Tracing Closed Curves with Epicycles

An Application of the Discrete Fourier Transform

Semester III

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Contents

1 Contributions

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Abstract. In this paper we introduce the fundamental concepts of Fourier series and transforms (continuous and discrete), delving into their mathematical formulations, properties and applications. We then present an elegant way to use the discrete Fourier transform to approximate periodic functions and define a system of epicycles that can be animated to trace out closed curves. We also include a python program that performs the discrete Fourier transform to animate tracing of curves.

2 Introduction

In the 1800s, Joseph Fourier, discovered the idea of Fourier series while working on Heat equations. While coming across a step function, he proposed a striking question. How can we express this function as a sum of sine waves? Upon hearing this, it is obvious to think of it as impossible, considering the discontinuity of the step function and on the contrary, the continuous nature of sine waves. However this idea revolutionised mathematics and physics.
Fourier dealt with the problem of describing the evolution of the temperature $T(x,t)$ of a thin wire of length π, stretched between $x = 0$ and $x = \pi$, with a constant zero temperature at the ends: $T(0,t) = 0$ and $T(\pi,t) = 0$. He proposed that the initial temperature $T(x,0) = f(x)$ could be expanded in a series of sine functions:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

Although Fourier did not give a convincing proof of convergence of the infinite series given above, he did offer the conjecture that convergence holds for an "arbitrary" function. Subsequent work by Dirichlet, Riemann, Lebesgue, and others, throughout the next two hundred years was needed to delineate precisely which functions were expandable in such trigonometric series. Part of this work entailed giving a precise definition of function (Dirichlet), and showing that the integrals in value of b_n are properly defined (Riemann and Lebesgue). Fourier analysis is the study of how general functions can be decomposed into trigonometric or exponential functions with definite frequencies. There are two types of Fourier expansions:

- Fourier series: If a function is periodic, then it can be written as a discrete sum of trigonometric or exponential functions with specific frequencies.
- Fourier transform: A general function that is not necessarily periodic can be written as a continuous integral of trigonometric or exponential functions with a continuum of possible frequencies.

The reason why Fourier analysis is so important in physics is that many of the differential equations that govern physical systems are linear, which implies that the sum of two solutions is again a solution. Therefore, since Fourier analysis tells us that any function can be written in terms of sinusoidal functions, we can limit our attention to these functions when solving the differential equations. And then we can build up any other function from these special ones. This is a very helpful strategy, because it is invariably easier to deal with sinusoidal functions than general ones.

3 Review of Literature

1. James W. Cooley and John W. Tukey - An Algorithm for the Machine Calculation of Complex Fourier Series. (English) Math. Comput. 19, 297-301 (1965).
2. Coppel, W. A. J. B. Fourier – on the occasion of his two hundredth birthday. (English) Zbl 0172.00701 Am. Math. Mon. 76, 468-483 (1969).

Fourier's work on the conduction of heat has stimulated the most diverse developments in pure mathematics. The object of the pages which follow is to trace these developments in outline. Fourier's other contributions to mathematics such as his work on the theory of equations and linear inequalities will not be discussed.

3. Dickinson, Bradley W.; Steiglitz, Kenneth Eigenvectors and functions of the discrete Fourier transform. (English) Zbl 0563.65022 IEEE Trans. Acoust. Speech Signal Process. 30, 25-31 (1982).

A method is presented for computing an orthonormal set of eigenvectors for the discrete Fourier transform. The technique is based on a detailed analysis of the eigenstructure of a special matrix which commutes with the DFT. It is also shown how fractional powers of the DFT can be efficiency computed and possible applications to multiplexing and transform coding are suggested.

4. Heideman, Michael T.; Johnson, Don H.; Burrus, C. Sidney Gauss and the history of the fast Fourier transform. (English) Zbl 0577.01027 Arch. Hist. Exact Sci. 34, 265-277 (1985).

The fast Fourier transform (FFT) has become well known as a very efficient algorithm for calculating the discrete Fourier Transform of a sequence of N numbers. The DFT is used in many disciplines to obtain the spectrum or frequency content of a signal, and to facilitate the computation of discrete convolution and correlation. Indeed, published work on the FFT algorithm as a means of calculating the DFT, by J. W. Cooley and J. W. Tukey in 1965, was a turning point in digital signal processing and in certain areas of numerical analysis. They showed that the DFT, which was previously thought to require N^2 arithmetic operations, could be calculated by the new FFT algorithm using only $NlogN$ operations. This algorithm had a revolutionary effect on many digital processing methods, and remains the most tidely used method of computing Fourier transforms.

5. North American GeoGebra Journal Volume 11, Number 1, ISSN 2162-3856 - TRACING CLOSED CURVES WITH EPICYCLES: A FUN APPLICATION OF THE DISCRETE FOURIER TRANSFORM - Juan Carlos Ponce Campuzano, School of Mathematics and Physics, The University of Queensland, Australia

The discrete Fourier transform has many applications in our modern digital world. In particular, it allows us to approximate periodic functions by means of trigonometric polynomials which provides the required information to define a system of epicycles that can be animated to trace out closed curves. In this paper the author presents a method in GeoGebra to create artistic animations consisting of systems of epicycles tracing out closed curves. The geometric construction presented here can also be used as an introductory learning activity to study the discrete Fourier transform from a geometric point of view.

6. North American GeoGebra Journal Volume 10, Number 1, ISSN 2162-3856 - TRIGONOMETRIC INTERPOLATION USING THE DISCRETE FOURIER TRANSFORM - Juan Carlos Ponce Campuzano, School of Mathematics and Physics, The University of Queensland, Australia

The Fourier transform (and all its versions discrete/continuous/finite/infinite), covers deep and abstract mathematical concepts and can easily overwhelm with detail. In this paper the author provides some intuitive ideas of how the discrete Fourier transform and its versions with low frequencies works and how we can use it to approximate real periodic functions and parametric closed curves by means of trigonometric interpolation.

4 The Concept Of Frequency Content

4.1 Amplitude, Frequency and Phase

Consider the functions $A \sin \mu x$ and $B \cos \mu x$.
We already know about the functional behavior of $\sin x$ and $\cos x$. These vary between +1 and -1 and repeat themselves every 2π radians. The functions $A \sin \mu x$ and $B \cos \mu x$ are somewhat more generalised.

The constant A is called the **Amplitude**. The *amplitude* is simply the constant that scales the height of the sine and cosine functions and causes them to vary between +A and -A.

The $RadialFrequency$ denoted by μ is a measure of how often the functions repeat themselves. Note that $\sin x$ repeats every 2π radians whereas $A \sin \mu x$ repeats every $2\pi/\mu$ radians.

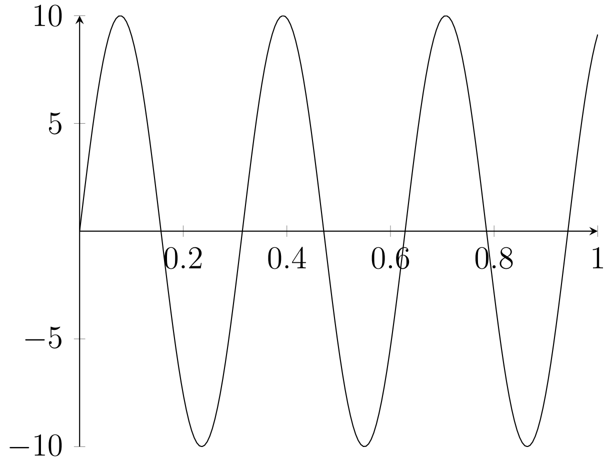
Another way of representing frequency is the number of cycles or complete revolutions of the radius vector. When measured in cycles the frequency is called **Circular Frequency** and is denoted by ω

$$2\pi\omega = \mu$$

The **Period** T of $A \sin \mu x$ or $B \cos \mu x$ is defined as the number of x units required to complete one cycle or 2π radians.

$$T = 1/\omega = 2\pi/\mu$$

Example 1 : Consider $f(x) = 10 \sin 20x$, where x is in centimeters.



Here $A = 10$, $\mu = 20$ radians/cm, $T = 2\pi/\mu = 0.3142$ cm, $\omega = 1/T = 3.183$ cycles/cm.

The term **Phase** is used to be able to shift the functions to the right or left so that $x = 0$ may take on other values. For example, $\sin(2\pi\omega x - \phi)$ is the same as $\sin 2\pi\omega x$ shifted to the right by ϕ.

4.2 Frequency Content

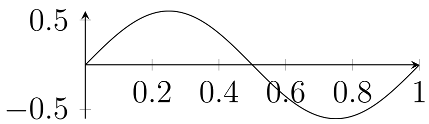
Fourier analysis is the study of representing arbitrary functions by adding together various sine and cosine functions. For a real appreciation of Fourier analysis it is necessary to understand the concept of frequency content of a function.

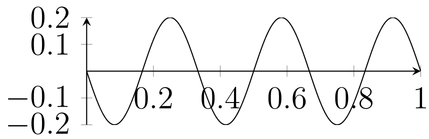
Example 2 : Consider the function

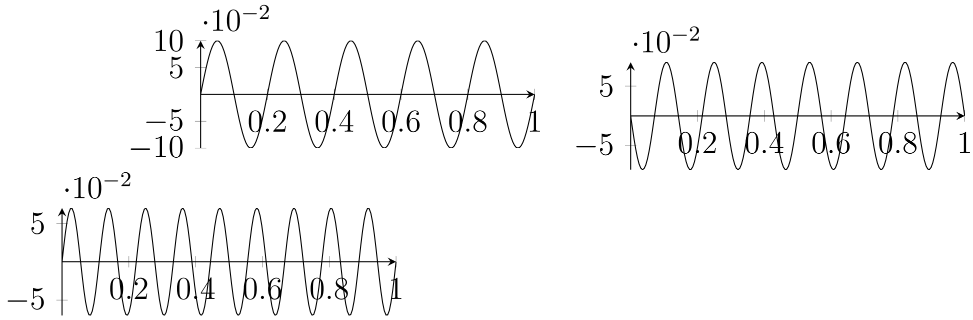
$$f(t) = \sum_{k=1}^{5} A_k \sin 2\pi\omega_k t$$

where $B_1 = 0.60, \quad \omega_1 = 1$
$A_2 = -0.20, \quad \omega_1 = 3$
$A_3 = 0.10, \quad \omega_1 = 5$
$A_4 = -0.09, \quad \omega_1 = 7$
$A_5 = 0.07, \quad \omega_1 = 9$

