Lawvere 18/5 78.

Algebraic theory of thermostatics.

Refresh our memory of thermo "dynamics", as we were taught in 2nd year university

Truesdell & Bh

. . .

Describe two alg. theories, and a certain morphism.
Shall refer back to a certain exact sequence. If A is a commutative ring and ∂_V and ∂_θ are two commuting derivations of A. Then we can make the following construction (A is a comm. $\mathbb{R}$-algebra).

$$\mathbb{R} \longrightarrow A \xrightarrow{\langle \partial_V, \partial_\theta \rangle} A \times A \xrightarrow{\partial_\theta \pi_V - \partial_V \pi_\theta} A$$

V'th factor

θ'th factor

This is a complex (compositions are zero), because the ∂'s commute.

What is A supposed to represent?
Variable quantities associated with some piece of substance.
In particular

∂_V is partial derivative w.r.to volume

∂_θ " " " " temperature

Among these quantities, there a central ones:
In A there are two distinguished elt's $\lambda, \varkappa \in A$.

λ: latent heat (at const. volume) $\varkappa$: specific heat (at constant volume)

Have also $p \in A$, $\theta \in A$ (pressure, and temperature, resp)

$$A \xrightarrow{\langle \partial_V, \partial_\theta \rangle} A \times A$$
$$? \longmapsto \langle \lambda, \varkappa \rangle$$

is there something? This was Carnot's wrong assumption.
Think of $\langle \lambda, \varkappa \rangle$ as a diff form. Is it [exact]?

It can be adjusted to be one ; there is an E s.t.

$$E \longmapsto \langle \lambda - p, \varkappa \rangle \qquad \mathbb{E} = \text{energy}$$

$$\langle \partial_v, \partial_\theta \rangle$$

Can have an integrating factor, namely $\frac{1}{\theta}$

$$S \longmapsto \left\langle \frac{\lambda}{\theta}, \frac{\varkappa}{\theta} \right\rangle \qquad S = \text{entropy.}$$

Carnot : Q s.t. $\dot{Q} = \lambda \dot{v} + \varkappa \cdot \dot{\theta}$. ($Q$ = heat content).

($\dot{Q}$ is meaningful, Q is not). But there is E

(energy) ; $\dot{E} = \lambda \dot{v} + \varkappa \dot{\theta} - p \dot{v}$

p is a mechanical quantity (macroscopic).

. . –

Since above is a complex, we get

$$\boxed{\begin{aligned} \partial_\theta(\lambda - p) &= \partial_v(\varkappa) \\ \partial_\theta\left(\frac{\lambda}{\theta}\right) &= \partial_v\left(\frac{\varkappa}{\theta}\right) \end{aligned}}$$

Basic axiom of one of my theories.

Unary op's $\partial_\theta, \partial_v$, nullary op's p, θ

A model for the(se) alg. theories is a possible substance (at least FP models). There may be nilpotents

. . .

Thermostatics since the temperature is constant over the whole body.

What are these two theories (three, rather : there is a basic theory underlying both).

Commutative ring with two commuting derivations with chosen elt. θ invertible $\frac{1}{\theta}$

and $\partial_\theta \theta = 1 \quad \partial_v \theta = 0$

But v as a chosen elt. does not occur (it so happens).

This is the common theory. The two theories

The "energetic" theory and the "calorimetric" theory

and a morphism energetic $\leftarrow\!\!\#$ calorimetric

$$\begin{tikzcd}[column sep=small]
 & \boxed{\begin{array}{c}\text{Commut rng with 2 comm. der} \\ \text{chosen elt: } \theta,\ \frac{1}{\theta} \\ \partial_\theta \theta = 1,\ \partial_v \theta = 0\end{array}} \arrow[dl, -{Triangle[open]}] \arrow[dr] & \\
\text{energetic theory} & & \text{calorimetric theory} \arrow[ll, "\#" near end]
\end{tikzcd}$$

λ, κ, p. Which functions are they of v and θ?

