Solutions to Problem Set 1

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Problem 1

(a) If we consider Red as the event that the chosen ball is red, firstBag as the event in which the chosen ball is drawn from the first bag and secondBag as firstBags complement(or we can say the ball is from the second bag), we can write

$$\begin{aligned} Pr(Red) &= Pr(Red|firstBag)Pr(firstBag) + Pr(Red|secondBag)Pr(secondBag) \\ &= \frac{1}{2}(0.3 + 0.6) = 0.45. \end{aligned}$$

(b) From the formula of conditional probability we have

$$\begin{aligned} Pr(secondBag|Red) &= \frac{Pr(Red \cap secondBag)}{Pr(Red)} \\ &= \frac{Pr(Red|secondBag)Pr(secondBag)}{Pr(Red)} \\ &= \frac{0.6 * 0.5}{0.45} = 0.66. \end{aligned}$$

Problem 2

First consider the following events defined on the sample space:
Positive: The event that the result of test is positive.
Negative: The event that the result of test is negative(obviously it's *Positive*s complement).
Cancer: The event that patient has cancer.
NoCancer: The event that patient does not have cancer(it's *Cancer*s complement).
We've been given the following information about these events:

$$\begin{aligned} Pr(Positive|NoCancer) &= 0.05. \\ Pr(Positive|Cancer) &= 0.99. \\ Pr(Cancer) &= 0.07. \end{aligned}$$

And the question wants us to find the value of $Pr(Cancer|Positive)$.
Now using $Bayes$'s formula we can see that

$$
\begin{aligned}
Pr(Cancer|Positive) &= \frac{Pr(Positive|Cancer)Pr(Cancer)}{Pr(Positive|NoCancer)Pr(NoCancer) + Pr(Positive|Cancer)Pr(Cancer)} \\
&= \frac{0.99 * 0.07}{0.05 * 0.93 + 0.99 * 0.07} = 0.598.
\end{aligned}
$$

Problem 3

We define the following events on the sample space:
$tail$: The event that the result of coin toss is tails.
ith: The event that we have used the ith coin for the experiment.
We want to find $Pr(kth|tail)$.
Using conditional probability formula we know that

$$Pr(kth|tail) = \frac{Pr(kth \cap tail)}{Pr(tail)}.$$

If we consider all events of form ith where i can be any integer ranging from 1 to n we can see that these are a partition of sample space. Using this we derive that

$$Pr(kth|tail) = \frac{Pr(kth)Pr(tail|kth)}{\sum_{i=1}^{n} Pr(kth)Pr(tail|kth)}$$

Knowing that for any i, $Pr(ith) = 1/n$ and $Pr(tail|ith) = i/n$ we can calculate $Pr(kth|tail)$:

$$Pr(kth|tail) = \frac{(1/n)(k/n)}{1/n^2 \sum_{i=1}^{n} i} = \frac{2k}{n^2 + n}.$$

Problem 5

(Throughout this answer we use H to represent a heads outcome and T to represent a tails outcome) Use the following algorithm:
Assign HT to player A and TH to player B. Flip the coin repeatedly until we see a head right after a tail(TH) and declare player B the winner or a tail right after a head(HT) and declare player A the winner.
It's easy to see that the probability of each person winning is $p(1-p)$ and are equal.

Problem 6

Define E_k as the event in which exactly k urns have balls with different colors. It is obvious to see that if k is odd then $Pr(E_k) = 0$. If k is even then there exist an integer m such that $k = 2m$. In this case we have:

$$|S| = \frac{20!}{2^{10}}$$

$$|E_{2m}| = \binom{10}{2m}(\prod_{i=11-2m}^{10} i^2)(\binom{10-2m}{5-m}(\frac{10-2m!}{2^{5-m}})^2.$$

$$Pr(E_{k=2m}) = \frac{|E_{2m}|}{|S|} = \frac{\binom{10}{2m}(\prod_{i=11-2m}^{10} i^2)(\binom{10-2m}{5-m}(\frac{10-2m!}{2^{5-m}})^2}{\frac{20!}{2^{10}}}$$

(To calculate E_{2m} we first choose k urns to have different colored balls. Then we place k red balls and k green balls in these k urns. Now we fill the remaining $10-k$ urns in a way that each urn has two balls of the same color. To do that we first choose half of the urns to only have red balls. Now we fill the $5-m$ urns with $10-2m$ red balls and then do so for the green balls and their $5-m$ urns.)