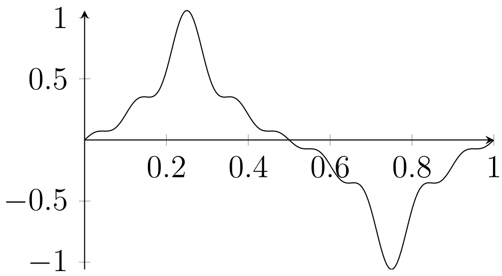
The individual sine terms are as follows:







But when added together as $f(t)$, we get :



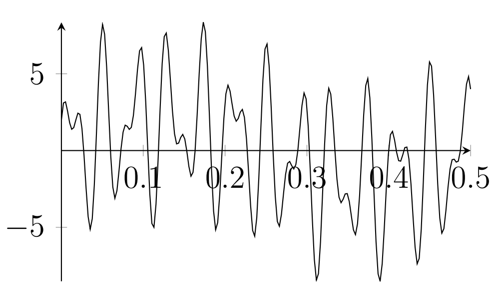
In the previous example we considered the sum of only sine functions. However the more general situation is given by linearly combining both sine and cosine functions as per the following equation :

$$f(t) = \sum_{k=1}^{n} A_k \cos 2\pi\omega_k t + B_k \sin 2\pi\omega_k t. \tag{1}$$

Note that $f(t)$ will have its unique shape which will not resemble any of the trigonometric functions used to form it. The study of what functions $f(t)$ can by used by using (1) is the cornerstone of Fourier series. As an example, consider

$$f(t) = \sum_{k=1}^{3} A_k \cos 2\pi\omega_k t + B_k \sin 2\pi\omega_k t \tag{2}$$

where $A_1 = 0$, $B_1 = 2$, $\omega_1 = 2$, $A_2 = -1$, $B_2 = 4$, $\omega_2 = 25\, A_3 = 3$, $B_3 = 0$, $\omega_3 = 40$ The graph of f(t) is given below:



The frequency content of a function, described by (1) is a measure of all the frequencies used in the summation, in which mode they are used (sine or cosine), and how much of each is used.

The **frequency content** of f is the set of 3-tuples (A_k, B_k, ω_k), k=1,n. A_k is called the pure cosine content of frequency ω_k. Similarly, B_k is called the pure sine content of frequency ω_k.

As in the above example, the set

$$\{(0,2,2), (-1,4,25), (3,0,40)\}$$

is the frequency content of equation (2)

4.3 Complex representation of frequency content

Recall that, by Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, the following can be obtained:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \tag{3}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{4}$$

Thus by using equations (3) and (4), we can rewrite (1) as :

$$f(t) = \sum_{k=1}^{n} \left[\frac{A_k}{2}(e^{2\pi i\omega_k t} + e^{-2\pi i\omega_k t}) + \frac{B_k}{i2}(e^{2\pi i\omega_k t} - e^{-2\pi i\omega_k t}) \right] \tag{5}$$

$$\implies f(t) = \sum_{k=1}^{n} (C_k e^{2\pi i\omega_k t} + C_{-k} e^{-2\pi i\omega_k t})$$

where

$$C_k = \frac{A_k - iB_k}{2},$$

$$C_{-k} = \frac{A_k + iB_k}{2}.$$

Equation (3) is the complex representation of Equation (1). The term $e^{2\pi i\omega_k t}$ is a rotating unit vector of circular frequency ω_k, whereas $e^{-2\pi i\omega_k t}$ is a rotating unit vector of circular frequency $-\omega_k$. These can be thought of as rotating in the opposite directions. The coefficients C_K and C_{-k} are the complex conjugates of each other. We can rewrite Equation (5) in a more compact form :

$$f(t) = \sum_{k=-n}^{n} C_k e^{2\pi i\omega_k t} \tag{6}$$

where

$C_k = \frac{A_k + iB_k}{2}, \quad k < 0,$

$C_0 = 0,$

$C_k = \frac{A_k - iB_k}{2}, \quad k > 0,$

$\omega_k = -\omega_k, \quad k < 0.$

5 The Fourier Series

Till now, we discussed the effect of adding together sine and cosine terms of various amplitudes and frequencies but we have not yet talked about some specific formulae to calculate the A_k, B_k, ω_k terms. In this section, given some function $f(t)$, we can find real numbers A_k, B_k, ω_k such that

$$f(t) = \sum_{k}^{\infty} A_k \cos 2\pi\omega_k t + B_k \sin 2\pi\omega_k t. \tag{7}$$

5.1 Dirichlet Conditions

In general, given any $f(t)$ we cannot guarantee that the series of equation (7) will converge to f or that the A_k, B_k, ω_k terms will even exist. The Dirichlet Conditions which when satisfied guarantee that a function f will have a Fourier series frequency content. The Dirichlet conditions are:

1. f is periodic with period T; that is $f(t+T) = f(t)$
2. f is bounded.
3. In any one period the function may have at most a finite number of discontinuities and a finite number of maxima and minima.

These are sufficient conditions but not necessary ones. That is to say, if these conditions are satisfied, then the function has Fourier series frequency content but there are still other functions that may not satisfy the conditions that also have Fourier series frequency content.

5.2 Fourier series frequency content

Assume that the Dirichlet conditions are satisfied for all considered functions.

The frequency ω_K is given by

$$\omega_k = \frac{k}{T}, \quad k = 0, 1, 2 \dots \tag{8}$$

The pure cosine frequency terms are given by

$$A_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt, \quad k = 1, 2, 3 \dots \tag{9}$$

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)$$

The pure sine frequency terms are given by

$$B_k = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi kt}{T}dt, \quad k = 1, 2, 3 \dots \tag{10}$$

$$B_0 = 0$$

When the constants A_k, B_k, ω_k are given by equations (8) $-$ (10), we say that the function f has **Fourier series frequency content**.

DERIVATION:

Note the following "orthogonality" relations

$$\int_{-T/2}^{T/2} \cos\frac{2\pi nt}{T}\cos\frac{2\pi mt}{T}dt = \begin{cases} 0, & n \neq m, \\ T/2, & n = m \neq 0 \end{cases} \tag{i}$$

$$\int_{-T/2}^{T/2} \sin\frac{2\pi nt}{T}\sin\frac{2\pi mt}{T}dt = \begin{cases} 0, & n \neq m, \\ T/2, & n = m \neq 0 \end{cases} \tag{ii}$$

$$\int_{-T/2}^{T/2} \sin\frac{2\pi nt}{T}\cos\frac{2\pi mt}{T}dt = 0 \tag{iii}$$

Consider our function :

$$f(t) = \sum_{n=1}^{\infty} A_n\cos 2\pi\omega_n t + B_n\sin 2\pi\omega_n t.$$

To obtain the cosine terms, firstly multiply $f(t)$ by $\cos\frac{2\pi kt}{T}$

$$\implies f(t)\cos\frac{2\pi kt}{T} = \sum_{n=1}^{\infty}(A_n\cos 2\pi\omega_n t + B_n\sin 2\pi\omega_n t)\cos\frac{2\pi kt}{T}$$

Now, integrate the resulting expression from $-T/2$ to $T/2$

$$\implies \int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt = \int_{-T/2}^{T/2}\Bigg(\sum_{n=1}^{\infty} A_n\cos 2\pi\omega_n t\cos\frac{2\pi kt}{T} \\ +B_n\sin 2\pi\omega_n t\cos\frac{2\pi kt}{T}dt\Bigg)$$

where, k=1,2...

Using the orthogonality relations we see that for $k = n$ we have

$$\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt = \frac{A_k T}{2}$$

For $n = k = 0$ we have a special case

$$\cos\frac{2\pi nt}{T} = \cos\frac{2\pi kt}{T} = 1,$$

$$\sin\frac{2\pi nt}{T} = 0$$

and we obtain

$$\int_{-T/2}^{T/2} f(t)dt = \int_{-T/2}^{T/2} A_0 dt = TA_0$$

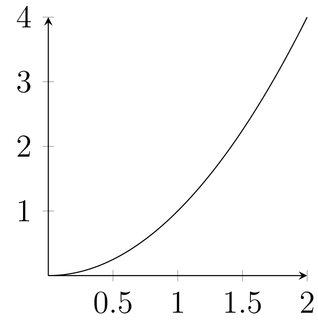
The sine terms are obtained in a similar way by multiplying both sides of $f(t)$ by $\sin\frac{2\pi kt}{T}$ and then integrating from $-T/2$ to $T/2$.

5.3 Periodic Functions

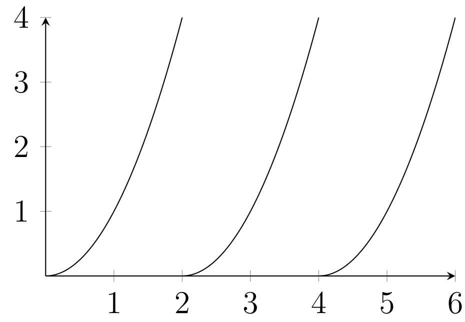
The first Dirichlet condition requires that our function be periodic. However, there are many times when we are interested in the Fourier series of a function that is not periodic. In the real world we are usually interested in functions over a finite domain. To solve this problem we simply extract that portion of f as needed and generate a new *protracted* periodic function from this extracted portion. We then apply Equations (8) – (10) to this protracted function.

Example :

Let $f(t) = t^2$ over the closed interval [0,2]



This function is obviously continuous over this interval but its protracted function as shown below is discontinuous at 0,2,4...



