Solutions to Problem Set 3

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Probability and Applications

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Problem 1

(a) We can obtain marginal distribution of X, by summing the probabilities in each column and marginal distribution of Y by summing each row. More formally,

$$Pr(X = x) = \sum_{y \in supp(Y)} p(x, y), \quad Pr(Y = y) = \sum_{x \in supp(X)} p(x, y).$$

Y\X	1	2	3	4	5	X
1	0.04	0.01	0	0.01	0.04	Pr(Y=1)=0.1
2	0.12	0.03	0	0.03	0.12	Pr(Y=2)=0.3
3	0.08	0.02	0	0.02	0.08	Pr(Y=3)=0.2
4	0.16	0.04	0	0.04	0.16	Pr(Y=4)=0.4
Y	Pr(X=1)=0.4	Pr(X=2)=0.1	Pr(X=3)=0	Pr(X=4)=0.1	Pr(X=5)=0.4	

(b) For X, Y to be independent, we must have

$$\forall x \in supp(X), y \in supp(Y) \;\; p(x, y) = P_X(x)P_Y(y).$$

Where $P_X(x)$, $P_Y(y)$ are marginal distributions of X, Y and $p(x, y)$ is the joint distribution of them. Observing the table provided in (a), this holds for all values of x, y. Therefor, X, Y are independent.

(c) In general,

$$P_{X|Y}(x|y) = \frac{p(x, y)}{P_Y(y)}.$$

Using marginal distribution of Y in the above formula will lead to the following results.

$$
\begin{aligned}
P_{X|Y}(x|y=1) &= 10p(x,y=1),\\
P_{X|Y}(x|y=2) &= \frac{10}{3}p(x,y=2),\\
P_{X|Y}(x|y=3) &= 5p(x,y=3),\\
P_{X|Y}(x|y=4) &= \frac{5}{2}p(x,y=4).
\end{aligned}
$$

If we substitute $p(x=i, y=j)$ from the table to these equations, it can be seen that they will all show the same probability distribution for different values of X. This means that, regardless of i,

$$
\begin{aligned}
P_{X|Y}(x=1|y=i) &= 0.4,\\
P_{X|Y}(x=2|y=i) &= 0.1,\\
P_{X|Y}(x=3|y=i) &= 0,\\
P_{X|Y}(x=4|y=i) &= 0.1,\\
P_{X|Y}(x=5|y=i) &= 0.4.
\end{aligned}
$$

Now, note that,

$$E[X|Y=y] = \sum_{x \in supp(X)} x P_{X|Y}(x|Y=y).$$

And because for all values of y, $P_{X|Y}(x|Y=y)$ has the same distribution, $E[X|Y=y]$ is also the same for all values of y and is equal to

$$E[X|Y=y] = 1 \times 0.4 + 2 \times 0.1 + 0 + 4 \times 0.1 + 5 \times 0.4 = 3.$$

It seems like that knowing the value of Y does not change the probabilities for X. Thus, it is probable that X, Y are independent.

(d)

$$E[X^2+Y^2|XY<10] = \sum_{xy<10} (x^2+y^2)\frac{p(x,y)}{Pr(XY<10)}.$$

By adding up joint probabilities for all x, y that $xy < 10$ it can be seen that, $Pr(XY < 10) = 0.3$. So,

$$
\begin{aligned}
E[X^2+Y^2|XY<10] = \frac{10}{3}&\sum_{xy<10} (x^2+y^2)p(x,y)\\
&= \frac{10}{3}(0.04(1+1) + 0.13(1^2+2^2) + 0.17(1^2+4^2)\\
&+ 0.03(2^2+2^2) + 0.05(1^2+5^2) + 0.07(2^2+4^2)) = 21.
\end{aligned}
$$

(e) It is easy to see that,

$$p_{X^2}(x) = Pr(X^2 = x) = p_X(\sqrt{x}), \;\; p_{Y^2}(y) = Pr(Y^2 = y) = p_Y(\sqrt{y}).$$

Also,

$$supp(X^2) = \{1, 4, 9, 16, 25\}, \;\; supp(Y^2) = \{1, 4, 9, 16\}.$$

Because X, Y are independent, X^2, Y^2 are also independent and mass function of their sum can be obtained using the convolution theorem.

