Primitive cohomology of smooth projective complete and non-complete intersections

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Abstract

Work in progress.

Let X be a smooth projective manifold. Let $F_1, ..., F_r$ be a set of homogeneous polynomials, all of them of the same degree (sufficiently high) defining X as a subscheme of the projective space. Let c be the codimension of X. Pick $G_1, ..., G_c$, c generic linear combinations of $F_1, ..., F_r$. The complete intersection $Z_0 := V(G_1, ..., G_c)$ contains X as an irreducible component. Let

$$Z_t := V(G_{1,t}, ..., G_{c,t})$$

be a 1-parameter smoothing of Z_0. Our aim is to compare the intermediate primitive cohomology of X (for a certain polarisation) with the intermediate cohomology of Z_t.

If $\dim(X) \leq 3$ we find a natural embedding of the intermediate primitive cohomology of X into the intermediate cohomology of Z_t. For $\dim(X) \geq 4$ this embedding does not exist in general. We find a necessary and sufficient condition: we define two polynomials P and Q on the Chern classes of the tangent bundle of X and on the polarisation given by the embedding, and the cohomology embedding holds if and only if Q is a multiple of P in the cohomology ring of X.

The condition is satisfied inmediately if X is a complete intersection, but also if the codimension of X is sufficiently low. This may be seen as a supporting evidence of Hartshorne conjecture on smooth varieties of small codimension being complete intersections, and perhaps as a tool to address it.

In the case that the condition is not satisfied, the piece of the primitive cohomology that does not embed is shown to embed into the intermediate primitive cohomology of a manifold of dimension 4 less that the original one, setting up in this way an inductive scheme to analyze it.