Since the functions (or protracted functions) are periodic, it allows us to shift the interval of integration by an arbitrary amount and not affect the value of the integral. To show this we first note that by using the fact $f(t+T) = f(t)$:

$$\int_{-T/2+\delta}^{-T/2} f(t)dt = \int_{-T/2+\delta}^{-T/2} f(t+T)dt = \int_{T/2+\delta}^{T/2} f(\tau)d\tau = -\int_{T/2}^{T/2+\delta} f(\tau)d\tau$$

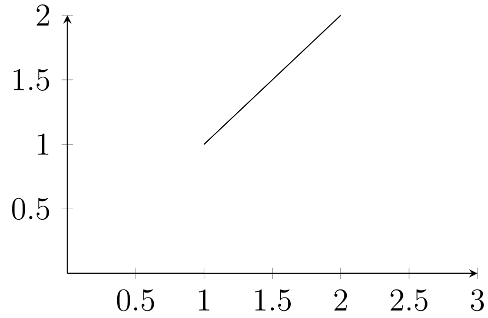
where $\tau = t + T$
Now we write

$$\int_{-T/2+\delta}^{T/2+\delta} f(t)dt = \int_{-T/2+\delta}^{-T/2} f(t)dt + \int_{T/2}^{T/2+\delta} f(t)dt + \int_{-T/2}^{T/2} f(t)dt$$
$$= \int_{-T/2}^{T/2} f(t)dt$$

The pure sine and cosine term equations implied that the periodic function $f(t)$ was centered about $t = 0$ and that we performed the required integration between $-T/2$ and $T/2$. But the above result relaxes this condition and $t = 0$ need not be the centre of the period of $f(t)$. Furthermore the limits need not exactly be $-T/2$ to $T/2$.
The following example clears the usage.

5.4 Sawtooth wave

Let $f(t) = t$ on the closed interval [1,2].



Firstly determine the frequency content of the function. We have $T = 1$, $-T/2 = 1$ and $T/2 = 2$.
Now we obtain the pure cosine terms using

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi nt}{T}dt = 2\int_1^2 t\cos 2\pi ntdt.$$

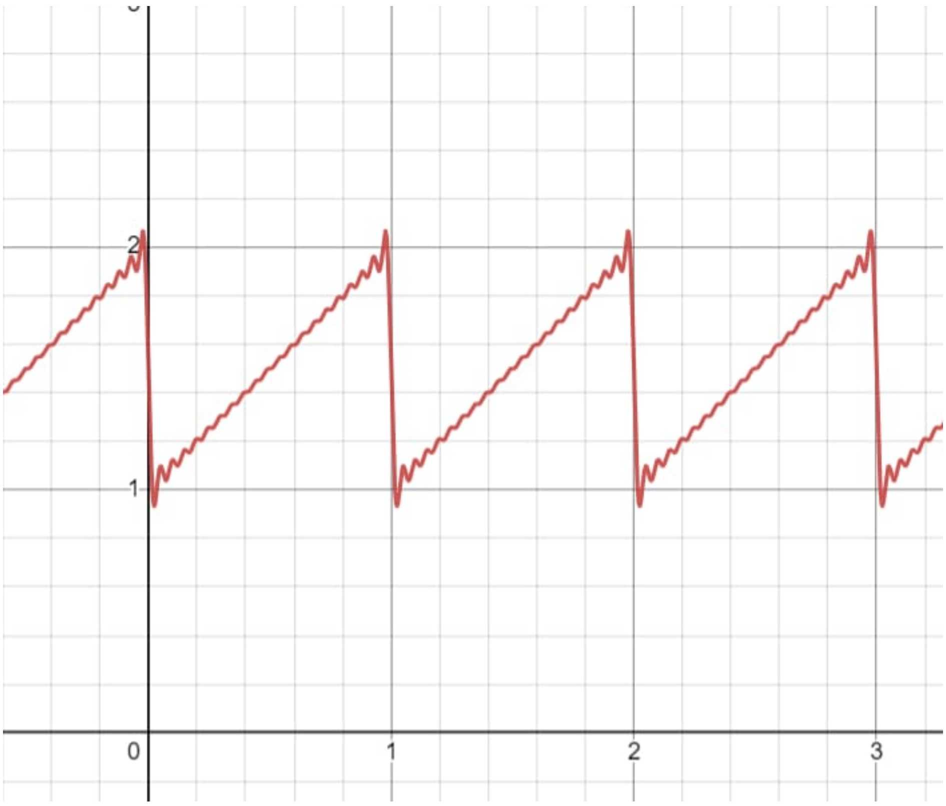
$$A_n = \left(\frac{\cos 4\pi n - \cos 2\pi n}{2(\pi n)^2}\right) + \left(\frac{2\sin 4\pi n - \sin 2\pi n}{\pi n}\right)$$

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \int_1^2 tdt = 1.5$$

Similarly for the pure sine terms

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} f(T)\sin\frac{2\pi nt}{T}dt = 2\int_1^2 t\sin(2\pi nt)dt$$

$$B_n = \left(\frac{\sin 4\pi n - \sin 2\pi n}{2(\pi n)^2}\right) + \left(\frac{2\cos 4\pi n - \cos 2\pi n}{\pi n}\right)$$



6 Complex form of Fourier series

Recall that we used Euler's equations to write the general frequency content of a function in its complex form :

$$f(t) = \sum_{k=-n}^{n} C_k e^{2\pi i \omega_k t} \tag{11}$$

where

$C_k = \frac{A_k + iB_k}{2}, \quad k < 0,$

$C_0 = 0,$

$C_k = \frac{A_k - iB_k}{2}, \quad k > 0,$

$\omega_k = -\omega_k, \quad k < 0.$

When we use the same Euler's equations to the Fourier series frequency content we obtain

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{2\pi int/T}, \tag{12}$$

where

$C_n = \frac{A_n+iB_n}{2}, \quad n < 0$

$C_0 = A_0$

$C_n = \frac{A_n-iB_n}{2}, \quad n > 0$

$\omega_n = \frac{n}{T}, \quad n = ... - 2, -1, 0, 1, 2, ...$

The above system of equations are quite similar. Other than the infinite range of the index n, the only real difference is the C_0 term. The reason is that frequency content as discussed in previous sections did not have a special formula for ω_n, whereas for Fourier series frequency content we do. Thus the C_0 term corresponds to a 0 frequency or constant term. The equations above give us the complex Fourier coefficients C_K in the terms of pure cosine and pure sine. We now derive a more direct expression for the C_k terms. To do this we first require the following complex orthogonality relation:

$$\int_{-T/2}^{T/2} e^{2\pi int/T} e^{-2\pi int/T} dt = \begin{cases} 0, & n \neq m \\ T, & n = m. \end{cases} \tag{13}$$

We first multiply both sides of Equation (12) by $e^{-2\pi int/T}$ and the integrating the result from $-T/2$ to $T/2$

$$\int_{-T/2}^{T/2} f(t) e^{-2\pi int/T} dt = \int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} C_n e^{2\pi int/T} e^{-2\pi int/T} dt, \quad m = ... - 1, 0, 1, ...$$

Now using the Equation (13) we see

$$C_n = \frac{1}{T} f(t) e^{-2\pi int/T} dt, \quad n = ... - 1, 0, 1, ... \tag{14}$$

Thus to calculate the complex Fourier series of a function we can proceed in one of the two ways. We can first calculate the rectangular form and then use equation to obtain C_n terms. The other approach is to calculate these terms directly from equation (14).

7 Properties of the Fourier series

Theorem 1 : Linearity

If both functions $f(t)$ and $g(t)$ have Fourier series frequency content given by the sets $\{A_n, B_n, w_n\}$ and $\{C_n, D_n, w_n\}$ respectively, then the function $h(t) = af(t) + bg(t)$ has a Fourier series frequency content given by the set $\{aA_n + bC_n, aB_n + hD_n, w_n\}$.

Corollary 1

If both $f(t)$ and $g(t)$ have complex Fourier series coefficients given by F_k and G_k, respectively, then the function $h(t) = af(t) + bg(t)$ has a Fourier series frequency content given by $aF_k + bG_k$

Theorem 2 : First shifting theorem

If $f(t)$ has complex frequency content given by the set $\{F_k, \omega_k\}$ and a is a constant, then the complex frequency content of the function $f(t - a)$ is give by the set

$$\{F_k e^{-2\pi ika/T}, \omega_k\}$$

Corollary 2

If $f(t)$ has complex frequency content given by the set $\{A_k, B_k, \omega_k\}$ and a is a constant, then the frequency content of the function $f(t - a)$ is give by the set

$$\{A_k \cos\frac{2\pi ka}{T} - B_k \sin\frac{2\pi ka}{T}, A_k \cos\frac{2\pi ka}{T} + B_k \sin\frac{2\pi ka}{T}$$

8 Fourier series of complex functions

Thus far we have considered the function f to be real valued. We now explicitly consider the situation when f is a complex function. That is, when f can be written as

$$f(t) = f_1(t) + if_2(t)$$

where $f_1(t)$ and $f_2(t)$ are both real valued functions. Assume that f_1 and f_2 are both periodic with a period T and possess Fourier series frequency content. In this case, f also has frequency content which can be derived as follows:

$$f_1(t) = \sum_{k=-\infty}^{\infty} A_{1,k} cos\frac{2\pi kt}{T} + B_{1,k}\sin\frac{2\pi kt}{T}$$

$$f_2(t) = \sum_{k=-\infty}^{\infty} A_{2,k} cos\frac{2\pi kt}{T} + B_{2,k}\sin\frac{2\pi kt}{T}$$

where

$$A_{1,k} = \frac{2}{T}\int_{-T/2}^{T/2} f_1(t) cos\frac{2\pi kt}{T}dt$$

$$B_{1,k} = \frac{2}{T}\int_{-T/2}^{T/2} f_1(t) sin\frac{2\pi kt}{T}dt$$

$$A_{2,k} = \frac{2}{T}\int_{-T/2}^{T/2} f_2(t) cos\frac{2\pi kt}{T}dt$$

$$B_{1,k} = \frac{2}{T}\int_{-T/2}^{T/2} f_1(t) sin\frac{2\pi kt}{T}dt$$

We now wish to find constants A_k and B_k such that

$$f(t) = \sum_{k=-\infty}^{\infty} A_k cos\frac{2\pi kt}{T} + B_k\sin\frac{2\pi kt}{T}$$

Now, the **Linearity Theorem** given above gives that

$$A_k = A_{1,k} + iA_{2,k}$$

and

$$B_k = B_{1,k} + iB_{2,k}$$

$$\implies A_k = \frac{2}{T}\int_{-T/2}^{T/2} f_1(t) cos\frac{2\pi kt}{T}dt + i\frac{2}{T}\int_{-T/2}^{T/2} f_2(t) cos\frac{2\pi kt}{T}dt$$

and

$$B_k = \frac{2}{T}\int_{-T/2}^{T/2} f_1(t) sin\frac{2\pi kt}{T}dt + i\frac{2}{T}\int_{-T/2}^{T/2} f_1(t) sin\frac{2\pi kt}{T}dt$$

9 Additional properties of the Fourier series

Theorem 3 (Second shifting theorem)