$$Z = X^2+Y^2 \implies p_Z(z) = \sum_{x \in supp(X^2)} p_{X^2}(x)p_{Y^2}(z-x) = \sum_{x \in supp(X^2)} p_X(\sqrt{x})p_Y(\sqrt{z-x}).$$

Problem 2

(a) First, we prove that if $E[|X-a|]$ is at its minimum, $a = E[X]$. Then, to find the proper value of a, we will just calculate $E[X]$.
If we choose a such that $E[|X-a|]$ is at its minimum, then $E[(X-a)^2]$ is also at its minimum and vice versa. Note that,

$$E[(X-a)^2] = E[X^2] - 2aE[X] + a^2.$$

Where $E[X^2]$, $E[X]$ are constants. To minimize $E[(X-a)^2]$ we can solve

$$\frac{d}{da}(E[(X-a)^2]) = 0 \implies -2E[X] + 2a = 0 \implies a = E[X].$$

As $X \sim Uniform(0, A)$, its expected value is $A/2$. Hence, $a = \frac{A}{2}$ will result in minimum value of $E[|X-a|]$.

(b) With a similar argument, $a = E[X]$ for $X \sim exponential(\lambda)$. Obviously, as is for any exponential random variable, $E[X] = \frac{1}{\lambda}$. Thus, the proper value of a is $\frac{1}{\lambda}$.

Problem 3

(a) Due to symmetry, the desired probability is $\frac{1}{3}$.

(b) $X_1 \sim Uniform(0, 1)$, $X_2 \sim Uniform(-1, 0)$, $Y = X_1 + X_2$. Because X_1, X_2 are independent, $f_Y(y)$ is the convolution of their distribution functions.

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x)f_{X_2}(y-x)dx = \int_0^1 f_{X_2}(y-x)dx.$$

Evaluating the last integral for $y \in (-1, 0)$, $y \in (0, 1)$ will yield

$$f_Y(y) = \begin{cases} 1+y & \text{if } y \in (-1,0) \\ 1-y & \text{if } y \in (0,1) \end{cases} .$$

(c) The probability of $\theta \in [0, 2\pi)$ is 1.

$$\int_0^{2\pi} (a + bcos(\frac{\theta}{2})d\theta) = 1 \implies 2a\pi = 1 \implies a = \frac{1}{2\pi}.$$

Also, $E[\theta] = \frac{\pi}{2}$.

$$\int_0^{2\pi} (\frac{\theta}{2\pi} + b\theta cos(\frac{\theta}{2})d\theta) = \frac{\pi}{2} \implies \pi + b(2\theta sin(\frac{\theta}{2}) + 4cos(\frac{\theta}{2}))\Big|_0^{2\pi} = \frac{\pi}{2} \implies 8b = \frac{\pi}{2} \implies b = \frac{\pi}{16}.$$

(d) If r is the distance between the random point and point $\Theta = 0$ on the circle and R is the radius, then

$$r = 2Rsin(\frac{\Theta}{2}), \;\; supp(r) = [0, 2R].$$

If we denote the cumulative distribution function of Θ and r by F_Θ, F_r respectively,

$$\forall x \in (0, 2\pi) \; F_\Theta(x) = \int_0^x f(\theta)d\theta = \frac{x}{2\pi} + \frac{\pi}{8}sin(\frac{x}{2}),$$

$$F_r(x) = Pr(r \leqslant x) = Pr(2Rsin(\frac{\Theta}{2}) \leqslant x) = Pr(\Theta \leqslant 2sin^{-1}\frac{x}{2R})$$
$$= F_\Theta(2sin^{-1}\frac{x}{2R}).$$

Thus,

$$\forall x \in (0, 2\pi) \;\; F_r(x) = \frac{1}{\pi}sin^{-1}\frac{x}{2r} + \frac{\pi x}{16R}.$$

Probability density function is now obtained by differentiation.

$$\forall x \in (0, 2\pi) \;\; f_Y(y) = \frac{1}{\pi R\sqrt{4 - \frac{y^2}{R^2}}} + \frac{\pi}{16R}.$$

