Course Summary: Ordinary differential equations

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I took this differential equations course in my third semester.

1 First Order Equations

Some description of what a first order differential equation is comes here.

1.1 Homogeneous first order linear differential equations

These equations are of the form

$$\frac{dy}{dt} + a(t)y = 0.$$

Here y is a function of t. To solve these equations use the method below:

$$\frac{\frac{dy}{dt}}{y} = -a(t) \implies \frac{d\ln|y|}{dt} = -a(t) \implies \ln|y| = -\int a(t)dt + c_1$$
$$\implies |y| = \exp\,(-\int a(t)dt + c_1) = c\exp\,(-\int a(t)dt)$$
$$\implies |y\exp(\int a(t)dt)| = c.$$

Since c is a constant and $|y\exp(\int a(t)dt)|$ is a continuous function of t, by the intermediate value theorem it can be deduced that

$$y(t) = c\exp\,(-\int a(t)dt. \tag{1}$$

This equation is regarded as the general solution of the homogeneous linear differential equation. Given the initial value $y(t_0) = y_0$, to find the specific solution of the equation, do the integration from t_0 to t .The solution will be

$$y(t) = y_0\exp\,(-\int_{t_0}^{t} a(s)ds).$$

1.2 Non-homogeneous first order linear differential equations

These equations are of the form

$$\frac{dy}{dt} + a(t)y = b(t).$$

In order to solve these more general equations we try to write them as $\frac{d?}{dt} = b(t)$. To do so, we multiply the equation by μ, the integrating factor. So now,

$$\mu(t)y' + \mu(t)a(t)y = \mu(t)b(t).$$

If we choose a function μ such that $\mu' = \mu(t)a(t)$, then

$$\frac{d}{dt}\mu(t)y(t) = \mu(t)b(t) \implies \mu(t)y(t) = \int \mu(t)b(t)dt + c \implies y = \frac{1}{\mu(t)}(\int b(t)\mu(t)dt + c).$$

Now to actually find such μ, we have to solve $\mu'(t) = a(t)\mu(t)$ which is a first order linear homogeneous equation. From (1), we know that $\mu(t) = c_1 \exp(\int a(t)dt)$. Let $c_1 = 1$. So Now

$$y = \frac{1}{\mu(t)}(\int \mu(t)b(t)dt + c) = \exp(-\int a(t)dt)(\int \mu(t)b(t)dt + c). \tag{2}$$

Equation (2) is the general solution of these types of differential equations. If the initial condition $y(t_0 = y_0)$ is given, integrate from t_0 to t, just as we did for the homogeneous types.

1.3 Separable equations

A separable equation here, is a first order differential equation that can be written in the form

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}.$$

(f, g are continuous functions)
Solving these equations also involves a similar technique.

$$y'f(y) = g(t) \implies \frac{d}{dt}F(y(t)) = g(t) \implies F(y(t)) = \int g(t)dt + c. \tag{3}$$

Where $F(y) = \int f(y)dy$.
Another way to see these equations is that, a differential equation is called separable if we can 'separate' all ys and xs in the equation and write it as

$$f(y)dy = g(t)dt.$$

Once we do this, we can integrate both sides with respect to their own variable. Thus, $\int f(y)dy = \int g(t)dt$ which yields the same result.

1.4 Bernoulli Equations

These equations are of the form

$$y' + p(x)y = q(x)y^n.$$

And using a change of variables, they can be turned into a linear equation.
To solve Bernoulli equations, we first divide both sides by y^n.

$$\frac{y'}{y^n} + p(x)\frac{1}{y^{n-1}} = q(x)$$

Let $u = \frac{1}{y^{n-1}}$. Therefor, $\frac{du}{dt} = u' = \frac{(1-n)y'}{y^n}$. Replacing $\frac{y'}{y^n}$ by $\frac{u'}{1-n}$ in the original equation will result in

$$\frac{u'}{1-n} + p(x)u = q(x).$$

This is now a first order linear differential equation we know how to solve for u. Using the results from (2),

$$u = \frac{1}{(1-n)\mu}(\int \mu(x)q(x)dx + c)$$

where $\mu = e^{\int p(x)dx}$. Finally the solution can be computed

$$y = \sqrt[n-1]{\frac{1}{u}}. \tag{4}$$

1.5 Homogeneous Equations

A First order differential equation is said to be homogeneous, iff it can be written in the form

$$y' = F(\frac{y}{x}).$$

A quick way to find out whether an equation is homogeneous or not is to multiply both x and y by a constant factor n. If the resulting equation is the same as the original one, then that equation is homogeneous.
To solve equations of this type, we use a change of variable $u = \frac{y}{x}$. Thus, $y' = u'x + u$. Now,

$$x\frac{du}{dx} = F(u) - u \implies \frac{dx}{x} = \frac{du}{F(u)-u} \implies \ln x = \int \frac{du}{F(u)-u}.$$

The former is a separable equation we know how to solve from (3). By solving it, we obtain u and then from $y = ux$, y can be found.

1.6 Exact Equations

We can solve all differential equations which can be put in the form

$$\frac{d}{dt}\phi(t,y) = 0$$

for some function $\phi(t,y)$. To wit, we can integrate both sides with respect to t to obtain

$$\phi(t,y) = C$$

for some constant C. Now we can solve it for y to get the desired function $y(t)$.
Consider a first order differential equation. It has the form

$$M(t,y) + N(t,y)y' = 0. \tag{5}$$

We want to find a function $\phi(t,y)$ such that $\frac{d}{dt}\phi(t,y) = M(t,y) + N(t,y)y'$. From the chain rule, we know that $\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}$. Comparing this with the previous equality, we can see that $M(t,y) = \frac{\partial\phi}{\partial t}$, $N(t,y) = \frac{\partial\phi}{\partial y}$.
For an arbitrary first order differential equation to be an exact equation (i.e a desired ϕ exists), it is necessary that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

If this equality holds, and also the domain of M, N is a simply connected subset of $\mathbb{R}^2$(i.e is without a hole), then a function ϕ with the aforementioned property exists.
In order to find this $\phi(t,y)$, integrate $M(t,y)$ with respect to t and treat y like a constant. So,

$$\phi(t,y) = \int M(t,y)dt + h(y).$$

Now, partial differentiation with respect to y would result in

$$\frac{\partial\phi}{\partial y} = N(t,y) = \frac{\partial\int Mdt}{\partial y} + h'(y) \implies h'(y) = N(t,y) - \frac{\partial\int Mdt}{\partial y}.$$

From this, $h(y)$ and consequently $\phi(t,y)$ can be obtained.
In some cases where equation (5) is not exact, it might be possible to make it exact by multiplying it in an integrating factor μ. If such a function exists, then the following equations should be exact.

$$\mu M(t,y) + \mu N(t,y)y' = 0$$

Therefore,

$$\frac{\partial\mu}{\partial y}M(t,y) + \mu\frac{\partial M}{\partial y} = \frac{\partial\mu}{\partial t}N(t,y) + \mu\frac{\partial N}{\partial t}.$$

We can check for existence of μ in two special cases. When μ is solely a function of t or of y. For example, if such $\mu(t)$ exists, then the above equality will turn to

$$\mu\frac{\partial M(t,y)}{\partial y} = \frac{d\mu}{dt}N(t,y) + \mu\frac{\partial N(t,y)}{\partial t}.$$

Hence,

$$\frac{d\mu}{dt} = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N}\mu = \mu R(t).$$

If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}}{N}$ is a function of t alone, then this equality is meaningful and we can find an integrating factor $\mu(t) = \exp(\int R(t)dt)$.