If the function $f(t)$ has complex frequency content given by the set $\{C_k, \omega_k\}$ then the complex frequency content of the function

$$f(t)e^{2\pi iat/T}$$

where a is a constant is given by the set $\{C_{k-a}, \omega_k\}$

Theorem 4

If the function $f(t)$ has a frequency content given by the set $\{A_k, B_k, \omega_k\}$, then the frequency content of the function

$$f(t)e^{2\pi iat/T}$$

where a is a constant is given by the set $\{\alpha_k, \beta_k, \omega_k\}$ where,

$$\alpha_k = \frac{1}{2}[A_{k+a} + A_{k-a} + i(B_{k+a} - B_{k-a})]$$

$$\beta_k = \frac{1}{2}[B_{k+a} + B_{k-a} + i(A_{k-a} - A_{k+a})]$$

Theorem 5

If f is a real valued function with frequency content given by the set $\{A_k, B_k, \omega_k\}$ and a is a constant then (i) $f(t)\cos(2\pi at/T)$ has frequency content given by the set

$$\{\frac{A_{k+a} + A_{k-a}}{2}, \frac{B_{k+a} - B_{k-a}}{2}, \omega_k\}$$

(i) $f(t)\sin(2\pi at/T)$ has frequency content given by the set

$$\{\frac{B_{k+a} + B_{k-a}}{2}, \frac{A_{k-a} - A_{k+a}}{2}, \omega_k\}$$

10 Fourier series of odd and even functions

A function is called even if and only if $f(-t) = f(t)$, and similarly called odd if and only if $f(-t) = -f(t)$

We start by splitting the Fourier series frequency contents given in Equation (9)

$$A_k = \frac{2}{T}\int_{-T/2}^{0} f(t)\cos\frac{2\pi kt}{T}dt + \frac{2}{T}\int_{0}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt$$

If we now substitute $-t$ for t and $-dt$ for dt in the first integral, we obtain

$$A_k = \frac{-2}{T}\int_{T/2}^{0} f(-t)\cos\frac{-2\pi kt}{T}dt + \frac{2}{T}\int_{0}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt$$

Now inverting the limits on the first integral to change the sign of the integral and noting that $f(-t) = f(t)$ and $cos(-2\pi kt/T) = cos(2\pi kt)$ we find

$$A_k = \frac{2}{T}\int_{0}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt + \frac{2}{T}\int_{0}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt$$

$$A_k = \frac{4}{T}\int_{0}^{T/2} f(t)\cos\frac{2\pi kt}{T}dt$$

Now for the B_k coefficients,

$$B_k = \frac{2}{T}\int_{T/2}^{0} f(t)\sin\frac{2\pi kt}{T}dt + \frac{2}{T}\int_{0}^{T/2} f(t)\sin\frac{2\pi kt}{T}dt$$

Again substitute $-t$ for t and $-dt$ for dt in the first integral

$$B_k = \frac{-2}{T}\int_{T/2}^{0} f(-t)\sin\frac{-2\pi kt}{T}dt + \frac{2}{T}\int_{0}^{T/2} f(t)\sin\frac{2\pi kt}{T}dt$$

Now inverting the limits on the first integral to change the sign of the integral and noting that $f(-t) = f(t)$ and $sin(-2\pi kt/T) = -sin(2\pi kt)$ we find

$$B_k = \frac{-2}{T}\int_{0}^{T/2} f(t)\sin\frac{2\pi kt}{T}dt + \frac{2}{T}\int_{0}^{T/2} f(t)\sin\frac{2\pi kt}{T}dt$$

$$\implies B_k = 0$$

The evaluation of the Fourier series coefficients is significantly simpler for an even function. All the B_k coefficients vanish and the A_k terms are now calculated by performing the integration from 0 to $T/2$ and then doubling the value of the results.
The following theorems summarize the results.
Theorem 6(Even Function)

If $f(t)$ is an even periodic function, then its Fourier series coefficients are given by

$$A_0 = \frac{2}{T}\int_0^{T/2} f(t)dt,$$
$$A_n = \frac{4}{T}\int_0^{T/2} f(t)\cos\frac{2\pi nt}{T}, \quad n = 1, 2, 3...$$
$$B_n = 0$$

for all n.

Theorem 7(Odd Function)

If $f(t)$ in an odd periodic function, then its Fourier series coefficients are given by

$$B_0 = 0$$
$$B_n = \frac{4}{T}\int_0^{T/2} f(t)\sin\frac{2\pi nt}{T}, \quad n = 1, 2, 3...$$
$$A_n = 0$$

for all n.

10.1 Illustration

Let us illustrate this with an example in which we determine the A_k and B_k terms of the function given by

$$f(t) = \begin{cases} \sin 2\pi t, & 0 \le t \le \frac{1}{2}, \\ 0, & otherwise. \end{cases}$$

The interval of interest is $[-\frac{1}{2}, \frac{1}{2}]$. Therefore to take advantage of the previous theorems, we must perform some mathematical manipulations. We begin by considering the function

$$g(t) = \begin{cases} \cos 2\pi t, & \frac{-1}{4} \le t \le \frac{1}{4}, \\ 0, & otherwise. \end{cases}$$

The interval of interest is $[-\frac{1}{2}, \frac{1}{2}]$. Obviously this is the same function that we previously considered shifted to the left by an amount $\frac{1}{4}$. This is an even function, we only need to consider the pure cosine α_k terms as per Theorem 6. Thus

$$\alpha_k = \frac{4}{T}\int_0^{T/2} g(t)\cos\frac{2\pi nt}{T} = 4\int_0^{1/4} \cos(2\pi t)\cos(2\pi kt)dt.$$

Now going through the required moves we obtain

$$\alpha_0 = \frac{1}{\pi},$$

$$\alpha_k = \frac{\sin[(1+k)\pi/2]}{(1-k)\pi} + \frac{\sin[(1-k)\pi/2]}{(1+k)\pi}$$

We now use the First Shifting Theorem to obtain the coefficients of $f(t) = g(t-a)$. First we note that $\alpha_0 = 1/\pi$ is not a function of k and thus $A_0 = \alpha_0 = 1/\pi$. Now for the values of $k \neq 1$ we have

$$A_k = \left\{ \frac{\sin[(1+k)\pi/2]}{(1-k)\pi} + \frac{\sin[(1-k)\pi/2]}{(1+k)\pi} \right\} \cos\frac{\pi k}{2}$$

This can be simplified by using a variation of the formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$:

$$A_k = \frac{\sin[(1+k)\pi/2]\cos(\pi k/2)}{(1-k)\pi} + \frac{\sin[(1-k)\pi/2]\cos(\pi k/2)}{(1+k)\pi}$$

$$A_k = \frac{1}{2}\left\{ \frac{\sin[(1+2k)\pi/2] + \sin(\pi/2)}{(1-k)\pi} \right\} + \frac{1}{2}\left\{ \frac{\sin[(1-2k)\pi/2] + \sin(\pi/2)}{(1+k)\pi} \right\}$$

If we note that $\sin(\pi/2) = 1$,$\sin[(1+2k)\pi/2] = -\cos(1+k)\pi$, and $\sin[(1-2k)\pi/2] = -\cos(1-k)\pi$, then we obtain

$$A_k = \frac{1-\cos(1+k)\pi}{2\pi(1+k)} + \frac{1-\cos(1-k)\pi}{2\pi(1-k)}$$

As for the pure sine B_k terms, Corollary 2 implies

$$B_k = \left\{ \frac{\sin[(1+k)\pi/2]}{(1+k)\pi} + \frac{sin[(1-k)\pi/2]}{(1-k)\pi} \right\} \sin\frac{\pi k}{2}$$

As can be seen when k is even, the term $\sin(k\pi/2)$ vanishes and similarly when k is odd, the $\sin[(1+k)\pi/2]$ terms vanishes as well as the $\sin[(1-k)\pi/2]$ term. The only situation we need explore is

$$\left\{ \frac{sin[(1-k)\pi/2]}{(1-k)\pi} \right\} \sin\frac{\pi k}{2}$$

for $k = 1$. When k=1 we have the indeterminate form $\frac{0}{0}$ and thus we must apply L'Hopital's rule for the limiting case of k=1. Doing this we find that $B_1 = 1/2$.

11 Fourier Transform

The Fourier Series showed us how to rewrite any periodic function into a sum of sinusoids. The Fourier Transform is the extension of this idea to non-periodic functions: a general function that is not necessarily periodic (but that is still reasonably well-behaved) can be written as a continuous integral of trigonometric or exponential functions with a continuum of possible frequencies. Methods based on the Fourier Transform are used in virtually all areas of engineering and physical science.

11.1 Definition

Given a function, we define the *Fourier transform pair* as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\,dt$$

(15)

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{2\pi i\omega t}\,dt$$

(16)

Equation (15) is called the (direct) Fourier transform and Equation (16) is known as the inverse Fourier transform. We say $F(\omega)$ is the Fourier transform of $f(t)$ and that $f(t)$ is the inverse Fourier transform of $F(\omega)$. Notationally, we write

$$F(\omega) = F[f(t)],$$
$$f(t) = F^{-1}[F(\omega)].$$

We assume that each equation implies the other. That is to say, if we obtain $F(\omega)$ from Equation (15), the Equation (16) will uniquely return $f(t)$. Similarly, if we obtain $f(t)$ from Equation (16), then Equation (15) will return $F(\omega)$ uniquely.

11.2 Existence of the Fourier Transform

The Fourier Transform exists if either:

1. On any finite interval
(a) $f(t)$ is bounded,
(b) $f(t)$ has a finite number of minima and maxima,
(c) $f(t)$ has a finite number of discontinuities;

2. or $f(t)$ is absolutely integrable, that is

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

11.3 Derivation from the Fourier Series

We show that the Fourier transform follows directly from the Fourier series as the period T approaches infinity. Our starting point is the complex form of the Fourier series:

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{2\pi ikt/T}, \tag{17}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi ikt/T}\, dt.$$

(18)

We recall that the constants C_k are a measure of the "amount" of the discrete frequencies ($\omega = k/T$) that are combined to represent the periodic function $f(t)$. We also note that although we have an infinite number of these discrete frequencies, they are all a multiple of a basic frequency $\omega_0 = 1/T$. As the period T increases, this basic frequency decreases and, therefore, the discrete frequencies become closer together until in the limit ($T \to \infty$) they equal a continuous spectrum. That is,

$$\lim_{T\to\infty} \omega_k \; = \omega, \quad k = ... - 1, 0, 1, ...$$

We also note that

$$\Delta\omega_k = \omega_{k+1} - \omega_k = \frac{k+1}{T} - \frac{k}{T} = \frac{1}{T}.$$

Now let us multiply both sides of Equation (18) by T to obtain

$$TC_k = \int_{-T/2}^{T/2} f(t)e^{-2\pi ikt/T}\,dt$$

and thus in the limit as $(T \to \infty)$ we have

$$F(\omega) = \lim_{T\to\infty}(TC_k) \;= \int_{-\infty}^{\infty} f(t)e^{-2\pi i\omega t}\,dt$$

.