(e)

$$E[r] = E[2Rsin(\frac{\Theta}{2})] = 2RE[sin(\frac{\Theta}{2})] = 2R\int_0^{2\pi} f_\Theta(\theta)sin(\frac{\theta}{2})d\theta$$
$$= \frac{4R}{\pi}.$$

Problem 4

(a) Let $u = xy$, $v = \frac{x}{y}$. Solving this system of equations result in

$$x = w_1(u, v) = \sqrt{uv}\;, \;\; y = w_2(u, v) = \sqrt{\frac{u}{v}}.$$

We now calculate the Jacobian of these two functions.

$$J = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \\ \frac{1}{2}\sqrt{\frac{1}{uv}} & \frac{-1}{2v}\sqrt{\frac{u}{v}} \end{vmatrix} = \frac{-1}{4v}\sqrt{\frac{v}{u}}\sqrt{\frac{u}{v}} - \frac{1}{4v} = -\frac{1}{2v}$$

Thus, $|J| = \frac{1}{2v} \neq 0$. If we denote joint distribution of u, v by $g(u,v)$,

$$\forall u \in [1,\infty), v \in (0,\infty) \ \ g(u,v) = \frac{1}{2v} f_{X,Y}(\sqrt{uv}, \sqrt{\frac{u}{v}}) = \frac{1}{2u^2 v}.$$

(b)

$$f_U(u) = \int_{-\infty}^{\infty} g(u,v)dv = \int_{v=\frac{1}{u}}^{u} \frac{1}{2u^2 v} dv = \frac{1}{2u^2} \ln v \Big|_{\frac{1}{u}}^{u} = \frac{\ln u}{u^2}$$

$$f_V(v) = \int_{-\infty}^{\infty} g(u,v)du = \int_{v}^{\infty} \frac{1}{2u^2 v} du = \frac{1}{2v^2}$$

The above marginal distributions are given for all $u \in [1,\infty)$, $v \in (0,\infty)$ and are equal to zero for all other values of u, v.

Problem 6

Because $X_1 \sim \mathit{exponential}(\lambda_1)$, $X_2 \sim \mathit{exponential}(\lambda_2)$, and they are independent, their joint probability density function is easily derived by multiplying their respective marginal density functions.

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \begin{cases} \lambda_1\lambda_2 e^{-\lambda_1 x_1 - \lambda_2 x_2} & x_1, x_2 > 0 \\ 0 & o.w \end{cases}$$

Let $Z = X_1/X_2$. If we denote the cumulative distribution function of Z by F_Z,

$$\begin{aligned} F_Z(z) &= Pr(Z \leqslant z) = Pr(\frac{X_1}{X_2} \leqslant z) = \iint\limits_{\frac{X_1}{X_2} \leqslant z} f(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 \\ &= \int_{x_2=0}^{\infty}\int_{x_1=0}^{zx_2} \lambda_1\lambda_2 e^{-\lambda_1 x_1 - \lambda_2 x_2} dx_1 dx_2 = -\lambda_2 \int_{x_2=0}^{\infty} e^{-\lambda_2 x_2}(e^{-\lambda_1 z x_2} - 1) dx_2 \\ &= -\frac{\lambda_2}{\lambda_2 + z\lambda_1} + 1 \\ &\implies F_Z(z) = 1 - \frac{\lambda_2}{\lambda_2 + z\lambda_1} \implies f_Z(z) = \frac{\lambda_1\lambda_2}{(\lambda_2 + z\lambda_1)^2}. \end{aligned}$$

Finally,

$$Pr(X_1 < X_2) = Pr(Z < 1) = F_Z(1) = 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1} = \frac{\lambda_1}{\lambda_2 + \lambda_1}$$

Problem 7

(a) We will find the density function of a continuous memoryless random variable. For any such random variable, X,

$$Pr(X > s+t|X > s) = Pr(X > t) \implies Pr(X > s+t) = Pr(X > s)Pr(X > t)$$
$$\implies 1 - F(s+t) = (1 - F(s))(1 - F(t)).$$

