TA Session 4: Duration Models

Microeconometrics with Joan Llull
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Overview

Duration Data

Duration Data

- *Duration data*: data on a variable that measures the length of time spent in a state before transition to another state
- `TA4.dta`: college dropouts data (single-record data, one obs. per individual)

	id	duration	event	sex	grade	part_time
1	1	41	0	1	2	0
2	2	8	1	0	4	1
3	3	41	0	1	3	0
4	4	4	1	1	4	1
5	5	47	0	0	1	0
6	6	44	0	1	2	0
7	7	39	0	1	1	0
8	8	4	1	0	5	0
9	9	21	1	0	1	0
10	10	41	0	1	4	0

- `event`: the event of interest, 1 = dropout, 0 = censored
- Empirical concern: the spell length may be incompletely observed (censored, individuals leave the study before the spell ends).

Duration Data

- Set the duration data structure based on variable `duration`. Commands begin with `st` (survival-time).

  ```
  .  stset duration, failure(event=1) id(id)
  ```

id	duration	event	_t0	_t	_d	_st
1	41	0	0	41	0	1
2	8	1	0	8	1	1
3	41	0	0	41	0	1
4	4	1	0	4	1	1
5	47	0	0	47	0	1

- Variables newly generated by the command:
  `_t0`: analysis time when record begins (the calendar time could be different for different individuals)
  `_t`: analysis time when record ends
  `_d`: 1 if failure, 0 if the spell is censored
  `_st`: 1 if the record is to be included in analysis; 0 otherwise

Continuous Duration vs. Discrete Duration

The key difference: grouping

- Continuously distributed durations
  - Time index is sill "discrete", you have natural numbers $t = 1, 2, ...$, not something like $t = 1.4142$.
  - Continuous means time is in its fairly precise unit, consecutively observed, not grouped.
- Discretely distributed durations: grouped data
  - When the measurements are in aggregated time intervals, it can be important to account for the discreteness in the estimation.
  - In grouped duration data, each duration is only known to fall into a certain time interval, such as a week, a month, or even a year.
  - Why we can't address this discreteness using the continuous duration model: explained later in section Discrete Duration.

Continuous Duration

Estimation Approaches

1. **non-parametric**: letting the data speak for itself and making no assumption about the functional form of the survivor function, the effect of covariates are not modeled either.
2. **semi-parametric**: no parametric form of the survivor function is specified, yet the effect of the covariates is still assumed to take a certain form (to alter the baseline survivor function that for which all covariates are equal to zero). The Cox(1972) model is the most popular semiparametric model.
3. **fully parametric**: analogous to a Tobit model with right-censoring, has the limitation of heavy reliance on distributional assumptions (in order for the parameter estimates to be consistent).

Censoring

- One important problem of survival data is that they are usually censored, as some spells are incompletely observed. In practice, data may be
  - **right-censoring/censoring from above**: we observe spells from time 0 until a censoring time c, the unknown end lies in (c, ∞).
  - **left-censoring/censoring from below**: the spells are incomplete with an unknown end lies in $(0, c)$. For example when we talk about unemployment spell, this individual ends unemployment before her entering the study.
  - **interval censoring**: the censored spell ends between two known time points $[t_1^*, t_2^*)$.
- The survival analysis literature has focused on right-censoring.

Assumption

- Each individual in the sample has a completed duration T_i^* and censoring time C_i^*. What we observe for each spell is the minimum of T_i^* and C_i^*.
- For standard survival analysis methods to be valid, the censoring mechanism needs to be one with **independent (noninformative) censoring**.
- This means that parameters of the distribution of C^* are not informative about the parameters of the distribution of the duration T^*.

