
- $[U\ W\ V^T]$=svd(C)
- U is the eigenvector arrranged in descending eigenvalues, W contains a diagonal matrix with descending eigenvalues.

- K-L transform: first basis -- data projected on which will have the maximum variance, the second basis captures the second maximum variance that is orthogonal to the first one. Once the variances are captured by some bases, the subsequent bases can be discarded.
- This helps us to achieve dimensionality reduction.

- Given your gallery of images
  - compute its principal components eigenvector u
    - this is just a set of other images that are used as a basis for representing the images in the gallery
  - determine an $M' << M$ such that the first M' principal components u are a "good" representation for the gallery
    - can be chosen based on the scores (eigenvalues) computed by the PCA
  - represent each image Γ in the gallery by its projection on these M' principal components $\longrightarrow$ Eigenvalue or weight
    - this is just the dot product of the image and the principal components. $\omega_k = u_k^T(\Gamma - \Psi) \;\; for\; k = 1, \ldots, M', \ldots, M$
    - each image now represented by M' numbers

$$\Omega^T = [\omega_1, \omega_2 \ldots . \omega_{M'}]$$





$$\lambda_1 \geq .. \geq \lambda_{M'} \geq .. \geq \lambda_M$$

$$\omega_k = u_k^T(\Gamma - \Psi) \;\; \textit{for}\; k = 1,\ldots,M',\ldots,M$$

Reconstruction

- If we compute M principal axes, then we can reconstruct any image exactly from its principal components representation:

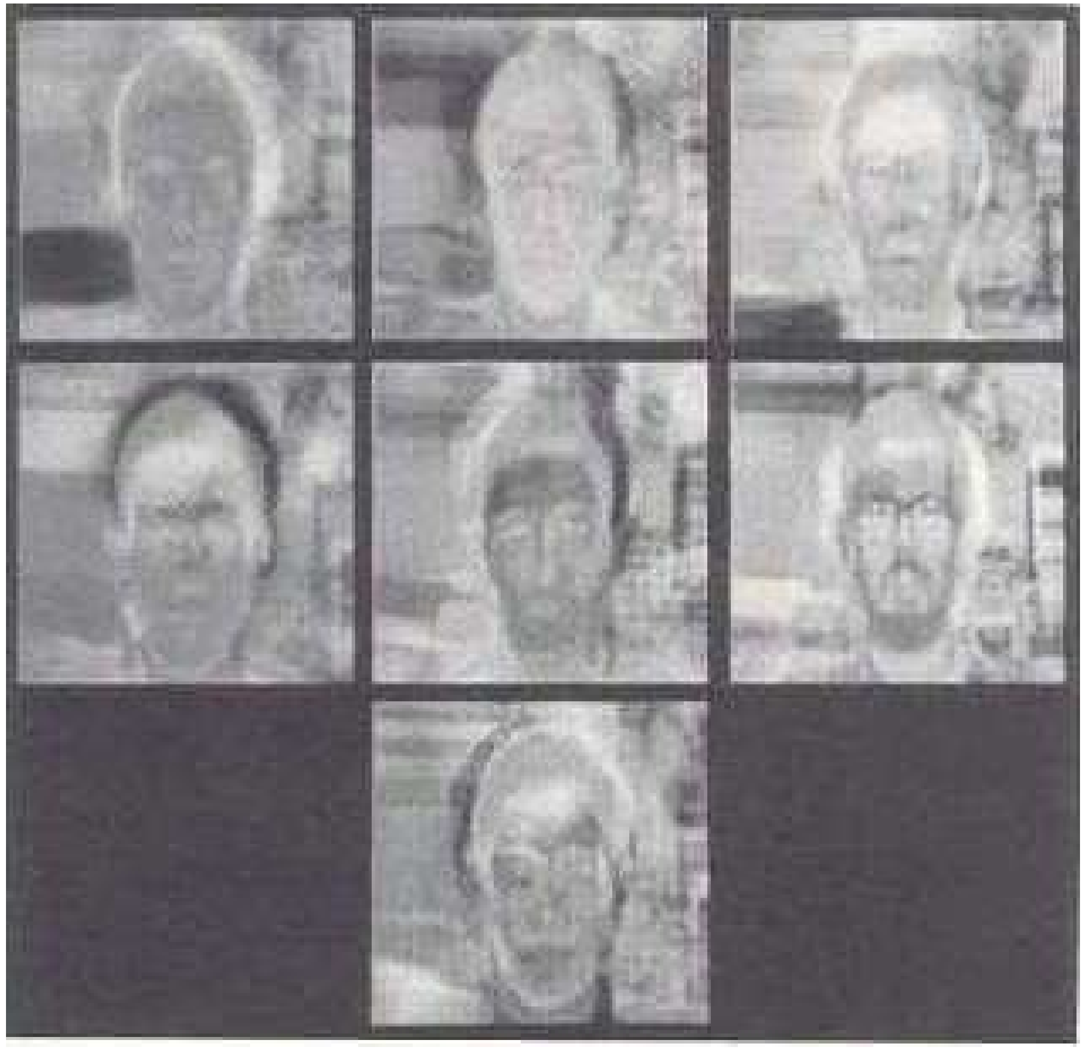
$$\Phi_f = \sum_{i=1}^{M} \omega_i u_i \qquad \Gamma_f = \Phi_f + \Psi$$

- This is just another basis for the M-vector that represents the image
  - the original basis is the natural one - (1,0,0,...0), (0,1,0,...) ...
  - the principal axes represent just a rotation of the original high dimensional coordinate system

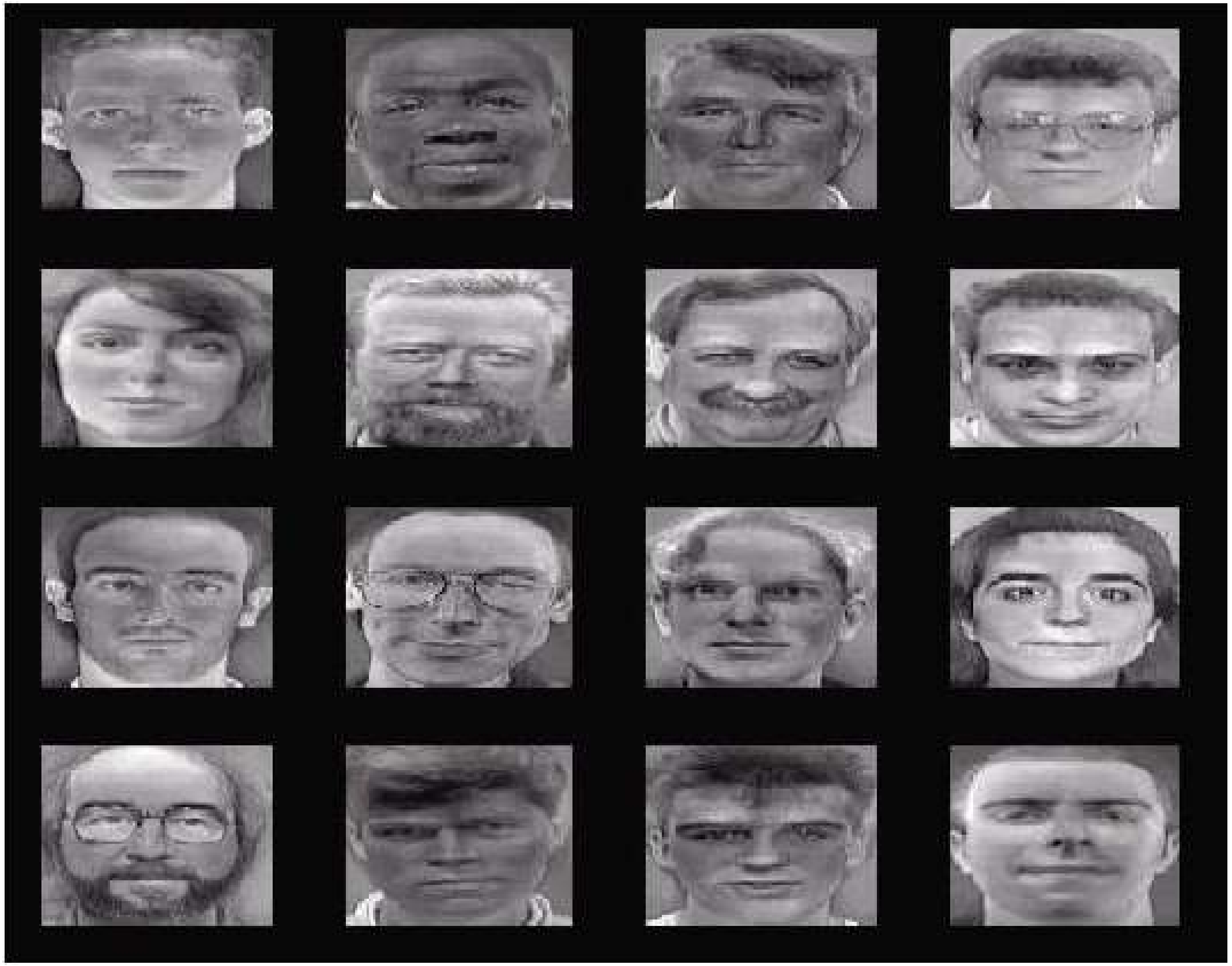
See lecture: Camera model example => World coordinate Vs. Camera Coordinate