Now if we define $R(x) = 1 - F(x)$, and replace s with x and t with h,

$$R(x+h) = R(x)R(h) \implies \frac{R(x+h) - R(x)}{h} = R(x)\frac{R(h) - 1}{h}.$$

Now using the definition of derivative and the fact that $R(0) = 1$, if we take the limit of the above equation as $h \to 0$ it follows that,

$$\lim_{h\to 0} \frac{R(x+h) - R(x)}{h} = \lim_{h\to 0} R(x)\frac{R(h) - R(1)}{h} \implies R'(x) = R(x)R'(0).$$

Let $R'(0) = -\lambda$. $R(x)$ and $F(x)$ are derived by solving this differential equation.

$$R(x) = e^{-\lambda x} \implies F(x) = 1 - e^{-\lambda x} \implies f(x) = \lambda e^{-\lambda x}$$

Thus, $X \sim exponential(\lambda)$.

(b) Because X_is are independent,

$$\begin{aligned} F_Y(y) &= 1 - Pr(Y > y) = 1 - Pr(X_1, X_2, \cdots, X_n > y) \\ &= 1 - Pr(X_1 > y)Pr(X_2 > y)\cdots Pr(X_n > y) = 1 - (1 - F_X(y))^n \\ &= 1 - e^{-\lambda n y} \implies \forall y \geqslant 0: \quad f_Y(y) = \lambda n e^{-\lambda n y}. \end{aligned}$$

(c) Let X_i be the lifetime of ith battery and $X = max(X_1, \cdots, X_n)$. $E[X]$ is desired.

$$F_X(x) = Pr(X < x) = Pr(X_1, X_2, \cdots, X_n < x) = (F(X_1))^n = (1 - e^{-\lambda x})^n$$
$$\implies f_X(x) = n\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{n-1}.$$

Hence,

$$E[X] = n\lambda \int_0^\infty x e^{-\lambda x}(1 - e^{-\lambda x})^{n-1} dx.$$

Or we can use another formula for $E[X]$ and get

$$E[X] = \int_0^\infty 1 - F_X(x)dx = \int_0^\infty 1 - (1 - e^{-\lambda x})^n . dx$$

Problem 8

In order to have distributions of velocities along each axix, we must first calculate variance for normal random variables V_x, V_y, V_z(all three are normal random variables with mean value of 0 and standard deviation of σ and have similar distributions.). We use E_0 to calculate it.

$$E_0 = E[\frac{1}{2}mV^2] = \frac{1}{2}mE[V_x^2 + V_y^2 + V_z^2] = \frac{1}{2}m(E[V_x^2] + E[V_y^2] + E[V_z^2]) = \frac{3m}{2}E[V_x^2].$$

Because $E[V_x^2] - E[V_x]^2 = \sigma^2$ and $E[V_x] = 0$,

$$E[V_x^2] = \sigma^2 \implies E_0 = \frac{3m\sigma^2}{2} \implies \sigma = \sqrt{\frac{2E_0}{3m}}.$$

So we now know σ and thus know the distribution of velocities along each axis.
If we denote the standard normal random variable by N(normal with $\mu = 0$, $\sigma = 1$),

$$V_x = V_y = V_y = \sigma N \implies V_x^2 = V_y^2 = V_y^2 = \sigma^2 N^2.$$

Thus,

$$V^2 = \sigma^2(N_1^2 + N_2^2 + N_3^2).$$

Because N_is are independent, we can easily calculate their joint density function by multiplying their density function.

$$\forall x, y, z \in \mathbb{R} \;\; f_{N_1,N_2,N_3}(x, y, z) = \frac{1}{\sqrt{8\pi^3}} e^{-\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2}}$$

Now, $F_{V^2}(x) = Pr(V^2 < x) = Pr(N^2 + N^2 + N^2 < \frac{x}{\sigma^2})$, and the latter can be calculated by integration on f_{N_1,N_2,N_3} over a sphere with radius $\frac{\sqrt{x}}{\sigma}$. So,

$$F_{V^2}(t) = \iiint\limits_{x^2+y^2+z^2<\frac{t}{\sigma^2}} e^{-\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2}} dxdydz$$