This establishes Equation (15). In Equation (17) if we multiply and divide the right-hand side by T, we obtain

$$f(t) = \sum_{k=-\infty}^{\infty} TC_k e^{2\pi ikt/T}\frac{1}{T} = \sum_{k=-\infty}^{\infty} TC_k e^{2\pi ikt/T}\Delta\omega_k.$$

In the limit as $(T \to \infty)$ we have

$$\begin{aligned} TC_k &= F(\omega), \\ \omega_k &= \frac{k}{T} = \omega, \\ \Delta\omega_k &= d\omega. \end{aligned}$$

The summation may be considered to approach an integral and thus we obtain Equation (16).

12 Fourier Transform Frequency Content

We now give the Fourier transform a frequency content interpretation. To do this, we first rewrite Equation (16) in the form

$$f(t) = \int_0^\infty [C(\omega) cos 2\pi\omega t + S(\omega) sin 2\pi\omega t]\, d\omega$$

(19)

in which case we define the *Fourier transform frequency content* as the infinite set of 3-tuples

$$\{C(\omega), S(\omega), \omega\}.$$

$C(\omega)$ is called the pure cosine frequency content of frequency ω. We obtained Equation (19) by rewriting Equation (16) as

$$f(t) = \int_{-\infty}^\infty F(\omega)(cos 2\pi\omega t + sin 2\pi\omega t)\, d\omega$$

$$f(t) = \int_0^\infty \{[F(\omega) + F(-\omega)] cos 2\pi\omega t + i[F(-\omega) - F(-\omega)] sin 2\pi\omega t)\, d\omega$$

.

Thus we see

$$C(\omega) = F(\omega) + F(-\omega), \tag{20}$$

$$S(\omega) = i[F(\omega) + F(-\omega)]. \tag{21}$$

Proceeding, we can also show

$$C(\omega) = F(\omega) + F(-\omega),$$

$$C(\omega) = \int_{-\infty}^\infty f(t) e^{-2\pi i\omega t}\, dt$$

+

$$\int_{-\infty}^{\infty} f(t)e^{2\pi i\omega t}\,dt$$

$$C(\omega) = 2\int_{-\infty}^{\infty} f(t)(\frac{e^{-2\pi i\omega t} + e^{2\pi i\omega t}}{2})\,dt$$

$$C(\omega) = 2\int_{-\infty}^{\infty} f(t)cos(2\pi\omega t)\,dt$$

(22)

Similarly, we can show that

$$S(\omega) = 2\int_{-\infty}^{\infty} f(t)sin(2\pi\omega t)\,dt$$

(23)

In Equations (22) and (23) when the lower limit of integration is zero, then $C(\omega)$ and $S(\omega)$ are called the Fourier cosine transform and Fourier sine transform, respectively.

If we compare the Fourier transform frequency content $\{C(\omega), S(\omega), \omega\}$ to the Fourier series frequency content ($\{A_k, B_k, \omega_k\}$), we note the similarities. The Fourier series frequency content is an infinite set consisting of a discrete frequency frequency spectrum, whereas the Fourier transform frequency content is an infinite set made up of a continuous frequency spectrum.

13 Properties of the Fourier Transform

13.1 Linearity

The Fourier transform is a linear transform. That is, suppose we have two functions $f(t)$ and $g(t)$ with Fourier transforms $F(\omega)$ and $G(\omega)$, respectively. Then the Fourier transform of any linear combination of f and g can be easily found:

$$F[\mathrm{c}_1 f(t) + c_2 g(t)]{=}\mathrm{c}_1 F(\omega) + c_2 G(\omega) \tag{24}$$

In this equation, c_1 and c_2 are any complex numbers.

13.2 Scaling Property

Let the function $g(t)$ have Fourier transform $G(\omega)$. If $g(t)$ is scaled in time by a non-zero constant λ, we write it $g(\lambda t)$. The resultant Fourier transform is given by:

$$F[\mathrm{g}(\lambda t)] = \frac{1}{|\lambda|}G(\frac{\omega}{\lambda}) \tag{25}$$

13.3 Duality

Suppose the function $g(t)$ has a Fourier transform given by $G(\omega)$. Then the Fourier transform of $G(t)$ is given by:

$$F[G(t)]=g(-\omega) \tag{26}$$

13.4 First Shift Property

If the function $g(t)$ has a Fourier transform given by $G(\omega)$, then the Fourier transform of the function $g(t-a)$ is given by:

$$F[g(t-a)] = G(\omega)e^{2\pi i\omega a} \tag{27}$$

This theorem tells us that a phase shift in the time domain results in a sinusoidal type modulation in the frequency domain.

13.5 Second Shift Theorem

If the function $g(t)$ has a Fourier transform given by $G(\omega)$, then the Fourier transform of the function $g(t)e^{2\pi iat}$ is given by:

$$F[g(t)e^{2\pi iat}] = G(\omega - a) \tag{28}$$

This theorem tells us that a sinusoidal type modulation in the time domain gives rise to a phase shift in the frequency domain.

13.6 Derivative of the Transform

If both Fourier transforms of $g(t)$ and $t^n g(t)$ exist, then they are related as follows:

$$(\text{-}2\pi i)^n F[\mathrm{t}^n g(t)] = \frac{d^n G(\omega)}{d\omega^n} \tag{29}$$

13.7 Transform of the Derivative

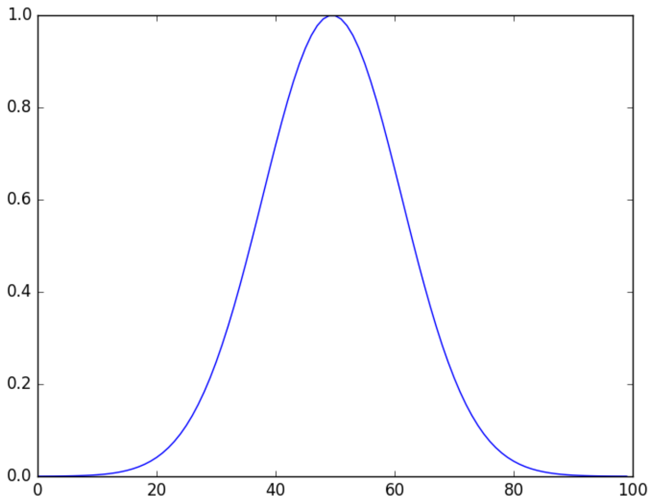
The Fourier transform of the nth derivative $g^{(n)}(t)$ of a function $g(t)$ is related to the transform of $g(t)$ by the following equation:

$$F[g^{(n)}(t)] = (2\pi i\omega)^n G(\omega) \tag{30}$$

14 The Gaussian Function

We consider the Fourier transform of the Gaussian function as an illustrative example. The Gaussian function is described mathematically as

$$f(x) = e^{-ax^2}. \tag{31}$$



This function is quite common in statistics and probability theory and is also known as the normal distribution. The Gaussian integral, which is the integral of the Gaussian function over the entire real line is equal to $\sqrt{\frac{\pi}{a}}$. An interesting property of this Gaussian function is that its Fourier transform is also a Gaussian function. This property is known as self-reciprocity.

By definition we have

$$F(\omega) = \int_{-\infty}^{\infty} e^{-ax^2} e^{-2\pi i\omega x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-a(x^2+2\pi i\omega x/a)}\, dx$$

If we now write $(x^2 + 2\pi i\omega x/a)$ as $(x + i\pi\omega/a)^2 + (\pi\omega/a)^2$, then we have

$$F(\omega) = e^{-\pi^2\omega^2/a} \int_{-\infty}^{\infty} e^{-a(x+i\pi\omega/a)^2}\, dx$$

.

Next we make the change of variable $s = x + i\pi\omega/a$ which implies $ds = dx$ (since $d\omega = 0$, and we obtain

$$F(\omega) = e^{-\pi^2\omega^2/a} \int_{-\infty}^{\infty} e^{-as^2}\, ds$$

$$= \sqrt{\frac{\pi}{a}} e^{-\pi^2-\omega^2/a}.$$

Thus

$$F[e^{-\pi x^2}] = \sqrt{\frac{\pi}{a}} e^{-\pi^2-\omega^2/a}.$$

When $a = \pi$, then we have complete self-reciprocity:

$$F[e^{-\pi x^2}] = e^{-\pi\omega^2}$$

.

15 Convolution

The convolution of two functions $f(x)$ and $g(x)$, denoted as $f(x) * g(x)$, is defined as follows:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\xi)g(x-\xi)\, d\xi$$

(32)

The concept of convolution is inherent in almost every field of the physical sciences and engineering. For example, in mechanics it is known as the superposition or the Duhamel integral. Mathematically speaking, convolution is a law of composition that combines two functions to yield a third. The convolution of functions is associative:

$$f(x) * [g(x) * h(x)] = [f(x) * g(x)] * h(x).$$

Convolution is also commutative; that is,

$$f(x) * g(x) = g(x) * f(x)$$

Finally, convolution is distributive with respect to addition:

$$f(x) * [g(x) + h(x)] = f(x) * g(x) + f(x) * h(x),$$

and

$$[g(x) + h(x)] * f(x) = g(x) * f(x) + h(x) * f(x).$$

We now consider the Fourier transform of the convolution of two functions which turns out to be a surprisingly elegant result.

15.1 Convolution Theorem

If both $f(t)$ and $g(t)$ have Fourier transforms given by $F(\omega)$ and $G(\omega)$, respectively, then:

$$F[f(t) * g(t)] = \mathrm{F}(\omega)G(\omega) \tag{33}$$

This theorem tells us that the Fourier transform of the convolution of two functions is simple the product of the two individual transforms. The analogue of this theorem is the product theorem which tells us that the Fourier transform of the product of two functions is the convolution of the two individual transforms. This is stated more formally in the following theorem.

