Lawvere 6/9 - 71

Expanding on the introduction I gave on friday.

Title of the course indicates that I would like to understand analysis, where I have had some initial successes. I hope you will participate in developing further.

Problem of understanding analysis.

To clear things a bit, I shall quote Mao: 4 essays, in particular On Contradiction (50 pages long), and <u>Where Do Correct Ideas Come From</u> (3 pages). Read the latter: ...

<u>Sums up</u> Dialectical Materialist Theory of Knowledge.

Where did our mathematical ideas come from

basic law: law of unity of opposites - makes things move

Contradictions have a main aspect, and another aspect. In any given situation, <u>one</u> contradiction is the main one.

Analysis - To form a correct idea of it, we must:

<u>sum up the main contradictions in it</u>

This is the progressive viewpoint on the axiomatic method the main contradiction must be into a mathematical idea. Having made the axioms, further progress takes place.

There is also a reactionary line: new axioms drop into the minds of great mathematicians; the work of the mathematicians is then "to play games with these axioms".

History of mathematics comes into it

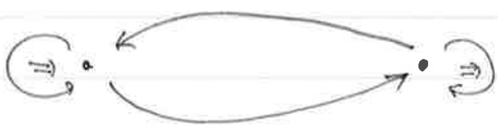
⟦Mejlbo: Unusual use of the word contradiction⟧ Lawvere: Introduced by Hegel; more general than the notion of <u>inconsistency</u>

Adjoint functors a very frequent form of occurrence of contradiction

Not all contradictions are of that form, though

Archimedes, in the Era of Slave Society. But analysis, as we now know it, started around 1650; intensive development, Newton Leibniz, Euler, Bernoulli, ..., Fourier, ..., Weierstrass, Cauchy, Riemann; .. Cantor, Dedekind; .. Poincare, Hilbert, Russell (Cantor's set theory arised out of his study of Fourier series, thus giving rise to "set theory", "type theory" .. )

Also mention Gauss Green Stokes; ~~lead~~ one of the main circles of ideas, that lead to homology, which in turn led to the notion of category (Eilenberg-MacLane) (which I know better; will go into more detail): adjoint functors

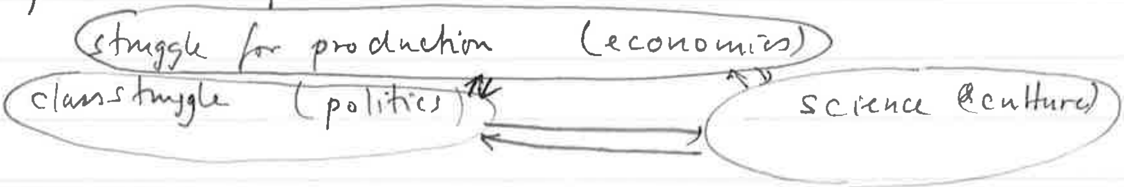


2-dimensional categories (Ehresmann, Benabou)
topos (Grothendieck, Giraud
Closed Categories (Kelly)

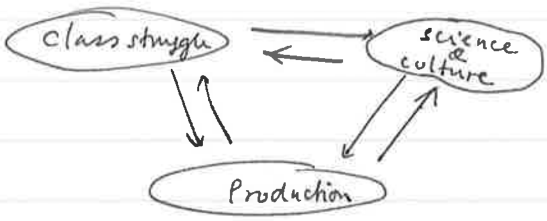
intimately related.

Some development of category theory carried out by people who want to sum of up mathematical practice (Mac Lane), who later (MacLane) took a more dialectical viewpoint on it.
Most Post-Cantor mathematics have been looked at by people who tried to sum up, usually by means of category theory. I hope to do similar thing to classical mathematics
Logic? Logic has not really addressed itself to that problem.

All of society is an organic whole




Ideas of science arise in the struggle for production (struggle with nature




At times when class struggle is acute, the question of science and ideology plays an important role.

All this mathematics we have seen has been developed in class societies (Slave, Feudalist, Capitalist).
Modern analysis was initiated at the time of the burgeoisie revolution (revolution of people living in towns, led by capitalists)
At that time the burgeoisie was the advanced class; their ideas therefore were grasped by the masses: people can find out things (not God alone). So burgeoisie did propaganda <u>for</u> science, <u>against</u> superstition

Development of analysis coincides with rise of burgeoisie; because of the struggle for production (industrial revolution).
Celestial mechanics was important to overthrow clericalism, thus having a purely <u>ideological function</u> (apart from being of use for navigation)
How did burgeoisie carry out the revolution (though a minority)?
Using the workers and peasants - ~~and~~ but after the success, new alliance with feudalism: constitutional monarchy ([ "House of Lords" ] in England. This movement was

also reflected in culture. — This is not my main concern. I am intersted in the place when the picture divedes into 2 : Cantor

Capitalism has built in a tendency away from competition, towards monopolization. Up till Cantor, competition was the leading aspect; around 1890 monopoly became the leading aspect. Monopoly capitalism leads to overproduction, leads to imperialism.

1871: the Paris Commune. Was reflected strongly in science and culture. Superstition is again promoted: IBM will do astrology for you; production is promoted as bad ("pollution").

They also try to prevent science to go forward, because they want to prevent the clarity that: imperialism should be overthrown. So they try to freeze the contradiction, by means of fascism. This is only possible by means of ideological propaganda. Example: The science teacher, at the side, stating that this is all not really serious.

Up till 1871, science was done mainly for production. After that, mainly for class struggle reasons; at this time all the pseudo-sciences were invented (psychology, .. anthropology) — You must understand this in a dialectical way; I am not saying that every anthropologist is a fascist. Likewise: abstract mathematics.

The mathematical part, to start next time, will start with the notion of abstract set. There are two lines on abstraction. The notion of abstract set is more useful than the notion of 'set' given by Zermelo-Fraenkel. (better summing up). – Shall develop: exponentiation, recursion, type theory ... ; constructing Reals; metric spaces.

'Closed categories' is such a good summing up that it even will include the notion of metric spaces, even though it was initially designed for homological algebra.
We shall try to keep in mind the concrete problems of analysis.

[[ Tait: Russell, .. Zermelo did sum up ; their followers did not ]]
FWL: Set theory does not start with history, but from nothing

$0 \; - - - - \; \underset{p}{\bullet} \; \cdots$

and goes on forever ~~sum~~ accumulating so much $P$ at each stage as possible. They try to prove that the process will go on (large cardinals)
This idea comes from somewhere.
When was the problem of large cardinals considered:

1911–13
1930's
1960's

There are also two lines on category theory, the bad one (which is very popular because cat. theory is so simple) is to make it just another one of the abstract games.

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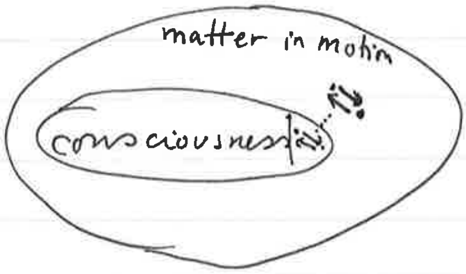
General remark about mathematical method
Basic position: all that exists is matter in motion
Different parts of matter in motion can be in contradiction to each other.
One form of m. i. m. is consciousness




In consciousness, other contradictions between forms of matter in motion is reflected into consciousness (because of the struggle of the people who have the consciousness; the reflection takes place for definite reasons

The contradiction between consciousness, and rest of matter in motion, is again reflected into consciousness: philosophy.
{{ (The leading aspect is in this particular case consciousness; the other is matter. Usually matter is the leading aspect (materialist position)).
Idealism divides into two parts: objective and subjective
The obj. idealism is Platonism; Hegel
Subj. idealism takes what at first seems to be a materialist position, namely empiricism (neutral monism). Perceptions are idealistic. Empiricism leads to clericalism, mysticism.
Engels: Dialectics of nature
Lenin: and Empirio-Criticism (stating: Mach ⇒ Berkeley).
In math.: Intuitionism, Constructivism.

dialectical materialism: unity of opposites is the basic reason for matter in motion; therefore, unity of opposites will exist in consciousness
(dialectical idealism: Hegel).
idealist dialectics

Even though the obj. idealist seem to us under the most [[ biggest ]] claims . . . reached the most correct ~~best~~ result ;
even though obj idealism is wrong, it has captured the
reflected into mathematics : platonism vs intuitionism

The opposite of 'dialectics' is 'metaphysics', more specially 'mechanical materialism' (a specific historic form of it, Laplace)
This position cannot account for qualitative change

Mechanical idealism - may say more about that later

Mathematical method ?


I can only list some features which it certainly has:

ABSTRACTION

So we study systems of interlocking contradiction, temporarily divorced from matter
The mathematical method of abstraction uses the principle of the main contradiction : i.e. regard the tertiary contradictions as identities (example : 2-dim category theory ; homotopies, Taylor series.

UNITY OF OPPOSITES

MAIN CONTRADICTION

With this in mind introduce the notion of
abstract set : a piece of matter totally divorced from matter, or having no internal contradictions (having no structure) (except = and ≠). It will have external contradiction, but these depend only on the cardinality

Two sets with same cardinality cannot be distinguished (except metaphysically). These are what in logic are called sets of individuals

WHEN A THING goes through a development, its development is reflected into internal contradictions

reflected in math, as follows, example: the power set of an abstract set has an internal contradiction: inclusions $\subseteq$. Abstract sets that are constructed by some process will have some internal contradictions (structure)

Things which "do not have" internal contradictions admit only quantitative change

quantitative change may develop into qualitative change

"Classical" set theory maintains: all qualitative change come that way (e.g. Gödel completeness thm)

The theory of sets that I will describe does not claim that there is only one way of developing: $\bigcup$ and $\mathcal{P}$

$\varepsilon$

Given two abstract sets: consider external contradictions: mappings $X \xrightarrow{f} Y$

The axiom of choice is related to the "abstractness": there is no ~~thing~~ structure to prevent sections.

Between mappings, there is also contradiction: composition
Equality of mappings [[ by extensionality ]], the contradiction between $(fg)h$ and $f(gh)$ we declare these

contradiction trivial (in contrast to 2-dim category theory).

In $X \to Y$, $X$ is the leading aspect

You cannot formulate the notion of adjointness between maps between abstract sets

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

except: $f \circ g = id_X$, $g f = id_Y$ (because $=$ is the only relation

Notion of bijection.

Category $\mathbb{S}$

$$\mathbb{S}/X \cong \mathbb{S}/Y$$

iff $X \cong Y$ inside $\mathbb{S}$ (cannot prove it at this stage)

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The "one-element set" $1$ can only be characterized up to iso: [[ terminal object ]]

$$X \xrightarrow{\exists !} 1$$

Philosophical remark about $1$: once we have it, we can also consider maps <u>out</u> of it — they are highly non-trivial.

$$1 \xrightarrow[y]{} Y \quad ; \text{ notation } \quad y \in Y.$$

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Develop further properties of abstract sets, & mappings. Good principle to keep in mind is that abstract sets have a purpose: they can in principle be applied to something, i.e. represent in a given situation more concrete [[phenomena]]

Abstract mappings
In particular, abstract sets should be applicable to themselves; e.g., there should be for any two abstract sets A, B, an abstract set which "index" the set of mappings from A to B.

Abstract sets and mappings form a category (Eilenberg MacLane, 1945 TAMS, (or Pareigis .. [[say]]).
So every mapping f has a definite abstract set as domain, as well as a definite abstract set as codomain (codomain not the same as image; essential). If Codom(f) = dom(g), then fg is also a mapping ...
Associative law. Identity maps for each [[abstract set]]
Possibility of "non-object" approach, replacing objects by their associated identity maps.
Two different notions are involved
1) equality between mappings with same domain and same codomain (e.g. $(fg)h = f(gh)$ )
2) equality of objects, e.g. 'codom (f) = dom (g)'

Ought to be a 1 order language in which we cannot talk about objects as being equal - on the other hand, how to state when maps can be composed.
Every thing we define will only be defined up to these ~~iso~~ isomorphisms. Two possibilities neither of which are good: either not introduce symbols for, say $B^A$; or introduce them - they are neither unique, nor arbitrary, but something

in between. Logicians seem not to take the problem serious.

Particular properties of the category of sets. Summing up the properties in axioms. E.g.:
existence of object $1$ ⟦terminal⟧. The notation $1$ is a case of ambiguity – we don't have to think of $1$ as the cardinal number.

$1 \xrightarrow[y]{} Y$ allows us to forget about our original elements, and take this as notion of element: $y \in Y$. Also "applying $f$ to $x$" becomes a special case of composition

$$\begin{tikzcd} & 1 \arrow[dl, "x"'] \arrow[dr, "xf"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

„evaluating $f$ at $x$".

⟦Extensionality⟧ for mappings gives a further condition on the category: if $f \neq g : X \longrightarrow Y$, then there is a map $1 \xrightarrow{x} X$ s.t. $x.f \neq x.g$:
"$1$ is a generator for $\mathcal{S}$".
This axioms distinguishes $\mathcal{S}$ from many other categories which we intend to look at:
Other important examples of categories which are "very much like" the category of sets (= topos) (cf. Kock & Wraith):
e.g. $\mathcal{S} \times \mathcal{S}$;
objects: ordered pairs of abstract sets
morphisms: " " " mappings.
$1$ is not a generator in this category – consider e.g. the pair $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and consider $Y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
There are two morphisms from $X$ to $Y$; whereas the terminal object is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which has no maps to $X$. However, consider $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; these two objects together form a

family of generators. $\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ are in this category the (only) subobjects of $1$; (they form a boolean algebra) So in $\mathcal{S}\times\mathcal{S}$, subobjects of $1$ generate.

Second example: $\mathcal{S}^{\circlearrowleft}$:

objects: pairs $X, t$ where $t$ is an endomorphism of $X$
morphism: a morphism will be the following picture in $\mathcal{S}$ itself

$$X^{\circlearrowright t} \xrightarrow{f} Y^{\circlearrowright u} \quad \ni \quad tf = fu.$$

morphisms in this category compose (by using associativity of composition in $\mathcal{S}$ three times).

This is again a topos; $1$ and its subobjects together does not generate; it has a generator – this is an infinity axiom for $\mathcal{S}$!

Things with internal contradictions are constructed out of things that don't.
'Endomorphism' (external self-contradiction) is reflected ~~gives rise~~ to an internal contradiction – e.g. the question whether $x$ equals $xt$, or equals $xtt$ ...

The method is to study the internal contradictions externally

Sometimes one can determine the internal contradiction completely by external means (e.g. for the categories of sets, groups, and topological spaces).

Going faster from now. Lawvere 15/9-71

From the outside to the inside :

ex: external contradiction $1 \xrightarrow{y} Y$ - has an internal reflection, the element of $Y$ described.

Meaning also: people who are conscious which act with matter, internal contradictions are formed (ideas in consciousness).

$X \xrightarrow{f} Y$ - reflects contradictions inside $X$ as well as inside $Y$; inside $Y$ the contradiction "being a value of $f$" or "not being a value of $f$" ("contradiction of additive type"):

$$\begin{tikzcd} & 1 \arrow[dl, dashed, "x"'] \arrow[d, "y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

meaning existence of an $x$

For the more general [toposes] I described is more ambiguous (has other truth-values than 'true' and 'false').

Notion of "$f$ being surjective". - the reflected internal contradiction in $Y$ is then not new [trivial]. (though: a function from $Y$ into cardinal numbers, "cardinality of counterimage).

There are several notions which in the $\mathcal{S}$ case coincide

Inside $X$: we have the following internal contradiction of "multiplicative type" $x \underset{f}{\equiv} x'$ iff $xf = x'f$.

Is this the only internal contradiction on $X$ that $f$ can induce?

If the above contradiction "vanishes", $f$ is injective.

Probably you can define no further contradictions on $X$ in the case of injective $f$

Can be reflected as a precise mathematical questions, via the structure functor: Consider morphisms of "sets equipped with a subset". Defines a category; which functors exist from this category to itself.

Justification for drawing the picture for an injective map.

"$f$ is a subobject (subset) of $Y$" means by def. that $f$ is injective and $Y = \mathrm{codom}(f)$.

The class of subobjects of $Y$ have lots of internal contradictions (therefore don't form a set). One of the internal contradictions is the notion of <u>inclusion</u>

$$\begin{tikzcd} \cdot \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & \cdot \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

$f \subseteq f'$ iff $\exists h$ s.t. $f = h.f'$. This is a reflexi[ve] and transitive relation on subobjects of $Y$. There derived equivalence relation: $f \equiv f' \Leftrightarrow f \subseteq f'$ and $f' \subseteq f$ is not equality.