- However, we don't need to use all of the principal axes to obtain good reconstructions of the image.
- The mathematical procedure that determines the principal axes uses an eigenvector analysis, and associates a "score" with each axis $\leftarrow$ eigenvalue
  - these scores correspond to the amount of variation in the image set that the axis corresponds to and are the eigenvalues of the procedure
  - The scores generally go to zero "quickly". For a face database, we can generally reconstruct a 512x512 face using only 80-100 principal axes with very small error.

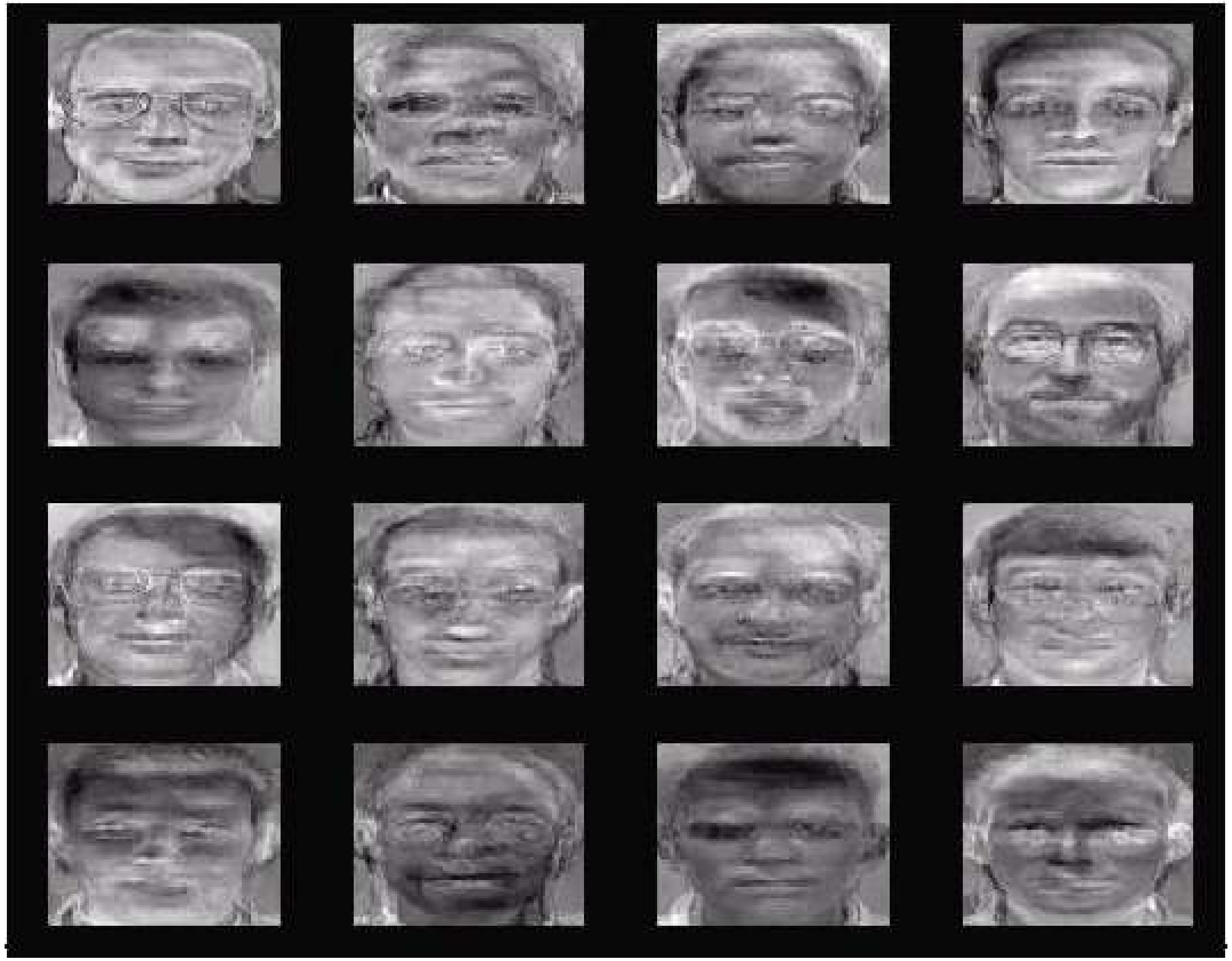
$$\Phi_f = \sum_{i=1}^{M'} \omega_i u_i \qquad \Gamma_f = \Phi_f + \Psi$$



