Branching Process

Branching process is a model for a society of individuals, each with random number of offsprings.

Galton-Watson Process:

A GW process assumes that the number of children of each individual are i.i.d random variables.

$X \sim (P_1, P_2, \dots)$ is the (common) distribution of the number of children.

Z_n is the size of the population of generation n.

We can see that $\text{supp}(Z_n) = \mathbb{N}_0$ and that $(Z_n)_n$ is a Markov chain.

Very often we are interested in the probability of extinction:

Extinction $= \{\exists n : Z_n = 0\}$ → The event of population size reaching 0 in finite steps.

$$\Pr[\text{Extinction}] = q = \lim_{n \to \infty} \Pr[Z_n = 0]$$

Changes in Population Size

If the size of the current population is $\underline{n}$, what is the probability that in the next generation, there are exactly $\underline{m}$ individuals? This probability is denoted by $P(n,m)$.

· Let $X \sim (P_0, P_1, \dots)$ be the distribution of the number of children.

$X_1, X_2, \dots, X_n, X$ are i.i.d random variables indicating the number of children of each individual in the current generation.

We have:

$$P(n,m) = \Pr[X_1 + \dots + X_n = m].$$

To evaluate this probability, we use the generating function f:

$$f(z) = \mathbb{E}_X[z^X] = \sum_{i=0}^{\infty} P_i z^i.$$

Then its easy to see that

$$f(z)^n = \sum_{m=0}^{\infty} P(n,m)\, z^m$$

Extinction

Theorem: [Necessary condition]

In any branching process, $\lim_{n\to\infty} E[Z_n]=0 \Rightarrow Pr[\text{Extinction}]=1$.

Proof:

$$1-Pr[Z_n=0]=Pr[Z_n\geq 1]\leq\sum_{i=1}^{\infty}Pr[Z_n\geq i]=E[Z_n]$$

$$Pr[\text{Extinction}]=\lim_{n\to\infty}Pr[Z_n=0]\geq 1-\lim_{n\to\infty}E[Z_n]=1$$

→ In general, the other direction may not be true; that is, if we almost surely go extinct, it doesn't mean that the expected population size goes to zero.

→ Therefore to study extinction, we can focus on the long-term expected size of the population: $\lim_{n\to\infty} E[Z_n]$.

Theorem: [Expected population size]

$$E[Z_n]=E[X]^n=\mu^n$$

Proof:

Let $X_1, X_2, \dots$ be i.i.d. and $X_i \sim X$.

We have $Z_{n+1}\sim\sum_{i=1}^{Z_n}X_i$.

Therefore, $E[Z_{n+1}]=E[Z_n]E[X]=\mu E[Z_n]$

$\Rightarrow E[Z_n]=\mu^n$ (Assuming $Z_0=1$)

Distribution of Population

We saw that $\mathbb{E}[Z_n] = \mu^n$. The natural follow up is to ask what is the distribution of Z_n?

To further simplify the matters, we investigate $Z_{n+1} | Z_n = k$, i.e., the distribution of the population given the size of the previous generation.

Recall from our discussion on P.2 that

$$f(z) = \mathbb{E}_X[z^X] \Rightarrow f(z)^k = \sum_{n=0}^{\infty} P(k,n) z^n$$

where $P(k,n) = \Pr[Z_{n+1} = n | Z_n = k]$.

So we can write

$$f_{Z_{n+1}|Z_n=k} = f_X^k.$$

Now note that $Z_{n+1} = \sum_{i=1}^{Z_n} X_i$. So we can write:

$$f_{Z_{n+1}} = \sum_{k=0}^{\infty} \Pr[Z_n = k] f_{Z_{n+1}|Z_n=k} = \sum_{k=0}^{\infty} \Pr[Z_n = k] f_X^k = f_{Z_n} \circ f_X$$

$$\Rightarrow f_{Z_n} = f_X^{(n)} = \underbrace{f_X \circ f_X \circ \cdots \circ f_X}_{n}$$

Extinction : II

Remember that we defined the extinction probability q as

$$q = \Pr[\text{Extinction}] = \lim_{n\to\infty} \Pr[Z_n = 0]$$

→ Notice that q is a fixed point for the generating function of X:

$$q = f_X(q)$$

This is because

$$q = \lim_{n\to\infty} \Pr[Z_n = 0] = \lim_{n\to\infty} f_{Z_n}(0) = \lim_{n\to\infty} f_X^{(n)}(0)$$

$$\Rightarrow f_X(q) = f_X\left(\lim_{n\to\infty} f_X^{(n)}(0)\right) = q$$

$\boxtimes$

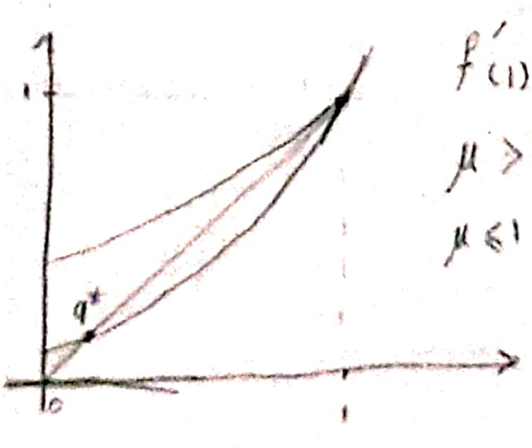
But we also have $f(1) = 1$. So 1 is always a fixed point for f_X.

We also know that f_X is increasing and convex:

$$f_X' = \sum_{i=1}^{\infty} i P_i z^{i-1} \geq 0 \quad \forall z. \Rightarrow f \text{ is increasing}$$

$$f_X'' = \sum_{i=2}^{\infty} i(i-1) P_i z^{i-2} \geq 0 \quad \forall z \Rightarrow f \text{ is convex}$$

Knowing these we can see that f_X intersects $y = x$ in the interval $[0,1]$ either once or twice:



$f'(1) = \mathbb{E}[X] = \mu$ (two fixed points)

$\mu > 1 \Rightarrow \exists\ 0 < q^* < 1$

$\mu \leq 1 \Rightarrow \Pr[\text{Extinction}] = 1$ (one fixed point)

Extinction: II (Cont.)

As we saw f_X has one or two fixed points in $[0,1]$.
If $f_X'(1) = \mathbb{E}[X] = \mu \leq 1$, then $q=1$ is the only fixed point, hence the extinction happens with probability 1.
What if $\mu > 1$ and we have another fixed point, $q^* < 1$?
What is the probability of extinction now?

Theorem: $[q = q^*]$.

If f_X has a fixed point $q^* < 1$, then the probability of extinction (q) is q^*.

Proof:

$$q = \lim_{n\to\infty} \Pr[Z_n = 0] = \lim_{n\to\infty} f_X^{(n)}(0).$$

But notice that

$$f_X(0) = P_0 \leq q^* \Rightarrow f_X^{(2)}(0) \leq q^* \Rightarrow \cdots \Rightarrow \forall n\ f_X^{(n)}(0) \leq q^*$$

f_X is increasing

$$\Rightarrow q \leq q^*$$
$$\Rightarrow q = q^*$$

Phase Transition

$\sum_{i=0}^{\infty} Z_i$

(Subcritical) $\mu < 1 \Rightarrow q = 1$, $\mathbb{E}[Z] < \infty$

(Critical) $\mu = 1 \Rightarrow q = 1$, $\mathbb{E}[Z] = \infty$

(SuperCritical) $\mu > 1 \Rightarrow q < 1$, $\mathbb{E}[Z] = \infty$