Assignment 2: Probabilistic Graphical Models

Machine Learning

Fall 2019

Learning Objectives

- TODO

1 Motivation and Context

- We've learned how probabilities can be used to describe uncertainty in the world
- We've learned how Bayes' rule can be used to reason about hypotheses, models, or other things that cannot be directly observed.

2 Product Rule and Marginalization for Random Variables

Recall: Product Rule and Marginalization for Events

Last assignment we learned about two very powerful techniques for computing the probability of events.

- The first technique we learned was the product rule (or conjunction rule), which states that for any two events $\mathcal{A}$ and $\mathcal{B}$,

$$\begin{aligned} p(\mathcal{A}, \mathcal{B}) &= p(\mathcal{A})p(\mathcal{B}|\mathcal{A}) \\ &= p(\mathcal{B})p(\mathcal{A}|\mathcal{B}) \ . \end{aligned} \tag{1}$$

- The second technique we learned was marginalization. This technique states that for any two events $\mathcal{A}$ and $\mathcal{B}$,

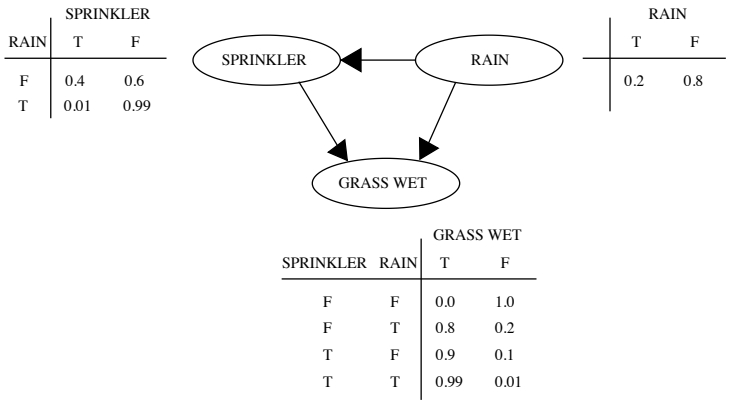
$$p(\mathcal{A}) = p(\mathcal{A}, \mathcal{B}) + p(\mathcal{A}, \neg\mathcal{B}) \tag{2}$$

It turns out that these rules can be applied to

3 Conditional Independence of Random Variables

4 Bayesian Networks

4.1 Simple Example



RAIN	SPRINKLER T	SPRINKLER F
F	0.4	0.6
T	0.01	0.99

RAIN T	RAIN F
0.2	0.8

SPRINKLER	RAIN	GRASS WET T	GRASS WET F
F	F	0.0	1.0
F	T	0.8	0.2
T	F	0.9	0.1
T	T	0.99	0.01

- Link to external resources
- D-separation

> **External Resource(s)**
>
> – Read d-Separation without Tears.
>
> – Pieter Abbeel Lecture (not sure how clear this is)
>
> – This one seems pretty good

- State the main conditions
- Do some exercises to determine when things are conditionally independent

> **Exercise 1**
>
> The alarm problem (need to find this one from CSE250A) (This has the description of the same network). More detail on the same network.

5 Generative versus Discriminative Models

TODO: write some intro here

5.1 Discriminative Models: a Look Back at Logistic Regression

Let's take a minute and think back to the logistic regression model for binary classification that we learned about in module 1. Given an input point $\mathbf{x_i}$, the logistic regression

utilized a weight vector $\mathbf{w}$ to compute the probability that the corresponding output y_i was 1 using the $\sigma(\mathbf{w}^\top \mathbf{x_i}) = \frac{1}{1+e^{-\mathbf{w}^\top \mathbf{x_i}}}$ (recall that σ is known as the sigmoid function and serves to squash its input into a number between 0 and 1, which can serve as a valid probability). While we didn't quite have the vocabulary for it then, what we really doing at the time was computing a conditional probability. We can think of Y_i as a random variable that represents the output that corresponds to the the input point $\mathbf{x_i}$ (Y_i is either 0 or 1 since we are dealing with binary classification). We can also think of the input as a random variable $\mathbf{X_i}$ (thinking of the input in this way will be helpful later in this section). In this way, we can think of what the logistic regression algorithm is doing as computing the following conditional probability:

$$p(Y_i = 1 | X_i = \mathbf{x_i}) = \sigma(\mathbf{w}^\top \mathbf{x_i}) \ . \tag{3}$$

We then defined a loss function that would let us find the best weights, $\mathbf{w}$, given a training set of corresponding input output pairs $(\mathbf{x_1}, y_1), (\mathbf{x_2}, y_2), \ldots, (\mathbf{x_n}, y_n)$. The details of how we did this are not important to the point we are trying to make now, so it'll suffice to say that learning in a logistic regression model meant tuning the conditional distribution of the outputs (the Y_i's) given the inputs ($\mathbf{x_i}$'s) to fit the training data the best. This type of model is what is known as a *discriminative model* (the Wikipedia article on discriminative models has more details if you are interested).

✔ Understanding Check

TODO

5.2 Generative Models

While the approach outlined above is totally logical, it is not the only way to approach supervised machine learning. Using a simple application of Bayes' rule, we can derive a whole new approach to the problem! Since we are interested in predicting Y_i given some inputs $\mathbf{x_i}$ it of course makes sense, for example for a binary classification problem, to want to determine $p(Y_i = 1 | \mathbf{x_i})$. Instead of modeling that distribution directly, we can instead use Bayes' rule to transform this probability distribution.

TODO: not sure if we should introduce a convention along the lines of $p(\mathbf{x_i}) = p(X_i = \mathbf{x_i})$.

$$\begin{aligned} p(Y_i = 1 | X_i = \mathbf{x_i}) &= \frac{p(X_i = \mathbf{x_i} | Y_i = 1) p(Y_i = 1)}{p(X_i = \mathbf{x_i})} \\ &= \frac{p(X_i = \mathbf{x_i} | Y_i = 1) p(Y_i = 1)}{p(X_i = \mathbf{x_i} | Y_i = 1) p(Y_i = 1) + p(X_i = \mathbf{x_i} | Y_i = 0) p(Y_i = 0)} \end{aligned} \tag{4}$$

What these equations are telling us is that if we have a model of the probability of the output being 1 *a priori* ($p(Y_i = 1)$) along with a model of the inputs $\mathbf{x_i}$ given the output Y_i ($p(Y_i | \mathbf{x_i})$), then we have all the information we need to compute $p(Y_i = 1 | \mathbf{x}_i)$. In a way this amounts to adopting the perspective that the hidden output Y_i causes the input X_i (see Figure 1).

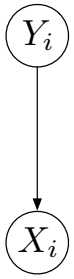


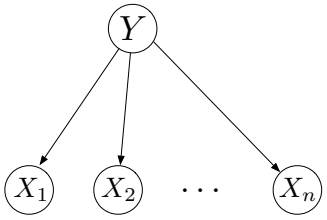
Figure 1: The graphical model corresponding to a probabilistic generative model in which the latent variable Y_i is thought of as a causally generating X_i.

The natural question you might ask yourself is *why?* Here are some potential advantages of using probabilistic generative models.

- Suppose you found out that $p(Y_i = 1)$ changed for some reason (any thoughts on when this might happen? Post here on NB). Incorporating this change into a probabilistic graphical model would be very straight forward (just modify $p(Y_i = 1)$ in Equation 4).

✔ Understanding Check

TODO



6 Your First Generative Model: Naïve Bayes

7 Probabilistic frameworks for Fairness in ML

8 Compas Model of Recidivism