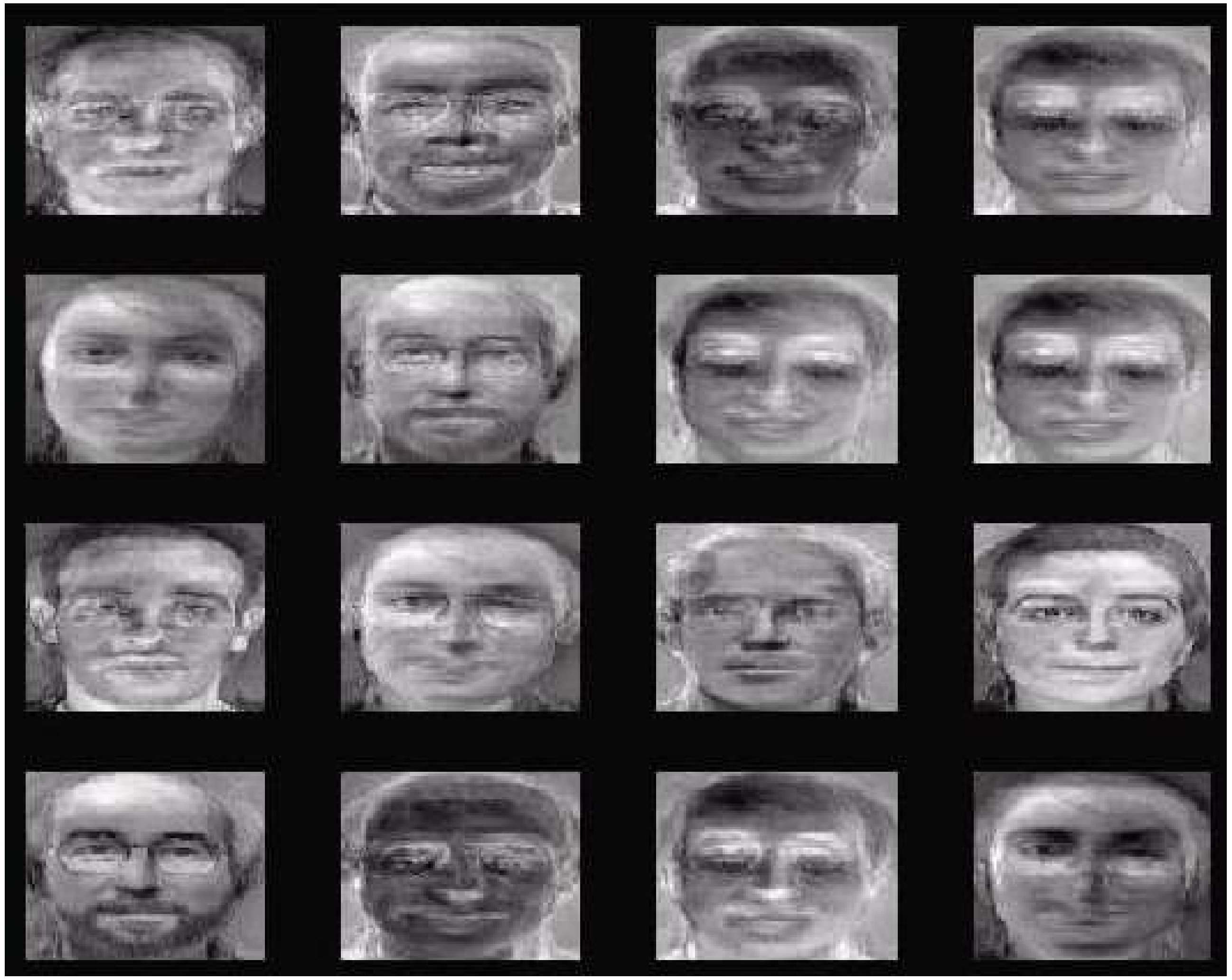
Original 16 Faces

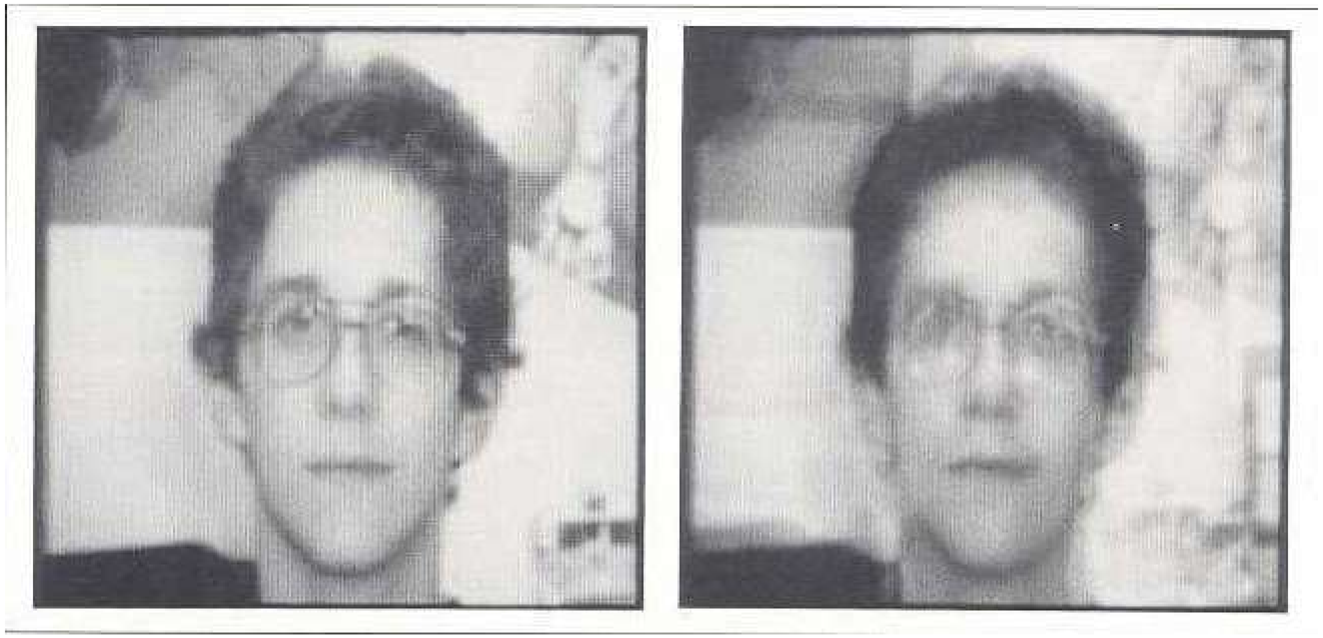


16 Eigenfaces (Pick 7 Largest)

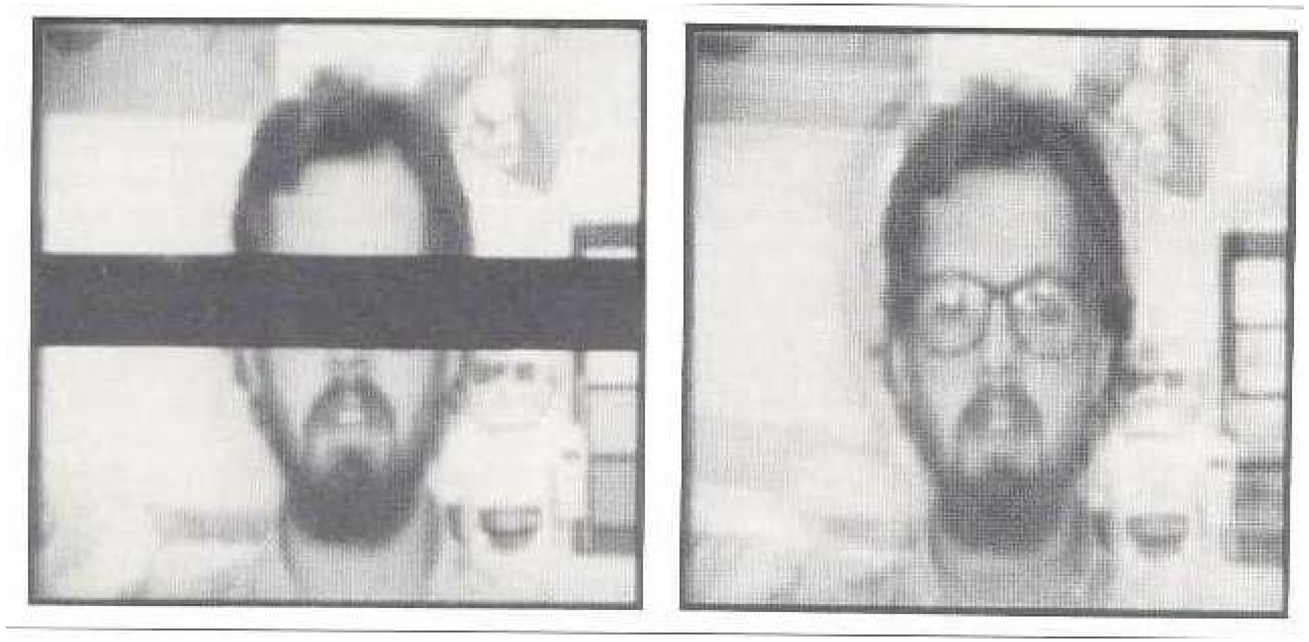


Reconstruction Based on 7 PCA





Lost details



Recognition

- Given an unknown image
  - compute its projection onto the principal component basis
    - this is a set of M'numbers representing the unknown image $\Omega^T = [\omega_1, \omega_2 .... \omega_{M'}]$
  - compare this M'-tuple against each of the database image M'-tuples
    - simple L^2 norm $\| \quad \|^2$
    - sometimes each component is weighted by the associated eigenvalue **~~Lamda~~**

