Lawvere 23 juli 66

Mainly conjectures.

$\mathfrak{X}$ right complete, $\mathbb{C}$ small

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[dr, "A"'] & \mathbb{S}^{\mathbb{C}^{opp}} \arrow[d, dashed, "\exists!"] \\ & \mathfrak{X} \end{tikzcd}$$

right continuous (unique up to natural equivalence).

So $\mathbb{S}^{\mathbb{C}^{opp}}$ is the free right complete category

Large theories in the cat of cat's: every possible limit will be an algebraic operation.

$\mathbb{S}^{\mathbb{C}^{opp}}$ is in some sense the power set; so one can use it for higher order theories. Grothendieck topology is ordinary topology, but with ~~a~~ new power set functor.

Why power set? Think of the closed cat. of ordered set

Then the analogue of $\mathbb{S}^{\mathbb{C}^{opp}}$ is

$$2^{\Lambda^{op}}$$

(2 analogue of $\mathbb{S}$); power set if Λ discrete

"Triple" on ordered sets. (not really)

$$\mathbb{C} \xrightarrow{f} \mathbb{C}' \qquad \mathbb{S}^{\mathbb{C}^{opp}} \underset{\square}{\overset{adj}{\rightleftarrows}} \mathbb{S}^{\mathbb{C}'^{opp}}$$

adj instead of $\square$ makes it covariant. (Corresponds to covariant power set with direct image).

$\mathbb{S} \circlearrowleft P$ (Power set with direct image, is a monad).

The algebras: right complete lattices. (the morphisms preserve arbitrary sup's). Free alg's: sets with relations.

This suggests the following: The Kleisli is just putting in more functions. Take e.g. P_0 (set of finite subsets).

E.g. cat of cat.

$\mathbb{C} \to \mathbb{D}$. But a generalized functor is

$$\text{Cocontinuous}\left(\mathcal{S}^{\mathbb{C}^{opp}}, \mathcal{S}^{\mathbb{D}^{opp}}\right) \cong \mathcal{S}^{\mathbb{C} \times \mathbb{D}^{opp}}$$

Objects in here are simply pairings (sense of Cohn). Composition is a generalized matrix multiplication.
This has been looked at in ring theory. Morita-theory: these things are just bimodules. Freyd showed the same to hold for theories. – Or take top. spaces and sheaves.

Every monad in sets had an algebraic part. The composition $P \circ P_0$ carries a monad structure; it describes Rcled lattices; (distributive law $x \wedge \bigvee y_i = \bigvee x \wedge y_i$ required).
Close connection with Hausdorff^{opp} and Rcled

$$\mathcal{H}^{opp} \underset{top}{\overset{}{\rightleftarrows}} \text{Rcled}$$

top

This was to prepare the way for a similar monad in the cat. of cat's, to get topos.

Given two cats, both relative over some common closed cat, and an object S common to both.

$$\begin{tikzcd} \mathcal{R}^{op} \arrow[dr, "\mathcal{R}(-,\mathcal{S})"'] & \\ & \text{Cat} \arrow[ul, "\text{Cat}(-,\mathcal{S})"'] \end{tikzcd}$$

will automatically be adjoint. The monads should have some special properties („analytical")

$$\begin{tikzcd}[column sep=small]
\mathrm{OrdVect} \arrow[dr, bend left=0, "\mathrm{OrdV}(-,R)"'] & \\
& \mathrm{Top} \arrow[ul, shift right=2, "\mathrm{Top}(-,R)"']
\end{tikzcd}$$

Gives the integration theory monad on Top.

The previous monad ~~is~~ is

$$\mathfrak{X} \rightsquigarrow R(\mathcal{S}^{\mathfrak{X}}, \mathcal{S})$$

I don't know what the algebras are. If $\mathfrak{X}$ has small hom sets, I have the Yoneda mapping

$$\mathfrak{X}^{\mathrm{opp}} \longrightarrow \mathcal{S}^{\mathfrak{X}} \quad ;$$

restricting along this, I obtain a functor

$$R(\mathcal{S}^{\mathfrak{X}}, \mathcal{S}) \longrightarrow \mathcal{S}^{\mathfrak{X}^{\mathrm{opp}}}$$

The same procedure for the measure case gives

$$\mathcal{O}(C(X,R),R) \longrightarrow \mathrm{Meas}(X,R)$$

The Riesz repr. theorem gives conditions when this is an equivalence.

Is there a Riesz theorem for

$$R(\mathcal{S}^{\mathfrak{X}}, \mathcal{S}) \longrightarrow \mathcal{S}^{\mathfrak{X}^{\mathrm{opp}}}$$

$$\int f\, dm = \varinjlim\left((-,\mathcal{S}) \xrightarrow{m} \mathcal{S} \right)$$

(the canonical limit).

One can speak of the small part of a monad in the cat of cats

$$\tilde{T}\cancel{R}\mathfrak{X} = \bigcup_{\substack{\mathbb{C} \to \mathfrak{X} \\ \text{small}}} R(\mathcal{S}^{\mathbb{C}}, \mathfrak{X}) \subseteq T\mathfrak{X}$$

$\hat{T}$ is a monad

I conjecture the alg's ~~of~~ over this is right complete categories (with chosen limit functors).

$$\begin{tikzcd}
\mathrm{Dir}(\mathcal{X}) \arrow[r] \arrow[dr, "\text{factors}"] & \mathbb{S}^{\mathcal{X}} \\
& \mathbb{S}^{\mathcal{X}^{op}} \arrow[u, "\text{conjugation (Isbell)}"'] \arrow[r, dashed, "\xi"'] & \mathcal{X}
\end{tikzcd}$$

An algebra ξ . . .

<u>Presentations</u>. Given a pair of small cats

$$\mathbb{R} \rightrightarrows \mathbb{S}^{\mathbb{C}^{opp}} \xrightarrow[\substack{\text{coeq} \\ \text{in } \mathbb{R}}]{} \mathcal{X}$$

Grothendieck topology

For small cat's: take the sheaves ~~with~~ for the canonical topology. – is the triple.

The triple in the cat of cat's has the following property, that

$$\mathbb{S}^{(\mathbb{S}^{\mathcal{X}^{op}})^{op}} \longrightarrow \mathbb{S}^{\mathcal{X}^{opp}}$$ then

$\mu_{\mathcal{X}} \dashv \eta_{\mathcal{X}} T$; so the μ is determined by η

$$\begin{tikzcd}
\mathbb{S}^{opp} \arrow[d, shift right, "2^{()}"'] \\
\mathbb{S} \arrow[u, shift right, "2^{()}"'] \arrow[loop right]
\end{tikzcd}$$

we get complete atomic Boolean alg's as alg's.