15.2 Product Theorem

If both $f(t)$ and $g(t)$ have Fourier transforms given by $F(\omega)$ and $G(\omega)$, respectively, then the Fourier transform of their product is given by:

$$F[f(t)g(t)] = \mathrm{F}(\omega) * G(\omega) \tag{34}$$

We now point out that in the previous results the functions f and g (as well as F and G) may be complex, in which case the profuct $f(\xi)g(x-\xi)$ must be performed according to the product law for the complex number system.

16 The Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for functions that are discrete and periodic in nature. To determine either the Fourier series or Fourier transform of a function we must evaluate an integral. This means that we can only consider functions that can be described analytically. In the real world we rarely find such nice functions and, therefore, must turn to a digital computer for help. A digital computer, however, does not digest functions but instead, sequences of numbers that represent functions. In other words, we digitise an arbitrary function to obtain a sequence of numbers that can be handled by a computer.

In this section, we discuss the discrete Fourier transform (DFT) which is an operation that maps a sequence $\{f(k)\}$ to another sequence $\{F(j)\}$.

16.1 Nth Order Sequences

A finite sequence of N terms, or Nth order sequence, is defined as a function whose domain is the set of integers $\{0, 1, 2, ..., N-1\}$ and whose rage is the set of terms $\{f(0), f(1), ..., f(N-1)\}$. Formally, we have an Nth order sequence in the set of ordered pairs

$$\{(0, f(0)), (1, f(1)), (2, f(2)), ..., (N-1, f(N-1))\}.$$

In this paper we follow the common practice of denoting the sequence as $\{f(k)\}$ and the kth term of the sequence as $f(k)$. The terms are, in general, complex numbers.

16.2 Definition

Given any bounded Nth order sequence $\{f(k)\}$, the *discrete Fourier transform pair* is defined as

$$F(j) = \frac{1}{N}\sum_{k=0}^{N-1} f(k)e^{-2\pi ikj/N}$$

(35)

$$\text{f(k)} = \sum_{j=0}^{N-1} F(j)e^{2\pi ikj/N} \tag{36}$$

If we define the *weighting kernel* W_N as

$$W_N = e^{2\pi i/N},$$

then the preceding equations become

$$\bar{F}[\{f(k)\}] = F(j) = \frac{1}{N}\sum_{k=0}^{N-1} f(k)W_N^{-kj}$$

(37)

$$f(k) = \sum_{j=0}^{N-1} F(j)W_N^{kj}, \quad j = 0, 1, ..., N-1 \tag{38}$$

Equation (35) is called the discrete Fourier transform and Equation (36) is known as the inverse discrete Fourier transform.

17 Properties of the DFT

17.1 Reciprocity

The discrete Fourier transform possesses complete reciprocity, that is, it is unique.

17.2 Periodicity

The discrete Fourier transform $\{F(j)\}$ and the discrete inverse Fourier transform $\{f(k)\}$ are both periodic with periodicity N; that is:

$$F(j+N) = F(j)$$

and

$$f(k+N) = f(k)$$

17.3 Negative Indices

If the sequences $\{f(k)\}$ and $\{F(j)\}$ are discrete Fourier transform pairs then:

$$\begin{aligned} F(-j) &= F(N-j), \quad j \in \{0,1,...,N-1\}, \\ f(-k) &= f(N-k), \quad k \in \{0,1,...,N-1\}. \end{aligned}$$

17.4 Linearity

Suppose we have two functions $f(k)$ and $g(k)$ with discrete Fourier transforms $F(j)$ and $G(j)$, respectively. Then the discrete Fourier transform of any linear combination of f and g is given by the following equation:

$$\bar{F}[\mathrm{c}_1 f(k) + c_2 g(k)] = \mathrm{c}_1 F(j) + c_2 G(j). \tag{39}$$

Here, c_1 and c_2 are any complex numbers.

17.5 First Shift Theorem

If the discrete Fourier transform of the Nth order sequence $\{f(k)\}$ is $\{F(j)\}$, then the discrete Fourier transform of the shifted sequence $\{f(k-n)\}, n \in \{0,1,...,N-1\}$, is given by $\{F(j)W_N^{-jn}\}$.

17.6 Second Shift Theorem

If the discrete Fourier transform of the Nth order sequence $\{f(k)\}$ is $\{F(j)\}$, then the discrete Fourier transform of the sequence $\{f(k)W_N^{nk}\}$ is given by $\{F(j-n)\}, n \in \{0,1,...,N-1\}$.

17.7 Transform of a Transform

If the Nth order sequence $\{F(j)\}$ is the discrete Fourier transform of the sequence $\{f(k)\}$, then

$$\bar{F}[\bar{F}[\{f(k)\}]] = \tfrac{1}{N}\{f(-k)\} = \tfrac{1}{N}\{f(N-k)\} \tag{40}$$

18 The Fast Fourier Transform

In 1965, J.W. Tukey and J.W Cooley published an algorithm that under certain conditions, tremendously reduces the number of computations required to compute the discrete Fourier transform of a sequence. This algorithm is known as the fast Fourier transform or the FFT. It is considered one of the most significant contributions to numerical analysis of the 20th century. Basically, the fast Fourier transform is a clever computational technique of sequentially combining progressively larger weighted sums of data samples so as to produce the discrete Fourier transform.

Let us begin by assuming that we have an Nth order sequence {f(k)} with discrete Fourier transform {F(j)}. Furthermore, we assume that N is an even integer and thus we can form the two new subsequences:

$$f_1(k) = f(2k),$$
$$f_2(k) = f(2k+1), \quad k = 0, ..., M-1,$$

where $M = N/2$.

Note that

$$f_1(k+M) = f(2(k+M)) = f(2k+N) = f(2k) = f_1(k),$$

$$f_2(k+M) = f(2(k+M)+1) = f(2k+N+1) = f(2k+1) = f_2(k),$$

and, therefore, we see that both $\{f_1(k)\}$ and $\{f_2(k)\}$ are periodic sequences with periodicity M.

Since $\{f_1(k)\}$ and $\{f_2(k)\}$ are Mth order sequences, we can determine their discrete Fourier transforms:

$$F_1(j) = \frac{1}{M}\sum_{k=0}^{M-1} f_1(k)W_M^{-kj}$$

$$F_2(j) = \frac{1}{M}\sum_{k=0}^{M-1} f_2(k)W_M^{-kj}$$

, $j = 0, ..., M-1$

We note that by Property 16.2, both $\{F_1(j)\}$ and $\{F_2(j)\}$ are periodic with periodicity M. Now let us consider the discrete Fourier transform of the Nth order sequence $\{f(k)\}$:

$$F(j) = \frac{1}{N}\sum_{k=0}^{N-1} f(k)W_N^{-kj}$$

.

Splitting the summation we can rewrite the preceding equation as

$$F(j) = \tfrac{1}{N}\textstyle\sum_{k=0}^{M-1} f(2k)W_N^{-2kj} + \displaystyle\frac{1}{N}\sum_{k=0}^{M-1} f(2k+1)W_N^{-(2k+1)j}$$

.

However, we note that

$$W_N^{-2kj} = e^{-2\pi ikj/(N/2)} = W_M^{-kj},$$

$$W_N^{-(2k+1)j} = e^{-2\pi ij(2k+1)/N} = W_M^{-kj}W_N^{-j}.$$

Therefore, the previous equation becomes

$$F(j) = \tfrac{1}{N}\textstyle\sum_{k=0}^{M-1} f_1(k)W_M^{-kj} + \displaystyle\frac{W_N^{-j}}{N}\sum_{k=0}^{M-1} f_2(k)W_M^{-kj}$$

, j = 0,...,N-1.

Thus,

$$F(j) = \frac{F_1(j)}{2} + \frac{W_N^{-j} F_2(j)}{2}, \quad j = 0, ..., N-1.$$

Because $\{F_1(j)\}$ and $\{F_2(j)\}$ are periodic (with period M) we have

F(j) = $1_{\overline{2[F_1(j)+F_2(j)W_N^{-j}], \quad j=0,...,M-1.}}$(43)

$$F(j+M) = \frac{1}{2}[F_1(j) - F_2(j)W_N^{-j}].$$

We note that, to calculate the discrete Fourier transform of $\{f(k)\}$ requires N^2 complex operations (additions and multiplications), whereas to calculate the discrete Fourier transform of $\{f_1(k)\}$ or $\{f_2(k)\}$ required only M^2 or $N^2/4$ complex operations. When we use Equation (43) to obtain $\{F(j)\}$ from $\{F_1(j)\}$ and $\{F_2(j)\}$, we require $N + 2(N^2/4)$ complex operations. In other words, we first require $2(N^2/4)$ operations to calculate the two Fourier transforms $\{F_1(j)\}$ and $\{F_2(j)\}$, and then we require the N additional operations prescribed by Equation (43). Thus we have reduced the number of operations from N^2 to $N + N^2/2$. For the smallest value of N (i.e, $N = 4$) this results in a factor of 0.75. For large N, this factor approaches a factor of $\frac{1}{2}$.

Before proceeding, we make note of the important fact that when $N = 2$ we divide the second order sequence $\{f(k)\} = \{f(0), f(1)\}$ into two first order sequences $\{f_1(k)\} = \{f(0)\}$ and $\{f_2(k)\} = \{f(1)\}$. However, since a first order sequence is its own transform [i e., $F_1(0) = f_1(0)$ and $F_2(0) = f_2(0)$], we do not require any complex multiplications or additions to obtain these transforms. Therefore, using Equation(43) for this case would require only $N = 2$ operations to obtain $F(j)$.

Now suppose that N is divisible by 4. Then the subsequences $\{f_1(k)\}$ and $\{f_2(k)\}$ can be further subdivided into four $M/2$ order sequences:

$$g_1(k) = f_1(2k)$$
$$g_2(k) = f_1(2k+1), \quad k = 0, 1, ..., M/2 - 1$$
$$h_1(k) = f_2(2k)$$

$$h_2(k) = f_2(2k+1)$$

Therefore, we can use Equation (43) to obtain the Fourier transforms $\{F_1(j)\}$ and $\{F_2(j)\}$ with only $M + M^2/2$ complex operations and then use these results to obtain {F(j)}. We observe that this requires $N + 2(M + M^2/2) = 2N + N^2/4$ operations.