<u>Exercises</u> 1) if $f = h.f'$ and $f'$ is injective, then $h$ is unique (if it exists) 2/ $f$ injective, $f'$ arbitrary, then $h$ is injective

Suppose $f'$ subobj. of $Y$ with domain $X'$, $h$ is a subobj of $X'$. Then $h f'$ is a subobj of $Y$; $h$ is <u>not</u> (in general) a subobject of $Y$ (for technical reasons); so a "subobject of a subobject of $Y$" is not a subobject of $Y$.

I denoted $1 \xrightarrow{y} Y$ by $y \in Y$ ("elementhood") There is also the notion of membership

$$\begin{tikzcd} 1 \arrow[rr, dashed] \arrow[dr, "y"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd}$$

write $y \in f$ ; so

$y \in f$ iff $y \subseteq f$ and $\mathrm{dom}(y) = 1$ and $f$ injective.

So elementhood is a special case of membership which is ~~also~~ a special case of inclusion. Can claim historical justification — Banach e.g. wrote $y \subseteq f$ for $y \in f$.
Mejlbo: Dedekind did not make the distinction either.

If $f \subseteq f'$, if $y \in f$ then $y \in f'$

<u>Proof</u>

$$\begin{tikzcd} 1 \arrow[r, dashed] \arrow[dr, "y"'] & X \arrow[d, "f"] \arrow[r, dashed] & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

Is the converse true? Not in general. There are many different axioms that will imply this. One of them is "axiom of choice":
<u>AC</u> Any surjective map has a section.
(In some sense this is implied by the notion of abstractness).

"There exists a set which can index such and such.."

$$\begin{tikzcd} \exists\,\text{section} \arrow[r, Rightarrow] & \text{surjective} \arrow[r, Rightarrow] & \text{epimorphism} \arrow[ll, bend left, Rightarrow, "AC"] \end{tikzcd}$$

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Reading list:

Philosophy: Mao On Practice

On Contradiction

Where do correct ideas come from

Engels Dialectics of Nature

Lenin Materialism and Empirio Criticism

Math: Lawv.: Elementary Theory of the Cat of sets

(PNAS.USA ; extended version: Chicago Notes

(rather out of date)).

-"- : Quantifiers & Sheaves, Proc of 1970 Internat Congress

-"- : Adjointness in Foundations, 1967

Kock & Wraith: Elementary Toposes.

Benabou: - notes

Benabou: Several writings on the subject of bicategories, in particular in Springer Lecture Notes Vol. 47.

Day: Closed Functor Categories SLN, Midwest Cat. Sem. No 4.

Busemann: Geometry of Geodesics

Federer: Geometric Measure Theory.

Hausdorff: Mengenlehre.

Back to question of maps: $X \xrightarrow{f} Y$ arbitrary map

1) Reflected internal contradiction in $X$, namely an equivalence relation

the "vanishing" of this contradiction means $f$ injective.

2) Reflected int. contradiction in $Y$ are richer "$y \xrightarrow{t} y'$" should mean: any mapping from $|f^{-1}(y)|$ to $|f^{-1}(y')|$.

- where inverse image is defined as follows

$$\begin{tikzcd} {|f^{-1}(y)|} \arrow[r] \arrow[d, "f^{-1}(y)"'] \arrow[dr, phantom, "///"] & 1 \arrow[d, "y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

[pull-back].

in particular, for elements $1 \xrightarrow{x} X$, $y = x.f \Rightarrow x \in f^{-1}(y)$

Various derived contradiction, e.g. we can say that $y \underset{f}{\le} y'$ if there is a $t : y \to y'$ which is injective
In particular, we can derive the contradiction
$f^{-1}(y)$ empty or not empty.
But this is only a derived (secondary) contradiction.
To say that $f$ is <u>surjective</u> just means that this <u>secondary</u> contradiction vanishes.

What is the totality of induced contradictions can be made into a mathematical problem. We study this by looking at the category $\mathcal{S}^2$ : objects are arbitrary mappings

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd}$$

and as morphisms

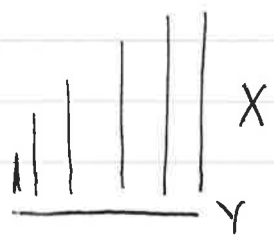
$$\begin{tikzcd} X \arrow[r, "\varphi_0"] \arrow[d, "f"'] \arrow[dr, phantom, "///"] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\varphi_1"'] & Y' \end{tikzcd}$$

commutative squares

obvious how to compose such squares.

"Find a complete set of invariants for isomorphism types in this category"
Can be solved if we can find canonical representatives.
How can this be done, since the objects have no structure?
By putting more structure on them, making them ⟦rigid⟧.

Example: $Y$ is a cardinal number — conceive it as an ordinal, i.e. put a ⟦well-ordering⟧ on $Y$. , $\underset{0}{\le}$
Also require, for $y \underset{0}{\le} y'$, require $y \underset{f}{\le} y$
Also, the fibres should each have a well ordering. (induced well ordering) (unique)
Then any abstract $X \to Y$ is isomorphic to <u>one</u> of these "structured" mappings.



Assertion: such are canonical forms of mappings [obj's in $\mathbb{S}^{\mathbf{2}}$]

Earlier I mentioned $\mathbb{S}\times\mathbb{S}$ (may also be denoted $\mathbb{S}^2$) as well as $\mathbb{S}^{\mathbf{2}}$ (invariants for this cat. were investigated by Freyd in Batelle 1968 (SLN 99).

$$\mathbb{S},\quad \mathbb{S}^{\mathbf{2}},\quad \mathbb{S}\times\mathbb{S},\quad \mathbb{S}^{\mathbf{2}}$$

Also, for a fixed set $B$, form $\mathbb{S}/B$: objects are maps to $B$; morphisms are commutative triangle.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \arrow[d, phantom, "="] & Y \arrow[dl] \\ & B & \end{tikzcd}$$

There is a functor "take codomain" (Then $\mathbb{S}/B$ can be defined as pb

$$\begin{tikzcd} \mathbb{S}/B \arrow[r] \arrow[d] & \mathbb{S}^{\mathbf{2}} \arrow[d, "\text{"take codomain"}"] \\ \mathbb{1} \arrow[r, "B"'] & \mathbb{S} \end{tikzcd}$$

).

Objects in $\mathbb{S}/B$ could be thought of as $B$-indexed families of sets. However, this latter notion suggests rather something like $B \longrightarrow \mathbb{S}$, so to do this one should "pull $B$ out of $\mathbb{S}$ to make it into a discrete category". This is an <u>external</u> thing.
Replacement scheme would say that every definable such family comes from a map (internal) $X \longrightarrow B$. If for <u>every</u> such thing we could internalize, this would be an "inaccessibility of $\mathbb{S}$"

We can work much better with $\mathbb{S}/B$. Look geometrically at it; then $X \to B$ implies smoothness, the $B \to \mathbb{S}$ point-of-view does not, so has a deeper structure; so there is also a <u>qualitative</u> difference between large and small: internal things have more structure.

The equation $y^2 = x^3 + k$ defines a curve

$$\begin{array}{c} X = \{ \langle x, y, k \rangle \mid \text{equation holds} \} \\ \big\downarrow \\ B = \{ k \} \end{array}$$

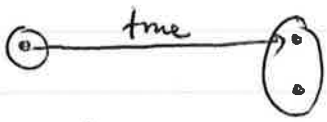
is a family of curves indexed by $B$, but they vary smoothly with $k$, so are richer than just a $B$-indexed family of curves.

The topos-idea is a geometrization of higher order logic. The main contradiction in it is between higher order and logic, and will constitute the definition of the notion of topos.

logic: there is an object $\Omega$, and a particular elt.

$$1 \xrightarrow{\text{true}} \Omega$$

(e.g. in $\mathcal{S}$, $2 = \Omega$




).

with the basic property: any subset $i$ has a unique [[characteristic function]] $\varphi$

$$\begin{tikzcd} X' \arrow[r] \arrow[d, tail, "i"'] & 1 \arrow[d, "\text{true}"] \\ X \arrow[r, "\varphi"'] & \Omega \end{tikzcd}$$

$$i = \varphi^{-1}(\text{true}).$$

Everything we need to know about logic will follow from this! (Maybe a bit exaggerated). Exercise: $\Omega$ is determined up to unique iso.

Exercise: $\Omega$ exists in $\mathcal{S}^{\mathcal{C}}$. What is it?

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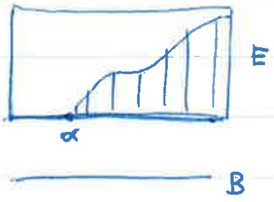
Forgot one in the list of literature: Pareigis: Kategorien und Funktoren (does not over-emphasize the notion of abelian categories) Also Fréchet, in Rendiconti Circ. de Palermo 1906 (notion of metric space).

Back to question of "structure of a map" $E \xrightarrow{p} B$; I was not quite correct




For cardinal form, $\alpha$ should be the cardinal of the complement of the image of $p$. - Similar things happens also later on, so one should also impose a continuity condition.

- So the problem is still unsolved - but we don't use this in the rest of the course

Universality of contradiction. Look at evidence for it in math. Category theory is more explicitly dialectical than rest of math. Why do I say 'contradiction' and not 'structure'. (or rather 'internal contradiction'). For basic kinds of structure, they all form categories (observation 25 years old). But the structures themselves are categories.

Example: The map $E \to B$ defines a category whose objects are the elements of $B$: a map $b \to b'$ being an ordinary map $E_b \to E_{b'}$. Likewise the structure induced on $E$ has the structure of a category: $\equiv$ ("there is an arrow $e \to e'$ if $e \underset{p}{\equiv} e'$ in $E$

Constructibility in $\mathcal{S}$ - ~~what show~~ when sets are equipped with some external structure. Might have to do with "structure of a map" - Is composite of two constructibles again constructible?

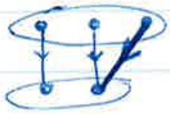
Is it a meaningful property that the "canonical form" of a map be constructible. (To make precise sense to it, take $\mathcal{S}$ to be category of ZF-sets (in a model, where $V \neq L$)).
Research Problem? And what should the continuity conditions be?

Notion of topos; main internal contradiction is higher order vs. logic
'Logic' sums up the contradiction between
subset and characteristic map

$$\begin{tikzcd} \cdot \arrow[d, tail] \arrow[r, phantom, "\leftarrow \text{Logic}"] & 1 \arrow[d, "\text{true}"] \\ X \arrow[r, "\varphi"'] & \Omega \end{tikzcd}$$

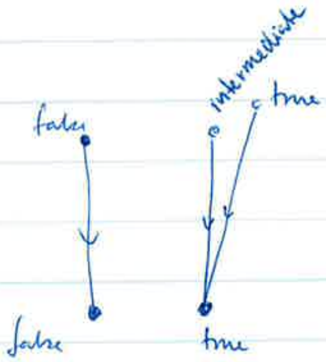
In $\mathcal{S}$, there is an internal logic: $\Omega = 2$.

In $\mathcal{S}^{\Omega}$, $\Omega =$

In $\mathcal{S}^{2}$, $\Omega =$  the [bold arrow] being the true

characteristic function $\varphi$ of $\partial$

$$\begin{tikzcd} X_0' \arrow[r, tail] \arrow[d] & X_0 \arrow[d, "p"] \\ X_1' \arrow[r, tail] & X_1 \end{tikzcd} \xrightarrow{\varphi}$$




$$\begin{cases} x_1\varphi_1 = \text{true iff } x_1 \in X_1'. \end{cases}$$

$$\begin{cases} x_0\varphi_0 = \text{true iff } x_0 \in X_0' \\ x_0\varphi_0 = \text{intermediate iff } x_0 \notin X_0' \text{ and } x_0 p \in X_1' \\ x_0\varphi_0 = \text{false iff } x_0 \notin X_0' \text{ and } x_0 p \notin X_1'. \end{cases}$$

For the $\mathcal{S}^{\circlearrowright}$ :


What is a subobject?
A sub"set" stable under $t$ :
$X' \subseteq X$.

If $\Omega$ would only have two elements $t_{\Omega}$ would have to be the identity
But elements outside $X'$ may move inside $X'$ by applying $t$.

$$\Omega = \{-\infty, \ldots, \ldots, -2, -1, 0 = \text{true}\}$$

with endomorphism

$$-n \rightsquigarrow -(n-1)$$
$$0 \longrightarrow 0$$
$$-\infty \rightsquigarrow -\infty.$$

for a subset $X' \subseteq X$ (stable under $t$), $\varphi$ is going to be
$x \rightsquigarrow -n$ iff $n$ is the smallest number such that $x t^n \in X'$.

Next time, to given $\varphi : X \longrightarrow \Omega$, construct subobject of $X'$.
Generalize this to non-monic mappings.

$$\begin{tikzcd} E \arrow[d, "p"'] & \\ X \arrow[r, "[\![p]\!]"'] & \Omega \end{tikzcd}$$

(going to mean essentially char. funct. of the image of $p$).

$x [\![p]\!] = \text{true}$ iff $x$ "$\varepsilon$" $p$
$x \cdot \varphi = \text{true}$ iff $x \varepsilon \{x \mid \varphi\}$ for the monic case.

Lawvere 27/9-1971.

$\mathcal{S}$, $\mathcal{S}\times\mathcal{S}$, $\mathcal{S}^{2}$, $\mathcal{S}/B$.

are closely related to the category of sets;
they are toposes.

A few more examples: we can consider various subcategories of $\mathcal{S}^{2}$, e.g. cat of $(X, t)$ where $t$ is invertible (notation for these categories:

$$\mathcal{S}^{\mathbb{Z}} \subsetneq \mathcal{S}^{\mathbb{N}}.$$

In $\mathcal{S}^{\mathbb{Z}}$, $\Omega$ is just a two-element set, but can. generator is infinite

Also

$$\mathcal{S}^{\mathbb{Z}_2} \subseteq \mathcal{S}^{\mathbb{Z}}$$

where $t$ is of order two (or one), $\Omega$ is a two-elt. set, can generator is finite.

Also included in $\mathcal{S}^{\mathbb{Z}}$ is

$$(\mathcal{S}^{\mathbb{Z}})_{fin}$$

(rather an *image* of $\mathcal{S}^{\mathbb{Z}}$, in a technical sense), consisting of $(X, t)$

where all orbits of $t$ are finite.

This latter topos has the property that it has generators, but they are not projective.

In $\mathcal{S}^{2}$, $1$ is not projective; $1$ "gives" only fix points for the axioms; $(\mathbb{N}$, successor$)$ "gives" arbitrary elements of objects of $\mathcal{S}^{2}$. It is a generator in $\mathcal{S}^{2}$, and it is projective (A is called projective iff every epimorphic f is A-surjective)

The point is that $(\mathcal{S}^{\mathbb{Z}})_{fin}$ has a canonical set of generators, but these are not projective ($(\mathcal{S}^{\mathbb{Z}})_{fin}$ has *no* nonempty objects which are projective). The set of generators are

$$\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \ldots$$

One can, however, say that if $X \xrightarrow{f} Y$ is surjective and $Z_n \xrightarrow{y} Y$, there is a diagram

$$\begin{tikzcd} Z_m \arrow[r] \arrow[d] & Z_n \arrow[d, "y"] \\ X \arrow[r, "f^*"'] & Y \end{tikzcd}$$

There is a general pattern: if $M$ is a monoid (i.e. an abstract set $M$ equipped with

$$1 \xrightarrow{\eta} M \xleftarrow{\mu} M \times M$$

such that ...

Then consider $\mathcal{S}^M$; its objects are sets $X$ together with an action

$$M \times X \xrightarrow{a} X$$

such that ...

$\mathcal{S}^M$ will be a topos; $M$ will be a projective generator in it; $\Omega$ can be identified with the set of those subsets of $M$ which are invariant under translation.

Another example is $\mathcal{S}^\omega$; an object is a sequence of abstract sets and a sequence of mappings:

$$X_0 \xrightarrow[t_0]{} X_1 \xrightarrow[t_1]{} X_2 \longrightarrow \cdots \quad ;$$

will also be a topos (much like $\mathcal{S}^{\mathbf{2}}$). Have a common abstraction; if $P$ is a partially ordered set, i.e.

$$p \le p \qquad (\text{"unit law"})$$

$$p \le q \wedge q \le r \Rightarrow p \le r$$

$\mathcal{S}^P$ is ...

it is a topos, and has a family of projective generators (indexed by the elements of $P$); they will be subobjects of $1$.

$\Omega_p$ = set of all subsets of $P$ consisting of elements beyond $p$ and which are "order ideals".

We shall later see generalizations in the direction $(\mathcal{S}^{\mathbb{Z}})_{fin}$.

It is a principle for how we should axiomatize set theory: it should apply as far as possible to these examples also.

The objects we have considered are after all just sets which are moving a bit. We have intuition (from experience) of set-"nature" about these "sets". (which are mathematically interesting structures).

