Lawvere 19/5 78.

Can categories be some help in learning thermomechanics?

To day: non-static thermomechanics,

I wanted to learn this material — a continuation of categorical dynamics

Mystifying concept of entropy. Oppose this, not only by seeing the political reason, but also by understanding. Recent Nobel price granted to the reactionary and obscurantist ...

Main contributors to concept of entropy, during last 15 years:

B. Coleman, W. Noll., D. Owen, W. Day (Archive for Rat. Mech & Anal.).

Edited by my old teacher Truesdell. Main references: Noll's selected papers (ed. Springer Verlag). .. Day: Thermodynamics of simple materials with fading memory.

Non equilibrium thermomechanics of ... paint, honey, rubber, glass

..

Coleman & Owen trying to sum up .. by inventing new mathematical structure. Main properties, I think, can be summed up via the categorical notion of fibration.

Basic problem addressed. In thermostatics (yesterday)

$$\theta \cdot dS = dE - W \qquad \underbrace{W = -p\,dV}_{\text{a diff form, "work"}}$$

i.e.

$\exists \theta, \exists S$ so that this eq'n holds.

i.e $dE - W$ has an integrating factor θ.

Also $\exists \psi$ so that $p = -\partial_V \psi$ (ψ = free energy)

$$S = -\partial_\theta \psi$$

$$\psi = E - \theta S.$$

In thermodynamics, the best we can hope for $\overbrace{}^{\text{is ? } \exists S}$ so that

$\theta\, dS \geq dE - W$.

In thermostatics, θ etc. are globally constant (one temperature for the body, but can of course be applied for vanishing small portions of air, .. large body by varying quantities .. sound ~~of a~~ in air.

But [[often]], non-static.

Principle of large body being made of $\overset{\text{inf}}{\vee}$ small parts gives rise to the word : simple material.
So the theory we write down is a theory for vanishing small portions

Constitution relations

Fund. principles : conservation of moment (Euler, Cauchy)

In such problem

cons-of energy

How can a body / response to external force / $\overset{\text{thermo}}{\text{radiation}}$

What makes one body different from another ?

So physics of ~~materials~~.

By contrast $\Theta dS \geq dE - W$ does restrict the kinds of response one can have: what makes honey different from glass. What kind of materials can there possibly be ?
Restriction on possible constitution relation

Constitution relation $\equiv$ stress tensor as a particular functional of the internal state of the body (mechanical response) and the <u>thermo</u>- response as a functional of internal state.

i.e. what function of temp. gradient is the heat flux (conduction)
(or internal energy flux)

Most simply : proportionality (Fourier): heat flux proportional to heat gradient.

But .. heat conductivity may depend on internal state.

A constitution relation is a pair of such functionals.

$$\underset{\sim}{q} \cdot \operatorname{grad} \Theta < 0 \qquad \underset{\sim}{q} \text{ heat } \cancel{\text{conduction}} \text{ flow}$$

(or > 0 depending on ...).

Internal state may depend on the whole history of strain and internal energy. Stress and strain are dual 6-dim vector spaces: Strain is the variable Riemannian metric measuring distance from point [to its neighbour points]. The internal motion is reflected as a variable metric.

Dual pairing between stress and strain : internal working
[is name for value].

Potential importance of topos theory : to develop theory for generic particle.
Not the particle of philosophers: point mass. (3 degrees of freedom).
but vanishing small part (of body) has 6 degrees of freedom.

(Stress tensor T is also called tension, whence the word 'tensor').
Discovered by Cauchy
Pression = $\frac{1}{3}$(trace of tension T).

Finding entropy function is equivalent to finding free energy function
$\exists \psi$: $d\psi \leq W$ not involving the temperature.