Problem 7

We use mathematical induction on variable n to prove this equality.
For the basis of induction, considering $n = 2$ the inequality will become $Pr(E_1E_2) \leqslant Pr(E_1) + Pr(E_2) - 1$. Knowing that $Pr(E_1E_2) = Pr(E_1) + Pr(E_2) - Pr(E_1 \cup E_2)$ and that $Pr(E_1 \cup E_2) \leqslant 1$ this can be derived.
Now assume that the inequality holds for any $m < n$. We prove that it also holds for $m = n$. So we have to prove

$$Pr(E_1E_2 \cdots E_n) \geqslant Pr(E_1) + Pr(E_2) + \cdots + Pr(E_n) - (n-1).$$

From the induction hypothesis we know that

$$Pr(E_1E_2 \cdots E_{n-1}) \geqslant Pr(E_1) + Pr(E_2) + \cdots + Pr(E_{n-1}) - (n-2).$$

We also can understand that

$$Pr(EE_n) \geqslant Pr(E) + Pr(E_n) - 1.$$

Let $E = E_1E_2 \cdots E_{n-1}$ and combine the above inequalities. This will yield the inequality that we wanted

$$Pr(E_1E_2 \cdots E_n) \geqslant Pr(E_1) + Pr(E_2) + \cdots + Pr(E_n) - (n-1).$$

Problem 8

Without loss of generality assume that A is the origin. This means that point C is a and B is $a+b$ (because of the distances given). Define p_i as the probability of stopping at B, assuming we've started from point i of the axis. Thus the problem is to find the value of p_a. Trivially, $p_0 = 0$ and $p_{a+b} = 1$. To calculate $p_i (i \leqslant a+b)$, let F be the event in which the first move from point i is to the positive side(the side where B is). Also let E be the event that we stop at B. We have

$$p_i = Pr(E) = Pr(F)Pr(E|F) + Pr(F^c)Pr(E|F^c)$$

Replacing $Pr(F) = Pr(F^c) = 0.5$ will lead to the following equality

$$p_i = 1/2p_{i+1} + 1/2p_{i-1}$$

And so there exists an α such that

$$p_i - p_{i-1} = p_{i+1} - p_i = \alpha.$$

Hence for any i the difference between p_{i+1} and p_i is α. Starting from $p_0 = 0$ we can see that

$$p_1 = \alpha, p_2 = 2\alpha, \cdots p_{a+b} = (a+b)\alpha = 1.$$

From this it is seen that $\alpha = \frac{1}{a+b}$. Therefore $p_a = \frac{a}{a+b}$.

Problem 9

We use induction to prove that the desired probability for n baskets is $p_n = \frac{a}{a+b}$.
Basis: Trivially $p_1 = \frac{a}{a+b}$.
Assume that $p_{n-1} = \frac{a}{a+b}$. We will show that $p_n = \frac{a}{a+b}$.
We can see that

$$p_n = Pr(white) = Pr(white|isNative)Pr(isNative)+Pr(white|isNotNative)Pr(isNotNative.)$$

Where *white* is the event of the selected ball being white, isNative is the event of selected ball being from the initial $a+b$ balls of the last basket and isNotNative is the complement of isNative. The above equality will result in

$$p_n = Pr(white) = \frac{a}{a+b}\frac{a+b}{a+b+1} + \frac{p_{n-1}}{a+b+1} = \frac{a}{a+b}$$