Also, the axiomatization should be relativizable, i.e. if we apply one of the above constructions to $\mathcal{E}$ rather than to $\mathcal{S}$, we should again get a [[topos]]

"higher order vs logic" — is roughly the notion of topos

vs.

number theory — "analysis"

Logic: There is a map $1 \xrightarrow{\text{true}} \Omega$. It has two properties

1) for every map $\varphi : X \longrightarrow \Omega$, there is a [[pull-back]]

$$\begin{tikzcd} X \arrow[r, "\varphi"] & \Omega \\ \{X|\varphi\} \arrow[u, "i"] \arrow[r] & 1 \arrow[u, "\text{true}"'] \end{tikzcd}$$

i.e.

$$\forall A\ \forall x : A \longrightarrow X \qquad x.\varphi = \text{true}_A \qquad \Rightarrow \qquad [x \in \{X|\varphi\}]$$

meaning in a unique way
i.e. $\exists! \bar{x}\ [\bar{x}.i = x]$

2) for every $E \xrightarrow{p} X$,
there is a (unique, in fact) $\varphi : X \longrightarrow \Omega$ such that
for all $\psi : X \to \Omega$, there is a map

$$\begin{tikzcd} E \arrow[rr] \arrow[dr, "p"'] & & \{X|\psi\} \arrow[dl] \\ & X & \end{tikzcd}$$

iff $\varphi \leq \psi$ — where the inequality means

$\varphi \leq \psi$ for $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \Omega$ means that

$$\forall A\ \forall x : A \longrightarrow X \quad [\, x.\varphi = \mathrm{true}_A \implies x.\psi = \mathrm{true}_A \,]$$

or, equivalently, that

$$\{X \mid \varphi\} \subseteq \{X \mid \psi\}.$$

The map asserted to exist for $E \xrightarrow{p} X$ will be denoted

$$X \xrightarrow{[\![p]\!]} \Omega$$

What does "higher order" mean? Basically that we can construct higher types. This actually divides into two

$$A \times (-) \rightleftarrows (-)^A$$

Each of these two themselves "divide into two."

Product For every pair of objects $Y_1$ and $Y_2$, there is a diagram

$$\begin{tikzcd} & Y_1 \\ Y_1 \times Y_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & Y_2 \end{tikzcd}$$

likewise for any $X$, there is a map $\delta_X : X \longrightarrow X \times X$.

Also, for a pair of maps

$$f_1 : X_1 \longrightarrow Y_1 \quad , \quad f_2 : X_2 \longrightarrow Y_2$$

there is a map

$$f_1 \times f_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$$

The contradiction between proj's and diagonals is an internal reflection of an external contradiction ..

projections and diagonal ... there should be a 1-1 correspondence

$$\frac{X \longrightarrow Y_1 \times Y_2}{X \longrightarrow Y_1 \qquad X \longrightarrow Y_2}$$

where the passage should be effected by the proj's and the diagonal.

Can be expressed by just stating that the projections have a universal mapping property : ...

Can then deduce $\delta$ and $f_1 \times f_2$ ....

Exercise $(f_1 \times f_2) \cdot (g_1 \times g_2) = f_1 \cdot g_1 \times f_2 \cdot g_2$ whenever it makes sense

using $x . \langle f_1, f_2 \rangle = \langle x f_1, x f_2 \rangle$

Also $\langle x_1, x_2 \rangle . (f_1 \times f_2) = \langle x_1 f_1, x_2 f_2 \rangle$.

All this can be expressed by saying that

the diagonal functor has a right adjoint $\times$

$$\mathcal{S} \underset{\Delta}{\overset{\times}{\rightleftarrows}} \mathcal{S} \times \mathcal{S}$$

(think on '$\mathcal{S}$' as ambiguously denoting the category of abstract sets, or as denoting an arbitrary topos).

$\times$ and $\Delta$ are unified by the $\pi$'s and the $\delta$.

$$X \xrightarrow{\delta_X} X \times X \qquad Y_1 \times Y_2 \qquad (Y_1, Y_2) \xleftarrow{\pi = \langle \pi_1, \pi_2 \rangle} \langle Y_1 \times Y_2, Y_1 \times Y_2 \rangle$$

The other half of the higher order goes into a similar kind of "unity of opposites"

$$\mathcal{S} \underset{A \times (\ )}{\overset{(\ )^A}{\rightleftarrows}} \mathcal{S}$$

$$X \xrightarrow{\lambda^A} (A \times X)^A \qquad Y \xleftarrow{\epsilon_Y^A} A \times Y^A \qquad \text{("evaluation")}.$$

Properties of this can be written as [[4]] algebraic equations, but also on the form: there is a 1-1 correspondence

$$\frac{X \longrightarrow Y^A}{A \times X \longrightarrow Y}$$

When 'higher order' interacts with 'logic', a lot of things break loose.

Mayoh: can 1) and 2) also be framed as an adjointness

Lawv: Yes, by means of representability of ~~adjoint~~ partial maps (logic) (for the higher order, may strengthen to existence of $\Pi_f$ [[ Weyl-extension ]]).

Lawvere 29/9 1971

Example: Consider endomappings $t$, with the property that for each $x$, there is an $n$ such that $x t^{n+1} = x t^n$. It has the same $\Omega$ as the full $\mathcal{S}^{\mathbb{Q}}$. But in this smaller subcategory $(\mathcal{S}^{\mathbb{Q}})_{\text{terminating}}$, the subobjects of $\Omega$ generate. Typical subobject of $\Omega$:

$$T_n = \{-n, -(n-1), \ldots, -1, 0\}$$

You have maps between the $T_n$'s

$$T_n \rightarrowtail T_{n+1}$$

all embedded in $\Omega$ in a canonical way, except $T_0$ which have two different embeddings ("true" and "false").

The process by which we get $(\mathcal{S}^{\mathbb{Q}})_{\text{terminating}}$ from $\mathcal{S}^{\mathbb{Q}}$ is a case of a general process, "biggest possible quotient where the ~~objects become~~ given set of objects become generators).

$$(\mathcal{S}^{\mathbb{Q}})_{\text{term}} \rightleftarrows \mathcal{S}^{\mathbb{Q}}$$

$$(\mathcal{S}^{\mathbb{Z}})_{\text{fin}} \rightleftarrows \mathcal{S}^{\mathbb{Z}}$$

The functors coming back are just "take set of those ell's that have finite orbit."

In an arbitrary topos $\mathcal{E}$, for $B \in \mathcal{E}$, $\mathcal{E}/B$ will be a topos.

Exercises 1) For $\mathcal{E} = \mathcal{S}^{\mathbb{Q}}$, $B = (\mathbb{N}, \text{successor})$, then $\mathcal{E}/B \simeq \mathcal{S}^{\omega}$

More generally, for a commutative monoid with cancellation $M$, let $P(M)$ be the ordered set of $M$'s elements, ordered by divisibility; then

$$* \qquad \mathcal{S}^M/M \simeq \mathcal{S}^{P(M)}$$

The definition of $\mathcal{S}^{\omega}$ was a little unsatisfactory, because it involved an external infinite family. However, the exercise shows that it can be defined internally. So it is a 1-order passage from $\mathcal{S}$ (equipped with an $N$) to $\mathcal{S}^{\omega}$. This is possible because of replacement: every external family can be internalized. For our point of view, the correct definition of $\mathcal{S}^{\omega}$ is

$$\mathcal{S}^{\circlearrowright}/\mathbb{N}$$

2) Take $\mathcal{E} = \mathcal{S}^{\mathbb{Z}}$, $B = \mathbb{Z}$; then $\mathcal{E}/B \simeq \mathcal{S}$:

$$\mathcal{S}^{\mathbb{Z}}/\mathbb{Z} \simeq \mathcal{S}$$

Same thing holds for any other group (of course a special case of the general statement in $*$).

<u>Remark</u> $\mathcal{S}^{\mathbb{Z}}$ can be deduced from $\mathcal{S}^{\circlearrowright}$ by a general process. In general, the inner logic $\Omega$ of a topos $\mathcal{E}$ is intuitionistic. However, there is a way of forcing it to be classical (by "double negation"); get new topos $\mathcal{E}_{\neg\neg}$. Claim

$$(\mathcal{S}^{\circlearrowright})_{\neg\neg} \simeq \mathcal{S}^{\mathbb{Z}}.$$

Conjecture: for a monoid $M$

$$(\mathcal{S}^{M})_{\neg\neg} \simeq \mathcal{S}^{Gr(M)}$$

where $Gr(M)$ is the groupification of $M$:

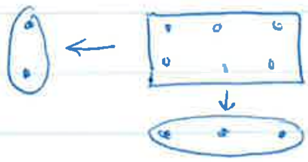
$$\begin{tikzcd} M \arrow[r] \arrow[dr, "\forall"'] & Gr(M) \arrow[d, dashed, "\exists!"] \\ & G \end{tikzcd}$$

"Reversible dynamics "is the same" as classical logic".

More about prod's and exponentiation

Purpose of abstract sets is to index real things (or less abstract sets). In particular, abstract sets should be able to index things we produce abstract sets; e.g. $A \times X$ indexes pairs of elements of the completely abstract sets $A$ and $X$.

(All we have assumed about the product is its universal property; so therefore the product of a set with two elements with a set ~~of~~ with three elements is a 6-element set, but arranged in a certain way, namely as a $2 \times 3$ rectangle.



Things that come about by a process will have internal contradictions reflecting that process.

What is the use of having products? Precisely the fact that the product is entirely determined by knowing into it; but once it is there, we have a qualitatively new possibility, namely to consider maps out of the product:

PRODUCTS in $\mathcal{S}$ $\Rightarrow$ ALGEBRA CAN BE DONE INTERNALLY in $\mathcal{S}$

Same type of remarks apply to exponentiation

$$Y^A$$

a set big enough containing enough elements to index all the maps from $A$ to $Y$.

For any map $A \times X \xrightarrow{f} Y$ and any $1 \xrightarrow{x} X$, there is a map $f_x : A \longrightarrow Y$ (easy to construct using the machinery of finite products). So any binary operation indexes some of the maps from $A$ to $Y$. But such an $f$ may be deficient in indexing all the maps from $A$ to $Y$. It may also be deficient in indexing them too often. The idea of $Y^A$ is that it not deficient in either respect; it is the best set, indexing maps $A \longrightarrow Y$; it is the best one in the sense ...

$$A \times Y^A \xrightarrow{\epsilon^A_Y} Y$$

Now things can be done internally. The leading aspect of "higher order": we can now consider functionals,

$$Y^A \longrightarrow Z$$

So

EXPONENTIATION in $\mathcal{S}$ $\Rightarrow$ FUNCTIONAL ALGEBRA is internal in $\mathcal{S}$.

($\mathcal{S}$ can in fact be any category with exponentiation).
May even consider higher things

$$(Y^A)^{(Z^B)} \longrightarrow C \qquad \text{etc.}$$

You can almost say "analysis is possible" - except that analysis involves numbers as well.

Tait: Once you have exponentials, are product then still necessary?
Lawv: There is a trend in logic to be metaphysical, i.e. one-sided, instead of 2-sided. Like when intellectual "marxists" say "there is nothing but economics" (and neglect the non-leading aspect, the superstructure.
Also, products are still there for practical purposes. Also if you only have exponentiation, you have to write down much more axioms. In Principia Mathematica, there is only the implication, and [therefore] the axioms become unperspicuous. In closed category theory, Eilenberg and Kelly tried to get through with hom alone, but did not really succeed. (had to introduce $\otimes$).
How to formulate cartesian-ness of the non-existent product

(Kock: That is what the combinators S and K do).

Remark: The interesting thing — even though the "functional algebra" has interesting consequences: can for instance express double dualization:

$$** \qquad X \longrightarrow D^{(D^X)}$$

the existence of this map follows from what we assumed about product (symmetric!) and exponentiation.

$$\frac{\dfrac{X \longrightarrow D^{(D^X)}}{D^X \times X \longrightarrow D}}{X \times D^X \xrightarrow[e]{} D}$$

$D^X$ is in fact functorial in both variables (namely contravariantly in the exponent). Then the ** also follows from the adjointness of

$$D^{(\ )}$$

to itself on the right (namely as the adjunction map).

— The interesting way in which higher order and logic interact is the fact that we can form the power set of X:

$$\Omega^X$$

$$X \times \Omega^X \xrightarrow{\ \epsilon\ } \Omega$$

where

$$1 \xrightarrow[\langle x, \ulcorner\varphi\urcorner\rangle]{} X \times \Omega^X$$

has the property $x \in \{x \mid \varphi\}$ iff $\langle x, \ulcorner\varphi\urcorner\rangle . \in\ = \text{true}$.
So can write $[\![ x \in \varphi ]\!]$ for this composite. —
Maps $A \longrightarrow \Omega^X$ correspond (by basic contradiction of "higher order", ~~as~~ to $X \times A \longrightarrow \Omega$, and then to subobjects of $X \times A$. Next time discuss singleton map $X \xrightarrow{\{\cdot\}} \Omega^X$.

Lawvere 4. okt. 1971

The most obvious reason why we need products is that they occur in mathematics.

A set theorist may tell you that $\mathbb{R} = \Omega^{N}$, and that $\mathbb{R}\times\mathbb{R} = \Omega^{N}$

But they are not homeomorphic. (Brouwer).

There will be toposes, however, where all maps $\mathbb{R} \longrightarrow \mathbb{R}$ are continuous, so is not $\Omega^{N}$.

Secondary aspect of "higher order" is just the products.

Consider $1$, $\times$ and $\Omega$. Which results can be proved by using these things together? Various kinds of finite limits can then be constructed: equalizers, pull-backs, inverse images, intersections

Equalizers:

$$E \xrightarrow{i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

$if = ig$

and with a universal property: $x.f = x.g \Rightarrow \exists! \bar{x}\, [x = \bar{x}.i]$

Any equalizer is a subobject of $X$.

Pull-backs: for two maps with common codomain, $f, y$:

$$\begin{array}{ccc} & & C \\ & & \downarrow y \\ X & \xrightarrow{f} & Y \end{array}$$

commutative square with universal property. We have assumed one special case of existence of a pull-back, namely for $Y = \Omega$, $y = \text{true}$.

Proposition If in a pull-back

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ {\scriptstyle y'}\downarrow & & \downarrow{\scriptstyle y} \\ \cdot & \longrightarrow & \cdot \end{array}$$

$y$ is mono, then so is $y'$. In particular, pulling $1 \xrightarrow{\text{true}} \Omega$ back

4/10 e

gives a mono, since any map out of 1 is mono.

Consider a pull-back

$$** \qquad \begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "y'"'] & C \arrow[d, "y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Form

$$\begin{tikzcd} & & C \arrow[dr, "y"] & \\ P \arrow[r, "\langle y', f'\rangle"] & X \times C \arrow[ur] \arrow[dr] & * & Y \\ & & X \arrow[ur, "f"'] & \end{tikzcd}$$

Proposition $\langle y', f'\rangle$ is the equalizer of the diagram $*$, so if we have products and equalizers, we have pull-backs.
In particular, $y', f'$ is jointly monic, and the pull-back is a subobject of a product.

Taking $Y = 1$, 'product' is a special case of 'pull-back'.

If we have pull-backs, we can also construct equalizers (provided we have 1),

$$\begin{tikzcd} \text{pull-backs, } 1 \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{products} & \\ & \text{product, } \cancel{\text{equalizers}} \text{ pull-backs} \arrow[u, Rightarrow] \arrow[r, Rightarrow] & \text{equalizer} \end{tikzcd}$$

namely as intersection of the graphs. To explain the notion of 'intersection' and 'graph'. If in ** above, $f$ and $y$ are mono, then we can conclude that the map $P \longrightarrow Y$ is monic, so $P$ is intersection - subobject in $Y$ (meet in lattice of subobjects (actually only lower semi-lattice)). "Graph": For $f : X \longrightarrow Y$, can form $X \xrightarrow{\langle X, f\rangle} X \times Y$. Follow by projection to $X$ gives identity map, hence $\langle X, f\rangle$ is monic. So the graph is a subobject of the product, with domain $X$.

The pull-back (for given $f, g : X \longrightarrow Y$)

$$\begin{tikzcd}
P \arrow[r, "f'"] \arrow[d, "g'"] & X \arrow[d, "\langle X, g\rangle"] \\
X \arrow[r, "\langle X, f\rangle"'] & X \times Y
\end{tikzcd}$$

will have $f' = g'$, and that map will be the equalizer of $f$ and $g$.

