Notes on Optimization on Smooth Manifolds

Embedded submanifolds: first-order geometry

In optimization on manifolds, we aim to solve problems of the form

$$\min_{x \in \mathcal{M}} f(x),$$

where $\mathcal{M}$ is a smooth manifold and $f : \mathcal{M} \to \mathbb{R}$ is a smooth cost function. In this chapter, we will develop tools that would enable us to solve such problems. In particular, we need to formally define the notion of *smooth manifold* and *smooth function* on it. Furthermore, we will provide a notion of *differential* and *inner products* on such spaces. Defining these notions will enable us to generalize classic first-order optimization methods such as gradient descent to smooth manifolds, which are possibly non-linear.

In this chapter, we will

- have a quick review on Euclidean spaces and their properties,
- introduce the concept of (an embedded sub-) manifold,
- define smooth maps on these manifolds and extend the notion of differential to them,

Euclidean space

A **linear space** or a **vector space**, denoted by $\mathcal{E}$, is a set equipped with (and closed under) vector addition and scalar multiplication. Examples include $\mathbb{R}^n, \mathbb{R}^{n\times p}, \mathrm{Sym}(n), \mathrm{Skew}(n)$, with the last two being the spaces of real symmetric matrices of size n and the space of real skew-symmetric matrices of size n, respectively.

A **basis** for $\mathcal{E}$ is a maximally large set of (linearly) independent vectors $e_1, \cdots, e_n$. Any vector $x \in \mathcal{E}$ can be expressed as a unique linear combination of the basis vectors: $x = a_1 e_1 + \cdots + a_n e_n$ with a_is being real numbers.

Each basis induces a one-to-one mapping between $\mathcal{E}$ and $\mathbb{R}^n$: we write $\mathcal{E} \equiv \mathbb{R}^n$. Moreover, $\mathcal{E}$ inherits the usual **topology** of $\mathbb{R}^n$: we can define the **neighborhood** of $x \in \mathcal{E}$ to be an open subset of $\mathcal{E}$ that contains x.

For two linear spaces $\mathcal{E}, \mathcal{E}'$ of dimensions d, d' respectively, using the identification $\mathcal{E} \equiv \mathbb{R}^d, \mathcal{E} \equiv \mathbb{R}^{d'}$, a function $F : \mathcal{E} \to \mathcal{E}'$ is **smooth** if and only if it is smooth (infinitely differentiable) in the usual sense for a function from $\mathbb{R}^d$ to $\mathbb{R}^{d'}$.

The **differential** of F at x is a linear map $DF(x) : \mathcal{E} \to \mathcal{E}'$ defined by

$$DF(x)[v] = \lim_{t\to 0} \frac{F(x+tv) - F(x)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} F(x+tv)\bigg|_{t=0}.$$

This defines the differential using its relation to the directional derivatives. We can alternatively define the differential to be the (unique) linear map that satisfies

$$\lim_{h\to 0} \frac{|F(x+h) - F(x) - DF(x)[h]|}{|h|} = 0.$$

For a curve $c : \mathbb{R} \to \mathcal{E}$, we write $c'(t)$ to denote its velocity at t, $\frac{\mathrm{d}}{\mathrm{d}t} c(t)$.

We can also equip a linear space $\mathcal{E}$ with an **inner product**, $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$. Any linear space with an inner product is called a **Euclidean space**. Notice that the isomorphism between $\mathcal{E}$ and $\mathbb{R}^n$ induced by a particular basis also induces the standard inner product (and Euclidean norm) of $\mathbb{R}^n$ into $\mathcal{E}$.

To wrap up, we can think of a Euclidean space $\mathcal{E}$ (along with a fixed basis for it) as $\mathbb{R}^n$ where we have standard notions of topology, inner product, norm, differential, etc.

Embedded submanifolds of Euclidean spaces

In this section, we will

- define what mean by an *embedded submanifold* of a linear space,
- what is the *tangent space* and how to find it for embedded submanifolds,
- define the *embedded topology*, which allows us to extend notions such as open/closed set and neighborhoods for embedded submanifolds, and
- show how to create some *new manifolds* from existing ones.