Thus when we subdivide a sequence twice ($N > 4$ and N divisible by 4) we reduce the number of operations from N^2 to $2N + N^2/4$. The $2N$ term is the result of applying Equation (43) twice whereas the $N^2/4$ term is the result of transforming the four reduced sequences. For the case when $N = 4$ we note that we completely reduce the sequence to four first order sequences that are their own transforms and, therefore, we do not need the additional $N^2/4$ transform operations. The formula then becomes $2N$. Continuing in this way we can show that if N is divisible by 2^p (p is a positive integer), then the number of operations required to compute the discrete Fourier transform of the Nth order sequence $\{f(k)\}$ by repeated subdivision is

$$pN + \frac{N^2}{2^p}.$$

Again for complete reduction (i.e., $N = 2^p$) the $N^2/2^p$ term is not required and we obtain pN for the number of operations required. This results in a reduction factor of

$$\frac{pN}{N^2} = \frac{p}{N} = \frac{log_2 N}{N}.$$

The essence of the Cooley-Tukey FFT algorithm is to choose sequences with $N = 2^p$ and go to complete reduction. Since most sequences do not have such a convenient number of terms, we can always artificially add zeroes to the end of the sequence to reach such a value. This extra number of terms in the sequence is more than compensated for by the tremendous savings afforded by using he Cooley-Tukey algorithm.

19 Epicycles

19.1 Defining epicycles

Consider a single point P on the edge of a rotating disk. If we traced it out for a full cycle, that point describes a circle:

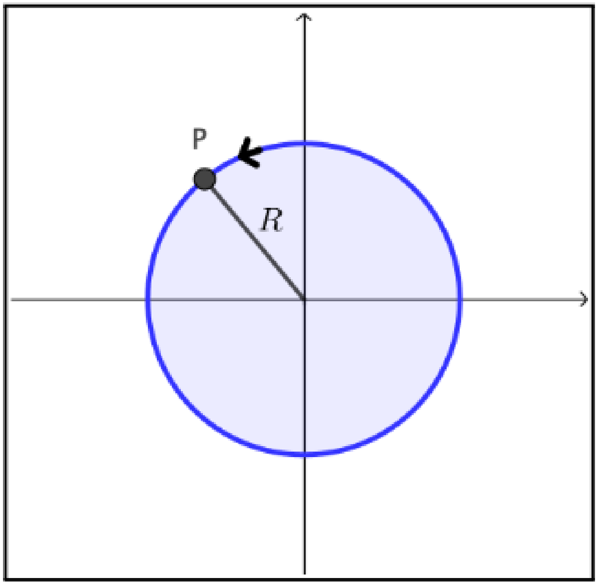


Figure 1: The point P with an anticlockwise rotation describes a circle.

We can represent the motion of this point P considering the center of the circle as the origin of the Cartesian coordinate system and using parametric equations:

$$x(t) = Rcos(\omega t)$$
$$y(t) = Rsin(\omega t)$$

where R is the radius of the disk and ω is the speed at which the disk is rotating. If ω is positive, the rotation is anticlockwise, and if it is negative, the rotation is clockwise.

19.2 Exploring epicycles further

Now let's look at something a bit more complex: Let's add another disk with radius half of the original disk and centred at P (point rotating on the original disk) but this time rotating clockwise (three times faster) with its own point Q rotating on its circumference. If we trace the locus of this second point, we can observe a shape resembling a flower, shown in Figure 2.

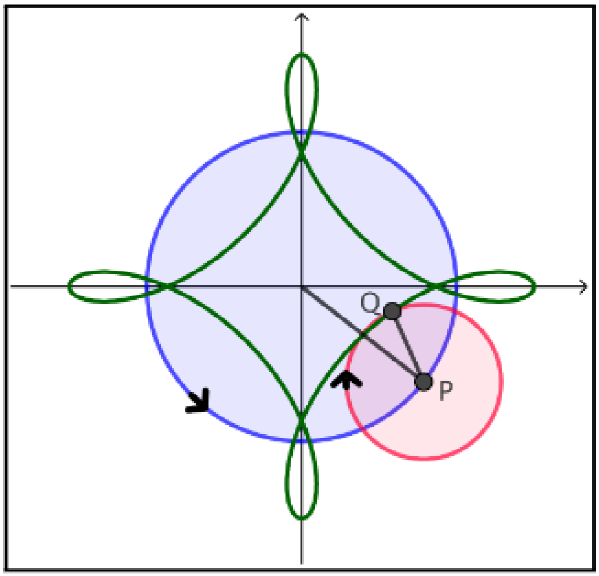


Figure 2: The point Q describes a flower curve

Luckily, it is not difficult to describe this new path: the overall position is just the sum of the contri- butions from each disk. More formally, the point Q at time t is the position at time t due to the first disk plus the position at time t due to the second disk. In other words

$$x(t) = R_1 cos(\omega_1 t) + R_2 cos(\omega_2 t)$$
$$y(t) = R_1 sin(\omega_1 t) + R_2 sin(\omega_2 t)$$

Here R_1 is the radius of the first, inner disk and R_2 is the radius of the second, outer disk. We'll be using subscript notation throughout to keep track of both the radii of the disks (R_k) and the speed that they're rotating at ω_k. Thus, in the example shown in Figure 2 we used the values $R_1 = 1$, $R_2 = \frac{1}{2}$, $\omega_1 = 1$, and $\omega_2 = 3$.

By adding extra disks and varying their speeds and sizes, you can get increasing complex curves, called epicycles. For example, the curve shown in Figure 3 is

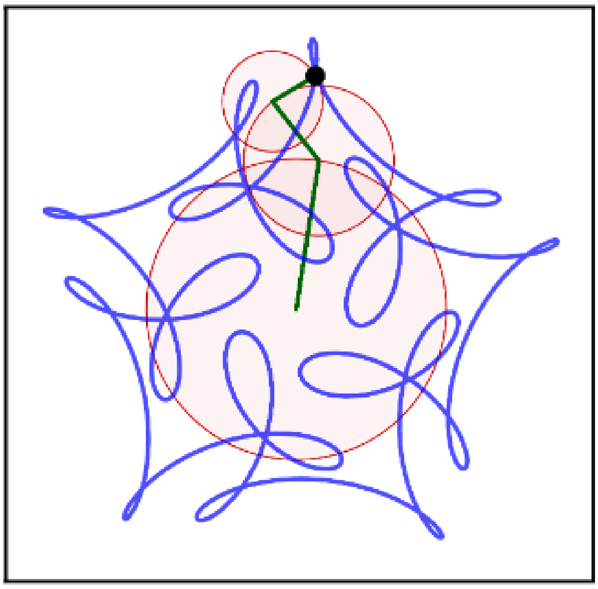


Figure 3: A more complex curve traced out by epicycles.

defined as

$$
\begin{aligned}
x(t) &= R_1 cos(\omega_1 t) + R_2 cos(\omega_2 t) + R_3 cos(\omega_3 t) \\
y(t) &= R_1 sin(\omega_1 t) + R_2 sin(\omega_2 t) + R_3 sin(\omega_3 t)
\end{aligned}
$$

where

$$
\begin{aligned}
R_1 = 1,\ R_2 = \frac{1}{2},\ R_3 = \frac{1}{3}, \\
\omega = 1,\ \omega = 6,\ \omega = -14
\end{aligned}
$$

In general, epicycles are curves defined by the equations:

$$
\begin{aligned}
x(t) &= \sum_{k=1}^{N} R_k cos(\omega_k t + \phi_k) \\
y(t) &= \sum_{k=1}^{N} R_k sin(\omega_k t + \phi_k)
\end{aligned}
$$

This new symbol ϕ_k indicates how much the disk k is initially rotated at time $t = 0$, and we called a phase offset. If we do not specify a phase offset, we can only describe epicycles where the circles start aligned.

At first glance, it seems that the geometric shapes described by epicycles are smooth closed curves. But, do all epicycles have to be curvy or closed? To answer this question, consider two disks with the same radius rotating in opposite directions. With this setup we can create a straight line:

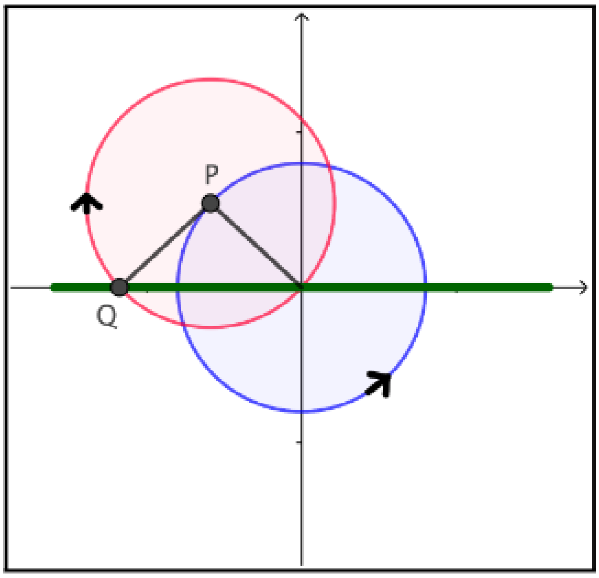


Figure 4: A more complex curve traced out by epicycles.

In this case, because the sizes of the circles are the same, the y components cancel each other out, and we get:

$$x(t) = Rcos(\omega t) + Rcos(-\omega t) = Rcos(\omega t) + Rcos(\omega t)$$
$$y(t) = Rsin(\omega t) + Rsin(-\omega t) = 0$$

Here $y(t)$ is always zero. This means that we draw a straight line along the x axis. Thus, we can create epicycles that are simple curves like a circle, a simple straight line, a flower curve, and crazy complex curves like the one shown in Figure 3). Furthermore, it seems like the more disks we have, the more complex the curves can get.

In this context, if we have enough disks, and we choose the right sizes for each of the disks, and have them spin at the appropriate speeds, can we create any closed shape/curve. Using epicycles, we can draw any closed shape, including Homer Simpson, as Ginnobili and Carman did back in 2008.

19.3 Tracing closed curves with epicycles

In the previous section we saw how to trace out continuous closed curves using rotating disks and varying their speed and sizes. The problem we want to solve now is: Given any continuous close curve/shape, is it possible to trace it out using epicycles?