Another important example of a pull-back is the equivalence relation generated by a map: the pull-back of that map with itself

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, dashed, bend left, "x_2"] \arrow[ddr, dashed, bend right, "x_1"'] & & \\
& E_f \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd} \qquad *$$

In general, ~~$p_1 \neq p_2$.~~ $E_f$ will be a subobject of $X \times X$; it will contain $\delta_X = \langle X, X\rangle : X \rightarrowtail X \times X$ ("reflexivity"). Likewise, it is symmetric: $E_f$ has an automorphism $s$, which you get by putting $x_1 = p_2$, $x_2 = p_1$ in $*$ (with $A = E_f$).
To state its transitivity is a somewhat bigger diagram.
Form pull-back:

$$\begin{tikzcd}
E_f^{(2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & E_f \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[dd, dashed, "f"] \\
E_f \arrow[r, "p_2"'] \arrow[d, "p_1"'] & X & \\
X \arrow[rr, dashed, "f"'] & & Y
\end{tikzcd}$$

The two outer composites are equal, giving rise to a map

$$t : E_f^{(2)} \longrightarrow E_f$$

"The proof of transitivity". Analyze now for any ["test-object"] $A$ the set of maps from $A$ to $E_f$ and $E_f^{(2)}$;

Maps $A \longrightarrow E_f^{(2)}$ will be indexed by triples $\langle x, x', x'' \rangle$ with $xf = x'f$, $x'f = x''f$

In any topos, a pair of maps $p_1, p_2$ which in this way is jointly monic, refl., symm., transitive comes about as pull-back of a map by itself. One uses also the exponentiation to prove it.

'Inverse image' is a special case of pull-back
Pull-backs have a symmetric aspect as well as an unsymmetric
Symmetric: products in $\mathcal{S}/Y$ are pull-backs in $\mathcal{S}$.
On the other hand, for $f: X \longrightarrow Y$, pulling back along $f$ gives rise to a functor

$$\mathcal{S}/X \xleftarrow{f^*} \mathcal{S}/Y$$

Consider also $\Sigma_f : \mathcal{S}/X \longrightarrow \mathcal{S}/Y$ ("composing with $f$"), then $\Sigma_f$ is a left adjoint of $f^*$. I shall later show that $f^*$ also has a right adjoint $\Pi_f$ ("relative infinite products", "Weyl extension", "universal quantification").

To prove that equalizers exist. Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$.
Want to form "$\{x \mid f = g\}$."
Form

$$(*) \qquad X \xrightarrow{\langle f, g\rangle} Y \times Y \xrightarrow{\ominus_Y} \Omega,$$

where $\ominus_Y$ is the characteristic map of the diagonal of $Y$, $Y \longrightarrow Y \times Y$ (which actually is the graph of the identity map of $Y$).
Call the propositional function $(*)$ "$f \ominus g$". The "extension" $\{x \mid f \ominus g\}$ [write $\{x \mid f = g\}$] will be the equalizer (i.e., get the equalizer of $f$ and $g$ as pull back of true along $(*)$) (or rather, the map NB will be so:

$$\begin{tikzcd}
\{x \mid f = g\} \arrow[r, "NB"] \arrow[d] \arrow[dr, phantom, "***"] & Y \arrow[r] \arrow[d, "\delta"] \arrow[dr, phantom, "**"] & 1 \arrow[d, "\text{true}"] \\
X \arrow[r] & Y \times Y \arrow[r, "\ominus_Y"'] & \Omega
\end{tikzcd}$$

where NB exists since $**$ is a pull back. Exercise: $***$ is a pull-back also.

Could also define the pull-back directly, using logic instead of going through the geometric argument

$$\begin{tikzcd} P = \{X \times C \mid \pi_X f \ominus \pi_C y\} \arrow[d] & & \\ X \times C \arrow[r, "f \times y"'] & Y \times Y \arrow[r, "\ominus_Y"'] & \Omega \end{tikzcd}$$

$\Omega$ can be viewed as the reflection of consciousness, a pale reflection of the whole $\underline{E}$ – which in turn interacts on $\underline{E}$ (giving rise to equalizers, f. example).

(Adjointness of functors: $\mathcal{a} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{B}$

$$1_{\mathcal{a}} \xrightarrow{\eta} U \circ F \qquad F \circ U \xrightarrow{\epsilon} 1_{\mathcal{B}}$$

such that certain tertiary contradictions that might arise actually vanish:

$$\begin{tikzcd} F \arrow[r, "F \circ \eta"] \arrow[rr, bend right, "id_F"'] & F \circ U \circ F \arrow[r, "\epsilon \circ F"] & F \end{tikzcd}$$

are equal.

$$\begin{tikzcd} U \arrow[r, "\eta \circ U"] \arrow[rr, bend right, "id_U"'] & U \circ F \circ U \arrow[r, "U \circ \epsilon"] & U \end{tikzcd}$$

are equal

Equivalent to the following statement about ~~external~~ internal contradictions

$$\mathcal{B}(FA, B) \cong \mathcal{a}(A, U(B))$$

natural in $A$ and $B$. Write

$$\frac{F(A) \longrightarrow B}{A \longrightarrow U(B)}$$

Examples (apart from those that has occurred) $Sets \underset{U}{\overset{F}{\rightleftarrows}} Monoids$, $F$ free word monoid

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Singleton $X \xrightarrow{\{\ \}_X} \Omega^X$

For any $1 \xrightarrow{x} X$ $x.\{\cdot\}$ with characteristic function $\varphi : X \to \Omega$ should have the property that for $1 \xrightarrow{y} X$, $y.\varphi = \text{true}$ iff $x = y$.
Forgetting about 1, we can perform the construction by the following adjointnesses

$$\frac{\dfrac{\dfrac{X \xrightarrow{\{\ \}} \Omega^X}{X \times X \xrightarrow{=_X} \Omega}}{X \rightarrowtail^{\delta_X} X \times X}}{X \xrightarrow{1} X \quad X \xrightarrow{1} X}$$

For arbitrary $A$ and $A \xrightarrow{x} X$ we then have

$x.\{\ \}$ (denoted $\{x\}$) : $A \longrightarrow \Omega^X$

so that $\{x\}$ is $\lambda$-transform of $X \times A \longrightarrow \Omega$, and for arbitrary $B \xrightarrow{\langle y, a\rangle} X \times A$

$$\langle y, a\rangle \in x \quad \text{iff} \quad a.x = y$$

(General notation:

$$\begin{tikzcd} Z \arrow[dr, "\bar{z}"'] & \\ & A \arrow[u, phantom] \end{tikzcd} \quad \big\downarrow i \qquad \text{Def: } z \in i \quad \text{iff} \quad \exists! \bar{z}\, [z = \bar{z}.i]$$
)

The same notation was used when defining $\{X | \varphi\}$

Note that $\mathcal{E}(X, \Omega)$ is a (trivial, i.e. partially ordered) category ($\mathcal{E}$ the topos we are working in). There is a functor

$$\mathcal{E}(X, \Omega) \xrightarrow[\{X| -\}]{} \mathcal{E}/X$$

has a left adjoint "$[\![\ ]\!]$", i.e. for $p : E \longrightarrow X$ get $[\![p]\!] : X \to \Omega$,
i.e. for any $\varphi : \Omega \longrightarrow X$

$$\begin{tikzcd} E \arrow[rr, "\exists"] \arrow[dr, "p"'] & & \{X|\varphi\} \arrow[dl] \\ & X & \end{tikzcd} \quad \text{iff} \quad [\![p]\!] \leq \varphi$$

The values of $\{X|-\}$ are always monic; we shall use in order to prove this:

<u>Axiom</u> Any $E \xrightarrow{p} X$ gives rise to $\{X|[\![p]\!]\}$

$$\begin{tikzcd} E \arrow[rr, "q"] \arrow[dr, "p"'] \arrow[drr, phantom, "///"] & & \{X|[\![p]\!]\} \arrow[dl, "i"] \\ & X & \end{tikzcd}$$

(adjunction map); $E \xrightarrow{q} \{X|[\![p]\!]\}$ should be surjective. Consider the equivalence relation generated by $p$. Now the axiom is that $E \to \{X|[\![p]\!]\}$ is the coequalizer of this equivalence relation.

(This is technical – just to minimalize axiomatics)

A consequence of this axiom, independently of $\Omega$, is (the proof uses $\Omega$):

NB Every map $p$ can be factored in the form $p = q.i$ where $q$ is a coequalizer (there exist a pair of maps of which it is the coequalizer), and $i$ is an equalizer (i being the equalizer:

$$\{X|[\![p]\!]\} \xrightarrow{i} X \underset{\to 1 \xrightarrow{true}}{\overset{[\![p]\!]}{\rightrightarrows}} \Omega \qquad ).$$

The corresponding statement in category of topological spaces and continuous maps because you have there bijections which are not homeomorphisms.

Likewise it fails for commutative rings.

<u>Exercises</u> Assume (NB. Prove that) every mono is an equalizer in the category $\mathcal{C}$ and every epi is a coequalizer.

In a topos, in fact, every mono is of form $\{X|\varphi\}$ (up to iso)

<u>Exercise</u>

Also prove that if NB holds, then epi-mono factorization is unique (up to isomorphism):

$$\begin{tikzcd}
E \arrow[r, two heads] \arrow[d, two heads] & \cdot \arrow[d, tail] \\
\cdot \arrow[r, tail] \arrow[ur, dashed, "\cong\ \exists!"] & \cdot
\end{tikzcd}$$

($\twoheadrightarrow$ denoting epics, $\rightarrowtail$ denoting monics).

Also prove monic & epic $\Rightarrow$ iso (under assumption of NB).

The factorization is called the image factorization.

More generally, if I have a commutative square

$$* \qquad \begin{tikzcd}
\cdot \arrow[r, "g"] \arrow[d, two heads, "f"'] & \cdot \arrow[d, tail, "f'"] \\
\mathbb{X} \arrow[r, "g'"'] \arrow[ur, dashed, "h"] & \cdot
\end{tikzcd} \qquad f.g' = g.f'$$

it may be "homotopically trivial", i.e. splits into two commutative triangles (dotted arrow).

[Suppose $y \in f'$. This means $y = u.f'$]

To define the dotted map $h$ on $1 \xrightarrow{x} \mathbb{X}$, first lift it (using axiom of choice) to $\bar{x} : 1 \longrightarrow \mathbb{X}$ — or more precisely, $h =$ $s.g$ where $s$ is a section for $f$. To prove

$$f.h = g \qquad \boxed{\text{exists by ax. of choice}}$$

$$h.f' = g'$$

(to prove the upper equation, multiply on the right by $f'$ which is monic i.e. right cancellable)

Clearly such $h$ is unique since $f'$ is monic

Claim: A square $*$ is "homotopically trivial" ($h$ exists (uniquely)), just under the assumption of NB, not using "axiom of choice".

This is what is called "diagram chase"

The axiom above does not say that coequalizers exist. (Only coequalizers for some maps). We shall prove in stages, using logic and number theory, that coequalizers in general exist.

Theorem

Given two maps

$$R \overset{m_1}{\underset{m_2}{\rightrightarrows}} E \, ,$$

their coequalizer $q$ exist if $\langle m_1, m_2 \rangle$ satisfy the three properties r s t (reflexive, symmetric, transitivity)

Proof Suffices to assume $\langle m_1, m_2\rangle$ mono; for we can take image factorization, ~~and~~ $\bar{m}$, and

Exercise $\bar{m}$ still satisfies r s t and the coequalizer of $m$ is the same as the coequalizer of $\bar{m}$. [If one of them exists] Can think of a map $A \xrightarrow{c} R$ as a proof that two maps should be identified, ~~…~~ namely $e_1 = c \cdot m_1$, $e_2 = c \cdot m_2$

So assume that $\langle m_1, m_2 \rangle$ is mono. (If we don't have r, s, t,) We have to use the natural numbers, which is introduced via another adjointness which we have not introduced yet
(The theorem is true, though, also for finite sets)

Let $(\ ) \underset{m}{\equiv} (\ )$ be the characteristic function of

$$R \xrightarrow{\langle m_1, m_2\rangle} E \times E \, ,$$

and let the transpose of that be

$$E \xrightarrow{\hat{m}} \Omega^E$$

(The ambiguity does not matter, because the presence of $s$).
$\hat{m}$ is described as follows on elements:

$$x \rightsquigarrow [x]_m$$

Take the image factorization of $\hat{m}$.

$$\begin{tikzcd} E \arrow[rr, "\hat{m}"] \arrow[dr, "q"'] & & \Omega^E \\ & Q \arrow[ur] & \end{tikzcd}$$

To prove that $q$ is coequalizer of $m_1, m_2$.

First prove $m_1 \cdot q = m_2 \cdot q$ ; enough to prove
$m_1 \cdot \hat{m} = m_2 \cdot \hat{m}$.
This follows from symmetry and transitivity

For the universal property:

$$
\begin{tikzcd}
R \arrow[r, shift left, "m_1"] \arrow[r, shift right, "m_2"'] & E \arrow[rr, "\hat{m}"] \arrow[dr, two heads, "q"'] \arrow[ddr, "h"'] & & \Omega^E \\
R_q \arrow[ur, shift left] \arrow[ur, shift right] & & Q \arrow[ur] & \\
& & Z &
\end{tikzcd}
$$

Let $R_q$ be the ~~coequalizer~~ equivalence relation of $q$ ; get $R \longrightarrow R_q$ by universal property of pull-back ($R_q$ being a pull-back)

Can also get $R_q \longrightarrow R$ , by

<u>Lemma</u>: Any MRST pair is the equivalence relation generated by a map, namely $\hat{m}$ (constructed as above).

So $q$ is coequalizer also of $m_1, m_2$.

Recapitulation on the definition of a topos $\mathcal{E}$.

Higher order (or cartesian closed) $\Delta$, $A\times(\ ) \dashv (\ )^A$ $\delta, \pi_0, \pi_1, \lambda, \varepsilon$ etc.

Consequences: The internal composition law: $B^A \times C^B \xrightarrow{\pi} C^A$

if $f: A\times X \longrightarrow B$, $g: B\times X \longrightarrow C$ then $\langle \ulcorner f \urcorner, \ulcorner g \urcorner \rangle \circ \pi = \ulcorner f\circ g \urcorner$

(e.g. for $X = I$) generally:

"$A\times X \xrightarrow{A\times\delta} A\times(X\times X) \cong (A\times X)\times X \xrightarrow{f\times X} B\times X \xrightarrow{g} C$" $= f\circ g$.

(recall: this is the usual $\lambda$-conversion)

the construction of $\pi$ is

$$\begin{tikzcd} A\times(B^A\times C^B) \arrow[r] \arrow[d, "ev_B^A\times C^B"'] & C \\ B\times C^B \arrow[ur, "ev_C^B"'] & \end{tikzcd}$$

$$\overline{\pi: B^A\times C^B \longrightarrow C^A}$$

Contravariant Functoriality of $A\times(-) \dashv (-)^A$

i.e. if $A' \xrightarrow{a} A$ then $Y^A \xrightarrow{Y^a} Y^{A'}$

$\sim$
$$\begin{tikzcd} A'\times Y^A \arrow[r] \arrow[d, "a\times Y^A"'] & Y \\ A\times Y^A \arrow[ur, "ev_Y^A"'] & \end{tikzcd}$$

Exercise: Verify that: $Y^a \circ Y^{a_1} = Y^{a_1\circ a}$.

Logic. $1 \xrightarrow{true} \Omega$ ($\Omega$ = the truth-value object)

s.t.

1)
$$\begin{tikzcd} A \arrow[drr, bend left] \arrow[dr, "x"'] & & \\ & \{x\mid\varphi\} \arrow[r] \arrow[d] & 1 \arrow[d, "true"] \\ & X \arrow[r, "\varphi"'] & \Omega \end{tikzcd}$$

i.e. $x\varphi = true$ iff $x \in \{x\mid\varphi\}$

2)
$$\begin{tikzcd} E \arrow[dr, "p"'] & & \\ & X \arrow[r, "\llbracket p \rrbracket"'] & \Omega \end{tikzcd}$$

s.t. $\dfrac{\forall\ \varphi: E \longrightarrow \{x\mid\varphi\} \quad \mathcal{E}/X}{\llbracket p \rrbracket \leq \varphi \quad \mathcal{E}(X,\Omega)}$

Consequences: Pull-backs, inverse images, intersections, equalizers, kernel pairs (equivalence relations induced by a map) – all these exist. (if Logic and finite products)

**Remarks.** Elementary Properties.

1)
$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "p.b."] & Y' \arrow[d] \arrow[r] & 1 \arrow[dd, "true"] \\ X \arrow[r, "f"] & Y \arrow[dr, "\varphi"] & \\ & & \Omega \end{tikzcd}$$

"$ch(f \circ \varphi) = f \circ ch(\varphi)$"

2)
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr] & \cdot \arrow[d] & \\ \cdot \arrow[r] & Y \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"] \arrow[dr, "\varphi_1 \wedge \varphi_2"] & \Omega \\ & \Omega & \Omega \end{tikzcd}$$

Internal description
$$\begin{tikzcd} 1 \arrow[r] \arrow[d, "\langle true, true \rangle"] & 1 \arrow[d, "true"] \\ \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \end{tikzcd}$$

$$\begin{tikzcd} Y \arrow[r, "\langle \varphi_1, \varphi_2 \rangle"] & \Omega \times \Omega \arrow[r, "\wedge"] & \Omega \\ & 1 \arrow[u, "\langle true, true \rangle"] \arrow[r] & 1 \arrow[u, "true"] \end{tikzcd}$$

then $\langle \varphi_1, \varphi_2 \rangle \circ \wedge = \varphi_1 \wedge \varphi_2$ (associated.)