Smooth embedded submanifold

Let $\mathcal{M}$ be a subset of a linear space $\mathcal{E}$. We say $\mathcal{M}$ is a *(smooth)* embedded submanifold of $\mathcal{E}$ if either of the following holds:

1. $\mathcal{M}$ is an open subset of $\mathcal{E}$. In this case, we call $\mathcal{M}$ an *open submanifold*. If $\mathcal{M} = \mathcal{E}$ then we also call it a *linear manifold*.
2. For a fixed integer $k \geqslant 1$ and for each $x \in \mathcal{M}$, there exists a neighborhood $\mathcal{U}$ of x in $\mathcal{E}$ and a smooth function $h : \mathcal{U} \to \mathbb{R}^k$ such that

   (i) For every $y \in \mathcal{U}$, $h(y) = 0$ if and only if $y \in \mathcal{M}$; and

   (ii) rank $Dh(x) = k$.

Such a function h is called a **local defining function for $\mathcal{M}$ at x**.

We call $\mathcal{E}$ the embedding space or the ambient space of $\mathcal{M}$.

Tangent space

Let $\mathcal{M}$ be a subset of $\mathcal{E}$. For all $x \in \mathcal{M}$ we define the **tangent space** to $\mathcal{M}$ at x as

$$\mathrm{T}_x\mathcal{M} = \{c'(0) | c : I \to \mathcal{M} \text{ is smooth around } 0 \text{ and } c(0) = x\}.$$

Basically, a vector $v \in \mathcal{E}$ is in $\mathrm{T}_x\mathcal{M}$ (is a **tangent vector**) if and only if there exists a smooth curve on $\mathcal{M}$ passing through x with velocity v.

Notice that $\mathrm{T}_x\mathcal{M} \subseteq \mathcal{E}$ is not necessarily a linear space. However, the following theorem which characterizes the tangent space of smooth manifolds shows that *the tangent space of a smooth embedded submanifold at any point is indeed a linear space.*

Characterization of $\mathrm{T}_x\mathcal{M}$: Let $\mathcal{M}$ be an embedded submanifold of $\mathcal{E}$. Consider a point x on $\mathcal{M}$. If $\mathcal{M}$ is an open submanifold, then $\mathrm{T}_x\mathcal{M} = \mathcal{E}$. Otherwise, $\mathrm{T}_x\mathcal{M} = \ker Dh(x)$ with h being any local defining function of $\mathcal{M}$ at x.

The above theorem implies that for any embedded submanifold $\mathcal{M}$, and for any point $x \in \mathcal{M}$, the set $\mathrm{T}_x\mathcal{M}$ is a linear subspace of $\mathcal{E}$ of some fixed dimension $\dim\ker Dh(x) = \dim\mathcal{E} - \operatorname{rank} Dh(x) = \dim\mathcal{E} - k$. The dimension of $\mathrm{T}_x\mathcal{M}$ is called the **dimension of $\mathcal{M}$** denoted by $\dim\mathcal{M}$.

Topology of an embedded submanifold

A subset $\mathcal{U}$ of $\mathcal{M}$ is *open* (respectively, *closed*) in $\mathcal{M}$ if $\mathcal{U}$ is the intersection of $\mathcal{M}$ with an open (respectively closed) subset of $\mathcal{E}$. This is called the **subspace topology**.

A **neighborhood** of $x \in \mathcal{M}$ is an open subset of $\mathcal{M}$ that contains x. A neighborhood of a subset of $\mathcal{M}$ is an open set of $\mathcal{M}$ that contains that subset.

Creating new manifolds

- Let $\mathcal{M}$ be an embedded submanifold of $\mathcal{E}$. Any open subset of $\mathcal{M}$ is also an embedded submanifold of $\mathcal{E}$ with same dimension and tangent spaces as $\mathcal{M}$.
- Let $\mathcal{M}$, $\mathcal{M}'$ be embedded submanifolds of $\mathcal{E}$, $\mathcal{E}'$. Then their Cartesian product, $\mathcal{M} \times \mathcal{M}'$ is an embedded submanifold of $\mathcal{E} \times \mathcal{E}'$ of dimension $\dim\mathcal{M} + \dim\mathcal{M}'$ such that

$$\mathrm{T}_{(x,x')}\mathcal{M} \times \mathcal{M}' = \mathrm{T}_x\mathcal{M} \times \mathrm{T}_{x'}\mathcal{M}'.$$

Smooth maps on embedded submanifolds

References

An introduction to optimization on smooth manifolds by Nicolas Boumal

(By: Saeed Hedayatian)