J: ?? w_i is normalized by dividing corresponding eigenvalue Lamda_i

1. Face Detection: Distance From Face Space

$$\varepsilon^2 = \left\| \Phi - \Phi_f \right\|^2 \qquad \Phi_f = \sum_{i=1}^{M'} \omega_i u_i$$

input face

Euclidean Distance

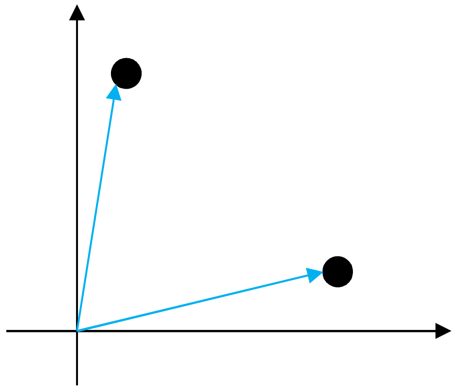
2. Face Classification

$$\Omega^T = [\omega_1, \omega_2 \dots . \omega_{M'}]$$

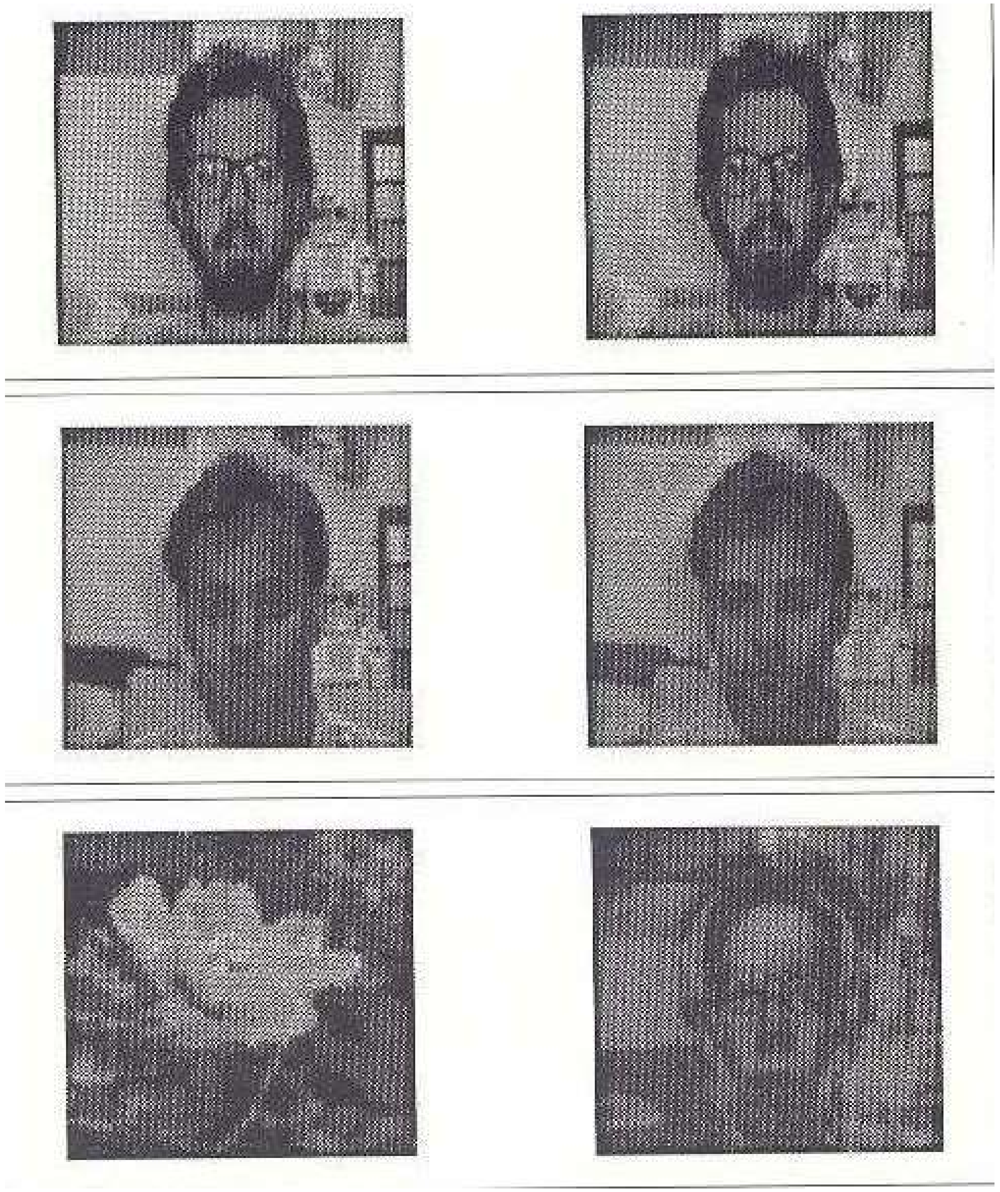
Minimize the Euclidian distance

$$\varepsilon_k = \left\| \Omega - \Omega_k \right\|^2 \quad 1 \le k \le M$$

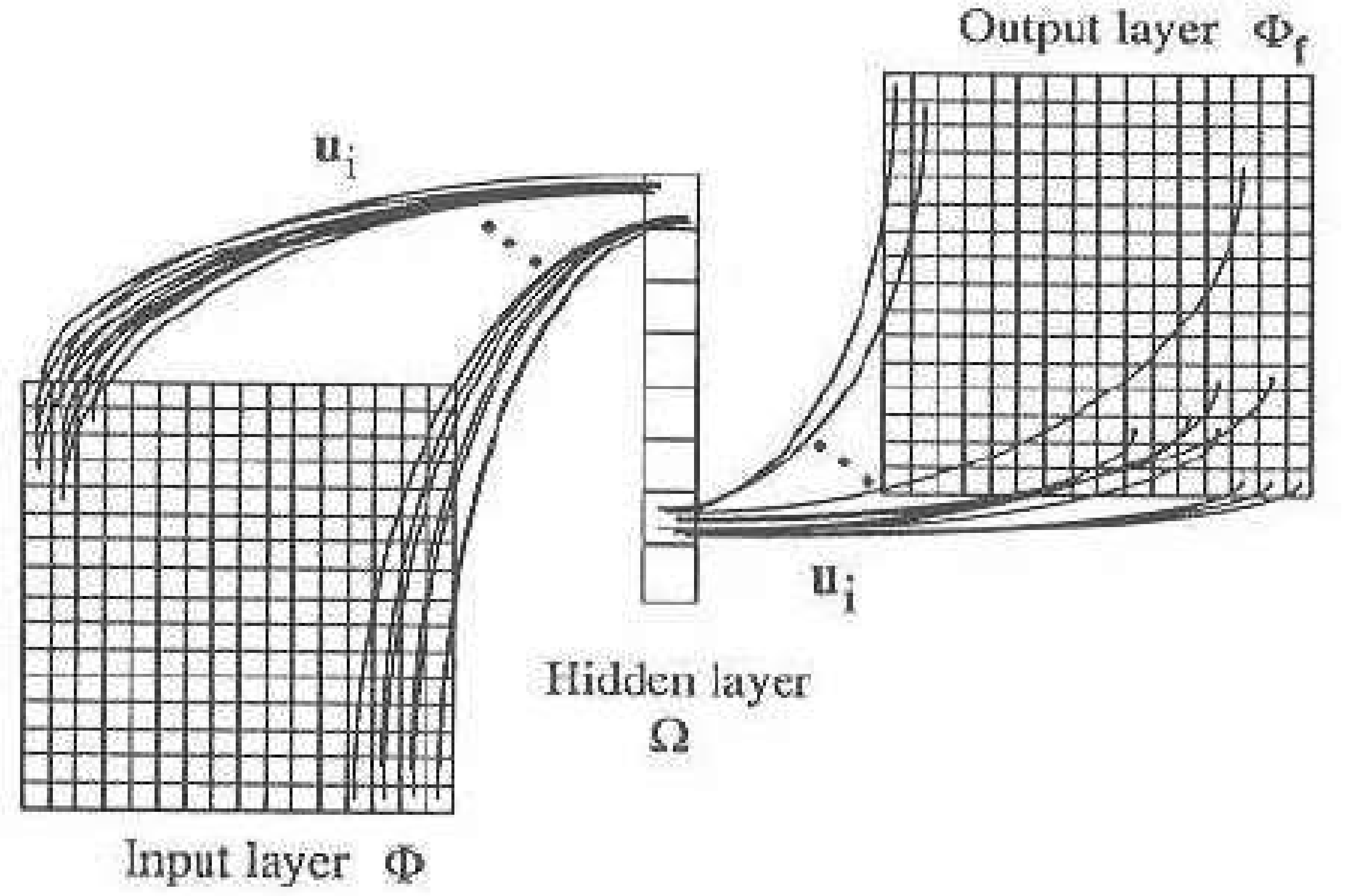
Novel face Known face class k