To start, instead of describing the continuous curve/shape that we want using parametric equations, let's say that we just specify some points, and then find a formula for connecting those points into the shape that we want. This means that our input is a finite set of points (x_k, y_k), and we want to find epicycles that best connect those points. That is, we want to determine the values of R_k, ϕ_k, and ω_k so the expression

$$x(t) = \sum_{k=1}^{N} R_k cos(\omega_k t + \phi_k)$$
$$y(t) = \sum_{k=1}^{N} R_k sin(\omega_k t + \phi_k)$$

comes as close to the points as possible. Then we can use the points to draw out the shape we want, and consequently we will have our epicycles. But how can we determine the values of R_k, ω_k, and ϕ_k?

Surprisingly, we can use the Discrete Fourier Transform (DFT) to solve this problem in a way that is not only elegant, but also very easy for computers to perform. In this case, instead of thinking about our points defined on the real plane, we can think about those points as complex numbers. Thus we can write

$$p_k = x_k + iy_k$$

and the closed curve

$$x(t) + iy(t) = \sum_{k=1}^{N} R_k cos(\omega_k t + \phi_k) + i \sum_{k=1}^{N} R_k sin(\omega_k t + \phi_k) \tag{44}$$

will come as close to the complex points pk as possible. Notice also that equation 44 can be rewritten, by using Euler's formula

$$e^{it} = cos(t) + isin(t)$$

as follows:

$$x(t) + iy(t) = \sum_{k=1}^{N} R_k e^{i\omega_k t + \phi_k}$$

Even better, if we allow X_k to be a complex number, then the formula above becomes:

$$\sum_{k=1}^{N} X_k e^{i\omega_k t}$$

Finally, rather than choosing N arbitrary speeds ω_k, we will make the speeds to be

$$0, 1x, -1x, 2x, -2x, 3x, -3x, \cdots \frac{N}{2}x, -\frac{N}{2}x$$

With this simplification, we can state our problem as follows: Find the complex numbers X_k to minimize the mean squared error between the complex points p_k and the curve

$$p_k = \sum_{k=0}^{N} X_k e^{i\frac{2\pi ik}{N}t + \phi_k} \tag{45}$$

As it turns out, the form above almost identically matches the form of the DFT and the process described above is known as trigonometric interpolation.

19.4 Bringing in the Discrete Fourier Transform

The Discrete Fourier Transform and its inverse are defined as:

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_k e^{-ik\frac{2\pi n}{N}t}, \quad k = 0, 1, \cdots, N-1 \tag{46}$$

and the inverse

$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_k e^{ik\frac{2\pi n}{N}t}, \quad n = 0, 1, \cdots, N-1 \tag{47}$$

which can be rewritten, by using Euler's formula, as follows

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_k cos(k\frac{2\pi n}{N}t) - sin(k\frac{2\pi n}{N}t)$$
$$x_k = \frac{1}{N}\sum_{n=0}^{N-1} X_k cos(k\frac{2\pi n}{N}t) + sin(k\frac{2\pi n}{N}t)$$

It is well-known that the DFT can be used for trigonometric interpolation. That is, the process of finding a function defined as a sum of sines and cosines of given periods, which goes through a given data set. In particular, we can use the DFT to approximate periodic functions by means of trigonometric polynomials using only a finite number of sampled function values, and the same procedure works for approximating parametric curves of the form

$$u(t)$$
$$v(t)$$

where $t \in [0, 2\pi]$, which can rewritten in its complex form as $z(t) = u(t) + iv(t), t \in [0, 2]$.

First, consider a set of N points belonging to the parametric curve

$$x = u_0 + iv_0, u_1 + iv_1, u_{N-1} + iv_{N-1}$$
$$= x_0, x_1, \ldots, x_{N-1}$$

Then we can apply the DFT to obtain the Fourier coefficient X_k which encodes the amplitude and phase of a sinusoidal wave with frequency k. This is the necessary information to create a system of epicycles that will trace out the parametric curve.

The radii of each circle is the amplitude of X_k, that is, its modulus.

Finally, using the IDFT, we can calculate the trigonometric polynomial to approximate the parametric curve $z(t)$.

The image shown in Figure 5 demonstrates a simple sum of complex numbers in terms of phas- es/amplitudes can be nicely visualized as a set of concatenated circles in the complex plane. Each radius is a vector representing a term in the sum:

$$\frac{1}{N}\sum_{n=0}^{N-1} X_k cos(k\frac{2\pi n}{N}t) + sin(k\frac{2\pi n}{N}t) \tag{48}$$

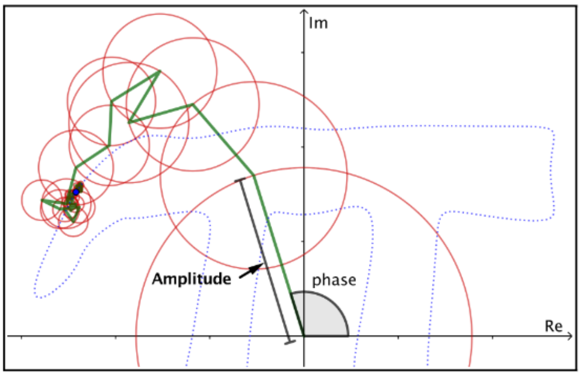


Figure 5: Geometric interpretation of the addition of complex numbers as vectors.

Adding the summands corresponds to simply concatenating each of these red vectors in complex space.

20 Implementation in Python

In the tracing of closed curves using python, we will only show the parts that are relevant to the mathematical aspects. The code that will now be shown is built on top of the code of aliemen, in particular, the animations, reading of SVGs and the functions to solve the integrals comes from the work of aliemen.

To write a computer program that can trace closed curves usinf DFT, we first need a way of representing the closed curves. In this code we use the SVG format to do that. Scalable Vector Graphics (SVG) is a web-friendly vector file format. As opposed to pixel-based raster files like JPEGs, vector files store images via mathematical formulas based on points and lines on a grid.

SVG is an XML-based vector image format for defining two-dimensional graphics, having support for interactivity and animation. The SVG specification is an open standard developed by the World Wide Web Consortium since 1999. SVG images are defined in a vector graphics format and stored in XML text files.

The use of SVG format allows us to convert the paths of a closed figure to be converted into discrete coordinate points using the `numpy` package.

Once we have the points for the closed curve, we can now readily apply the DFT to find the coefficients of the Fourier series that will give us the trignometric interpretation of said curve. The python code to do that is given below.

```
def get_fourier_coeff(func, T=[0, 1], N=4):
    indices = np.arange(-N, N+1)
    coeff = np.empty(indices.shape, dtype=complex)

    period = T[1] - T[0]
    for k, ind in enumerate(indices):
        solver = IS(lambda t:
        func(t)*np.exp(2j*np.pi*ind/period * t), T)
        coeff[k] = solver.get_approximation(200, n_gauss_param=6)

    return indices, coeff
```

We then evaluate the value of the parametric curves that we get from these coefficients at different points.

```
def fourier_eval(ind, coeff, t_eval, period=1):
    if isinstance(t_eval, float) or isinstance(t_eval, int):
        return sum([coeff_t * np.exp(2j*np.pi*ind_t/period
        * t_eval) for ind_t, coeff_t in zip(ind, coeff)])

    ret_values = np.zeros(t_eval.shape, dtype=complex)

    for i, t_t in enumerate(t_eval):
        ret_values[i] = sum([coeff_t * np.exp(2j*np.pi*
        ind_t/period * t_t) for ind_t, coeff_t in zip(ind, coeff)])
```

```
    return ret_values
```

The value from this function is then used to create the animation using the `matplotlib` package[1].

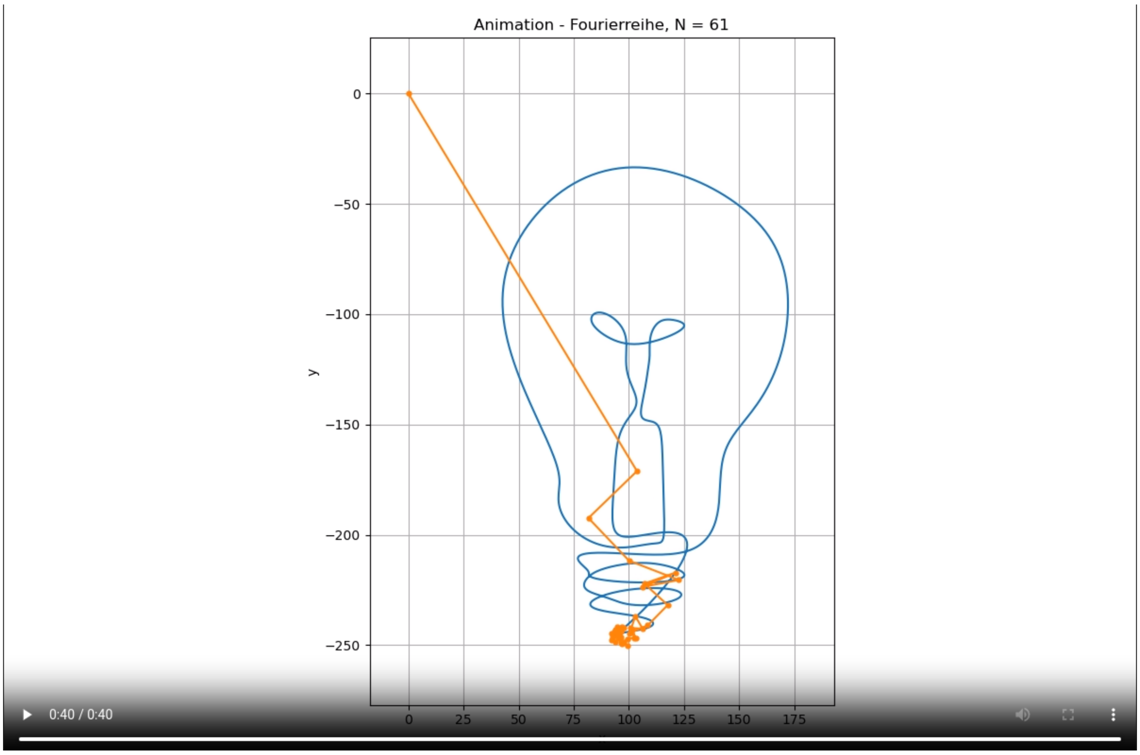


Figure 6: The result of animating the tracing of a light bulb shaped SVG

[1]To look at the full code, `https://github.com/confumbit/Visualization-of-the-Fourier-series|`

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