Nonparametric Approach

Estimation of survival functions:

1. Estimate the survivor or hazard function in the presence of independent censoring.
2. No regressors are included.

Key concepts of survival analysis

Function	Symbol	Definition	Relationship
Density	$f(t)$		$f(t) = \frac{dF(t)}{dt}$
Distribution	$F(t)$	$P(T \leq t)$	$F(t) = \int_0^t f(s)ds$
Survivor	$S(t)$	$P(T > t)$	$S(t) = 1 - F(t)$
Hazard	$h(t)$	$\lim_{h \to 0} \frac{P(t \leq T \leq t+h \mid T \geq t)}{h}$	$h(t) = \frac{f(t)}{S(t)}$
Cumulative hazard	$H(t)$	$H(t) = \int_0^t h(s)ds$	$H(t) = -\ln S(t)$

- For each t, $h(t)$ is the instantaneous rate of leaving per unit of time.

$$h(t) = \lim_{\Delta \to 0} \frac{\mathbb{P}\left(t \leq T \leq t + \Delta \mid T \geq t\right)}{\Delta}$$

and for "small" Δ,

$$\mathbb{P}\left(t \leq T \leq t + \Delta \mid T \geq t\right) \approx h(t) \cdot \Delta$$

Thus, the hazard function can be used to approximate a conditional probability in much the same way that the height of the density of T can be used to approximate an unconditional probability.

The Kaplan-Meier Estimator

- Kaplan–Meier estimator or product limit estimator of the survivor function

$$\hat{S}(t) = \prod_{j|t_j \leq t} \frac{\texttt{\#Spells at risk}(t_j) - \texttt{\#Spells ending}(t_j)}{\texttt{\#Spells at risk}(t_j)}$$

Kaplan–Meier survivor function

Time	At risk	Fail	Net lost	Survivor function	Std. error	[95% conf.	int.]
1	265	3	0	0.9887	0.0065	0.9653	0.9963
2	262	10	0	0.9509	0.0133	0.9170	0.9712
3	252	8	0	0.9208	0.0166	0.8810	0.9476
4	244	7	0	0.8943	0.0189	0.8506	0.9258
5	237	3	0	0.8830	0.0197	0.8378	0.9162

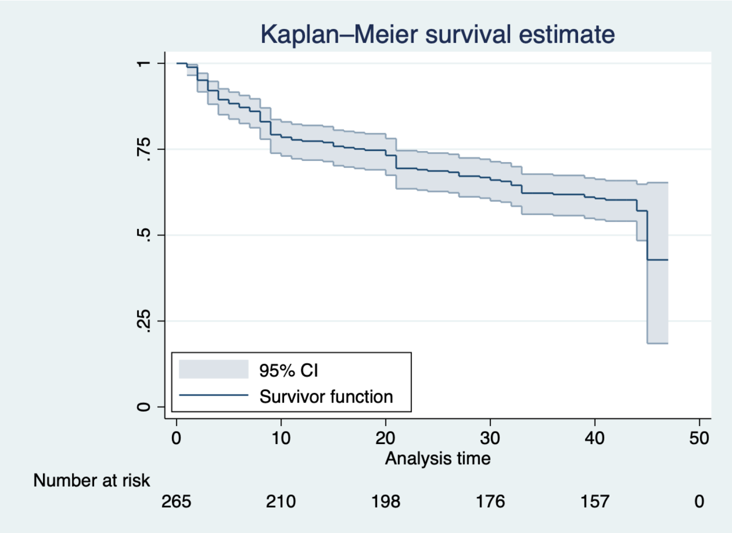
- `At risk`: at school; `Fail`: dropped out; `Net Lost`: censored
- **Example**:

  The probability of survival beyond $t = 1$ is $\frac{262}{265} \approx 0.9887$.