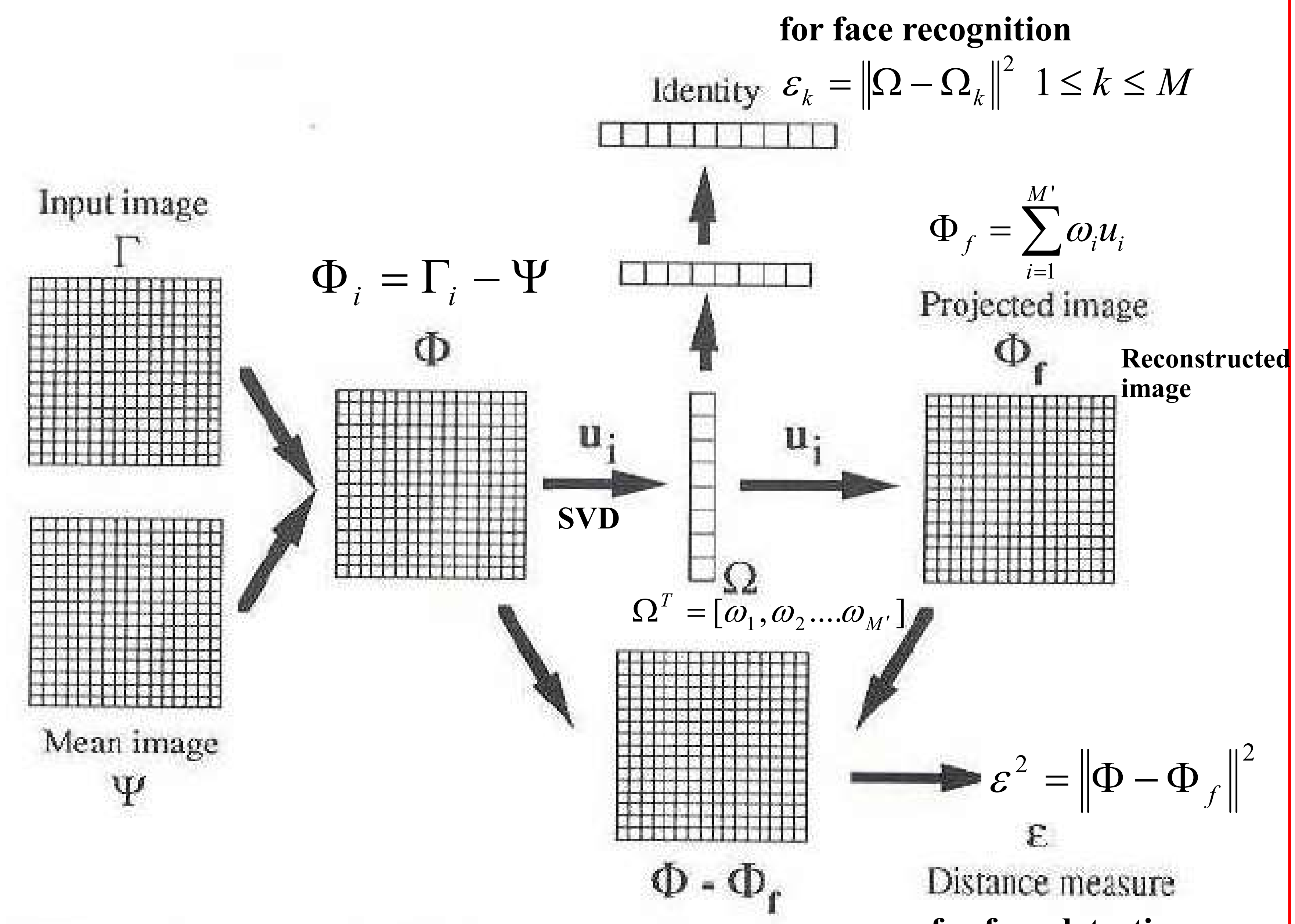

Same distance (magnitude), but different samples or vectors (directions)

Novel Input Image $\Rightarrow$ Reconstruction

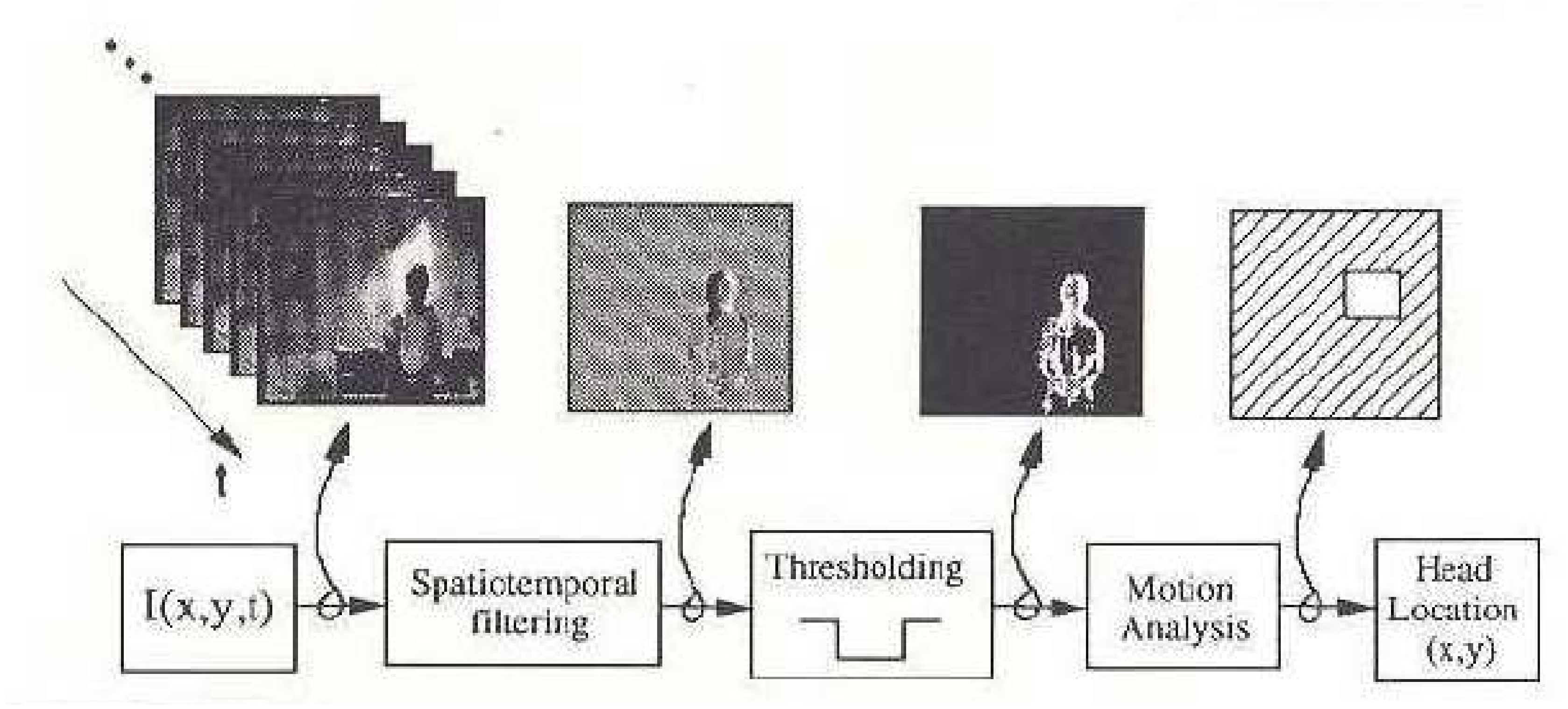


PCA by Neural Networks




for face recognition
Identity
$\varepsilon_k = \|\Omega - \Omega_k\|^2 \quad 1 \le k \le M$
Input image
Γ
$\Phi_i = \Gamma_i - \Psi$
Φ
$\Phi_f = \sum_{i=1}^{M'} \omega_i u_i$
Projected image
Φ_f
Reconstructed image
u_i
SVD
u_i
Ω
$\Omega^T = [\omega_1, \omega_2 \dots \omega_{M'}]$
Mean image
Ψ
$\varepsilon^2 = \|\Phi - \Phi_f\|^2$
ε
$\Phi - \Phi_f$
Distance measure
for face detection