3)
$$\begin{tikzcd} \Omega_{\text{order-relation}} \arrow[r] & \Omega \times \Omega \arrow[r, shift left, "\wedge"] \arrow[r, shift right, "\pi_0"'] & \Omega \end{tikzcd}$$

"$\Omega_{\text{order-rel.}} = Eq(\wedge, \pi_0)$"

**Exercise:** $\Omega$ is a Heyting algebra: i.e.
$$\frac{\alpha \wedge \varphi \le \psi}{\varphi \le \alpha \Rightarrow \psi} \qquad (\alpha, \varphi, \psi : X \rightrightarrows \Omega)$$

thus. $\alpha \wedge (\alpha \Rightarrow \psi) = \alpha \wedge \psi$

$$\begin{tikzcd} & \Omega \times \Omega \arrow[dl, "\langle \pi_0, \Rightarrow \rangle"'] \arrow[dr, "\wedge"] & \\ \Omega \times \Omega \arrow[rr, "\wedge"'] \arrow[rr, phantom, shift left=4, "\equiv"] & & \Omega \end{tikzcd}$$

— o —

$$\begin{tikzcd} R \arrow[r, "m_2"] \arrow[d, "m_1"'] \arrow[dr, phantom, "p.b."] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & X \end{tikzcd}$$

for $\langle m_1, m_2 \rangle$ there is $r, s, t$ then

**Theorem**

$R \underset{m_2}{\overset{m_1}{\rightrightarrows}} E$, $r, s, t$ $\Rightarrow$ the image of $m = \langle m_1, m_2 \rangle$ is the kernel pair of $E \xrightarrow{\tilde{m}} \Omega^E$ ($E \times E \to \Omega$, ch(image))

<u>Corollary</u> Any pair $m_1, m_2$ admitting r.s.t. has a <u>coequalizer</u> (and is the image of $E \xrightarrow{\tilde{m}} \Omega^E$

$$\begin{tikzcd} E \arrow[r, "\tilde{m}"] \arrow[dr, two heads] & \Omega^E \\ & Q \arrow[u] \end{tikzcd}$$

this was actually based on the following axiom

<u>Axiom</u>

$$\begin{tikzcd} R = E \times E \arrow[r, shift left, "m_1"] \arrow[r, shift right, "m_2"'] & E \arrow[r, "\mathrm{coeq}"] \arrow[dr, "\rho"'] & \{X \mid [\![\rho]\!]\} \arrow[d] \\ & & X \end{tikzcd}$$

$\longrightarrow$ coeq of $m_1, m_2$

Consequences: ......

Proof of the Theorem:

$$\begin{tikzcd} R \arrow[r, shift left, "m_1"] \arrow[r, shift right, "m_2"'] & X \arrow[r, "\tilde{m}"] & \Omega^X \\ T \arrow[u, dashed] \arrow[ur, shift left, "x_1"] \arrow[ur, shift right, "x_2"'] & & \\ A \arrow[u, "t"] & & \end{tikzcd}$$

1) $m_1 \circ \tilde{m} = m_2 \circ \tilde{m}$

2) if $x_1 \circ \tilde{m} = x_2 \circ \tilde{m}$ then $\exists !$ s.t.

ad 1) $m_1 \circ \tilde{m} = m_2 \circ \tilde{m} \iff X \times T \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} \Omega \iff \rightrightarrows X \times T$

let $t$ arb.: $\boxed{\langle x, t \rangle \varphi_i = [\![ t.x_i \equiv_m x_2 ]\!]}$

then $\boxed{t\,x_1 \equiv_m x \iff t\,x_2 \equiv x}$

$\Downarrow$ r, s, t

$\underline{t\,x_1 \equiv_m t\,x_2 \quad , \quad t = id}$

$x_1 \equiv_m x_2$ i.e. $\langle x_1, x_2 \rangle \in m$ / <u>unique (as w-ly $\langle m_1, m_2 \rangle$ mono</u>

<u>Quantifications.</u>

Justification.

$$\begin{tikzcd} \{X \mid \varphi\} \arrow[d, hook] \arrow[dr, two heads] & \\ X \arrow[r, "f"] \arrow[d, "\varphi"] & Y \arrow[d, dashed, "\exists_f[\varphi]"] \\ \Omega & \Omega \end{tikzcd}$$

$\exists_f[\varphi] =$ "chr$(\mathrm{Im}(\{X \mid \varphi\} \to X \to Y))$"

<u>abstract sets</u> $(\Omega = 2)$

$y \in f[\{x \mid \varphi\}]$ iff $\exists x [x f = y \wedge x\varphi = \text{true}]$

<u>Consider</u> in $\mathcal{L}$

$$\begin{tikzcd} & X = A \times Y \arrow[d, "f"', "/proj"] \\ \Omega & Y \arrow[l, "{\exists_a[\varphi(a,y)]}"] \end{tikzcd}$$

if
$$\begin{tikzcd} & Y \arrow[dl, "\psi"'] \\ \Omega & \end{tikzcd}$$
then $\dfrac{\exists_f[\varphi] \leq \psi}{\varphi \leq f\cdot\psi}$ (verify: Exercise)

Hence. $$E(X,\Omega) \underset{f\cdot(\,)}{\overset{\exists_f[\,]}{\rightleftarrows}} E(Y,\Omega)$$

$$\exists_f[\,] \dashv f\cdot(\,)$$

By the combinatorial property: $$\Omega^X \underset{\exists_f}{\overset{\Omega^f}{\leftrightarrows}} \Omega^Y \qquad \ulcorner\varphi\urcorner\,\Omega^f = \ulcorner f\varphi\urcorner$$

$$\dfrac{\dfrac{\exists_f : \Omega^X \longrightarrow \Omega^Y}{Y\times\Omega^X \longrightarrow \Omega}}{\rightarrowtail Y\times\Omega^X}$$
$\mathrm{im}''(\in_X \circ f\times\Omega^X)$

We have
$$\begin{tikzcd} & Y\times\Omega^X \\ \in_X \arrow[r, tail] & X\times\Omega^X \arrow[u, "f\times\Omega^X"'] \end{tikzcd}$$

$$\dfrac{\in_X \rightarrowtail X\times\Omega^X}{X\times\Omega^X \xrightarrow{ev_X} \Omega}$$

Exercise

$X \xrightarrow{f} Y \xrightarrow{g} Z$ : $\exists_{fg} = \exists_f\,\exists_g$

The power "set" functor:

Further properties:

Exercise
$$\begin{tikzcd} X \arrow[r, "\{\}_X"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & \Omega^X \arrow[d, "\exists_f"] \\ Y \arrow[r, "\{\}_Y"'] & \Omega^Y \end{tikzcd}$$

$$\dfrac{\Omega^{(\Omega^X)} \xrightarrow{\cup_X} \Omega^X}{\rightarrowtail X\times\Omega^{(\Omega^X)}}$$

$$\begin{tikzcd} \cdot \arrow[r, tail] & X\times\Omega^{(\Omega^X)} \\ \in_X \wedge \in_{\Omega^X} \arrow[u] \arrow[r, tail] & X\times\Omega^X\times\Omega^{(\Omega^X)} \arrow[u, "proj"'] \end{tikzcd}$$

Analysis: $x \in \cup\alpha$
$\exists_\varphi\; x\in\varphi \wedge \varphi\in\alpha$

Exercise:
$$\begin{tikzcd} \Omega^{(\Omega^X)} \arrow[r, "\cup_X"] \arrow[d, "\exists_{\exists_f}"'] \arrow[dr, phantom, "///"] & \Omega^X \arrow[d, "\exists_f"] \\ \Omega^{(\Omega^Y)} \arrow[r, "\cup_Y"'] & \Omega^Y \end{tikzcd}$$

Exercise
$$\begin{tikzcd} \Omega^{(\Omega^{(\Omega^X)})} \arrow[r, "\cup_{\Omega^X}"] \arrow[d, "\exists_{\cup_X}"'] \arrow[dr, phantom, "///"] & \Omega^{\Omega^X} \arrow[d, "\cup_X"] \\ \Omega^{(\Omega^X)} \arrow[r, "\cup_X"'] & \Omega^X \end{tikzcd}$$

Lawvere 13/10-71.

A. Kock found out that the axioms can be further weakened:
finite cartesian prod's <u>and</u> $\Omega^X$ ; $X^Y$ can then be defined.

$$\frac{Z \longrightarrow \Omega^X}{Z\times X \longrightarrow \Omega}$$

Then define $Y^X$ as a pull-back, namely viewing a map as a special case of a relation

$$\begin{array}{c} \Omega^{X\times Y} \\ \uparrow \\ Y^X \end{array}$$

on basis of our previous axioms, this subobj can be described as

$$\begin{array}{c} X\times Y\times Y^X \\ \uparrow \text{ "graph of } X\times Y^X \longrightarrow Y\text{"} \\ X\times Y^X \end{array}$$

take char map of that $X\times Y\times Y^X \longrightarrow \Omega$ and transpose:

$$Y^X \longrightarrow \Omega^{X\times Y} ;$$

associate to a map its graph (viewed as a relation).

To define $Y^X$ without mentioning exponentiation (except of form $\Omega^X$).
First note that if we have $\Omega^X$, then we also have

$$(\Omega^X)^Y \qquad (\cong \Omega^{X\times Y}).$$

$$\Omega^{X\times Y} \xrightarrow{\cong} (\Omega^Y)^X \xrightarrow{(s)^X} \Omega^X$$

where $s : \Omega^Y \longrightarrow \Omega$ is the characteristic map of $Y \rightarrowtail \Omega^Y$.

Jeg SKYLDER LILIAN 5 KR. + ~~Chr Frank~~. ~~Chr skylder mig~~ 5 ~~Mogens skylder~~ 5

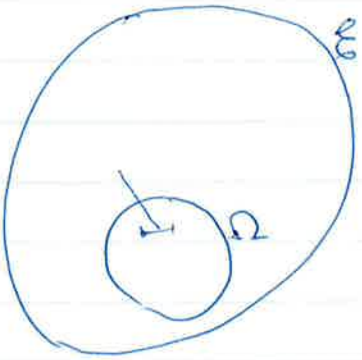
For $A \rightarrowtail Y$, $s_Y(\mathrm{ch}(A))$ is the "truth value that A has exactly one element. So therefore $Y^X$ can be <u>defined</u> as the pull-back

$$\begin{tikzcd} \Omega^{X\times Y} \cong (\Omega^Y)^X \arrow[r, "(s_Y)^X"] & \Omega^X \\ Y^X \arrow[u, tail] \arrow[r] & 1 \arrow[u, "\ulcorner true_X \urcorner"'] \end{tikzcd}$$

So we are gradually moving from geometry to logic ( "only power sets are assumed to exist).

(We still need the axiom of epi - mono factorization, it seems)
(Anders + Chris)

Can view [[The existence proof of $Y^X$]] as




Contradictions have now been reflected into $\Omega$

(Should now turn things around and use them on geometry)

This [[$Y^Y$]] is a good introduction to the notion of partial map.
<u>Notion of partial morphism</u>, described in a geometric way (i.e. in a category that has <u>pull-backs</u> (which is a geometric notion — comes from fibre bundles etc.)) Partial morphism from X to Y is a pair:

$$\begin{tikzcd} \dot{Y} \arrow[d, tail] \arrow[dr] & \\ X & Y \end{tikzcd}$$

To compose partial maps:

$$\begin{tikzcd} & & \cdot \arrow[dl] \arrow[dr] & & \\ & \cdot \arrow[dl, "i"'] \arrow[dr, "f"] \arrow[rr, phantom, "\text{p.b.}"] & & \cdot \arrow[dl, "j"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

by forming pull-back — pull-back of a mono is mono; composing monics give monics.

We say "$\langle i, f\rangle$ is defined at $x$" iff $x \in i$.
Write $f$ for $\langle i, f\rangle$. Then
$fg$ is defined at $x$ iff there exist $\bar{x}, \bar{y}$ so that $\bar{x}\cdot f = \bar{y}\cdot j$
iff $f$ is defined at $x$ and $xf \in j$
iff $f$ is defined at $x$ and $g$ is defined at $xf$.

A point that we must make: equivalence of subobj's is not equality; we similarly have to consider equivalent partial morphisms.

$$\begin{tikzcd} & \cdot \arrow[dl, "i"'] \arrow[dr, "f"] \arrow[dd, "\cong"] & \\ X & & Y \\ & \cdot \arrow[ul, "i'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

Using external set theory, we can make the equivalence classes of partial maps into a category (without external set theory, we could have made it a bi-cat.). E.g. the fact that composition is associative (up to 2-cells) is the fact that iterated pull-backs are pull-backs. — The 2-dimensional structure is very simple [[preordered]]
Call this category

$$\mathcal{E}_\partial$$

There is an obvious functor $\mathcal{E} \longrightarrow \mathcal{E}_\partial$ given by:

$$\begin{tikzcd} X \arrow[d] \\ Y \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & X \arrow[dl, "1"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

(identity on object).
To represent this notion, i.e. to have a right adjoint for this functor.

For abstract sets, this is possible

$*$ $\quad X \longrightarrow Y + \{\text{undefined}\}$ same information as

$**$
$$\begin{tikzcd} \dot{Y} \arrow[d] \arrow[dr] & \\ X & Y \end{tikzcd}$$

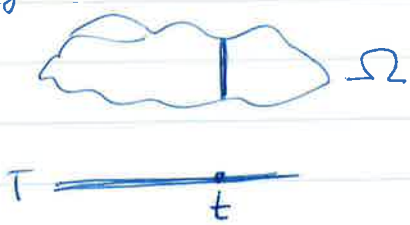
$Y \rightarrowtail (Y + \{\text{undefined}\})$ canonically given; to pass from $*$ to $**$, pull back along this canonical map.
Passing the other way depends on complements.

So you do have

$$\begin{tikzcd} \mathcal{S} \arrow[r, shift right, "\mathrm{Incl}"'] & \mathcal{S}_\partial \arrow[l, shift right, "R = \tilde{\ }"'] \end{tikzcd}$$

$R$ given by $R(Y) = Y + \{\text{undefined}\} = \tilde{Y}$ (note that Incl is the identity on objects). Incl is left adjoint to $R$.
In most categories, such will not exist. It will exist in a topos, but will be more complicated. (because we cannot form "the set of all elements where $f$ is not defined"). So in general, $\tilde{Y}$ must contain $Y$ and "all boundary points."

$\tilde{1}$ we already know; it clearly is $\Omega$, because a partial map to $1$ (from $X$) has precisely the information of a subobject of $X$.
$\Omega$ for sheaf categories for topological spaces is complicated.
"$\Omega(U)$ = set of open subsets of $U$". From the "espace etalé", "sheaf of germs"




what is $\Omega_t$? Complicated in this set-up. "Sheaf of germs of open sets" — Shall discuss all this later.

So even though $\sim$ is complicated, it is easy to prove that it is there

Shall first give example for $Y \in \mathcal{S}^P$ ($P$ a partially ordered set).

$$(\tilde{Y})_p = \sum_{R \subseteq p} \mathcal{S}^P(R, Y)$$

where $R \subseteq p$ is such that $R_q = \begin{cases} 0 \\ 1 \end{cases}$, and such that $R_q = 1 \Rightarrow p \leq q$ (i.e. the $R$'s are order filters).

Another example, namely for $\mathcal{S}^\Omega$; consider $Y \supset t$. To construct $\tilde{Y} \supset \tilde{t}$;

$$\tilde{Y} = \sum_{n=0}^{\infty} \mathcal{S}^\Omega(R_n, Y)$$

where $R_n = \{m \mid m \geq n\}$;

but $R_n \cong \omega$ - so since $\mathcal{S}^\Omega(\omega, Y) = Y$

$$\tilde{Y} = \omega \times Y$$

with $\tilde{t}$ given by ? (Exercise).

