69 Find the number that maximises $n/\varphi(n)$.

Given an upper bound N, we want to find the positive integer $n \leqslant N$ for which the quotient ${}^{n}/_{\varphi(n)}$ achieves its maximum (if the maximum is achieved more than once, we might pick the smallest such n or determine all).

The best method to solve this problem doesn't even involve a computer – except perhaps as a calculator if N is too large to do the calculations in your head.

A not-so-good method is to calculate $\varphi(n)$ for all n from 1 to N and compare the value ${}^{n}/_{\varphi(n)}$ with the largest found so far. If you know nothing about the totient function and determine $\varphi(n)$ by calculating $\gcd(n, k)$ for all $1 \leqslant k \leqslant n$, your programme will find the answer after running for several hours.

If you know how to quickly calculate all those totients, it is not hard to see that you don't need to scan the range at all.

The key is to know how $\varphi(n)$ depends on the prime factorisation of n, namely

$$\text{if } n = \prod_{k=1}^{r} p_k^{\alpha_k}, \text{ then } \varphi(n) = \prod_{k=1}^{r}\Big((p_k - 1)p_k^{\alpha_k - 1}\Big), \tag{69.1}$$

or, put differently

$$\varphi(n) = n \cdot \prod_{p\,|\,n}\Big(1 - \frac{1}{p}\Big), \tag{69.2}$$

where the notation $a \mid b$ is pronounced 'a divides b'. From this follows immediately

$$\frac{n}{\varphi(n)} = \prod_{p\,|\,n} \frac{p}{p-1}. \tag{69.3}$$

We see that the quotient depends only on the primes dividing n, not on their exponents in the prime factorisation of n; and that among all numbers having exactly k distinct prime divisors, the quotient is largest for those numbers divisible by the k smallest primes. If q is a prime not dividing n, then ${}^{(q \cdot n)}/_{\varphi(q \cdot n)} > {}^{n}/_{\varphi(n)}$.

So if n_k is the product of the k smallest primes, ${}^{n_k}/_{\varphi(n_k)}$ is maximal among all ${}^{n}/_{\varphi(n)}$ for $n < n_{k+1}$.

We must therefore find the k with $n_k \leqslant N < n_{k+1}$. For $N = 10^6$, we find $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 510\,510$, hence $n_7 = 510\,510$ is the answer.