Head Tracking and Locating System



Discussion

- **PCA Properties:**
  - Compression
  - Correlation
- **Normalization**
  - Histogram equalization
  - Geometric normalization

Normalize training data or testing data ?

- **Recognition Rate: Lighting Variation (96%) > (In-Plane) Orientation Variation (85%) > Scale Variation (64%) (out-of-plane rotation ??)**
  - Under lighting changes alone the neighborhood pixel correlation remains high.
  - Under size changes, the correlation from one image to another is largely lost

Same Class

If Having New Data to PCA?

Other (Face) Recognition Methods

- **Linear Discriminant Analysis (LDA)**
  - Fisher's Discriminant, Within + Between, one subject has many images

    PCA => linear discriminant, but considering between only, one subject has only one image
- **Non-Linear Discriminant**
  - Support Vector Machine: Kernel Function (to high dimensional space ?)

Applications

- Face recognition -- the eigenface method by Pentland and Turk .
  - Collect images of human faces in the same pose and label them with the name of the person;
  - find the first 7 principle components of the images -- resemble faces, called eigenfaces.
  - For each image, store the expansion coefficients for the corresponding eigenfaces, and when an unknown face arrives, find its expansion in terms of eigenfaces and assign to it the name of the closest image in eigenspace.

$$\Omega^T = [\omega_1, \omega_2 \ldots . \omega_{M'}]$$

Face Detection -

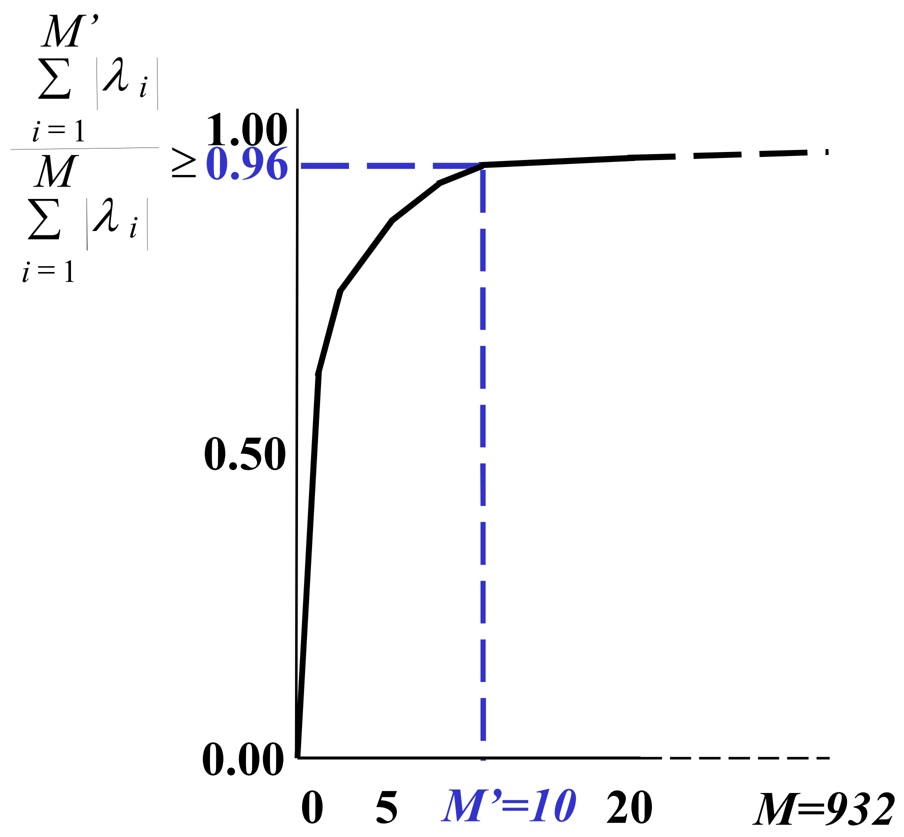
Principal Components

- Image compression: Use only main PCs and their coefficients in representation. Gabor wavelets are the main PC of natural images . Strategy used in Face detection algorithm of Schneiderman and Kanade.

❑Facial expression recognition

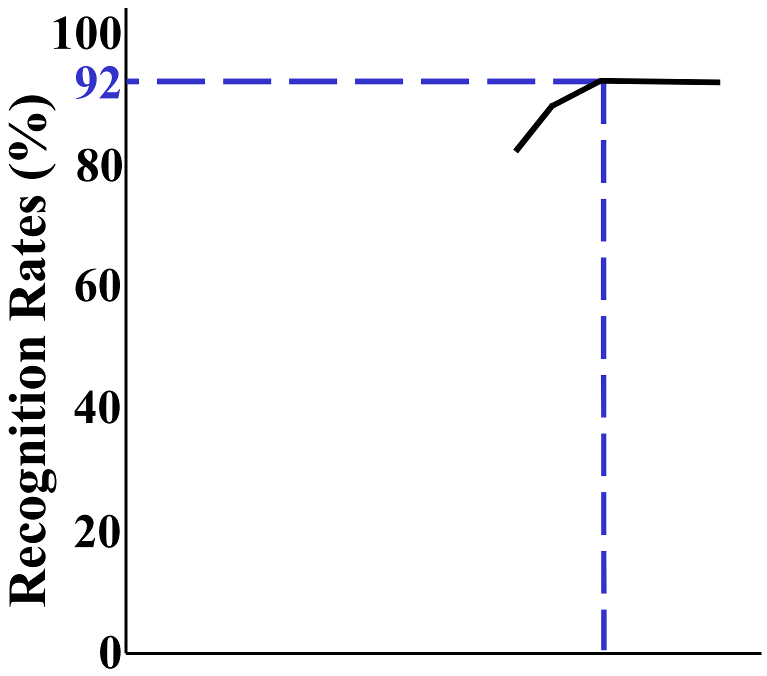
- Use flow base, not gray-value base, to ignore differences across individual subjects
- PCA properties:
  - Compression
  - Correlation

Computation of (horizontal) eigenflow number (upper facial expression)