Problem 10

Note that

$$E \searrow F \iff P(E|F) \leqslant P(E) \iff \frac{P(EF)}{P(F)} \leqslant P(E) \iff P(EF) \leqslant P(E)P(F).$$

(a) This statement is true. Because

$$A \searrow B \implies P(AB) \leqslant P(A)P(B) \implies B \searrow A.$$

(b) Consider the following events and their probabilities. Event A with $Pr(A) = 0.5$.
Event $B = A^c$ with $Pr(B) = 0.5$.
Event $D \subset B$ with $Pr(D) = 0.1$.
And finally event $C = D \cup A$ with $Pr(C) = 0.6$.
Now we have $A \searrow B$ because $Pr(AB) = 0 \leqslant Pr(A)Pr(B) = 0.25$. Also $B \searrow C$ because $Pr(BC) = Pr(D) = 0.1 \leqslant Pr(B)Pr(C) = 0.3$. But because $Pr(AC) = Pr(A) = 0.5 > Pr(A)Pr(C) = 0.3$ we do not have $A \searrow C$. Thus these A, B, C events are a counterexample to the given statement.

(c) Consider the following events and their probabilities.
Event A with $Pr(A) = 0.3$.
Event C with $Pr(C) = 0.3$ and $Pr(AC) = 0.05$.
Event $B = AC \cup (A \cup C)^c$ with $Pr(B) = 0.5$.
These events form a counterexample to the claim in this part of the question. We know that $A \searrow B$ and $B \searrow C$ because $Pr(AB) = Pr(BC) = 0.05 \leqslant Pr(A)Pr(B) = Pr(C)Pr(B) = 0.15$. But $A \cap C \searrow B$ does not hold because $Pr(ABC) = 0.05 > Pr(AC)Pr(B) = 0.05 * 0.5 = 0.025$.

Problem 12

We denote red balls with the set $R = R_1, R_2, \cdots, R_r$. We also use sets G and B to represent green and blue balls respectively. If we continue the game until no ball is left in the urn, sample space(S) of this problem will be $\{(x_1, x_2, \cdots, x_{r+g+b}) | x_i \in R \cup G \cup B\}$.

(a) We need to count all cases in which x_{r+g+b} (the last ball taken) is green. It's easy to see that there are $g(r+g+b-1)!$ cases with this property. The probability can now be calculated.

$$Pr(last\ is\ green) = \frac{g(r+g+b-1)!}{(r+g+b)!} = \frac{g}{r+g+b}$$

(b) To calculate the probability of red balls finishing before green or blue balls we have to count the elements of sample space that have an element of both B and G after the last element of R in the sequence. If we denote these cases with E we can see that

$$|E| = |S| - |\{No\ element\ of\ B\ after\ R_r\}| - |\{No\ element\ of\ G\ after\ R_r\}| \\ + |\{No\ element\ of\ B\ or\ G\ after\ R_r\}|.$$

And the desired probability is $\frac{|E|}{|S|}$.
Now we have

$$|S| = (r+g+b)!$$
$$|\{No\ element\ of\ B\ or\ G\ after\ R_r\}| = r(r+g+b-1)!$$

Due to the symmetry of permutations we know that number of cases in which after R_r(last red ball taken out of the urn) no element of B exists are the same as cases in which after B_b(last blue ball taken out of the urn) no element of R exists(a bijective mapping can be constructed by replacing R_rs positions with B_bs position). We also know that these two partition S. So each of them has exactly $\frac{(r+g+b)!}{2}$ elements. Same can be said about R_r and G_g. Now we can calculate $|E|$.

$$|E| = (r+g+b)! - 2(\frac{(r+g+b)!}{2}) + r(r+g+b-1)! = r(r+g+b-1)!$$

Hence

$$Pr(E) = \frac{r(r+g+b-1)!}{(r+g+b)!} = \frac{r}{r+g+b}$$

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