On Derivatives of Complex Functions

Any complex function $f : \mathbb{C} \to \mathbb{C}$ can also be thought of as a function $F : \mathbb{R}^2 \to \mathbb{R}^2$. But the notion of derivative for these two functions are different. Noticeably, $\frac{\mathrm{d}f}{dz}(z_0)$ is a single complex number (equivalently, a point in $\mathbb{R}^2$) while $DF(x_1, x_2)$ is a linear transformation from $\mathbb{R}^2$ into itself, which can be represented as a 2×2 matrix. Although they seem to be completely different mathematical objects, these two are related to each other in a subtle and beautiful way. In this note I try to explore the connections between these two notions.

Holomorphic functions

Let Ω be an open set in $\mathbb{C}$ and f a complex-valued function on it. The function f is said to be **holomorphic at point** $z_0 \in \Omega$ if the following limit converges:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

When this limit exists, we denote it by $f'(z_0)$ and call it the (complex) **derivative of f at** z_0.

A function f is said to be **holomorphic on** Ω if it is holomorphic at every point of Ω.

Notice that in this definition, h is a complex number and can approach 0 from any direction(side-note: check out the $\varepsilon - \delta$ definition of limit as well as the the one using convergence of sequences to see what we mean by *approach*).

Although the definition of holomorphic functions is very similar to a real, one-variable, differentiable function, there are important distinctions between the two. Here are some examples without proof:

- A holomorphic function is **infinitely many times complex-differentiable**.
- Every holomorphic function is **analytic**, in the sense that it has a power series expansion near every point. For this reason, holomorphic functions are sometimes called analytic.

Real-differentiable functions

Now consider a function $f : U \to \mathbb{R}^2$, where $U \subseteq \mathbb{R}^2$ is an open set. This function is said to be differentiable at a point $a \in U$ if there exists a linear transformation $Df(a) : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Df(a)[h]\|}{\|h\|} = 0.$$

The linear map $Df(a)$ is called the (total) **differential or derivative** of f at a.

This linear mapping is described in the standard basis of $\mathbb{R}^2$ by the **Jacobian matrix**:

$$Df(a) = J_f(a) = J(a) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) \end{bmatrix}$$

Complex-valued functions as mappings

We will now consider both interpretations of a complex-valued function simultaneously and investigate the relation between the two.

Let $f : \mathbb{C} \to \mathbb{C}$ be defined as

$$f(z) = f(x+iy) = u(x,y) + iv(x,y),$$

where $x, y \in \mathbb{R}$ and u, v are functions from $\mathbb{R}^2$ to $\mathbb{R}$. We can associate a mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ to this complex-valued function by letting

$$F(x,y) = (u(x,y), v(x,y)).$$

I call this the *real representation* of the complex-valued function f (I use this terminology because it sounds intuitive to me. I don't know if it is standard or not).

At the first glance, we see that the (complex) derivative of f at a point z_0 is a single complex number, $f'(z_0)$, whereas the (total) derivative of F at a point (x_0, y_0) is a 2×2 real matrix. However on a closer look, we can see that these two derivatives are related to each other.

To see this, let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Consider the definition of complex derivative when h is real, that is, $h = h_1 + ih_2$ with $h_2 = 0$. If we write $z = x + iy$ and $z_0 = x_0 + iy_0$, we find that

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1 + iy_0) - f(x_0 + iy_0)}{h_1} = \frac{\partial f}{\partial x}(z_0),$$

where $\frac{\partial}{\partial x}$ denotes the usual partial derivative with respect to variable x. To see how the limit of a complex function reduced to a (real) partial derivative, notice that we fixed y_0 and thought of f as a complex-valued function of a single, real variable x.

Now taking h to be a purely imaginary number and a similar argument yields

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0 + i(y_0 + h_2)) - f(x_0, iy_0)}{ih_2} = \frac{1}{i} \frac{\partial f}{\partial y}(z_0),$$

where $\frac{\partial}{\partial y}$ is the usual partial differentiation with respect to y variable.

From the two equations above, we get

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{1}{i} \frac{\partial f}{\partial y}(z_0).$$

If we write f as $f = u + iv$ and separate the real and imaginary parts, we see that the partial derivatives of u, v with respect to x, y exist and satisfy the following relations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$
$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the **Cauchy-Riemann** equations and they link complex and real analysis.

To summarize, if $f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic and $F(x, y) = (u(x, y), v(x, y))$ is the associated multivariable function, then we can write

$$DF(x, y) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial v}{\partial y} & -\frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

The converse of the above is also true:

Let $f = u + iv$ be a complex-valued function defined on an open set Ω. If u and v are *continuously differentiable* and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \frac{\partial f}{\partial z}$. (see the definition of $\frac{\partial f}{\partial z}$ in the last section)

Geometric Interpretation

Why is it the case that when we represent a holomorphic function as a mapping from $\mathbb{R}^2$ to itself, the Jacobian has a special form? Geometrically speaking, how can we justify the special form of the Jacobian?

First, let's look closely into the Jacobian matrix of the real representation of a holomorphic function $f(z)$. Cauchy-Riemann equations assert that it must be of the following form:

$$J = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Assuming $J \neq 0$, this matrix is orthogonal, meaning that it can be viewed as a (real) scalar times a rotation matrix:

$$J = \sqrt{a^2+b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & -\frac{b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Hence, the mapping identified with J performs a *rotation* and a *uniform scaling*.

Now, let's look at the complex derivative, $f'(z_0)$. This is a single complex number which can be represented as $r_0 e^{i\theta_0}$. Multiplying it by an arbitrary complex number $z = re^{i\theta}$ will give us

$$z_0 z = r_0 r e^{i(\theta_0+\theta)},$$

which is again, a rotation and a uniform scaling!

Taking a further step

Here are two side-notes.

Partial derivative with respect to complex variable

It is convenient to define two differential operators for complex-valued functions:

$$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y})$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})$$

Using this new notation, we can write the following theorem:

If a function f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0, \quad \frac{\partial f}{\partial z}(z_0) = f'(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Also, if write $F(x,y) = f(z)$, then F is differentiable in the sense of real variables, and

$$\det J_F(x_0, y_0) = |f'(z_0)|^2.$$

Going from $\mathbb{R}^2$ to $\mathbb{C}$

In this note, we mostly considered complex-variable functions $f : \mathbb{C} \to \mathbb{C}$ and associated with each of them a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ in a natural way. Here we want to discuss the opposite direction, given a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, can we create a natural *complex representation* for it? Meaning a function $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ such that $u = F_1$ and $v = F_2$.

The answer is positive. Given $F(x, y) = (F_1(x, y), F_2(x, y))$, let $z = x + iy$. Then it is easy to see that

$$x = \frac{z + \bar{z}}{2},$$
$$y = \frac{z - \bar{z}}{2}.$$

Now we can define f as a function of z to be

$$f(z) = F_1(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2}) + iF_2(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2}).$$

As we can see, the complex representation depends on both z and $\bar{z}$.

Remember that Cauchy-Riemann condition asserts that f is holomorphic, if $\frac{\partial f}{\partial \bar{z}} = 0$, which means that f does not depend on $\bar{z}$.

Conclusion

We knew intuitively that any complex valued function f can also be seen (in a natural way) as a mapping F from $\mathbb{R}^2$ into itself. Here we showed that the complex derivative of f is related to the normal (total) derivative of F through Cauchy-Riemann equations. Therefore, if a function is complex-differentiable (holomorphic) then the Jacobian matrix of it's real representation is orthogonal.

References

Complex Analysis by Stein and Shakarchi