TA Session 1: Exercises on Quantity Competition

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Roadmap

A review of the Cournot Model

Quantity Competition

- Static imperfect competition: firms choose a single strategic variable, quantity or price, once.
- Cournot (1838) analyses situations in which firms set quantities.
- We have n firms in the industry with each firm i sets q_i.
- Total output:

$$q = q_1 + \cdots + q_n \equiv q_i + q_{-i}$$

- Firm i's problem:

$$\pi_i = \max_{q_i} \; q_i \cdot P(q) - C_i(q_i)$$

$$\frac{\partial \pi_i}{\partial q_i} = P(q) + q_i P'(q) - C_i'(q_i) = 0 \tag{1}$$

where C_i is the cost function and P is the inverse demand.

$$(1) \Leftrightarrow \underbrace{\frac{P - C_i'}{P}}_{\substack{\text{Lerner Index} \\ \text{for firm } i}} = \underbrace{\frac{q_i}{q}}_{\substack{\text{market share} \\ \text{of firm } i}} \cdot \underbrace{\left(-\frac{qP'}{P}\right)}_{\substack{\equiv \frac{1}{\varepsilon},\ \varepsilon \text{ is the} \\ \text{elasticity of demand}}} \tag{2}$$

Example: The linear Cournot Model

- *Context*: an oligopoly facing a linear demand for a homogeneous product and producing at constant marginal costs.
- Linear inverse demand:

$$P(q) = a - b \cdot q, \quad a, b > 0$$

- Linear cost function:

$$C_i(q_i) = c_i q_i, \quad 0 \leq c_i < a, \forall i$$

- Firm i solves:

$$\begin{aligned} \max_{q_i} \; & q_i \left[a - b\left(q_i + q_{-i}\right)\right] - c_i q_i \\ & \Rightarrow \frac{\partial \pi_i}{\partial q_i} = a - 2bq_i - bq_{-i} - c_i = 0 \\ & \Leftrightarrow q_i(q_{-i}) = \frac{1}{2b}(a - bq_{-i} - c_i) \end{aligned} \tag{3}$$

(3) gives firm i's best response function.

Example: The linear Cournot Model

- At the Cournot equilibrium, (3) is satisfied for each of the n firms, each firm best responds to the choices of the other firms:

$$
\begin{aligned}
(3) \Rightarrow \sum_{i=1}^{n} q_i &= \frac{1}{2b}\left(na - (n-1)q - \sum_i c_i\right) \\
\Leftrightarrow q &= \frac{1}{2b}\left(na - (n-1)q - \sum_i c_i\right) \\
\Leftrightarrow q^* &= \frac{na - \sum_i c_i}{b(n+1)}
\end{aligned}
$$

q^* is the total industry output at the Cournot equilibrium. For firm i,

$$
\begin{aligned}
(3) \Rightarrow q_i^* &= \frac{1}{2b}\left[a - b(q^* - q_i^*) - c_i\right] \\
\Leftrightarrow q_i^* &= \frac{a - nc_i + \sum_{j \neq i} c_j}{b(n+1)}
\end{aligned}
$$

Example: The linear Cournot Model

- Given the quantity, it follows that firm i's equilibrium profits are

$$\begin{aligned}\pi_i &= q_i^* \left(P(q^*) - c_i\right) \\ &= \frac{\left(a - nc_i + \sum_{j \neq i} c_j\right)^2}{b(n+1)^2}\end{aligned} \tag{4}$$

  π_i decreases with c_i and increases with $\sum_{j \neq i} c_j$.
- In the linear Cournot model with homogeneous products, a firm's equilibrium profits increase when the firm becomes relatively more efficient than its rivals (i.e., all other things being equal, when its marginal cost decreases or when the marginal cost of any of its rivals increases).

Exercises on the Cournot Model

Exercise 5.3 of Tirole (1988)

There are three identical firms in the industry. The demand is $1 - Q$, where $Q = q_1 + q_2 + q_3$. The marginal cost is zero.

1. Compute the Cournot equilibrium.
2. Show that if two of the three firms merge (transforming the industry to a duopoly), the profit of these firms decreases. Explain.
3. What happens if all three firms merge?