To show that $\sim$ exists in general; idea: to form "set" of those relations whose values, if they are there, are singletons. So the construction from above can probably be applied

I know another construction:

$$\begin{tikzcd} Y \arrow[rr, tail, "\{\cdot\}"] \arrow[dr] & & \Omega^Y \\ & \tilde{Y} \arrow[ur, tail] & \end{tikzcd}$$

$\tilde{Y}$ being defined as an equalizer $\Omega^Y \rightrightarrows \Omega^Y$; for, the condition that $A \subseteq Y$ is a singleton or empty iff

$$A = \{y \mid A = \{y\}\}.$$

which is an equation. (for abstract sets! - but it is good in general).

no disjunction enters!

So therefore the two maps $\Omega^Y \rightrightarrows \Omega^Y$ are

1) the identity map
2) the $\lambda$-transform of the characteristic map of the graph of $\{\ \}_Y$.

Easy to see that $Y \xrightarrow{\{\cdot\}} \Omega^Y$ factors across $\tilde{Y} \longrightarrow \Omega^Y$, denote it $\eta : Y \longrightarrow \tilde{Y}$. It is monic

Has exactness properties as consequence; we shall show that push-outs exist – but assume that we have a push-out $*$

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, tail, "\text{monic}"'] \arrow[dr, phantom, "*"] & Y \arrow[d] \arrow[ddr, bend left, "\eta_Y"] & \\
X \arrow[r] \arrow[drr, bend right, "f"'] & Y' \arrow[dr, dashed] & \\
& & \tilde{Y}
\end{tikzcd}$$

Then $Y \longrightarrow Y'$ is also monic!
Further, $*$ is also a pull-back! Follows from the fact that the outer diagram already is a pull-back.

Program for next time: With help of partial map, construct the notion of "universal quantification" "infinite product", $\prod_f$

$$\begin{tikzcd}
E \arrow[d] & \prod_f(E) \arrow[d] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Will be a generalization of exponentiation.

$1, \times$ vs $\Omega$

strengthens to (by being resolved

$\prod$ vs $\tilde{(\ )}$

[illegible] Vacation

Lawvere 20/10-71

Sheaves [illegible] partial [illegible]

$$\begin{tikzcd} Y \arrow[rr, "\{\}"] \arrow[dr] & & \Omega^Y \\ & \tilde{Y} \arrow[ur] & \end{tikzcd}$$

A map $X \longrightarrow \Omega^Y$ may be viewed as a relation $X \nrightarrow Y$; it factors through $\tilde{Y}$ iff the relation is a partial map.

Exercise What is $\tilde{Y}$ in $\mathcal{S}^{\circlearrowleft}$

$\tilde{Y}^X$ "set of" all partial maps from $X$ to $Y$. Various subobjects of this will be of interest, e.g. "partial maps with finite domain"; note that $Y \longrightarrow 1$ gives rise to

$$\tilde{Y} \longrightarrow \tilde{1} = \Omega$$

called the 'domain map', dom.

$$\begin{tikzcd} X_f \arrow[r] \arrow[d] & Y \arrow[d, "\eta_Y"] \\ X \arrow[r, "f"] & \tilde{Y} \arrow[d, "\mathrm{dom}_Y"] \\ & \Omega \end{tikzcd}$$

$f \cdot \mathrm{dom}_Y$ will be characteristic map of the domain of $f$, $X_f$. – Of course, $\mathrm{dom}_Y$ is char fct of $Y \xrightarrow{\eta_Y} \tilde{Y}$.

~~Now use that the composite of two pull-backs is a pull-back~~

Now let $F$ be a subfunctor of the power set functor. Form the p.b.

$$\begin{tikzcd} F(X,Y) \arrow[r] \arrow[d] & \tilde{Y}^X \arrow[d, "(\mathrm{dom}_Y)^X"] \\ F(X) \arrow[r] & \Omega^X \end{tikzcd}$$

Like $\Omega$, $\tilde{Y}$ is always canonically a partially ordered object (and so is therefore $\tilde{Y}^X$):

$$\frac{\dfrac{A \longrightarrow \tilde{Y}^X}{X \times A \longrightarrow \tilde{Y}}}{\begin{tikzcd} \bullet \arrow[r] \arrow[d, tail] & Y \\ X \times A & \end{tikzcd}}$$

The order-relation here is "one partial map extends another" The order-relation itself is also <u>representable</u> by a subobject

$$\bullet \rightarrowtail \tilde{Y}^X \times \tilde{Y}^X$$

The notion of partial-map-representor will be useful in obtaining "infinite products" [[ Weyl - ~~extension~~ restriction ]]. Let $X \xrightarrow[f]{} Y$ be a map in $\mathcal{E}$. We have

$$\begin{tikzcd} \mathcal{E}/X \arrow[r, shift left=2, "\Sigma_f"] \arrow[r, shift right=2, "\Pi_f"'] & \mathcal{E}/Y \arrow[l, "f^*" description] \end{tikzcd}$$

($\Sigma_f$ is in some sense, for abstract sets, a summation; over $y$, $\Sigma_f(E)$ has the disjoint sum of all the fibres over such $x$ with ~~$f(x)=y$~~ $xf = y$.) ($f^*$ is essentially pull-back; logically, it corresponds to substitution; think of

$$\begin{tikzcd} c \arrow[r, tail] & E \arrow[d] \arrow[rr, "a"] & & A \arrow[d, "y"] \\ & X \times_Y A \arrow[rru] \arrow[d] & & \\ & X \arrow[rr, "f"] & & Y \end{tikzcd}$$

as a family of proofs over $Y$.

$$c \vDash f^*(A)(x) \quad \text{iff} \quad a \vDash A[y/xf]$$

(Similar interpretation for $\Sigma_f$) Means: "is a proof of".

$$\frac{b \vDash \sum_f (E)(y)}{b_0, b_1 \qquad b_0 \in X \quad b_1 \vDash E(b_0)}$$

$E(b_0)$ means here the pull-back

$$\begin{tikzcd} E(b_0) \arrow[r] \arrow[d] & E \arrow[d] \\ A \arrow[r, "b_0"'] & X \end{tikzcd}$$

(generalizing the notation for "the fibre over $b_0$ in $E$").

This is just to motivate the opposite thing which we are going to do now: $\Pi_f$; logical interpretation is universal ~~quan~~ quantification; geometric —"— is $\Gamma$, i.e., sections (relative):

$$\begin{tikzcd} E \arrow[d] & \left(\prod_f(E)\right) \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in the $\mathcal{S}$-case, $\left(\prod_f(E)\right)_y$ should be $\prod_{xf=y} E_x$. Its defining property in general, however is that it should be right adjoint to $f^*$

$$\boxed{\Sigma_f \dashv f^* \dashv \Pi_f}$$

i.e.

$$\frac{A \longrightarrow \Pi_f(E) \quad \text{(over } Y)}{f^*(A) \longrightarrow E \quad \text{(over } X).}$$

In geometry, one sometimes uses notation suppressing $f$

$*$ $\qquad A \longrightarrow \prod_{X/Y} E \qquad$ over $Y$

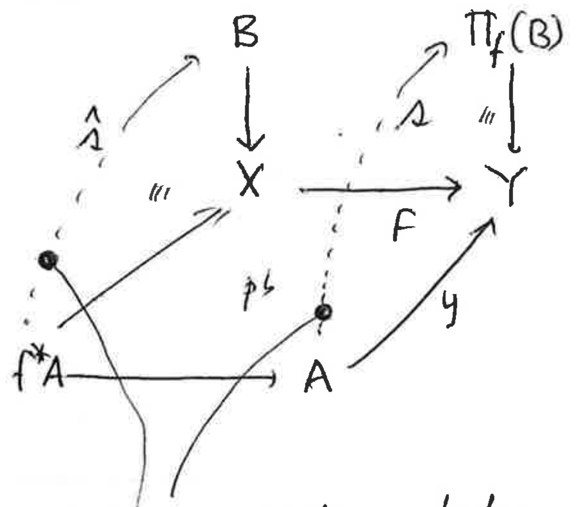
(in alg-geometry: Weil restriction). - in one-one corresp with

$X \times_Y A \longrightarrow E \qquad$ over $X$.

$*$ is also denoted $\quad \Gamma_Y\left(A, \prod_{X/Y}(E)\right)$

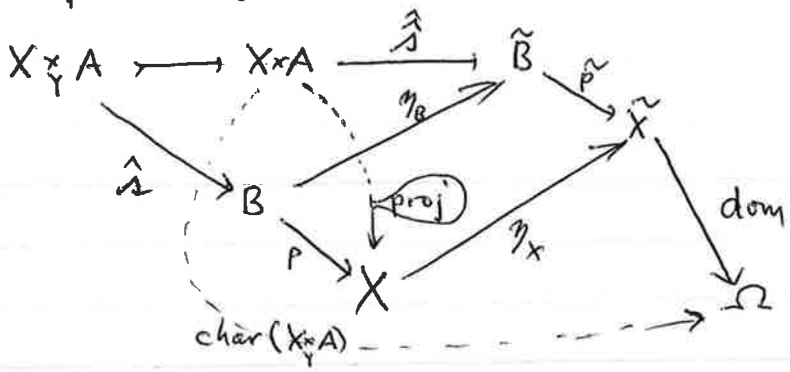
$$\Gamma_Y\left(A, \prod_{X/Y}(E)\right) \cong \Gamma_X\left(f^{-1}(A), E\right).$$

Putting things into one picture




1-1 correspondence between such sections.

Describe object of such $\hat{s}$




To suppress the dependence of the variable $A$, $y$

$$\begin{tikzcd} X \arrow[r, "\hat{\hat{\hat{s}}}"] \arrow[dr] & \tilde{B}^A \arrow[d, "\tilde{p}^A"] \\ & \tilde{X}^A \end{tikzcd}$$

Put $A = X$ , $y = f$

arrive at the following definition

$$\begin{tikzcd}
A \arrow[drr, bend left, "\hat{\hat{\hat{s}}}"] \arrow[ddr, bend right, "y"'] & & \\
& \Pi_f(B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pull-back}"] & \tilde{B}^X \arrow[d, "\tilde{p}^X"] \\
& Y \arrow[r, "f"'] & \tilde{X}^X
\end{tikzcd}$$

$$\dfrac{\dfrac{y.f : A \longrightarrow \tilde{X}^X}{X \times A \longrightarrow \tilde{X}}}{\begin{tikzcd} X \times A \arrow[r] & \tilde{X} \\ X \times_Y A \arrow[u, tail] \arrow[r] & X \arrow[u] \end{tikzcd}}$$

so I must take $f$ to be given by the 1-1 correspondences

$$\dfrac{\dfrac{Y \xrightarrow{f} \tilde{X}^X}{X \times Y \longrightarrow \tilde{X}}}{\begin{tikzcd} X \times Y \arrow[r] & \tilde{X} \\ X \arrow[u, tail, "\langle X, f\rangle"] \arrow[r, equal] & X \arrow[u, "\eta"'] \end{tikzcd}}$$

$$\boxed{\hat{\hat{\hat{s}}}\ \tilde{p}^X = y\, f} \quad \text{iff} \quad \boxed{s.p = \mathrm{proj}^{X \times_Y A}_X}$$

So we have gone from to

$1,\ \times,\ \exp,\ \Omega$

to the stronger form of contradiction

$f^*,\ \prod_f,\ \tilde{\ }$

Now adjointness implies continuity properties:
Since $\Pi_f$ exists, $f^*$ preserves all colimits that might exist; and natural no. object, and the notion of 'epic'.

So if $A \twoheadrightarrow Y$ is epic, then left hand column in

$$\begin{array}{ccc} f^*(A) & \longrightarrow & A \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

is also epic. "Pull back preserves epics" is the slogan that goes with this.

$\Pi_f$ (being a <u>right</u> adjoint) will preserve monics, hence subobjects.

$$\begin{array}{ccc} B & & \\ \downarrow & & \\ X & \xrightarrow[f]{} & Y \end{array} \quad \Longrightarrow \quad \begin{array}{c} \Pi_f(B) \\ \downarrow \\ Y \end{array}$$

This gives rise to an operation on characteristic functions

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \varphi \downarrow & & \downarrow \forall_f \varphi \\ \Omega & & \Omega \end{array}$$

and the adjointness now yields:

$$\frac{\psi \leq \forall_f(\varphi)}{f\psi \leq \varphi}$$

Is also represented <u>internally</u>:
$$\frac{\dfrac{\Omega^X \xrightarrow{\forall_f} \Omega^Y}{Y \times \Omega^X \longrightarrow \Omega}}{\rightarrowtail Y \times \Omega^X}$$

This subobject is taken to be universal quant of

$$\forall_{f \times 1}\,(\in^X_\Omega)$$

($\in^X_\Omega$ the evaluation map).

Lawvere 15/11 – 1971.

We have seen that 1) coequalizers of equivalence relations exist, and that moreover 2) an arbitrary coequalizer = coeq. of the equiv. rel. hull of the given relation.

Certain exactness properties : a) coequalizers are preserved by pull-backs (because of the existence of $\prod$ b) the kernel pair of coequalizer of an RST- relation R equals R ("equivalence relations are effective")

"2)" can be established without the natural numbers, by the machinery of higher order logic : Can form the object of eq. relations containing the given relation, and can form intersection of this family of relations, so that RST - hulls can be proved to exist by working from above, and not using $\mathbb{N}$.

$$\bigcap_{E \supset R} E .$$

Can we also define sums ~~from~~ by working from above, - ~~from~~ inside

$$\begin{tikzcd} A \arrow[r, tail] & A \times B \times \Omega \\ B \arrow[ur, tail] & \end{tikzcd}$$

(using 'true' and 'false').

"the smallest subobject of $A \times B \times \Omega$ containing A and containing B."

. . . .

To give some discussion concerning localization in commutative monoids
Suppose A is a commutative monoid in $\mathscr{E}$. Basic idea of localization is that of saturated submonoid $S \rightarrowtail A$ : a submonoid, so that $1 \in S$ and $a \cdot b \in S \iff a \in S \wedge b \in S$. ($a, b$ any maps $X \longrightarrow A$, "$\in S$" meaning : factoring through $S \rightarrowtail A$.) Can also be expressed as :

$$\begin{tikzcd} & & S \arrow[d] \\ \bullet \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "p.b"] & A\times A \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d] & A \\ S \arrow[r, tail] & A & \end{tikzcd}$$

The <u>fraction monoid</u> is obtained by considering a certain quotient of $A\times S$

$$R \rightrightarrows A\times S \twoheadrightarrow A_S$$

The relation should be the following:

$$\frac{a}{r} \equiv \frac{b}{s} \quad \text{iff} \quad \exists u \in S \; [(as)u = (br)u]$$

So R should be

$$\{\langle a,r,b,s,u\rangle \mid asu = bru\}$$

and the two maps to $A\times S$ ~~going int~~ doing as follows

$$\langle a,r,b,s,u\rangle \mapsto \langle a,r\rangle$$
$$\mapsto \langle b,s\rangle$$

R is an equivalence relation; better to think of it as a category

$$\frac{a}{r} \xrightarrow{u} \frac{b}{s} \quad \text{iff} \quad asu = bru \;;$$

these compose

$$\frac{a}{r} \xrightarrow{u} \frac{b}{s} \xrightarrow{v} \frac{c}{t}$$

namely to

$$\frac{a}{r} \xrightarrow{usv} \frac{c}{t}$$

Write $u \circ v$ to denote this composite $usv$ (not $uv$). The multiplication thus defined is associative; it makes .. into a $*$-category, since $asu = bru \implies bru = asu$. We don't have good identity maps, though; put

$\bigstar$ $u \equiv u'$ iff $\exists s \in S$ so that $us = u'$

divide out in this to get an actual category.

Example: $A = N$ (with multiplication)
$S = \{ n \geq 1 \}$

Can form $A_S$ - deserve the name of "non-negative rationals, $\mathbb{Q}_+$".

So the best set-up is to view the data ($\bigstar$ etc. ...) as defining a 2-dimensional category.

S consists ~~$\$$~~ precisely of the things that become units. (invertible);

$$\left( \exists_{\substack{b \in A \\ s \in S}} \left[ \left(\frac{a}{1}\right) \cdot \left(\frac{b}{s}\right) \equiv 1 \quad \text{in } A_S \right] \right) \iff a \in S$$

~~then $\exists b \in A, s \in S$~~

$\frac{a}{1} \cdot \frac{1}{a} \equiv \frac{1}{1}$; on the other hand, the left hand side implies

$$\exists u \in S \, [(ab)u = su]$$

but $su \in S$, so $a \in S$ (since $S$ is saturated).

This is one approach to the non-negative rationals. I am going to take a different approach. But first some general facts, we can prove in this context: Can construct the smallest saturated submonoid, by the method of collapsing from above. So can construct $[f]$, smallest sat. submonoid containing $f$.

