TA Session 2: Exercises on Price Competition

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Roadmap

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A Review of the Envelope Theorem

The Maximum Value Function

- Consider following unconstrained maximization problem with one choice variable x and one parameter θ:

$$V(\theta) = V(x^*(\theta), \theta) = \max_x \; f(x, \theta)$$

- $V(\theta)$ is a *maximum value function*, which is an objective function where the choice variables have been assigned their optimal values. These optimal values of the choices vairables are, in turn, functions of the exogenous variables and parameters of the problem. The value function indirectly becomes a function of the parameters only.
- Example: the profit function. Given K, the profit function of a competitive firm

$$\pi(K) = \pi(L^*(K), K) = \max_L \; f(L, K) - wL - rK$$

The Envelope Theorem for Unconstrained Optimization

- If the second-order condition is met, the following first-order necessary condition defines the solution:

$$f_x(x^*(\theta), \theta) = 0 \tag{1}$$

- If we differentiate V w.r.t. θ, its only argument, we get

$$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \underbrace{f_x(x^*(\theta), \theta)}_{=0 \text{ by } (1)} \cdot \frac{\partial x^*(\theta)}{\theta} + f_\theta(x^*(\theta), \theta) = f_\theta(x^*(\theta), \theta)$$

  Although θ enters the maximum-value function in two places, at the optimum, only the direct effect of θ on the objective function matters.

- At the optimum, as θ varies, with x^* allowed to adjust, the derivative $\frac{\mathrm{d}V}{\mathrm{d}\theta}$ gives that same result as if x^* is treated as a constant. This is the essence of the envelope theorem.

The Envelope Theorem for Constrained Optimization

- Again we have an objective function, one choice variable, and one parameter, except now we introduce a constraint to the problem.

Theorem (Envelope)

Let f and g be continuously differentiable functions of x and θ. For any given θ, let $x^*(\theta)$ maximize $F(x,\theta)$ subject to $g(x,\theta) \leq c$, and let $\lambda^*(\theta)$ be the associated value of the Lagrange multiplier. Suppose, further, that $x^*(\theta)$ and $\lambda^*(\theta)$ are also continuously differentiable functions, and that $g_1(x^*(\theta),\theta) \geq 0, \forall\theta$. Then the maximum value function defined by

$$V(\theta) = \max_x \; f(x,\theta) \;\; s.t. \;\; g(x,\theta) \leq c$$

satisfies

$$V'(\theta) = f_2(x^*(\theta),\theta) - \lambda^*(\theta) \cdot g_2(x^*(\theta),\theta)$$

- The optimal value function is the upper envelope of the family of value functions, in each of which the choice variable is held fixed. This is why the above theorem is referred to as the *envelope* theorem.

The Envelope Theorem for Constrained Optimization

Proof:

$$V(\theta) = \max_{x} \ f(x, \theta) \ \ s.t. \ \ g(x, \theta) \leq c$$

The Lagrangian for this problem is

$$\mathcal{L} = f(x, \theta) + \lambda \left[c - g(x, \theta)\right]$$

For any given value of θ, $x^*(\theta)$ and $\lambda^*(\theta)$ must satisfy the first-order necessary condition:

$$\mathcal{L}_1 \left[x^*(\theta), \theta\right] = f_1 \left[x^*(\theta), \theta\right] - \lambda^*(\theta) \cdot g_1 \left[x^*(\theta), \theta\right] = 0 \tag{2}$$

and the complementary slackness condition (which comes from duality):

$$\lambda^*(\theta) \cdot \{c - g\left[x^*(\theta), \theta\right]\} = 0 \tag{3}$$

It follows that

$$V(\theta) = f\left[x^*(\theta), \theta\right] \overset{(3)}{=} f\left[x^*(\theta), \theta\right] + \lambda^*(\theta) \{c - g\left[x^*(\theta), \theta\right]\} \tag{4}$$

The Envelope Theorem for Constrained Optimization

Proof *(continued)*:

Differentiating both side of (4) w.r.t. θ yields

$$V'(\theta) = \underbrace{\{f_1[x^*(\theta),\theta] - \lambda^*(\theta)\cdot g_1[x^*(\theta),\theta]\}}_{=0 \text{ by } (2)} \cdot x^{*\prime}(\theta) + f_2[x^*(\theta),\theta]$$
$$+ \underbrace{\lambda^{*\prime}(\theta)\{c - g[x^*(\theta),\theta]\}}_{\equiv\Delta} - \lambda^*(\theta)\cdot g_2[x^*(\theta),\theta]$$