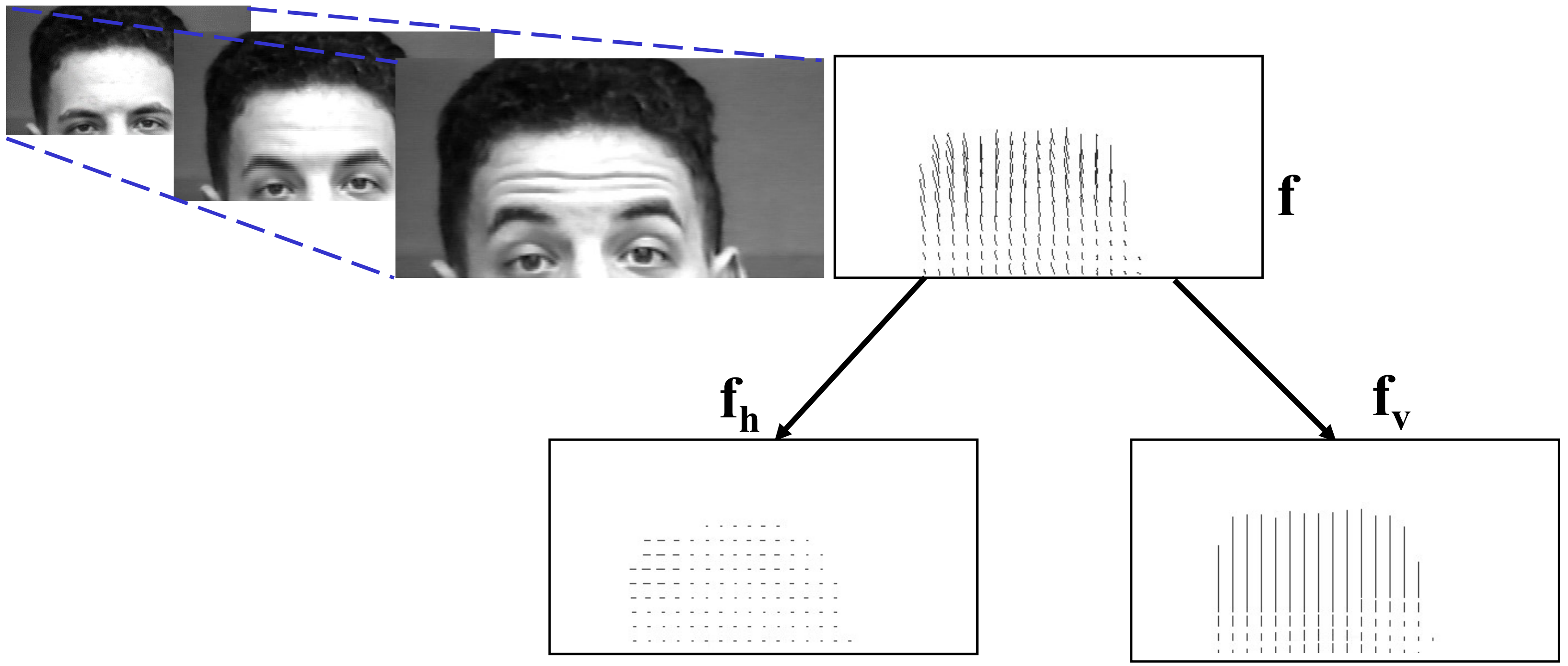
Number of Eigenvalues

$$\lambda_1 \geq .. \geq \lambda_{M'} \geq .. \geq \lambda_M$$



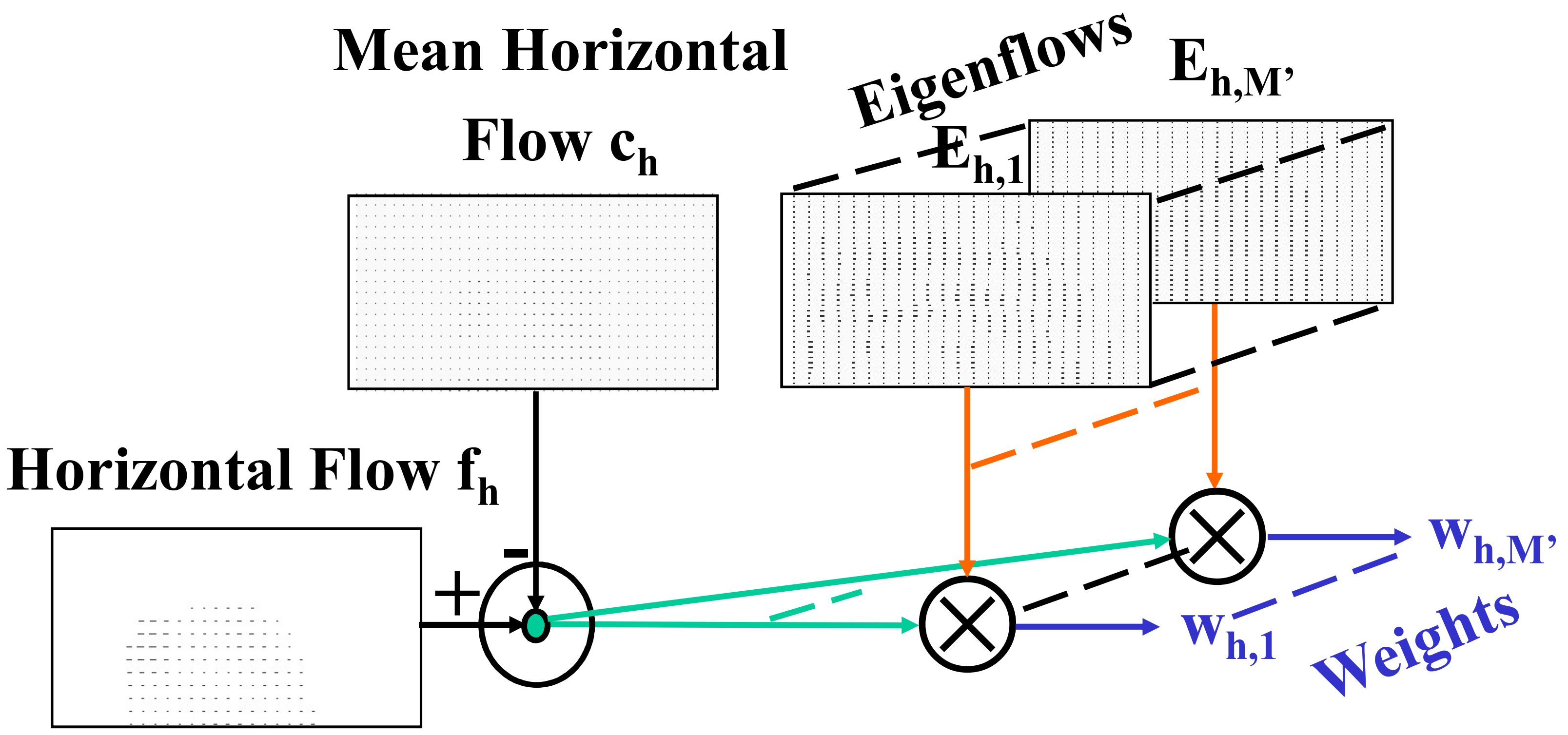
Dimensions of Eigenspace

Input flow image



Weight vector for horizontal flow components

(upper facial expression)