Claim:

$$g \in [f] \text{ iff } \exists n \in N \; \exists a \in A \quad f^n = ag$$

In a stronger form, can construct

$$A \longrightarrow \Omega^A$$
$$f \rightsquigarrow [f]$$

This construction has nice universal property; if $B$ is a comm. monoid, and $\varphi : A \longrightarrow B$ is a homomorphism, then $f^\varphi$ is invertible in $B$ implies that $g^\varphi$ is also invertible in $B$:

$$g \in [f]$$

iff

$$\exists n \in N \ \exists a \in A \quad f^n = ag$$

iff

$A \xrightarrow{\varphi} B$ any homomorphism between commutative monoids, then $f^\varphi$ invertible $\Rightarrow g^\varphi$ invertible

Call these three properties

$$f \leq g$$

The inverse image of the set of units under a homomorphism is always a ~~unit~~ saturated submonoid.

It is not true in general that saturated submonoids are related to primes. For rings, it is true, by a theorem of Hilbert

Lawvere 29/11-71

Some of the students are struggling with the word-algebra, and seem to approach success.

Last time I discussed localization. . . Program now is to define the Real Numbers, and do some of the theory of metric spaces. I can avoid any quotient formation when forming rationals, and reals.

The definition I shall give is based on the Dedekind-cut idea. A dedekind cut is a subset$^{*}$ of $\mathbb{Q}$ - (we are only interested, here, in non-neg. rationals and reals). - such that

$$* \qquad q \in x \;\wedge\; q \le q' \;\Rightarrow\; q' \in x.$$

Write $x \le q$ ~~or~~ $q \ge x$ for $q \in x$.

Need a couple of adjustments. What we need for metric spaces is $\mathbb{R}$ = all reals $x$ with $0 \le x \le \infty$ (i.e. including $\infty$), and equipped with ordering $\ge$, as well as addition.

Can think of $x$ rather as its char. function $x : \mathbb{Q} \to \Omega$. The condition $*$ is then just that $x$ is order preserving:

$$q \le q' \longrightarrow [q \in x \Rightarrow q' \in x].$$

$\mathbb{Q}$ is preordered, but $\Omega$ is part. ordered, hence the order-preserving maps $\mathbb{Q} \to \Omega$ is the same as the set of order-preserving maps $\mathbb{Q}/\sim \longrightarrow \Omega$, when ~~$\mathbb{Q}/\equiv$~~ $\mathbb{Q}/\sim$ is $\mathbb{Q}$ made partially ordered.

Exercise: $\Lambda, \Gamma$ equipped with preorder. Then

$$\mathcal{O}(\Lambda, \Gamma) \subseteq \Gamma^{\Lambda}$$

can be defined.

What is $\mathbb{Q}$? If $\mathbb{N}^* = \{n \mid n \ge 1\}$, then $\mathbb{Q} = \mathbb{N} \times \mathbb{N}^*$ with order-relation given by

$$\frac{a}{s} \leq \frac{b}{t}$$

iff

$$\exists k \in N \quad \text{such that} \quad a \cdot t + k = b \cdot s \,;$$

so the order-relation itself could be identified by a certain set of quintuples $a, s, b, t, k$ such that . . .

Also need the <u>strict</u> order-relation:

$$\frac{a}{s} < \frac{b}{t}$$

iff

$$\exists n \quad \text{such that} \quad a \cdot t + n + 1 = b \cdot s$$

Thus we can define the set of Dedekind cuts, $\overline{\mathbb{R}} = \mathcal{O}(\mathbb{Q}, \Omega)$

There is now a more serious point: we want to be able to consider rationals as reals

$$\underset{\leq}{\mathbb{Q}^{op}} \xrightarrow{\ \cdot\ } \underset{\geq}{\overline{\mathbb{R}}} = \mathcal{O}(\mathbb{Q}, \Omega)$$

in elementwise terms: $\quad q \rightsquigarrow (q' \rightsquigarrow [\![ q \leq q' ]\!])$

in more general terms: take transpose of

$$\mathbb{Q}^{op} \times \mathbb{Q} \xrightarrow[{[\![ \cdot \leq \cdot ]\!]}]{} \Omega$$

Then

$$\dot{q} \geq x \quad \text{iff} \quad q \in x$$

$\uparrow$
in the ordering of $\mathcal{O}(\mathbb{Q}, \Omega)$.

Could also have used instead the <u>strict</u> ordering

$$q \rightsquigarrow \bar{q} = (q' \rightsquigarrow [\![ q < q' ]\!])$$

We would like $\dot{q} = \bar{q}$. So we will identify them. How? In the set-case, one can of course just take the coequalizer of the embeddings $\dot{}$ and $\bar{}$ ; but this is not a good method in a general topos. A better way of doing it is to ~~consider~~ make a choice : use $\dot{q}$, exclude $\bar{q}$. But then we exclude things by logically subtracting them – that is no good either. It seems that the happy solution to this is to recognize that there is a further property of Dedekind cuts which $\dot{q}$ does satisfy but $\bar{q}$ does not, namely the property of being a sheaf. So official definition of $\mathbb{R} \subseteq \overline{\mathbb{R}}$ : The sheaves with respect to the Grothendieck topology on $\mathbb{Q}^{op}$ given by: $\bar{q} \subseteq \dot{q}$ is a covering.

How to define addition? $x+y$ How to define it in $\overline{\mathbb{R}}$, and how in $\mathbb{R}$? In $\overline{\mathbb{R}}$, need to specify which rationals $q$ are such that $q \geq x+y$. It has to go back to addition of rationals, via a convolution formula:

$$q \geq x+y \quad \text{iff} \quad \exists\, q_1, q_2 \text{ such that } q \geq q_1 + q_2 \text{ and } q_1 \geq x \text{ and } q_2 \geq y$$

Could write the right hand side $\displaystyle\int^{q \geq q_1+q_2} x(q_1) \wedge x(q_2)$

There is one problem with this. It does not give the right answer in one kind of case, namely where $x, y$ are irrational but $x+y$ is rational. For in this case

$\boxed{\text{i.e., of form } \dot{q}}$

The definition will in this case define the set $q > x+y$, not the set of $q$ such that $q \geq x+y$.

This also shows that we cannot just <u>exclude</u> $\bar{q}$'s in $\mathbb{R}$.

So we need a map $\overline{\mathbb{R}} \xrightarrow{\alpha} \mathbb{R}$ splitting the inclusion.

I propose to use: the "associated sheaf" – functor. So the correct definition, then, of $x+y$ is

associated sheaf of $x \oplus y$

where $x \oplus y$ is defined by the above convolution formula. (using the obvious $+$ on $\mathbb{Q}$ : obvious, because we did not

define $\mathbb{Q}^{\cdot}$ as a <u>quotient</u> of $\mathbb{N} \times \mathbb{N}^*$.
How is $\alpha$ defined?

$$** \qquad \alpha(x)(q) = \bigcup_{q<q'} x(q')$$

(where $\bigcup$ is one of the maps defined in an earlier exercise). – This process $**$ e.g. gives $\dot{q}$ out of $\overline{q}$.
Then $\mathbb{R} = \{x \mid x = \alpha(x)\} = \{x \mid x \geq \alpha(x)\}$ (since always $x \leq \alpha(x)$).
Also, for instance, the identical false map $\mathbb{Q} \longrightarrow \Omega$ is in $\mathbb{R}$; we denote it $\infty$ (recall that $x \leq y$ iff $x \geq y$ by def. of $\geq$). Then $\infty \geq x \quad \forall x$. – In fact, $\mathbb{R}$ is a complete ordered set:

$$\inf_I \cancel{\sup} (x_i) = \alpha\left(\bigcup_{i \in I} x_i\right)$$

$$\sup_I (x_i) = \bigcap_{i \in I} x_i$$

Crucial property of $\alpha$: $\alpha(x) \leq y \quad (y \in \mathbb{R})$ iff $x \leq y$
<u>or</u> equivalently

$$\frac{\alpha(x) \geq y \qquad (y \in \mathbb{R})}{x \geq y} \text{ iff}$$

Basic property of $\alpha$: $\alpha(x \cap y) = \alpha(x) \cap \alpha(y)$
In terms of $\geq$, $x \cap y$ is just: $\max(x,y)$

To define notion of metric space: an object $X$ together with a function $d : X \times X \longrightarrow \mathbb{R}$ so that $d(x,x) = 0 \quad \forall x$

$$d(x,y) + d(y,z) \geq d(x,z) \qquad \forall x,y,z.$$

Weaker than usual notion: do not require definiteness; distance need not be <u>finite</u>; and distance need not be symmetric. This is the framework through which we will discuss some basic constructions on metric spaces, i.e., without using classical logic: law of excluded middle, extensionality. The endavour has other justifications. Bear analogies to a certain kind of categories.

1-12-71

Lawvere 1. dec. 71

<u>Metric space</u>

$\underline{X}(x,y)$ (distance) real number $0 \leq \underline{X}(x,y) \leq \infty$

$\underline{X} : X \times X \longrightarrow \mathbb{R}$

$0 \geq \underline{X}(x,x)$.

$\underline{X}(x,y) + \underline{X}(y,z) \geq \underline{X}(x,z)$.

1/ Cat object then $\underline{X}(x,y)$ is an object

$1 \xrightarrow{i_x} \underline{X}(x,x)$

$\underline{X}(x,y) \times \underline{X}(y,z) \longrightarrow \underline{X}(x,z)$ composition law.

2/ Additive cat object then $\underline{X}(x,y)$ is an abelian group object.

$\mathbb{Z}$ free ab group on 1.

$\mathbb{Z} \xrightarrow{i_x} \underline{X}(x,x)$. additive homomorphism

$\underline{X}(x,y) \otimes \underline{X}(y,z) \xrightarrow[\text{additive comp. law.}]{} \underline{X}(x,z)$

3/ Poset obj (preordered obj)

$\underline{X} : X \times X \longrightarrow \Omega$

$T \underset{y}{\overset{x}{\rightrightarrows}} X$

$\underline{X}(x,y) = [\![ x \leq y ]\!]$

$\text{true} \Rightarrow \underline{X}(x,x)$.

$\underline{X}(x,y) \wedge \underline{X}(y,z) \Rightarrow \underline{X}(x,z)$

„$\Rightarrow$" means here the order rel. in $\Omega$.

Internal hom of $\mathbb{R}$.

$$\frac{x \geq [a, y]}{a + x \geq y}$$

in $S$: $[a, y] = \begin{cases} 0 & a \geq y \\ y - a & a < y \end{cases}$

$$\frac{x \geq [a, y]}{a + x \geq y}$$ means

$$[a, y] = \inf \{ x \mid a + x \geq y \}$$
$$= \sum_{\text{copr}} \quad \text{in } \underline{\text{cat}} \ \mathbb{R}.$$
$$= \underrightarrow{\lim}$$

$$0 \geq |x - y| = [a, y] + [y, a]$$
$$= \max \{ [a, y], [y, a] \}$$

in case of $\mathbb{R}$.

in case of any $\underline{X}$ ($\mathcal{V}$ cat).

$S_{\otimes}(\underline{X})(x, y) = \underline{X}(x, y) \otimes \underline{X}(y, x)$.

$S_{\times}(\underline{X})(x, y) = \underline{X}(x, y) \times \underline{X}(y, x)$

$\underline{X} \xrightarrow{f} \underline{Y}$ special case of $\mathcal{V}$ - ~~functor~~ valued functor.

$X \xrightarrow[\text{in } E]{f} Y$

$$\underline{X}(x, y) \xrightarrow[\substack{f_{x,y} \\ \text{in } \mathcal{V}}]{} \underline{Y}(f(x), f(y))$$

Lawvere 6/12-71.

Deep desires of improving the teaching in this room. Summed up:
Main problem in this class is that large part of the motion is wasted: the professor is telling the students things they don't want to hear, and the students are not forthrightly asking the questions which are on their minds.

Correct to say that I am disgusting; correct, because

1. Monopolization
2. Overproduction

these two aspects are demonstrated by insisting on complete silence.
these two aspects give rise to 1. Stink 2. Disgust.
-- which the people experience in their <u>relation</u> to production, of corn, or of ideas.

The way to oppose it is to sum up our experience, Summing up my experience: I have not been sufficiently 1. <u>serious</u> or 2. <u>forthright</u>
The base is forthrightness. I have sometimes said that I have proved things which I have <u>not</u> proved, or said that things were obvious that were not. Have not been serious in transforming the teaching process from its reactionary aspect of accomplishing nothing.
Have <u>you</u> been serious? Yes. Have shown 1. faith and daily reborn hope in future. 2.

G: Mathematics

Talking nonsense for a certain reason
Remember Hegel, who also talked nonsense for a very good reason as pointed out by Marx, Engels, Lenin
Hegel was 1 great bastion of objective idealist dialectics.

talked nonsense, because he was a man of the people but imprisoned in the university.

Second great bastion, from ~~[illegible]~~ which we can learn is category theory; it is its highest stage. We can learn from it because we have ~~[illegible]~~ long experience with it.

Should remember

.. here I enter the blackboard:

The substance of math resides in its Form rather than its substance — This statement I made 7 years ago, and which idealist philosophers have attacked, and which I until recently have been ashamed of. I believed this was idealist

Hegel also said the thing we ‖know‖ is false: "Ideas have supremacy over matter" — but even then Marx and Lenin said Yes to Hegel

reflects correctly what Hegel saw.

This statement while at first glance idealist, is more deeply: dialectical, and reflects a definite experience which we all have in mathematics: we know that 1. the form of a proof can be taken from one subject into another and 2. both constructions of the real no' lines are isomorphic (Dedekind constr: truth valued attributes of discrete quantities; Cauchy ~~seq.~~ constr, based on that which I have not seriously enough studied to say what it is in its essence).

This is the internal contradiction which gives rise to a deepened conscience of development: Category theory can change its substance from (obj) idealism; replace it by new content learning from the old form.

Must now take qualitative approach to quantity; the head must take full account to the role of the hands.

This is the legal part; the illegal part is: SMASH THE IMPERIALIST STATE. Our object here is concretely and .. bring about ..

Must reconnect ourselves with the base.

hope to bring about material change

General guidelines for summing up experience, in academic terms

Question: Is it so, that I am not engaging in illegal agitation, but in a legal attempt based on my deepest desires to sum up the experience of the students in order to give it back to them in the form of truth.

Lawvere 8/12-71.

...more fundamentally, I think I succeeded on Monday: never got of the problem of improving the teaching and understanding. Teachers are also students, and conversely.

Three day plan.

Typical example as how to sum experience.

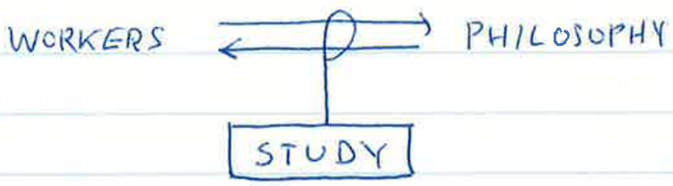
WORKERS STUDY PHILOSOPHY

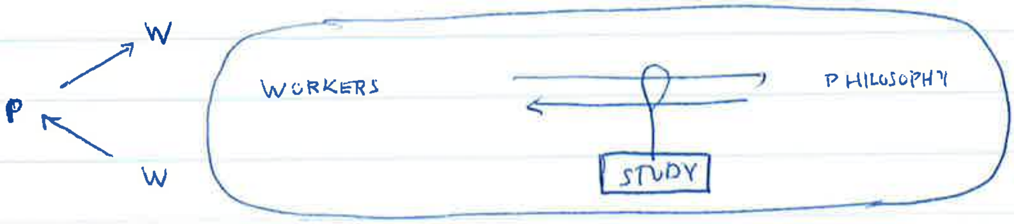
Summing up means report, but report the essentials. This original report may be divided into two. WORKERS, STUDY BY PHILOSOPHY! (by a comma). Then it sounds like an order. The two main things are

WORKERS PHILOSOPHY

The process of bringing these together takes place, independently of anyone's will. Then the reactionary line will , or notice that it is carried out but then giving the slogan in the air that it should be carried out but in a subjective way, but by subjectivism i.e. by hypnotism in television, say. Subjective, total, unity of man and machine (unprincipled unity): "workers = machines", "philosophy = humanity" ("humanity" taken alone does usually not mean the people, let alone the workers). Or saying "we should use teaching machines to achieve the unity".

There are lots and lots of people in the world, who organize to do the connection, engage in the actual practice, and STUDY:






takes place in wave upon waves, from lower level to higher level.

In certain publications, for instance on category theory, (for the time being), sum up in form of diagram. One does not in this magazine, but one has written it:

Workers
Study
Philosophy

Look at the question of workers. We are intellectuals, who have been detached from the workers. But more profoundly, we are workers, since we are engaged in the process of changing ... : teaching, learning, writing articles. We should be conscious of that, in order to [do it better].
Why are we here? Did we drop from the sky? There is a reason for our being here, even though we are detached from raising potatoes, repairing tractors, ... . We are coming from (family relations) with people which do these things. This is not a subjective connection, as certain people want us to believe.