This integral is easier to solve in spherical coordinate system. To convert it, let $\rho^2 = x^2 + y^2 + z^2$, θ, ϕ be coordinates in spherical system. Here, in this integral,

$$0 \leqslant \theta \leqslant 2\pi$$

$$0 \leqslant \phi \leqslant \pi$$

$$0 \leqslant \rho \leqslant r = \frac{\sqrt{t}}{\sigma}.$$

We rewrite the integral,

$$F_{V^2}(t) = \frac{1}{\sqrt{8\pi^3}} \int_0^{2\pi} \int_0^{\pi} \int_0^{r} e^{-\frac{\rho^2}{2}} \rho^2 \sin\phi d\rho d\phi d\theta.$$

This can be solved using bipartite integration. After some calculations we get,

$$F_{V^2}(t) = \frac{2}{\sqrt{2\pi}}(-re^{-\frac{r^2}{2}} - e^{-\frac{r^2}{2}} + 1).$$

Because $r = \frac{\sqrt{t}}{\sigma}$,

$$F_{V^2}(t) = \frac{2}{\sqrt{2\pi}}(-\frac{\sqrt{t}}{\sigma}e^{-\frac{t}{2\sigma^2}} - e^{-\frac{t}{2\sigma^2}} + 1).$$

Probability density function of V^2 is $f_{V^2} = \frac{\mathrm{d}}{\mathrm{d}t}F_{V^2}(t)$.

$$f_{V^2}(t) = \frac{e^{-\frac{t}{2\sigma^2}}(t + \sqrt{t}\sigma - \sigma^2)}{\sqrt{2\pi t}\sigma^3}$$

Problem 9

(a) Due to symmetry, mean distance vector in every step(and after N independent steps) is $(0, 0, 0)$.

(b) Here we use spherical coordinate system to represent points in space with a triple (r, ϕ, θ). In every step, displacement vector is (l_0, Φ, Θ), where l_0 is the constant distance, $\Phi \sim Uniform(0, \pi)$ and $\Theta \sim Uniform(0, 2\pi)$. If $R_i = (X_i, Y_i, Z_i)$ is the displacement vector in ith step in cartesian coordinate system, we have

$$X_i = l_0 sin\Phi_i cos\Theta_i,$$
$$Y_i = l_0 sin\Phi_i sin\Theta_i,$$
$$Z_i = l_0 cos\Phi_i.$$

If D_n^2 is the square distance from origin after n steps,

$$D_n^2 = (\sum_{i=1}^{n} X_i)^2 + (\sum_{i=1}^{n} Y_i)^2 + (\sum_{i=1}^{n} Z_i)^2 = \sum_{i=1}^{n}(X_i^2 + Y_i^2 + Z_i^2) + \sum_{i \neq j}\sum(X_iX_j + Y_iY_j + Z_iZ_j)$$
$$= n + l_0^2 \sum_{i \neq j}\sum(\sin\Phi_i \cos\Theta_i \sin\Phi_j \cos\Theta_j + \sin\Phi_i \sin\Phi_j \sin\Theta_i \sin\Theta_j + \cos\Phi_i \cos\Phi_j).$$

Because Θ_i, Φ_js are all independent of each other, mean of multiplication of functions of them is equal to multiplication of mean of these functions. So,

$$E[D_n^2] = n + l_0^2 \sum_{i \neq j}\sum(E[\sin\Phi_i]E[\cos\Theta_i]E[\sin\Phi_j]E[\cos\Theta_j] + E[\sin\Phi_i]E[\sin\Phi_j]E[\sin\Theta_i]E[\sin\Theta_j] +$$

Now it is easy to see that $E[\sin\Theta_i] = 0$. Hence, $E[\cos\Phi_i]E[\cos\Phi_j]$ is the only non-zero term in the sum. Also note that $E[\cos\Phi_k] = -\frac{2}{\pi}$ holds as $f_{\Phi_k}(\phi) = \frac{1}{\pi}$ for all $\phi \in (0, \pi)$. Therefor,

$$E[D_n^2] = n + l_0^2(\frac{4n(n-1)}{\pi^2}) = n + \frac{4n(n-1)l_0^2}{\pi^2}.$$

Problem 10

(a) $P(t, X)dX = \frac{1}{2}P(t - \Delta t, X + \Delta X)dX + \frac{1}{2}P(t - \Delta t, X - \Delta X)dX$