It only remains to show that

$$\lambda^{*\prime}(\theta)\{c - g[x^*(\theta),\theta]\} = 0 \tag{5}$$

Clearly, for any binding constraint ($g[x^*(\theta),\theta] = c$), (5) holds. For non-binding constraint ($g[x^*(\theta),\theta] < c$), (3) implies that $\lambda^*(\theta)$ is always zero, the derivative of a constant function is zero.

$$\mathrm{d}\lambda^* = \frac{\partial\lambda^*(\theta)}{\partial\theta}\mathrm{d}\theta \equiv 0 \Leftrightarrow \frac{\partial\lambda^*(\theta)}{\partial\theta} \equiv 0 \Rightarrow (5) \text{ holds}$$

Q.E.D.

Duality and the Envelope Theorem

Duality

- The utility maximization problem (UMP, the primal problem) for a consumer has budget B and faces the market price:

$$(Primal) \;\; V(p,B) = \max_q \;\; u(q) \quad s.t. \;\; p \cdot q \leq B \tag{6}$$

Lagrangian and FOC:

$$\begin{aligned} \mathcal{L}^m &= u(q) + \lambda^m (B - pq) \\ \mathcal{L}^m_q &= u'(q^m) - \lambda^m p = 0, \quad \lambda^m \cdot \mathcal{L}^m_\lambda = \lambda^m \cdot (B - pq^m) = 0 \end{aligned} \tag{7}$$

- The expenditure minimization problem (EMP, the dual problem) with the utility level $u^* = V(p,B)$:

$$(Dual) \;\; E(p,u^*) = \min_q \;\; p \cdot q \quad s.t. \;\; u(q) \geq u^*$$

Lagrangian and FOC:

$$\begin{aligned} \mathcal{L}^h &= pq + \lambda^h (u^* - u(q)) \\ \mathcal{L}^h_q &= p - \lambda^h u'(q^h) = 0, \quad \lambda^h \mathcal{L}^h_\lambda = \lambda^h (u^* - u(q^h)) = 0 \end{aligned}$$

Duality and the Envelope Theorem

Complementary Slackness

$$V[p, e(p, u^*)] = u^*$$
$$E[p, V(p, B)] = B$$

- The solutions for the quantity q in the primal and dual problems are determined by the tangency point of the same indifference curve and budget constraint line.
- If one problem is bounded, then the associated dual is also bounded.
- From their FOCs, we have that the solutions of λ^m and λ^h are reciprocal to each other: $\lambda^m = \frac{u'(q^*)}{p} = \frac{1}{\lambda^h}$. If one problem is bounded, the associated dual problem is also bounded: $\lambda^m > 0 \Leftrightarrow \lambda^h > 0$, the denominator doesn't go to zero in an economic context.

Duality and the Envelope Theorem

Complementary Slackness

- In the two problems, If one problem is unbounded (the constraint is not binding), the other problem is not feasible.
- Case 1: consider if the EMP: $q^h = \arg\min_q pq \ \ s.t. \ \ u(q) \geq u(q^m)$ is unbounded: $u(q^h) > u(q^m)$.

$$q^m \text{ is feasible but not chosen in the EMP} \Rightarrow pq^m \geq pq^h \tag{8}$$
$$u(q^h) > u(q^m) \text{ but not a solution to the UMP } \Rightarrow pq^h > B \tag{9}$$
$$(8)\&(9) \Rightarrow pq^m > B, \text{ the UMP is not feasible}$$

- Case 2: consider if the UMP: $q^m = \arg\max_q u(q) \ \ s.t. \ \ pq \leq pq^h$ is unbounded: $pq^m < pq^h$.

$$q^h \text{ is feasible but not chosen in the UMP} \Rightarrow u(q^h) \leq u(q^m) \tag{10}$$
$$pq^m < pq^h \text{ but not a solution to the EMP } \Rightarrow u(q^m) < u^* \tag{11}$$
$$(10)\&(11) \Rightarrow u(q^h) < u^*, \text{ the EMP is not feasible}$$

- This is called the *complementary slackness*, it asserts that there cannot be slack in both a constraint and the associated dual.

