SLR: Why $\hat{\beta}_1$ (estimator of β_1) is a linear combination of the errors?

$y_i = \beta_0 + \beta_1 X_i + e_i$

$$\hat{\beta}_1 = \frac{\sum (X_i-\bar{X})(y_i-\bar{y})}{\sum_{i=1}^{n} (X_i-\bar{X})^2}$$

$$= \frac{\sum_{i=1}^{n} [(X_i-\bar{X})y_i + (X_i-\bar{X})\bar{y}]}{S_{XX}}$$

$$= \frac{\sum_{i=0}^{n} (X_i-\bar{X})y_i + \sum (X_i-\bar{X})\bar{y}}{S_{XX}}$$

$$= \frac{\sum_{i=0}^{n} (X_i-\bar{X})y_i + \bar{y}\sum_{i=0}^{n} (X_i-\bar{X})}{S_{XX}}$$

$$= \frac{\sum_{i=0}^{n} (X_i-\bar{X})y_i}{S_{XX}}$$

$$= \frac{\sum_{i=0}^{n} (X_i-\bar{X})(\beta_0+\beta_1 X_i + e_i)}{S_{XX}}$$

Work on the numerator.

$$\sum_{i=0}^{n} (X_i-\bar{X})\beta_0 + \sum_{i=0}^{n} (X_i-\bar{X})\beta_1 X_i + \sum_{i=0}^{'} (X_i-\bar{X}) e_i$$

First Term:

$$\sum_{i=0}^{n} (X_i-\bar{X})\beta_0 = \beta_0 \sum_{i=0}^{n} (X_i-\bar{X}) = \beta_0 \cdot 0 = 0$$

Second Term:

$$\sum_{i=0}^{n} (X_i-\bar{X})\beta_1 X_i = \beta_1 \sum (X_i^2 - X_i\bar{X}) =$$

$$= \beta_1(\sum X_i^2 - \sum X_i\bar{X}) = \beta_1(\sum X_i^2 - \bar{X}\sum X_i)$$

$$= \beta_1[(\sum X_i^2) - \bar{X}\cdot n\bar{X}]$$

$$= \beta_1[(\sum X_i^2) - n\bar{X}^2]$$

$$= \beta_1 \sum (X_i-\bar{X})^2$$

$$= \beta_1 S_{XX}$$

$$[\sum (X_i-\bar{X})^2 = \sum (X_i^2 - 2X_i\bar{X} + \bar{X}^2)$$

$$= (\sum X_i^2) - (\sum 2X_i\bar{X}) + (\sum \bar{X}^2)$$

$$= (\sum X_i^2) - (2\bar{X}\cdot\sum X_i) + (n\bar{X}^2)$$

$$= (\sum X_i^2) - (2\bar{X}\cdot n\bar{X}) + n\bar{X}^2)$$

$$= (\sum X_i^2) - n\bar{X}^2).$$

$$\hat{\beta}_1 = \frac{0 + \beta_1 S_{XX} + \sum (X_i-\bar{X})e_i}{S_{XX}}$$

$$= \beta_1 + \sum\left[\frac{(X_i-\bar{X})}{S_{XX}} e_i\right] \quad w_i = \frac{X_i-\bar{X}}{S_{XX}}$$

$= \beta_1 + \sum w_i e_i$ → linear combination of errors

so $\hat{\beta}_1$ is a normal distribution by CLT