Recent development in
Categorical dynamics

Louvain 29/7 11

Many people have heard about SDG, carried out by Wraith, AK, Reyes, Bunge, …
To clarify the objective logic of cont. mechanics
Noll, Gabriel (gros toposes, taught me), Verdier (La Jolla),

Groth had idea of AB5. There ought to be a non-linear version.
ETCS was the 0-dim part of it.
Halifax — needed: elementary theory of toposes.
"Topological dynamics" (… & Hedlund).
Continuous, smooth, … (and many more) so categorical dynamics,

should fit in many different categories.
Use of nilpotents in describing differentiation in algebraic, analytic, …
Why not C^∞? United in: can be embedded in a cat with
tangent bundle is rep'ble, by a unique object,
Fundamental thing was: CCC. From hindsight: Hurewicz' ~~exponentials~~.
Volterras functionals, Bernoulli's calc of variations
Hurewicz' campaign for this, communicated it to Fox, who solved
it for sequential spaces. Hurewicz 1948-49 (Princeton) coined the
notion of k-space (compactly generated).
Gale (BAMS Vol. 1, 1950), 1955 Kelley's book, gave an exposition
Many people thought k was for "Kelley". I think I
defined CCC. 1958 Kan adjoint functors; he was aware of
∃ of fn. space in simplical sets. Since uniqueness,
can be used axiomatically, 1963 Cat of cat's defn using
exp.; Eilenberg-Kelly (La Jolla) Closed monoidal cat.
in particular, since × is "cartesian", hence the name. They
also noted that Heyting alg's are CCCs

Axiom ① $\mathcal{E}$ is a cart closed cat.

② $T\exists$ $T_0 \hookrightarrow T_1 \hookrightarrow T_2 \to$ amazingly tiny

Extra right adjoint $f^!$ in Groth.'s notation

$(\ )^!$ means "amazing!". or ATOM.

$$\begin{tikzcd} \mathcal{E}/_T \arrow[r, bend left, "f_!"] \arrow[r, "f_*"'] & \mathcal{E} \arrow[l, "f^*"'] \arrow[l, bend left, "f^!"] \end{tikzcd}$$

$f_!$, or Σ_T

f_* used notation $\prod_T$

By composing, get a fractional exponent $(\ )^{1/T}$

$$\frac{X^T \longrightarrow Y}{X \longrightarrow Y^{1/T}}$$

So T-ary functionals can be viewed (as "functions with different codomain"). Eilenberg-MacLane space.

Differential form. Undreamed of in classical logic

Shall use: $T_0 = 1$, $T_1 = T$, $T_2 = T^2/2!$

(beginning of Taylor series)

'Tiny' by Freyd + Yetter. Relative to the topos you are in.

(Every T can become tiny in a [bigger topos].)

Then also have $X^{B/T}$ arbitrary B.

Where do they come from? Imported in math. from Heraclitus "river twice". Intermediate position: Galilei, Newton, med. tiny motion

$$T_0 \xrightarrow{\ 0\ } T_1$$

Radical synthetic (Synthetic: derive algebraic from geometric, this program is progressing).

$T \xrightarrow{x} X$ is stepping into the river once, at x_0, but

"Il punto nella punta"

$$\begin{array}{ccc} T & \xrightarrow{\text{feminine}} & X \quad \text{punta} \\ {\scriptstyle 0}\uparrow & \nearrow_{\text{masculin}} & \quad \text{punto.} \\ 1 & & \end{array}$$

The fact that $1 \xrightarrow{0} T$ is not an iso is shown by: T^T has two points, with empty meet

$$1 \overset{\ulcorner 0 \urcorner}{\underset{\ulcorner id \urcorner}{\rightrightarrows}} T^T$$

Call X^T "tangent bundle, with $ev_0 : X^T \longrightarrow X$,

can get $T_x X$ for $x \in X$. "Tangent space at x". Has one

structure: $T^T \xrightarrow[ev_0]{} T$ tangent bundle of T

Take tangent space at x, call it R. It is a

monoid. This is essentially Euler's def of real no's.

It operates on T^T, by "speed-up"s.

So X^T has operations by R, so have ratios of velocities.

So every high school student can calculate when Achilles

overtakes the tortoise (provided $1-\lambda$ is invertible)

Principle in philosophy.

$$\begin{array}{ccc} X^T & \longrightarrow & Y^T \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

So the tangent space at $x \in X$

get $[\![T_x X \longrightarrow T_{f(x)} Y]\!]$

$(X^T)_x \longrightarrow (Y^T)_{f(x)}$ compatibly

$f_x(\lambda \cdot -) = \lambda \cdot f_x$ for any endomorphism of T fixing 0

If T spectrum of $k[\epsilon]$, then
$k[X]$ ~~spectr~~ has $(T^T)_0$ as spec. $\Big\}$ in alg. geom

a simple observation in [SDG].

"Infinitesimal generation":

$\varinjlim$'s of $\varprojlim$ and exponential $\wedge$ of (atoms)).