- Firm i's problem, $\forall i \in \{1, ..., n\}$:

$$\max_{q_i} \; q_i(1 - Q)$$

$$\frac{\partial \pi_i}{\partial q_i} = 0 \Leftrightarrow q_i = \frac{1}{2}(1 - q_{-i}) \Rightarrow \sum_{i=1}^{n} q_i = \frac{1}{2}\sum_{i=1}^{n}(1 - q_{-i}) \Leftrightarrow Q^* = \frac{n}{n+1}$$

$$\Rightarrow P^* = \frac{1}{n+1}, \;\; q_i^* = \frac{1}{n+1}, \;\; \pi_i = q_i^* P^* = \frac{1}{(n+1)^2} \tag{5}$$

1. Symmetric Cournot oligopoly, $n = 3$.

$$(5) \Rightarrow q_i^* = \frac{1}{4}, \;\; \pi_i = \frac{1}{16}, \;\; \forall i$$

3. Monopoly. The only firm in the industry solves the problem:

$$\begin{aligned} \max_Q \;\; Q(1-Q) \;\; &\Rightarrow \frac{\partial \pi}{\partial Q} = 1 - 2Q = 0 \\ &\Rightarrow Q^* = \frac{1}{2}, \;\; P^* = \frac{1}{2}, \;\; \pi^* = \frac{1}{4} \end{aligned}$$

2. Cournot duopoly. If two of the tree firms merge, we are left with two firms which are still ex ante symmetric in the sense that they have the same cost structure.

$$(5) \Rightarrow q_i^* = \frac{1}{3}, \ \ \pi_i = \frac{1}{9}, \ \ \forall i$$

Using expressions from (5), in this symmetric linear Cournot model, if we let the number of firms n decrease, we obtain the following comparative statics results:

- The market price increases;
- The individual quantity increases; the total quantity decreases; as a result, the market share of each firm increases;
- Individual profit increases; the merged company earns less than the sum of the two before merge: $\frac{1}{9} < \frac{1}{16} \times 2$; industry-wide profits increase:

$$\frac{\partial \left(\sum_i \pi_i\right)}{\partial n} = \underbrace{\frac{\partial \pi}{\partial Q^*}}_{=\frac{1-n}{n+1} \leq 0} \cdot \underbrace{\frac{\partial Q^*}{\partial n}}_{=\frac{1}{(n+1)^2} > 0} \leq 0$$

The equality only holds for $n = 1$.

Exercise 5.4 of Tirole (1988)

Consider a duopoly producing a homogenoeous product. Firm 1 produces one unit of output with one unit of labor and one unit of raw material. Firm 2 produces one unit of output with two units of labor and one unit of raw material. The unit costs of labor and raw material are w and r. The demand is $p = 1 - q_1 - q_2$, and the firms compete in quantities.

1. Compute the Cournot equilibrium.
2. Show that firm 1's profit is not affected by the price of labor (over some range). To prove this elegantly, use the envelope theorem. Explain.

1. Cournot duopoly:

$$\text{Firm 1: } \max_{q_1} \; q_1(1 - q_1 - q_2 - w - r)$$

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Leftrightarrow q_1 = \frac{1}{2}(1 - q_2 - w - r) \tag{6}$$

$$\text{Firm 2: } \max_{q_2} \; q_2(1 - q_1 - q_2 - 2w - r)$$

$$\frac{\partial \pi_2}{\partial q_2} = 0 \Leftrightarrow q_2 = \frac{1}{2}(1 - q_1 - 2w - r) \tag{7}$$

$$(6)+(7) \Rightarrow Q^* = \frac{2}{3} - w - \frac{2}{3}r$$
$$\Rightarrow q_1^* = \frac{1}{3} - \frac{1}{3}r, \quad q_2^* = \frac{1}{3} - w - \frac{1}{3}r, \quad p^* = \frac{1}{3} + w + \frac{2}{3}r$$

2. By the Envelope Theorem, w doesn't affect the objective function through the optimizer q_1.

$$\begin{aligned}\pi_1(q_1^*, w) &= \max_{q_1} \; q_1(1 - q_1 - q_2 - w - r) \\ &= q_1^*(1 - q_1^* - q_2(w) - w - r) \\ \Rightarrow \frac{\mathrm{d}\pi_1}{\mathrm{d}w} &= \frac{\partial \pi_1(q_1^*, w)}{\partial w} = q_1^*\left(-\frac{\partial q_2}{\partial w} - 1\right) \overset{(7)}{=} q_1^*(1-1) = 0\end{aligned}$$

Firm 2 is more labor intensive ($c_1 = w + r, c_2 = 2w + r$). An increase in w raises firm 1's cost but also weakens firm 2's strategic position.