Duality and the Envelope Theorem

Roy's Identity

- One application of the envelope theorem is the derivation of Roy's identity.
- Consider the primal problem (6):

$$\begin{aligned} V^m(p,B) &= u[q^m(p,B)] + \lambda^m(p,B)\cdot[B - p\cdot q^m(p,B)] \\ \Rightarrow \frac{\mathrm{d}V^m(p,B)}{\mathrm{d}p} &= [u'(q^m) - \lambda^m p]\cdot\frac{\partial q^m}{\partial p} + (B - pq^m)\cdot\frac{\partial \lambda^m}{\partial p} - \lambda^m q^m \\ &\overset{(7)}{=} -\lambda^m q^m \\ \Rightarrow \frac{\mathrm{d}V^m(p,B)}{\mathrm{d}B} &= [u'(q^m) - \lambda^m p]\cdot\frac{\partial q^m}{\partial B} + (B - pq^m)\cdot\frac{\partial \lambda^m}{\partial B} + \lambda^m \\ &\overset{(7)}{=} \lambda^m \end{aligned}$$

Taking the ratio of the two partial derivatives, we get

$$\frac{\frac{\partial V}{\partial p}}{\frac{\partial V}{\partial B}} = -q^m$$

This result is known as the Roy's identity, q^m is the Marshallian demand.

Exercise

- Prove the Shephard's Lemma using the envelope theorem (on the dual problem).[1]

[1] No submission needed. Note that in the Almost Ideal Demand System later in this course, you will need to use Roy's identity or Shephard's Lemma (one of them is suffice due to duality).

Price Competition: The Standard Bertrand Model

The Standard Bertrand Model

Duopoly

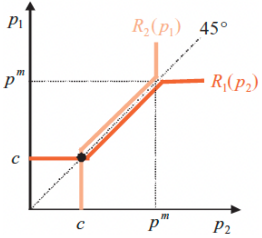
- Consider the simplest version of the Bertrand model: duopoly.
- There are two firms with homogeneous products and identical constant marginl costs c. Both firms set price simultaneously (and noncooperatively) to maximize profit.
- The firm with the lower price attracts all the market demand $Q(p)$, where p is the relevant price. Suppose, at equal prices, the market splits at α_1 and $\alpha_2 \equiv 1 - \alpha_1$. Then the firm i faces demand:

$$Q_i(p_i) = \begin{cases} Q(p_i) & \text{if } p_i < p_j \\ \alpha_i Q(p_i) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

- There is a unique pure-strategy equilibrium in which both firms set price equal to marginal costs: $p = c$. (To prove this result, it suffices to show that for all other price combinations, there is at least one firm that has an incentive to deviate.)

The Standard Bertrand Model

Reaction Functions



- Firm i's best response is to set a price p_i just below the price p_j of its rival so as to attract all consumers. This rule suffers two exceptions:
  1. For $p_j < c$, firm i chooses $p_i = c$ as undercutting firm j would entail losses.
  2. For $p_j > p^m = \arg\max_p (p - c)Q(p)$, where p^m is the monopoly price, firm i chooses $p_i = p^m$ as a price just below p_j would not maximize profits.
- The above graph confirms that $p_1 = p_2 = c$ is the unique Nash equilibrium of the game.
- The reaction functions of Bertrand duopolists are (weakly) upward-sloping: when one firm raises its price, the best response of the other firm is to raise its price too.

Exercise 5.1 of Tirole (1988)

Consider the asymmetric duopoly, where one of the two firms, say, firm 1, has constant unit cost $c_1 < c_2$. Prove the following conclusions:

1. that both firms charge price $p = c_2$ (actually, firm 1 charges an ε below c_2 to make sure it has the whole market), and
2. that firm 1 makes a profit of $(c_2 - c_1)Q(c_2)$, and firm 2 makes no profit (as long as $c_2 \leq p^m(c_1) = \arg\max_p (p - c_1)Q(p)$; otherwise, firm 1 charges $p^m(c_1)$)