Often, homogeneity implies linearity (i.e. addition).

A paper I have in Bell volume, concerning addition.

Another synthetic addition, ..

For much of what I do, T_2 could be anything, not nec $T\times T/2!$

Define $D_n = \{h \in T^T \mid h^{n+1} = 0\} \subset R$

In most books, T is identified with D_1. To choose an

iso $T \cong D_1$ is choosing a unit of time.

T is an infinitesimal time. T^T is 'pure' quantities, whatever

T is; all the D_i's are pure.

$T \longrightarrow D_1 \subseteq T^T$

By exp adjoint $T \times T \xrightarrow{\cdot} T$, monoid without unit

$D_1 \longrightarrow T^T$ is a monoid homo which has a retraction.

That is the origin of Newton's $(\dot{\ })$.

Parallel to that, have inclusion $D \subset D_2$ $\quad D_2 = \{t \mid t^3 = 0\}$

I use this coordinatization for exposition.

Achievement of all my SDG friends, all diff calculus

comes out right, for a dense class of spaces

Dynamics. Consider a 'configuration' space C

Consider

$$C^{T_1} \xleftarrow{\alpha^*} C^{T_2} \qquad \text{where } T_1 \xrightarrow{\alpha} T_2$$

Basic structure: a section ξ of α^*, "prolongation"

ξ is a specific information, unlike α^*.

Get a cat. $\mathcal{E}^\alpha$. If $\mathcal{E}$, $\mathcal{E}^\alpha$ is a topos $\mathcal{E} \rightleftarrows \mathcal{E}^\alpha$

Like when you have enriched monad / comonad.

But $(\;)^D \dashv (\;)^{1/D}$ is not enriched over $\mathcal{E}$, but over $\mathcal{S}$ (consisting of S with $S \longrightarrow S^T$ is iso)

Desired by Anders and Gonzalo. Think of $\mathcal{E}^\alpha$ as the "cat of laws of motion" 2nd order, as Fibonacci understood in the discrete case. $\mathcal{E}^\alpha$ (even when T is not an atom) is a type of alg structure: a T_2-tuple of T_1-ary operation

An elementary transformation in universal alg.

Assume you have a theory $\mathbb{A}$ which involves a W-tuple of T-ary op'ns on C. Can transform into W^T-tuple of unary op'ns on C^T

This functor does not preserve underlying set, U is replaced by U^T

Not full functor $Alg(\mathbb{A}, \mathcal{E}) \longrightarrow \mathcal{E}^M$

where M is the monoid, $T^T \leq W^T$ modulo eq'ns ...

In Dynamics, if $W = T_2$, $Alg(\mathbb{A}, \mathcal{E}) = $ 2nd order diff eq'ns

$\mathcal{E}^M = D$-action on X^D .. [equivariant]

$$Alg(\mathbb{A}, \mathcal{E}) \longrightarrow \mathcal{E}^M \qquad \text{full \& faithful}$$

implies a vector field on X^D.

M = monoid of unary op'ns on a bigger space.

Bool alg's - the content of $\vee\ \wedge$ in terms of unary op'ns on 3. ($3 = 2 \times 2 / 2!$). Similarly, .. motion of ultrafilter.

A third ex. ...

Also happens for finite discrete varieties
A basic result in univ alg!

[[There are enough unary op'ns on $R \times R$ (R a ring), to distinguish ring homo's out of $R \to R'$.]]

$\mathcal{E}^M$ has a further forgetful to the cat of vector field since a vector field is a prolongation along $1 \to T$.

When ξ is a law of motion, what is a lawful motion?

Is a morphism in a cat.

Clear what a morphism of vector field is.

There is a left adjoint to $Alg(A, \mathcal{E}) \longrightarrow \mathcal{E}^M$

"Algebra of time", a force moving time forward

Open problem!

Laws of motion are laws of becoming

An ex. would be "infinite anticipation of an infinitesimal law."

This far, we came in 1997 in Montreal.

I have recently come up with a new idea, the def. of force.

A law of motion = prolongation along $T_1 \xrightarrow{\alpha} T_2$

SDG may (by coordinatization) $\xi(x,v)(t)$ $t \in D_3$

$= x + vt + a(x,v) \cdot \frac{t^2}{2}$

So the information of the prolongation law is $a(x,v)$

A force law is an endomorph of C^{T_2} s.t.

$$\begin{tikzcd} C^{T_2} \arrow[loop right, "f"] \\ C^{T_1} \arrow[u, "\alpha^*"] \end{tikzcd} \qquad \alpha^* f = \alpha^*$$

They act on laws of motion ξ law of motion implies $f\xi$ is also a force law.

$\boxed{\text{Force laws act on motion laws}}$ composition

(So) force laws form a monoid that act

In engineering .. there are 17 different forcelaws.

Want one force? Just by composing.

A particular starting point (law), the "geodesic law of motion".

$\xi\lambda = \lambda\xi$ $[\lambda \in T^T]$, "straight line motion"

Unlike vector fields, there are no trivial 2nd order eq's.