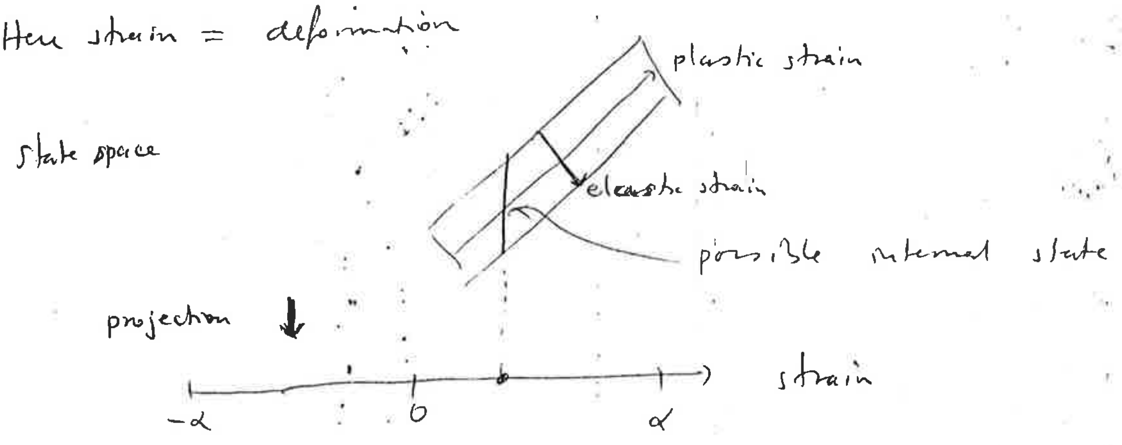
$$S \underset{def}{=} \frac{E - \psi}{\theta}$$

Simple example. Assume temperature constant. Isothermo deformation of 1-dimensional elastic-plastic material.
Elastic : stress = constant • strain (Hooke's law).

$$\int T(A) \cdot A = \text{work}.$$

Bridgeman 1950.
What do we mean by plastic? The stress for a plastic material is constant. It is yielding : the resistance does not increase ~~by~~ [with the deformation].
There is a plastic region and an elastic region [of state space].

State space to describe this. Consider the strain axis.
Here strain = deformation



Here we see a category : a discrete fibration

To any top space, X, can associate a category $\mathbb{X}$, the Moore category with obj's: points ; Morphisms: path's of arbitrary duration $[0, d]$ (d for duration). It comes equipped with a duration functor

$$\mathbb{X} \to (\mathbb{R}_+, +). \quad \text{(monoid).}$$

It is not a fibration, but a somewhat similar condition.

(Have functor $\text{top} \to \text{Cat}/\langle \mathbb{R}^+, + \rangle$.)

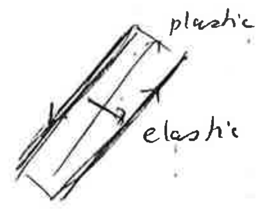
For locally connected 1st countable top. spaces, the restriction of the functor to that subcategory will be full & faithful.

The fibration-like property: suppose I have a morphism above and a factorization of it below; then there is a unique factorization of it above.

Shall not take all these paths - would be geometry, not physics. Consider subcategories of the Moore category which still satisfy the property of unique lifting of factorizations.

Instance of : basic mathematical structures are categories.

For the state space picture above: For the strain space, all paths are allowed. Above (in state space)



Consider only paths that run along "constant plastic strain" lines, except on the boundary lines (max or min.) you can go along that boundary (down or up, resp.)

Crucial thing which expresses the determinism is the fact that this is a discrete op-fibration

The stress has to be a function of the state. It is thus a functor into the Moore category of possible values of stress

$$T(s_e, s_p) = C\circ s_e$$

Can then construct the work functor

$$\text{states} \xrightarrow{W} \langle \mathbb{R}, + \rangle$$

$$w(\alpha) = \int \text{stress times } \overbrace{\text{strain}}^{\text{differential of}}$$

$$= \int_0^{d(\alpha)} T \cdot d\,\text{strain}$$

$$= \int_0^{d(\alpha)} c \cdot s_e \cdot d(s_e + s_p) = \int c\, s_e\, ds_e + \int c\, s_e\, ds_p$$

$$= \frac{1}{2} c (s_e)^2 + \qquad \geq 0$$

(for last term:
(plus times plus on lower line

minus " minus on upper line). So [call this] the free energy. It is a functor

$$\begin{tikzcd} \text{states} \arrow[r, "W"] \arrow[dr, "\psi"'] & (\mathbb{R}, +) \arrow[d, phantom, "\leq"] \\ & \text{Moore cat of reals} \arrow[u, "\Delta"'] \end{tikzcd} \quad (\text{difference})$$

such that
not that it commutes, but there is an inequality.
True in integrated as well as diff. form.
So .. free energy functor.

What is not easy in general: given such fib. and work functor - find free energy.
Clausius - Duhem. If you look only on "processes" = morphisms in state category endomorphism (cyclic process) and are sufficiently small; then the work is positive: Try to define

$$\psi_2(x,y) = \inf_{x \xrightarrow{\alpha} y} w(\alpha)$$

Find reference state x_0. Put

$$\psi(x) = \psi_2(x_0, x)$$

(x_0 is declared to have zero free energy).

will not be $-\infty$,
provided for all small endoprocesses, $\omega \geq 0$.