- Case 1: $p^m(c_1) \leq c_1 < c_2$.
  - No equilibrium. For Firm 1, any price above the monopoly price $p^m(c_1)$ will not maximize profit, so it has no incentive to raise the price above the monopoly price $\Rightarrow p_1 \leq p^m$. But it also has no incentive to price below marginal cost $\Rightarrow p_1 > c_1$. Contradicts, no profitable prices. Firm 2 will also never charge a price lower than its marginal cost c_2.
- Case 2: $c_1 < p^m(c_1) < c_2$.
  - Lucky firm 1 charges $p^m(c_1)$ and gets the whole market.
- Case 3: $c_1 < c_2 \leq p^m(c_1)$.
  - Given a higher marginal cost, firm 2 is fully motivated to keep prices as low as possible, the lowest profitable price for firm 2 is c_2. Firm 1 charges $c_2 - \varepsilon$, gets the whole market and still makes a profit: $\pi_1 = (c_2 - \varepsilon - c_1)Q(c_2 - \varepsilon)$.

Price Competition: Capacity Constraints

Capacity-then-price Game

- The assumption of constant (or even decreasing) unit costs is appropriate as long as capaticy is not fully utilized.
- In the short run, a firm has to respect its capacity choices as increasing output beyond capacity is often prohibitively costly.
- Consider the following **two-stage game**:
  - In stage 1, firms set capacities $\bar{q}_i$ simultaneously. In stage 2, firms set prices p_i simultaneously.
  - The marginal cost of capacity is c, once the capacity is installed, the marginal cost of production in the second stage is zero.
  - Firms are aware that their capacity choices may affect equilibrium prices, and can observe the competitors' capacity choices. (Think of a local farmers' market.)
- We are going to prove this result: given market demand $Q = a - p$,
  - **the equilibrium at the second stage of the game is such that both firms set the market-clearing price:** $p_1 = p_2 = a - \bar{q}_1 - \bar{q}_2$ (this price does indeed clear the market as it equalizes demand and supply, i.e., total capacity).

The Proportional-rationing Rule

- When allowing for capacity constraints, it is well possible that one firm sets its price so low that the quantity demanded at that price exceeds its supply (so this firm can always sell all its products).
- Suppose another firm on the market that offers the same product charges a reasonable higher price (and still profitable because there is positive residual demand). Then an important question arises: *who will be served at the low price, who will not?*
- We have to make an assumption on the rationing scheme.
- We assume that there is ***proportional rationing*** (Edgeworth, 1897), i.e., **all consumers have the same probability of being rationed**. Under this rule, the highest price charged is always the monopoly price.

The Profit-maximizing Capacity Choice

- Let's continue with the two-stage game, first consider the connection between the two stages.
- Giving linear demand $Q(p) = a - p$, and $c \in [\frac{3}{4}a, a]$

$$\pi = \max_q q(a-q)$$

$$\frac{\partial \pi}{\partial q} = 0 \Leftrightarrow q^* = \frac{a}{2} \Rightarrow \pi = \frac{a^2}{4}$$

- It's obvious that a firm never sets a very large capacity since capacity is costly. The profit-maximizing capacity choice must satisfy that: firm i's costs at stage 1 have to be lower than maximal revenues.

$$c \cdot \bar{q}_i \leq \pi_i = \frac{a^2}{4} \Rightarrow \bar{q}_i \leq \frac{a^2}{4c} \in \left[\frac{a}{4}, \frac{a}{3}\right]$$

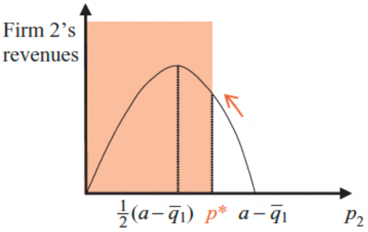
Price Setting Game for Given Capacities

- Assume proportional rationing. Suppose $p_1 < p_2$ is such that quantity $\bar{q}_1$ is insufficient to serve all consumers, i.e. $Q(p_1) > \bar{q}_1$.
- Some consumers are rationed and there is positive *residual demand* for firm 2:

$$\hat{Q}(p_2) = \begin{cases} 0 & \text{if } Q(p_2) < \bar{q}_1 \\ Q(p_2) - \bar{q}_1 & \text{if } Q(p_2) \geq \bar{q}_1 \end{cases}$$