The subcat of geodesic law takes the place of trivial laws

In particular, $\lambda f = f\lambda$ "spray law".

If you operate on a geodesic law ~~on a~~ by a spray,

you again get a geodesic law.

Is composition of force laws commutative? Not just

addition of vectors, not just addition of the $a(x,v)$-

terms (which is commutative).

Typical configuration space in continuum mechanics:

infinite ~~~~ dimensional spaces. The objective law should be the

same: body = B E = environment

A typical conf space: all possible placements of B in E,

i.e. E^B (not nec. monomorphism), $C = E^B$

Force (laws on $C = E^B$ - Two special classes; a law might
motion)

be induced from a law on E, just by exponentiation

"single particle" reasoning. eg geodesic (affine) law on E.

Or $B = E$. "body force" as they call it in cont mech

Internal force: if on B we have an (endomap of) $B \times T_2$ $\rightarrow$ $B \times T_1 \subset B \times T_2$ "H"

induces a H for 'Hooke', get H^{E^B} force law on E^B.

$B \times T_2 \circlearrowleft H$

Split into two parts $\qquad B \times T_2 \longrightarrow B \qquad$ Second order paths in B^B

$B \times T_2 \longrightarrow T_2 \qquad$ i'd on $\cancel{B} \times T_1$

$$H_B(\cancel{b})(b) = b + h_B(b)\tfrac{t^2}{2}$$

$h_B(b)$ is really 1^{st} derivative since ~~the~~ square of $t^2/2$ is 0

some kind of tensor (endomap of B) $\qquad B \longrightarrow T_2^{T_2}$

$$H_{T_2}(t)(b) = t + k(b)\cdot t^2/2$$

Should have considered first the case when $[\![B]\!]$ ~~(or t)~~ is 1.

So these laws are given by "messing with time". $\quad h$

Effect of k : adding $k \cdot v$ to the acceleration $a(x, v)$.

This is viscosity . Law of viscous motion.

If B is any obj. , k ... non constant viscosity , varying on B.

Constitutive rel'n H: $B \times T_2 \circlearrowleft H$

The h-part – I picked 'h' elasticity , Hooke tensor,

visco-elastic body.

I haven't mentioned mass, Bernoulli (engineer) et al they say "mass is nothing" ; they are saying that "mass is just another force"

Newton writes "mass is a force". We should take 'th serious. Where does it live . Not just on E, nor just on B.

eg Mach's principle. Recent stuff on "Higgs boson",

"Herzian Mechanics"

[[Discussion]]

Torsors · Line bundles

I forgot: do force laws commute? All internal forces do commute.

In particular, the 1-dim fam of viscosity laws, how do

they compose – the k's just add.

Endomaps of T_2 which fix T_1. Composing two of these

They add synthetically addition

Why is addition a module on mult. –

Easy if we assume that R is given apriori.

R is a (comm.) rig. Have $Mod_R(\mathcal{E})$, an AB5 cat if $\mathcal{E}$ is a topos.

Therefore have $\otimes_R$, Hom_R smc cat

A line bundle A is just an invertible module, $A \otimes_R B \cong R$,

so B must be A^*, i.e. $A \otimes A^* \xrightarrow[ev]{} R$ is invertible

Picard group of $\mathcal{E}$, R – invertible A's under $\otimes$,

also called H^1. In physics, we need quantities which

are not pure. Physical dimensions. Only q's of

same kind can be added, but of different dim can be

multiplied. The Pic group may just be $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

Length is different from mass — mass length time

A constant quantity of type A is just $R \to A$.

The rules of Hom ⊗ adjointness explain how transform

Is an additive category. Every map in it is a quantity.

Invertibility implies means locally trivially.

But Picard gp's are usually talked about f petit toposes

Look at gros toposes?

[[Filip Bar: .. non commutativity .. geom alg ..]]

I talk about comm/non-comm in the infinitesimal

Stone von Neumann: irred rep's of canon comm relations:

The only non-comm. is due to Leibniz rule.

The opposite of synthetic, but namely algebraic

If $\mathcal{A}$ is a comm alg. with a 2^{nd} order diff operator $\Box$

(There is a 2^{nd} order Leibniz rule) eg $\Box$ Laplacian

There is a binary op'n $*$ on $\mathcal{A}$ that can be defined

$$\tfrac{1}{2}[f\cdot\Box g - g\cdot\Box f] \qquad \text{or something}$$

k-bilinear, is a derivation in each variable separately, and symmetric. This is equiv to a metric on Spec A, by identifying tangent vectors with derivations

Can define

$$\mathcal{A} \underset{Q}{\overset{P}{\rightrightarrows}} \mathrm{Hom}_k(\mathcal{A},\mathcal{A})$$

$$Q_f(g) = f\cdot g$$
$$P_f(g) = f * g.$$

$[P_f, Q_g] = \cancel{k} Q_{f\cdot g}$ Canonical commutation rel'n.

(So whenever you have eg. wave eq'n ...)