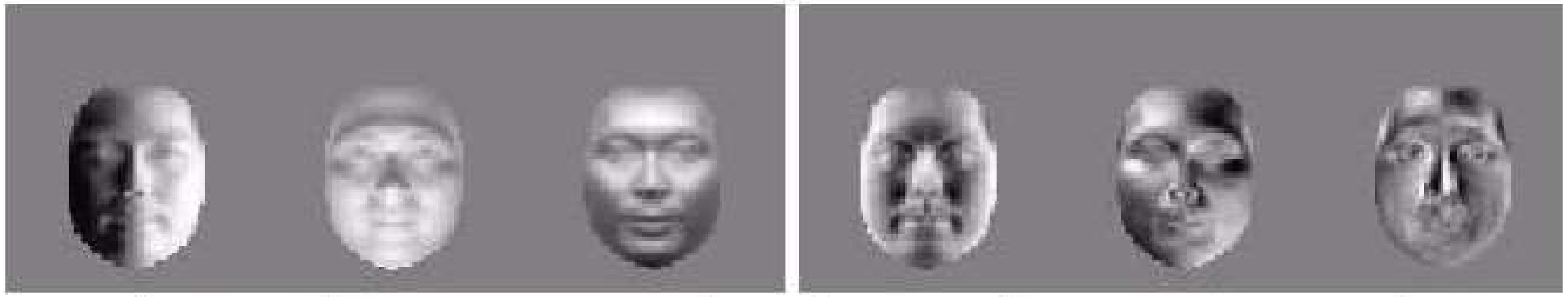


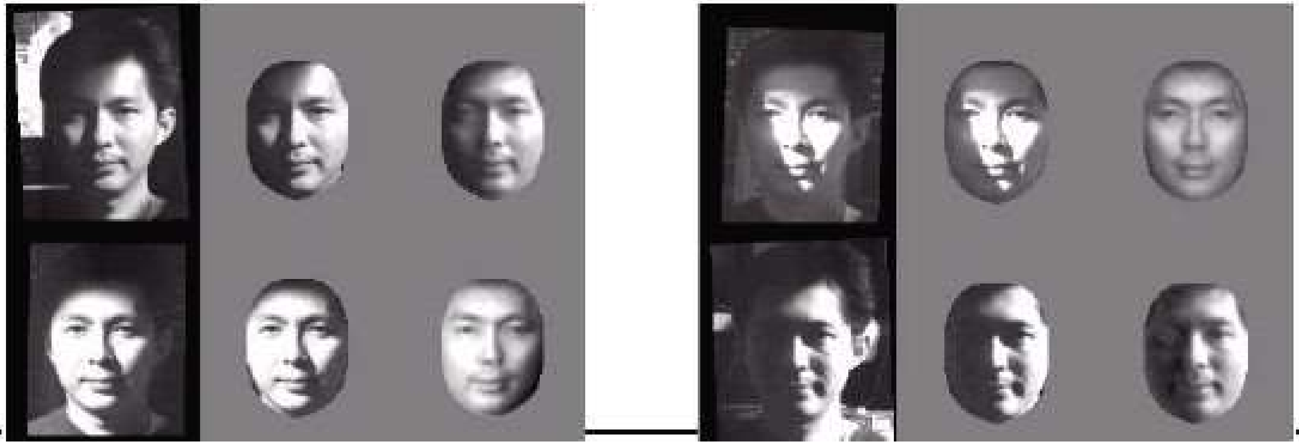
EigenButts

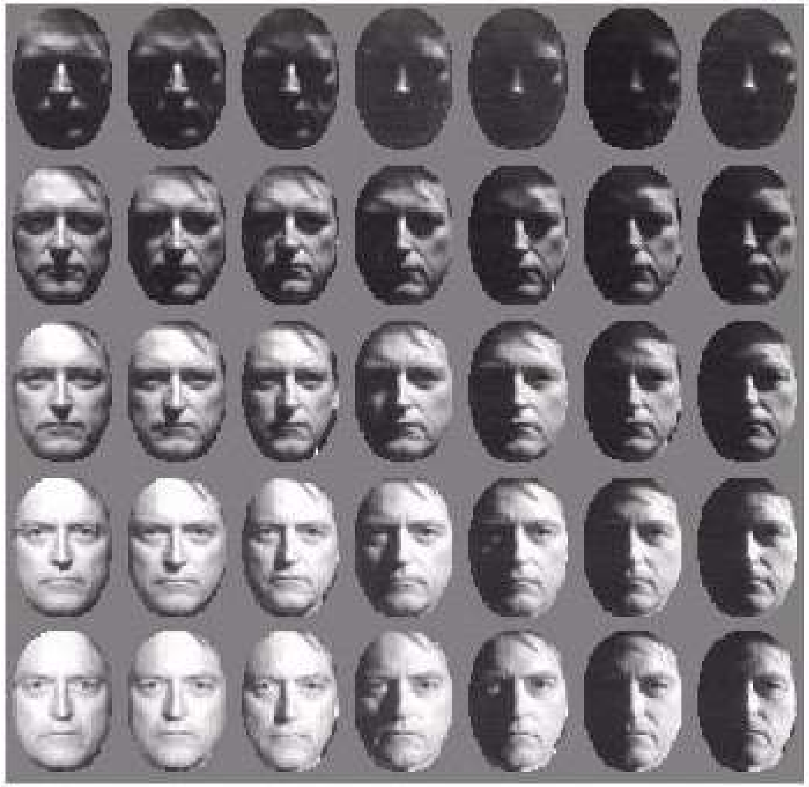
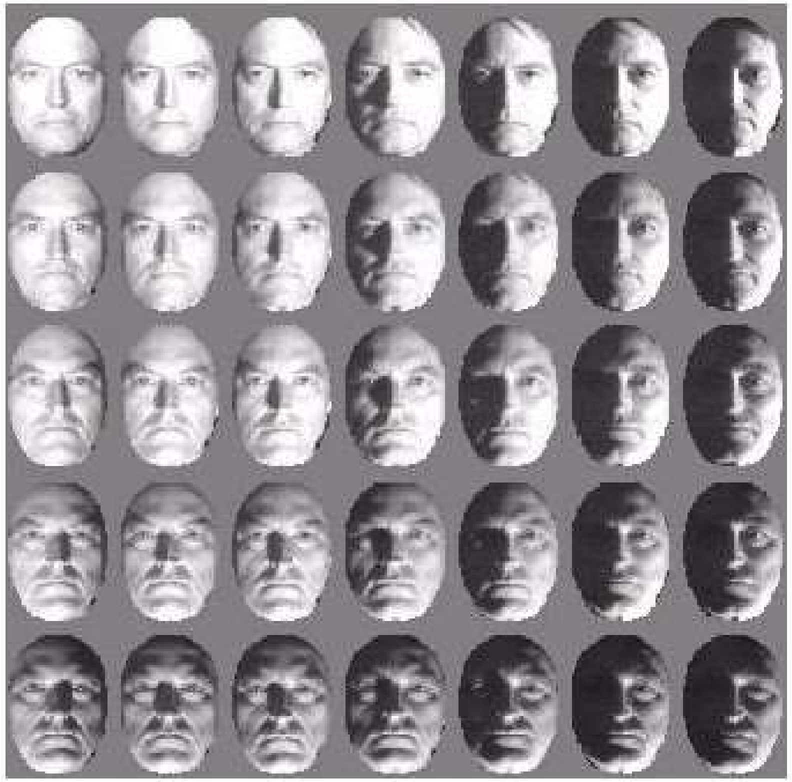
• Illumination Direction

- Suppose we have a set of images of a face with lighting from different directions, the principle components of these images are:

eigenvector





KPCA

PPCA

<u>GPCA</u>

SPCA

Dimensionality Reduction

Linear Vs Non-Linear
Supervised Vs. Unsupervised

- **PCA => PCA (Gaussian) + ICA (Non-Gaussian)**
  - One image per person for many persons
  - Consider relation between persons
- **LDA**
  - One set image per person for many persons
  - Consider relation within the same person and between different persons

- **LLE: Locally Linear Embedding**
- **ISOMAP**
- **Laplacision Eigenmap**

- **SVM: Linear and Non-Linear Discriminant**
- **Subspace => Kernel Function to High Dimension**

Factor Analysis

PCA Vs. ICA

- **PCA**
  - Gaussian Model
  - Orthogonal Vs. Orthonormal
- **ICA**
  - Non-Gaussian Model
  - Independent

References

- **PCA:**
  - DIP book

1. M. Kirby and L. Sirovich, “Application of the Karhunen-Loeve Procedure for the Characterization of Human Faces,” IEEE Transactions on PAMI, Vol. 12, No. 1, pp. 103-108, January 1990.
2. **M.A. Turk and A. Pentland, “Eigenfaces for Recognition,” Journal of Cognitive Neuroscience, Vol. 3, No. 1, pp. 71-86, 1991.**
3. H. Murase and S. Nayar, “Visual Learning and Recognition of 3-D Objects from Appearance,” IJCV, 14, pp. 5-24, 1995.
4. B. Moghaddam and A. Pentland, “Probabilistic Visual Learning for Object Recognition,” IEEE PAMI Vol. 19 No. 7, pp. 696-710, 1997.

- **KPCA**
- **PPCA**
- **GPCA**
- **SPCA**