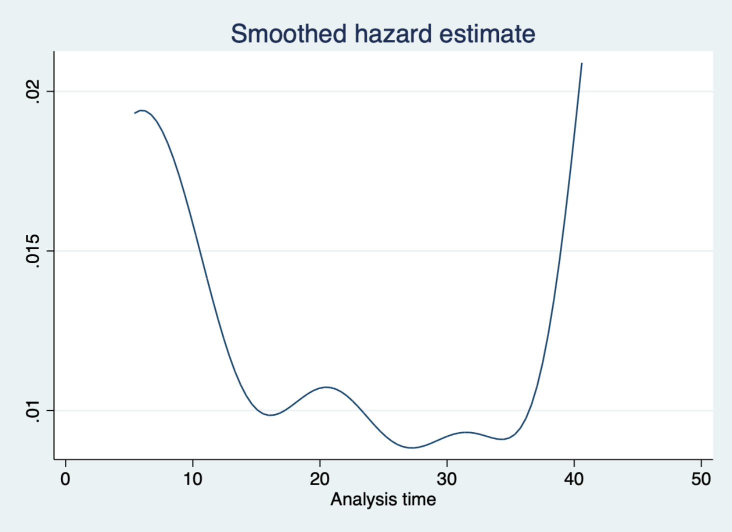
  The probability of survival beyond $t = 2$ is $\frac{262}{265} \times \frac{252}{262} = \frac{252}{265} \approx 0.9509$.

  ...

The Kaplan-Meier Estimator



The Kaplan-Meier Estimator



- Kernel smoothing: the weighted (kernel) average of neighboring observations

The Cox Proportional Hazards Model

- To estimate the role of individual observed heterogeneity while controlling for duration dependence, we consider the **Cox proportional hazards** regression model (Cox, 1972):

$$h(t|x) = \underbrace{h_0(t)}_{\text{baseline hazard}} \cdot \underbrace{e^{x\beta}}_{\text{relative hazard}}$$

The Cox model is semiparametric in the sense that $h_0(t)$ is estimated non-parametrically, and the scale up part $e^{x\beta}$ is assumed to be depending on regressors.

- The Cox model has no intercept since

$$h_0(t)e^{\beta_0 + x\beta} = \underbrace{h_0(t)e^{\beta_0}}_{\text{new baseline hazard}} \quad e^{x\beta}$$

Any intercept along with the regressors is not identified, since any value works as well as any other.

The Cox Proportional Hazards Model

Partial Likelihood Estimation

$$h(t|x) = h_0(t) \cdot e^{x\beta}$$

- Partial likelihood estimation (Cox, 1972, 1975)
    - For now, we consider only time-invariant regressors, but later we will relax this assumption.
    - "Partial": we estimate β without estimating $h_0(t)$.
    - Partial likelihood minimization $\rightarrow \hat{\beta}$
    - Nonparametric KM estimation $\rightarrow \hat{h}_0(t)$

The Cox Proportional Hazards Model

- Effects of regressors on the time until college dropout: the βs from $e^{x\beta}$

```
. stcox $x, nohr

Cox regression with Breslow method for ties

No. of subjects =   265                          Number of obs =    265
No. of failures =   107
Time at risk    = 8,087
                                                 LR chi2(6)    =  62.85
Log likelihood = -535.6177                       Prob > chi2   = 0.0000
```

_t	Coefficient	Std. err.	z	P>|z|	[95% conf.	interval]
female	.1059617	.2040423	0.52	0.604	-.2939538	.5058771
grade	.2892697	.087417	3.31	0.001	.1179355	.460604
part_time	1.210182	.2788914	4.34	0.000	.6635652	1.756799
lag	-.0138323	.0083869	-1.65	0.099	-.0302703	.0026057
stm	.1056626	.0201591	5.24	0.000	.0661515	.1451738
married	.9950366	.2631813	3.78	0.000	.4792107	1.510863

- The magnitude of these effects is not immediately clear. Why?