Depends on the substance, "constitutive relations"

Easiest way to present: one axiom

$$\partial_v \kappa = \theta \cdot \partial^2_\theta p$$

$$\lambda := \theta \cdot \partial_\theta p$$

This is one pres of calorimetric theory. Another is

$$\boxed{\begin{array}{l}\partial_\theta(\lambda - p) = \partial_v \kappa \\ \partial_\theta\left(\frac{\lambda}{\theta}\right) = \partial_v\left(\frac{\kappa}{\theta}\right)\end{array}}$$

Assume $\frac{1}{\partial_v p}$ exists. (always negative; pressure and volume are always inverse prop., not just for perfect gases)

Presentation of the other theory: specified nullary op's S and E and axiom

$$\partial_\theta E = \theta \cdot \partial_\theta S$$

$$\kappa \longmapsto \partial_\Theta E.$$
$$\lambda \longmapsto \Theta \cdot \partial_v S$$
$$p \longmapsto \cancel{\lambda - E} \quad = \lambda - \partial_v E$$

A better presentation is by using ψ "Gibbsian free energy"
(due to [Rietsch]) $\quad \psi = \partial_\Theta E = \Theta \cdot \partial_\Theta S$

$$p = \lambda - \partial_v E = \Theta \partial_v S - \partial_v E = -\partial_v (E - \Theta \cdot S)$$

because $\partial \cdot\cdot$ is zero

$$E - \Theta \circ S$$

is therefore the energy which is free, to be used for mechanical work.

~~ψ~~ Theory free on ψ with no axioms.

Interpretation of the ψ theory
$$\psi \longmapsto E - \Theta \cdot S$$

Vice versa
$$S := -\partial_\Theta \psi$$

...

so

$$-\psi \longmapsto \langle p, S \rangle \quad \text{in the exact sequence.}$$

$$\frac{1}{\partial_v^2 \psi}.$$

$$\boxed{\begin{array}{l} \partial_v \kappa = \Theta \cdot \partial_\Theta^2 p \\ \lambda := \Theta \, \partial_\Theta p \end{array}}$$

One example is the ideal gas
Here, $A =$ functions of v, Θ ; $\quad p \cdot v = R \cdot \Theta \quad \kappa = ?$
(specifies what kind of gas). Verify axioms of calorimetric theory

$$\Theta \cdot \partial_\Theta p = p$$

Conclude $\lambda = p$.
i.e. $\dot{E} = \kappa \cdot \dot{\Theta}$, or $\partial_v E = 0$ so E is a function of Θ only.

In the same way, you find also $\partial_V \varkappa = 0$ so $\varkappa$ is a function of Θ only. (Here Carnot was wrong: $\exists$ perfect gases where $\partial_\Theta \varkappa = 0$)

Can also deduce ...

2nd example "photon gas" Stefan, Bolzmann

Black body ration, radiation trapped in a cavity

$$p = \frac{1}{3} \frac{E}{V} = \text{function of } \Theta \text{ only.}$$

(Light exerts pressure) ($\frac{1}{3}$ comes from 3-dimensionality of space).

Again $\varkappa$ is to be determined. (~~We can't~~

$$\varkappa = c \cdot V \cdot \Theta^3 . \qquad c \text{ a constant}$$

$$p = B \cdot \Theta^4 \qquad B \text{ " "}$$

(led Planck to quantum mechanics).

$$\psi = - d \cdot v \cdot \Theta^4 \qquad D \text{ " "}$$

Back to perfect gas

$$\psi = \cdots \Theta + R \cdot \Theta \cdot \underbrace{\log\left(\frac{V}{V_0}\right)}$$ (V_0 a reference volume.)

Calculate speed of sound if we understand adiabatic process

Motivation: Processes. $\dot{\Theta}$: why the dot. ?

~~What~~ A process is a path through the m/ on which A functions. So algebraically, a ring homomorphism $A \longrightarrow B$ where B is another ring with one derivation $(\ )^{\cdot}$ operating ("B = functions of time"), satisfying

$$\dot{E} = \lambda \dot{v} + \varkappa \dot{\Theta} - p \dot{v}.$$

more generally $\boxed{\dot{f} = \partial_V f \cdot \dot{v} + \partial_\Theta f \cdot \dot{\Theta}}$

(chain rule); ("This is a process which leaves the parameters fixed for every f in A. The parameters are not mentioned here: electric potential ~~etc.~~, chemical potential, ...)

"$\dot{Q} = 0$"!

Adiabatic process: process in which i.e $\lambda \dot{v} + \varkappa \dot{\Theta} = 0$.

Morphism of alg. theories
Leads to a left adjoint

$$\begin{array}{c} \mathrm{Spec}(\bar{A}) \\ \downarrow \\ \mathrm{Spec}(A). \end{array}$$

our particular closed form in A becomes exact in $\bar{A}$.