

1996 1998 2000 2002
27.7 26.3 25.3 24.8
regression model, with the input
percentage. Take 1982 as the base
has input value $x = 4$, and so on.

SLR: Why $\hat{\beta}_1$ (estimator of β_1) is
a linear combination of the errors?

$$y_i = \beta_0 + \beta_1 x_i + e_i$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

$$= \frac{\sum_{i=1}^n [(x_i - \bar{x})y_i + (x_i - \bar{x})\bar{y}]}{S_{xx}}$$

$$= \frac{\sum_{i=1}^n (x_i - \bar{x})y_i + \sum_{i=1}^n (x_i - \bar{x})\bar{y}}{S_{xx}}$$

$$= \frac{\sum_{i=1}^n (x_i - \bar{x})y_i}{S_{xx}} + \bar{y} \frac{\sum_{i=1}^n (x_i - \bar{x})}{S_{xx}}$$

$$= \frac{\sum_{i=1}^n (x_i - \bar{x})y_i}{S_{xx}}$$

$$= \frac{\sum_{i=1}^n (x_i - \bar{x})(\beta_0 + \beta_1 x_i + e_i)}{S_{xx}}$$

Work on the numerator.

$$\sum_{i=1}^n (x_i - \bar{x})\beta_0 + \sum_{i=1}^n (x_i - \bar{x})\beta_1 x_i + \sum_{i=1}^n (x_i - \bar{x})e_i$$

First Term:

$$\sum_{i=1}^n (x_i - \bar{x})\beta_0 = \beta_0 \sum_{i=1}^n (x_i - \bar{x}) = \beta_0 \cdot 0 = 0$$

Second Term:

$$\sum_{i=1}^n (x_i - \bar{x})\beta_1 x_i = \beta_1 \sum_{i=1}^n (x_i^2 - x_i \bar{x}) = \beta_1 [\sum_{i=1}^n x_i^2 - \bar{x} \sum_{i=1}^n x_i]$$

$$= \beta_1 [\sum_{i=1}^n x_i^2 - \bar{x} \cdot n\bar{x}]$$

$$= \beta_1 [\sum_{i=1}^n x_i^2 - n\bar{x}^2]$$

$$= \beta_1 \sum_{i=1}^n (x_i - \bar{x})^2$$

$$= \beta_1 S_{xx}$$

$$\begin{aligned} \sum_{i=1}^n (x_i - \bar{x})^2 &= \sum_{i=1}^n (x_i^2 - 2x_i \bar{x} + \bar{x}^2) \\ &= \sum_{i=1}^n x_i^2 - \sum_{i=1}^n 2x_i \bar{x} + \sum_{i=1}^n \bar{x}^2 \\ &= \sum_{i=1}^n x_i^2 - 2\bar{x} \sum_{i=1}^n x_i + n\bar{x}^2 \\ &= \sum_{i=1}^n x_i^2 - 2\bar{x} \cdot n\bar{x} + n\bar{x}^2 \\ &= \sum_{i=1}^n x_i^2 - n\bar{x}^2 \end{aligned}$$

$$\hat{\beta}_1 = \frac{0 + \beta_1 S_{xx} + \sum_{i=1}^n (x_i - \bar{x})e_i}{S_{xx}}$$

$$= \beta_1 + \sum_{i=1}^n \left[\frac{(x_i - \bar{x})}{S_{xx}} e_i \right] \quad w_i = \frac{x_i - \bar{x}}{S_{xx}}$$

$$= \beta_1 + \sum_{i=1}^n w_i e_i \rightarrow \text{linear combination of errors}$$

so $\hat{\beta}_1$ is a normal distribution by CLT