The Cox Proportional Hazards Model

Effect Size

- If the jth regressor in $x = (x_1, x_2, ..., x_k)$ is increased by 1 unit,

$$h(t|x+\Delta) = h_0(t)e^{\beta_1 x_1 + \ldots + \beta_j(x_j+1) + \ldots + \beta_k x_k} = h_0(t)e^{x\beta + \beta_j} = e^{\beta_j} h(t|x)$$

- Therefore, changes in regressors can be interpreted as having a multiplicative effect on the original hazard (semi-elasticity), as

$$\frac{\partial h(t|x)}{\partial x_j} = h_0(t)\frac{\partial e^{x\beta}}{\partial x_j} = h_0(t)e^{x\beta}\beta_j = h(t|x)\beta_j$$

- The coefficients:
  - $\beta_{\texttt{female}} \approx 0.106 > 0$, *harzard rate* is higher for female students;
  - $\beta_{\texttt{grade}} \approx 0.289 > 0$, *harzard rate* is higher for college students with worse high school performance (high `grade`).
- **The effect size**:
  - *hazard ratio* for time-invariant variable `female` is $e^{0.106} \approx 1.112$;
  - A one unit increase in `grade` (high school grades before college, the lower the better) leads to the hazard rate being $e^{0.289} \approx 1.335$ times higher.

The Cox Proportional Hazards Model

Baseline

- Concern: the baseline

$$\begin{aligned} \Rightarrow h\left(t|x=0\right) &= h_0(t)\cdot e^{\beta_1\cdot 0+\cdots+\beta_k\cdot 0+0} \\ &= h_0(t)\cdot e^0 = h_0(t) \end{aligned}$$

- Problem: our $x =$ (`female`,`grade`,`part_time`,`lag`,`stm`,`married`), variable `stm` never goes to zero in our sample, $min($`stm`$) = 6$
- Solution: recenter the variable

```
.   generate stm6 = stm - 6
.   stcox $x stm6, shared(grade)
```

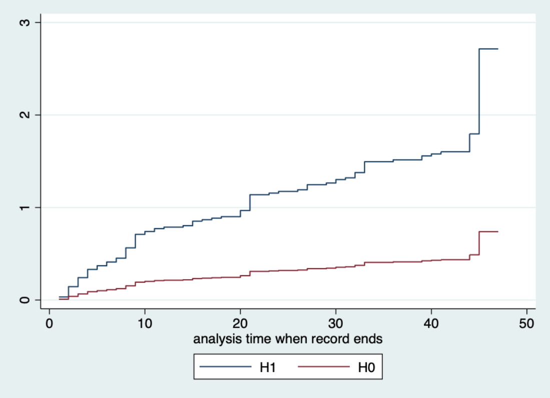
- Now the baseline survivor estimate (S_0) corresponds to a male full-time student, not married and `stm` $= 6$.

The Cox Proportional Hazards Model

- The cumulative hazard:

$$H(t|x) = \int_0^t h(s|x)ds = \int_0^t e^{x\beta} \cdot h_0(s)ds = e^{x\beta} \int_0^t h_0(s)ds = e^{x\beta} \cdot H_0(t)$$

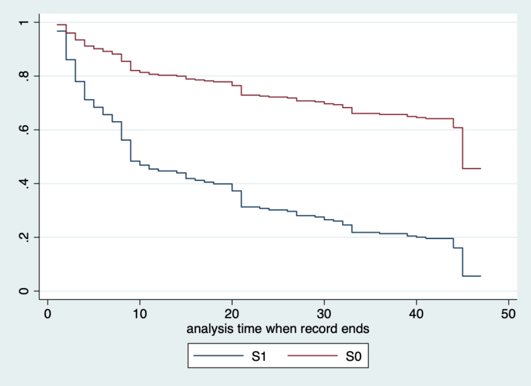
- After including one binary regressor (part-time student) whose estimate is $\beta_1 \approx 1.210$, we have $H(t|x=1) \approx e^{1.210} H_0(t) \approx 3.353 H_0(t)$.