- Given market demand $Q(p) = a - p$, firm 2's revenues are

$$p_2 \cdot \hat{Q}(p_2) = \begin{cases} 0 & \text{if } a - p_2 < \bar{q}_1 \\ p_2(a - p_2 - \bar{q}_1) & \text{if } a - p_2 \geq \bar{q}_1 \end{cases}$$



Exercise 5.2 of Tirole (1988)

Let the demand function be $Q = 1 - p$. Suppose that both firms' marginal cost (once the capacities are installed) is zero. Suppose further that $\bar{q}_1$ and $\bar{q}_2$ are lower than $\frac{1}{4}$. Show that under proportional rationing, both firms charged

$$p^* \equiv 1 - \bar{q}_1 - \bar{q}_2$$

and the profit functions:

$$\pi_i(\bar{q}_1, \bar{q}_2) = (1 - \bar{q}_1 - \bar{q}_2)\bar{q}_i$$

- To prove the above result, we proceed as follows: supposing that $p_1 = p^*$, we need to show that $p_2 = p^*$ is a best response.
  - By lowering its price than $p_1 = p^*$, firm 2 would increase the demand for its product but would not be able to serve this additional demand because it is capacity constrained. ($Q(p^*) = \bar{q}_1 + \bar{q}_2$ is already the largest total demand that the two firms can serve.)
  - Raising its price above $p_1 = p^*$ is also not profitable for firm 2, as illustrated in the figure on slide 20.

Exercise 5.2 of Tirole (1988)

Let the demand function be $Q = 1 - p$. Suppose that both firms' marginal cost (once the capacities are installed) is zero. Suppose further that $\bar{q}_1$ and $\bar{q}_2$ are lower than $\frac{1}{4}$. Show that under proportional rationing, both firms charged

$$p^* \equiv 1 - \bar{q}_1 - \bar{q}_2$$

and the profit functions:

$$\pi_i(\bar{q}_1, \bar{q}_2) = (1 - \bar{q}_1 - \bar{q}_2)\bar{q}_i$$

- We can now insert these stage-2 equilibrium prices in the profit functions and thus obtain reduced profit functions for stage 1, which only depend on capacities:

$$\tilde{\pi}_i(\bar{q}_1, \bar{q}_2) = (a - \bar{q}_1 - \bar{q}_2)\bar{q}_i - c\bar{q}_i = (1 - \bar{q}_1 - \bar{q}_2)\bar{q}_i$$

- If we reinterpret capacities as quantities, the objective function is the same as in the Cournot model, in which prices are not set by firms but where, for any quantity choice, the price clears the market.

Static Imperfect Competition: Concentration Indices

How to access market power?

- Market power can be defined as the ability to raise prices above the perfectly competitive level. So the first way to measure market power is to look at the difference between price and marginal costs.
- Lerner index: $L \equiv \frac{p-c'}{p}$
  - However, costs (and, in particular, marginal costs) are often not directly observable. And the Lerner index also ignores dynamic considerations (a firm profitably foregoes short-term gains to be able to raise margins in the future).
- Another way to capture the market power of firms is to look at *concentration indices*, which are statistics of the degree of concentration of the market.
  - *m*-firm concentration ratio: $I^m \equiv \sum_{i=1}^{m} \alpha_i$
    - However, interpreting such a measure is difficult since market conditions vary across markets.
  - Herfindahl index: $I^H \equiv \sum_{i=1}^{n} \alpha_i^2$
    - While the *m*-firm concentration ratio adds market shares of a small number of firms in the market (1 to m), the Herfindahl index considers the full distribution of market shares (all 1 to n).
    - The Herfindahl index provides a better measure of concentration as it captures both the number of firms and the dispersion of the market shares.

Exercise 5.6 of Tirole (1988)

(i) Show that under constant returns to scale and Cournot competition the ratio of total industry profit to total industry revenue is equal to the Herfindahl index divided by the elasticity of demand.

(ii) Show that under Cournot competition, the "average Lerner index" ($\sum_i \alpha_i L_i$) is equal to the Herfindahl index divided by the elasticity of demand.

Under constant returns to scale, $c(q) = c'(q) \cdot q$.
In the Cournot model, from the last TA session we know the profit is $\pi_i = \max_{q_i} \; [p - c_i'(q_i)] \cdot q_i$ and we have obtained the result that $\frac{p-c_i'}{p} = \frac{q_i}{q} \cdot \left(-\frac{q}{p} \cdot \frac{\partial p}{\partial q}\right) \equiv \alpha_i \cdot \frac{1}{\varepsilon}$.