In spite of the fact that we are in the university, we may therefore have a relation to philosophy. Where did the present day struggles in philsophy come from? Even the process of study is being mystified by certain people. We have been engaged in writing papers, have produced a few papers, did study.

What did I mean that Fourier analysis should be summed up

by means of the last stages of Hegel: category theory.
When I gave that call: it seemed like a revolutionary call which I did not precisely know to carry out, except the duality of frequency domain and time domain, heat vs. light.
Because of my previous failure to engage in struggle with you, I have doubts, but I am prepared to give the call in a more concrete way: Sum up Fourier. (Next week return to .., with this problem in mind, metric spaces).
In few weeks or months, we may hope to carry out the call.

Can see now a certain connection between Fourier and Norbert Wiener. We should look a little into history and philosophy. STRUIK (fairly reliable historian) says: " —— most .. mathematician —— Ecole Polytech (Poisson) —— Fourier, Cauchy —— : —— interested by appl of math to phys —— "
" Poisson : ...
" Fourier is primarily rem —— ... heat conduction
the source of all modern methods in mathematical physics.

.. met with vigourous opposition ..

Typical example .. IBM history chart says that Fouriers thesis was rejected "because it lacked proofs"
There is an old saying in English: The proof is in the pudding
1. Proof
2. Pudding
"Proof" is from something to conclusions: • ⟶ •
The recipe for a pudding is 'proved', not by its own [quality] but in the actual pudding.

Heat ⇄ Theory about heat

$\boxed{\text{Pudding}}$

You have motion of heat inside the pudding.

Liberate philosophy from the confines . .
and turn it into a sharp weapon ~~of~~ in the hands of the masses.

Let us go and look at the IBM history chart

Thermodynamics have never been deduced from statistical mechanics, yet. Some day it will. Mayoh

Cantor . . was started by looking into Fourier.

Lawrence 13/12-71.

Ask you to prepare a 1-2 page paper on how we can carry out some of the things . . : (mentioned wednesday). Take full initiative to sum up what we discussed wedn. — Give it to me on Wednesday. (In future, I shall assign exercises — if you have done some of the earlier exercises, I shall be glad to receive it.).

About heat : . . . ~~within~~ the motion of heat is macroscopic motion of microscopic motion . . . Convection, conduction, transport. Within a correct theory of heat, this dialectics must play a role. Question of motion of heat is connected with the question of Vitamin C. Linus Pauling came out this spring with a book on how Vitamin C cure the common cold. Small drug co's started producing Vit. C. ; the big co's came out through food-and-drug-administration, because of the sodium, all Vitamin C ~~were~~ tablets were recalled. U.S. imperialism will never solve the problem of the common cold — it is connected with .. obscurance of the motion of heat. In a modern monolithic plan .. workers are not allowed to adjust plans. Therefore draughts; they cause cold. This fact is explained away as "old grandmother's saying". Rapidly moving air can cool things. Conjecture about colds and draughts. Imperialism wants to obscure the connection between movement of heat and common colds.
Linus P. tried to prove, by Pearson-statistics, that the experiments carried out support his ~~the~~ line on Vitamin C. Mathematics ~~also~~ always comes back into it; it plays a leading role.

Should concentrate on Fourier and Fourier transform. ........
.."homogeneous chaos" (Wiener) — which is just what Imperialism wants to bring about in the minds of people.

Do you want to hear more about metric spaces?

bourgeois concocted the notion of randomness to obscure the existence of internal contradictions.

Fourier transform

$G$ a smooth group. Can consider measures on it, or functions on it, $f : G \longrightarrow \mathbb{R}$.
The Fourier transform of $f$, $\hat{f}$, has a value on every <u>representation</u> of that group.

$\hat{f}(X)$ where $X$ varies over

smooth representations of $G$ in vector spaces.

$$X : G \longrightarrow \mathcal{V}ect$$ ( $G$ viewed as a cat. with one object)

Call $X(e)$ $V$.
If $f \in \mathbb{R}[G]$

$$\begin{tikzcd} G \arrow[r, "\text{functor}", "X"'] \arrow[d] & \mathcal{V}ect \\ & V \\ \mathbb{R}[G] \arrow[uur, dashed, "\bar{X}"] & \\ 1 \arrow[uur, squiggly, "f"', "\hat{f}(X)"] & \end{tikzcd}$$

extend the functor $X$ to the additive functor $\bar{X}$.

Everything should be smooth (maybe we should be working in a topos which reflects the notion of smoothness). Then $\mathbb{R}[G]$ is much bigger than the discrete group ring; it will be the set of <u>measures</u> on $G$.

$\hat{f}(X) \in GL(V)$.

Whole point $f$ fixed; vary $X$ ... or conversely
$\hat{f} \in \underset{\underset{\varprojlim}{\text{or}}}{\prod} \mathrm{End}_V(X(e))$

Abstract essence (forget $G$, remember $\mathbb{R}[G]$).

$\mathcal{A}$ $(= \mathbb{R}[G])$ ; a $\mathcal{V}$- category

$\mathcal{V}^{\mathcal{A}}$ cat of rep's of "representations"

$\mathcal{X} \subseteq \mathcal{V}^{\mathcal{A}}$ subcat of irreducible rep's, say, ; $\mathcal{X}$ has a geometric structure, sometimes maybe even algebraic

$\mathcal{X}$ will be a $\mathcal{V}$-category, and an $\underline{E}$- cat. (underlying topos).

Problem: to choose $\mathcal{X}$. ?

In comm. algebra, $\mathcal{X}$ is spec $\mathcal{A}$, the indecomposable injectives.

$\mathcal{X} \subseteq \mathcal{V}^{\mathcal{A}}$ is really a profunctor between $\mathcal{A}$ and $\mathcal{X}$

$$\overset{f}{\mathcal{A}} \times \overset{x}{\mathcal{X}} \longrightarrow \mathcal{V}$$

frequency variable

turn it around

$$\mathcal{A} \xrightarrow{\hat{}} \mathcal{V}^{\mathcal{X}}$$

$$f \rightsquigarrow \hat{f}$$

$\mathcal{X}$ and $\mathcal{A}$ should both be $\mathcal{V}$-$\underline{E}$ -categories

How to make the geometry explicit ?

Where does the name spec come from ? (spectrum) Groth. was an analyst. The spectrum of an operator in Hilbert-space (set of eigenvalues)

PTO

= set of max ideals of the ring gen. by the operator.

Why is it called spectrum in Analysis

Norbert Wiener introduced that
- from light theory.

Lawvere 15/12-71.

Not yet ready to talk about Fourier transform.

There seemed to be lot of resistance last time to talking about metric spaces. I think, though, it is a good example of summing up .. experience of analysis. (Frechet - read his original paper in the library). I present a "higher level" summing up, using the method of closed categories; accounts for other things as well, as you know from the Closed Categories seminar. Frechet erred in assuming distance to be symmetric. Call $x \equiv y$ iff $d(x,y) < \infty$ - this is an equivalence relation if $d$ is symmetric. Examples concerning the question of symmetry (I mentioned process of symmetrizing) (for arbitrary closed cat's). Famous example of metric space (from probability theory). Given a measure space $S, \mathcal{B}, \mu : \mathcal{B} \longrightarrow \mathbb{R}$ $(\mathcal{B} \subseteq 2^S)$

Can make $\mathcal{B}$ into a metric space:

$$d(A,B) = \mu(A \setminus B).$$

So $d(A,B) = 0$ iff $A \subseteq B$ a.e. Usually, this metric is made symmetric by taking $A \setminus B \cup B \setminus A$ instead; important information is lost by this symmetrizing

Another type of example: Given a positive cone in a vector lattice [?]: consider the relation $\lambda \cdot x \leq y$ and define $d(x,y)$ as the sup (or inf?) of set of $\lambda$'s (positive reals) which does this.

Also examples going back to the $18^{th}$ century: Hilly ground; distance: time required to walk (it takes more time to go uphill than downhill).

[MacLane: J.A. Wilson in the 20's worked out a theory of non-symm.

metric spaces. 1929. II

Busemann developed .. metric spaces .. with distance defined in a geodesic manner. Unfortunately, he still assumes symmetry; should be reworked un-symmetrically.

Closed category $\mathbb{R}$ ; "strong" functor = Lipschitz map (constant $\leq 1$) between metric spaces. Metric spaces was a reasonable generalization because we also have to consider function spaces. This is a special case of the construction of functor-categories: $\underline{Y}^{\underline{X}}$ ; its points are: distance-decreasing maps $\underline{X} \longrightarrow \underline{Y}$ ; morphisms: transformations, but "enriched over $\mathcal{V}$" ; if $f, g : \underline{X} \longrightarrow \underline{Y}$ , then

$$\mathrm{nat}(f, g) \subseteq \prod_{x \in \underline{X}} \underline{Y}(f(x), g(x))$$

the naturality condition expresses that $\mathrm{nat}(f,g)$ is a certain equalizer – which in the partially ordered case $\mathbb{R}$ becomes an identity, – so we wind up with

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)) .$$

"THE NOTION OF NATURAL TRANSFORMATION generalize – specialize to NOTION OF METRIC IN FUNCTION SPACE"

$\mathcal{V}$ itself is a small $\mathcal{V}$-category here $(\mathcal{V} = \mathbb{R})$. (Can construct Yoneda embedding). $\mathrm{nat}(\ ,\ )$ is adjoint to tensor-product :

$$\frac{\underline{A} \otimes \underline{X} \longrightarrow \underline{Y}}{\underline{A} \longrightarrow \underline{Y}^{\underline{X}}}$$

so metric spaces themselves form a closed category

(Is there also one for ~~[illegible]~~ cartesian $\times$ instead of $\otimes$?)

Yoneda-embedding $\underline{X} \longrightarrow \mathbb{R}^{\underline{X}^{op}}$ (= a full & faithful $\mathcal{V}$-functor).
(we have to write $\underline{X}^{op}$ because we don't assume that distance is symmetric). 'Full & faithful' translates here to 'isometric embedding'.
Further development, due to Isbell: 'adequacy': $\underline{A} \xrightarrow{i} \underline{X}$ is called $\mathcal{V}$-adequate if

$$\underline{X} \longrightarrow \mathbb{R}^{\underline{X}^{op}} \longrightarrow \mathbb{R}^{\underline{A}^{op}}$$

is isometric; it is often possible to find such $\underline{A}$ with $\underline{A} = \mathbb{N}$, $d(n,m) = \infty \ \forall n,m$ - so get also Thm that a separable metric space can be isometrically embedded in $\mathbb{R}^{\mathbb{N}}$ which is more or less $\subseteq \ell^1$, so deduce the famous Polish Theorem: sep. metric space can be embedded in $\ell^1$.

<u>Kan-extensions</u> for $\mathcal{V}$-categories, which in this case should be called <u>Kan-Tietze extension</u>:

$$\begin{tikzcd} \underline{A} \arrow[r, "i"] \arrow[dr, "f"'] & \underline{X} \arrow[d, dashed, "\bar{f}"] \\ & \mathbb{R} \end{tikzcd}$$

there are two best ways of extending $f$; formula long ago given by analysts:

$$\bar{f}(x) = \varinjlim_{a} \underline{X}(a,x) \otimes f(a)$$
$$= \inf_{a} \left( d_{\underline{X}}(a,x) + f(a) \right)$$

Applied to integration: $\underline{A}$ = set of step functions on a measure space, or $\underline{A}$ = polynomial functions on unit interval. (such functions we know how to integrate). Then get two <u>extensions</u> to the whole function space, thus getting Darboux upper and lower integral

If these agree on $x \in \underline{X}$, call $x$ integrable (see picture ✱)
It makes sense in the general Kan-extension context.

Closed category sum up interactions in various fields of mathematics.

Further example: free category on a graph. (discussed this in connection with word algebra ⅋. in the case $\mathcal{V} = \mathcal{S}$).
For $\mathcal{V} = Ab$: free additive category on an additive graph (generalized tensor algebra).
What is a directed $\mathcal{V}$-graph? A set $X$ (or obj in a topos), to each pair $x, y \in X \times X$, an object $(x,y) \in \mathcal{V}$.
(So a graph = square $\mathcal{V}$-matrix).
Same basic fmla for free category: the hom in the free cat is a hom of $\otimes$-products

$$\hom(x,y) = \sum_{\substack{\text{finite sequences} \\ x, x_1, x_2, \dots, y}} (x,x_1) \otimes (x_1,x_2) \otimes \cdots \otimes (\ ,y)$$

So for an $\mathbb{R}$-graph, get

$$d(x,y) = \inf_{\text{all sequences}} (x,x_1) + (x_1,x_2) + \cdots (x_n,y)$$

which is the geodesic metric, or "the minimal cost" (or "minimal time" - which is the same according to Marx)

$\mathcal{V}$-profunctors are best introduced by first introducing the calculus of matrices. A profunctor from one metric space to another $\underline{X} \overset{\varphi}{\longrightarrow} \underline{Y}$: to every pair $x, y$ a number $\varphi(y,x)$ so that

$$\underline{Y}(y,y') + \varphi(y',x) \geq \varphi(y,x)$$
$$\varphi(y,x') + \underline{X}(x',x) \geq \varphi(y,x)$$

which is just the requirement needed to make a metric space out of $X+Y$ (with $d(y,x) = \infty$ for $y \in Y$, $x \in X$).
Every functor gives rise to a profunctor. How is it that (the profunctors form a 2-dim category) – so we can discuss the notion of adjointness of such profunctors:

$$\underline{X} \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \underline{Y}$$

What is the condition that $\varphi$ has an adjoint? – get an enlargement of $\underline{Y}$. Enough to consider $1 \to \underline{X} \to \underline{Y}$.
An adjoint profunctor is the same as a functor $\underline{X} \longrightarrow \hat{\underline{Y}}$; what is $\hat{Y}$, i.e. what is an adjoint pair $1 \underset{\varphi}{\overset{\psi}{\rightleftarrows}} \underline{Y}$

<u>Claim</u>: $\varphi \equiv$ functor into the Cauchy-completion of $\underline{Y}$.

For, $\varphi$ being ~~right~~ left adjoint means that we have

$$\psi \circ \phi \overset{\leq}{\longleftarrow} id_1$$

$$id_{\underline{X}} \overset{\leq}{\longleftarrow} \phi \circ \psi$$

So

$$0 = \psi \circ \phi = \inf_{y \in Y} (\psi(\cdot, y) + \varphi(y, \cdot))$$

Define $\underline{Y}_\epsilon$ to be

$$\{ y \mid \psi(\cdot, y) + \varphi(y, \cdot) \leq \varepsilon \}$$

For every $\epsilon > 0$, $\underline{Y}_\epsilon$ is non-empty. So for every $n \in N$ we can choose $y_n \in \underline{Y}_{\frac{1}{n}}$. Shall show that this is a Cauchy sequence. For this we look at the other adjunction

$$\inf(\varphi(y', \cdot) + \psi(\cdot, y)) = (\varphi \circ \psi)(y', y) \geq d_{\underline{Y}}(y', y)$$

$$\| $$

$$\varphi(y', \cdot) + \psi(\cdot, y)$$

So $\varphi(y_n, \cdot) + \psi(\cdot, y_m) \geq d(y_n, y_m)$; but the left hand side is bounded by $\frac{1}{n} + \frac{1}{m}$. So $\{y_n\}$ is a Cauchy-seq

If $\{z_n\}$ is another Cauchy sequence chosen by the same condition, then $\{y_n\}$ and $\{z_n\}$ are equivalent Cauchy-sequences.

Let $\hat{Y}$ be set of Cauchy-sequences, $y_\infty = \psi$ Put

$$d_{\hat{Y}}(y_\infty, y) = \psi(y)$$

$$d_{\hat{Y}}(y', y_\infty) = \varphi(y').$$

So the given profunctors <u>are</u> distances to points in the completion.

So even Cauchy completion can be dealt with by the dialectics of categories

Multiplicative $\mathcal{V}$-categories; for the metric-space-case

$$1 \xrightarrow{0} \underline{X} \qquad \underline{X} \otimes \underline{X} \xrightarrow{+} \underline{X}$$

$+$ being a $\mathcal{V}$-functor (= distance decreasing) says:

$$d(x,x') + d(y,y') \geq d(x+y, x'+y')$$

Banach ⟦Semigroup⟧. Call $d(x,0) = |x|$; conclude

$$|x| + |y| \geq |x+y|.$$

A closed $\mathcal{V}$-category is then here close to being a <u>normed abelian group</u>. (Karoubi claims that normed abelian group is interesting).

May define:

"Cauchy complete" = "any adjoint profunctor into it is a functor"

⟦Mac Lane: what about Hahn-Banach thm? ⟧

⟦ —"— : what light does it shed on the various theories of integral? ⟧