Exercise 5.5 of Tirole (1988)

This exercise illustrates the strategic considerations faced by a multimarket firm. It is inspired by the more general theroy of Bulow et al. (1985). There are two firms in a market. They produce perfect substitues at cost $C(q) = \frac{q^2}{2}$. The demand is $p = 1 - (q_1 + q_2)$.

1. Compute the Cournot equilibrium.
2. Suppose now that firm 1 has the opportunity to sell the same output on another market as well. The quantity sold on this market is x_1, so firm 1's cost is $\frac{(q_1+x_1)^2}{2}$. The demand on the second market is $p = a - x_1$. Consider the Cournot game in which firm 1 chooses q_1 and x_1 and firm 2 chooses q_2 simultaneously. Show that $q_1 = \frac{2-a}{7}$ and $q_2 = \frac{5+a}{21}$ over the relevant range of a. Show that for $a = \frac{1}{4}$, a small increase in a hurts firm 1. (Use the envelope theorem.) Interpret.

1. Symmetric Cournot duopoly, $i \in \{1, 2\}$:

$$\max_{q_i} \; pq_i - C(q_i) = \max_{q_i} \; (1 - q_i - q_{-i})q_i - \frac{q_i^2}{2}$$
$$\frac{\partial \pi_i}{\partial q_i} = 0 \Leftrightarrow q_i = \frac{1}{3}(1 - q_{-i}) \Rightarrow Q^* = \frac{1}{2}, \;\; q_i^* = \frac{1}{4}, \;\; p^* = \frac{1}{2}, \;\; \pi_i^* = \frac{3}{32}$$

2. Firms' problems:

$$\text{Firm 1: } \max_{q_1} \; (1 - q_1 - q_2)q_1 + (a - x_1)x_1 - \frac{(q_1 + x_1)^2}{2}$$
$$\frac{\partial \pi_1}{\partial q_1} = 0 \Leftrightarrow 1 - 3q_1 - q_2 - x_1 = 0 \tag{8}$$
$$\frac{\partial \pi_1}{\partial x_1} = 0 \Leftrightarrow a - q_1 - 3x_1 = 0 \tag{9}$$
$$(8)\&(9) \Rightarrow 1 - \frac{1}{3}a - \frac{8}{3}q_1 - q_2 = 0 \tag{10}$$

$$\text{Firm 2: } \max_{q_2} \ (1 - q_1 - q_2)q_2 - \frac{q_2^2}{2}$$

$$\frac{\partial \pi_2}{\partial q_2} = 0 \Leftrightarrow 1 - q_1 - 3q_2 = 0 \tag{11}$$

$$(10)\&(11) \Rightarrow q_1^* = \frac{2-a}{7}, \ \ x_1^* = \frac{8a-2}{21}, \ \ q_2^* = \frac{5+a}{21} \tag{12}$$

By the Envelope Theorem,

$$\frac{\mathrm{d}\pi_1}{\mathrm{d}a} = \frac{\partial \pi_1(q_1^*, a)}{\partial a} = -q_1^* \frac{\partial q_2}{\partial a} + x_1^*$$

$$(12)\&a = \frac{1}{4} \Rightarrow x_1^* = 0, \ \ \frac{\mathrm{d}\pi_1}{\partial a} = -\frac{1}{21 \times 4} < 0$$

When $a = \frac{1}{4}$, $x_1^* = 0$, firm 1 is indifferent between whether it sells in the other market or not. A smalll increase ε in a will raise the demand in market 2, firm 1 then raises its supply x_1 by $\frac{8}{21}\varepsilon$ and decrease its supply q_1 by $\frac{1}{7}\varepsilon$. As a result, its cost will be increased by $\frac{1}{2}(\frac{8}{21}\varepsilon - \frac{1}{7}\varepsilon)^2 = \frac{1}{2}(\frac{5}{21}\varepsilon)^2$. Firm 2 is fully aware of this and will increase its output q_2.

References

- Tirole, J. (1988). *The Theory of Industrial Organization*. MIT press. Chapter 5.4.