(i) It then follows that

$$\frac{\sum_i \pi_i}{p \cdot q} = \frac{\sum_i (p - c_i')q_i}{p \cdot q} = \sum_i \left(\frac{p - c_i'}{p} \cdot \frac{q_i}{q}\right) = \frac{1}{\varepsilon} \sum_i \alpha_i^2 = \frac{1}{\varepsilon} \cdot I^H$$

(ii) and

$$\sum_i \alpha_i L_i = \sum_i \left(\alpha_i \cdot \alpha_i \cdot \frac{1}{\varepsilon}\right) = \frac{1}{\varepsilon} \sum_i \alpha_i^2 = \frac{1}{\varepsilon} \cdot I^H$$

Exercise 5.7 of Tirole (1988)

Suppose that demand is linear ($Q = 1 - p$) and that there are two firms with constant marginal costs c_1 and c_2 such that $c_1 + c_2 = 2c$ (where c is a constant). Show that when the firms become more asymmetric (c_i moves away from c), Cournot competition yields a higher concentration index and a higher industry profit.

- The profit of firm 1 under Cournot competition:

$$\pi_1 = \max_{q_1}\ (p - c_1)q_1 = \max_{q_1}\ (1 - q_1 - q_2 - c_1)q_1$$

$$\text{Similarly, } \pi_2 = \max_{q_2}\ (1 - q_1 - q_2 - c_2)q_2$$

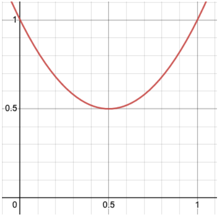
$$\texttt{TA2.m} \Rightarrow q_1 = \frac{1}{3} + \frac{2}{3}c - c_1,\ \ q_2 = \frac{1}{3} - \frac{4}{3}c + c_1,\ \ q_1 + q_2 = \frac{2}{3} - \frac{2}{3}c$$

- Suppose firm 1 is the low-cost firm: $c_1 \leq c \leq c_2$, when c_1 moves away from c: $c_1 \downarrow$, we have: $c_2 = 2c - c_1 \uparrow$, $q_1 \uparrow$, $q_2 \downarrow$, $\alpha_1 = \frac{q_1}{q_1+q_2} = \frac{q_1}{\frac{2}{3}-\frac{2}{3}c} \uparrow$, $\alpha_2 \downarrow$.

Exercise 5.7 of Tirole (1988)

Suppose that demand is linear ($Q = 1 - p$) and that there are two firms with constant marginal costs c_1 and c_2 such that $c_1 + c_2 = 2c$ (where c is a constant). Show that when the firms become more asymmetric (c_i moves away from c), Cournot competition yields a higher concentration index and a higher industry profit.

$$
\begin{aligned}
I^H &= \sum_i \alpha_i^2 = \alpha_1^2 + \alpha_2^2 = \alpha_1^2 + (1-\alpha_1)^2 = 2\alpha_1^2 - 2\alpha_1 + 1 \\
&\geq 0.5^2 + 0.5^2
\end{aligned}
$$



$f(x) = 2x^2 - 2x + 1$

Exercise 5.7 of Tirole (1988)

Suppose that demand is linear ($Q = 1 - p$) and that there are two firms with constant marginal costs c_1 and c_2 such that $c_1 + c_2 = 2c$ (where c is a constant). Show that when the firms become more asymmetric (c_i moves away from c), Cournot competition yields a higher concentration index and a higher industry profit.

- The total profit

$$\texttt{TA2.m} \Rightarrow \pi = \pi_1 + \pi_2 = (p - c_1)q_1 + (p - c_2)q_2$$
$$= 2c_1^2 - 4c \cdot c_1 + \frac{20}{9}c^2 - \frac{4}{9}c + \frac{2}{9}$$
$$\frac{\partial \pi}{\partial c_1} = 4c_1 - 4c$$

π is convex in c_1 and reaches its minimum at $c_1 = c$.

References

- Dixit, A. K., & Sherrerd, J. J. (1990). *Optimization in economic theory*. Second Edition. Oxford University Press.
- Tirole, J. (1988). *The Theory of Industrial Organization*. MIT press. Chapter 5