The Cox Proportional Hazards Model

- The survival function:

$$S(t|x) = e^{-H(t|x)} = e^{-e^{x\beta} H_0(t)} = \left[e^{-H_0(t)}\right]^{e^{x\beta}} = S_0(t)^{e^{x\beta}}$$

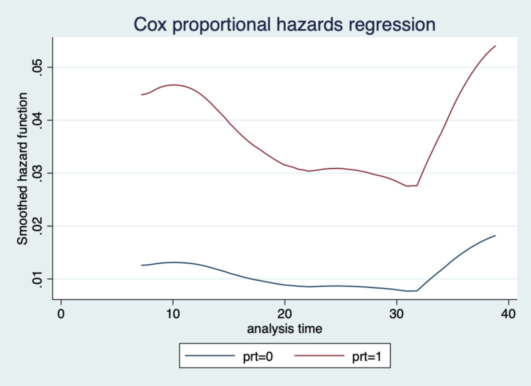


- Part-time students (S_1) survive much worse: $S(t|x=1) \approx S_0(t)^{e^{1.210}} \approx S_0(t)^{3.353}$, higher power $e^{x\beta}$ makes $S(t|x)$ more convex.

The Cox Proportional Hazards Model

- Hazards:

$$h(t|x=1) = h_0(t) \cdot e^{1.210} = h_0(t) \cdot 3.353$$



- The hazards are indeed proportional, and if graphed on a log scale they would be parallel.

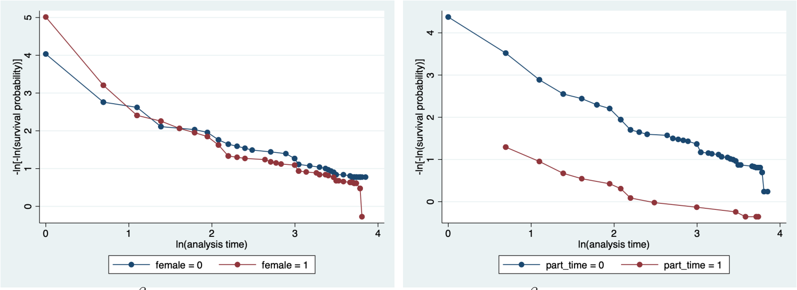
The Cox Proportional Hazards Model

Model Diagnostics

1. PH implies proportional integrated hazards:

$$H(t|x) = \int_0^t h(s|x)\mathrm{d}s = e^{x\beta}\int_0^t h_0(s)\mathrm{d}s = H_0(t)e^{x\beta}$$
$$\Rightarrow \ln H(t|x) = \ln H_0(t) + x\beta$$

Therefore under PH, the log-integrated hazard curves $\ln H(t|x)$ (the *log-log survivor curves*), should be parallel at different values of the time invariant regressors x (as there is no t in $x\beta$) → a graphical test on PH

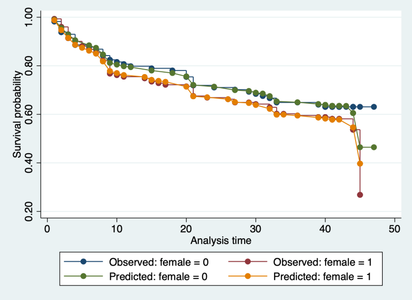


$e^{\beta_{\texttt{female}}} \approx 1.112$ $e^{\beta_{\texttt{part\_time}}} \approx 3.353$

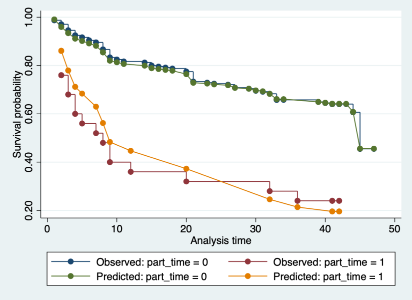
The Cox Proportional Hazards Model

Model Diagnostics

2. The predicted survivor function from Cox regression and the (nonregression) Kaplan-Meier estimate (observed) of the survivor function should be similar if PH is appropriate. → another graphical test on PH



`female`



`part_time`

The PH model is reasonable for `female` but does not do so well for `part_time`.

The Cox Proportional Hazards Model

Model Diagnostics

3. **A formal residual-based statistical test** on the key assumption of the Cox model: separable components, duration part $h_0(t)$ and regressors part $e^{x\beta}$.

   Under the standard PH assumption, there should be no time (duration/spell length) trend in the regressors part. Rejection of the null (no time trend / zero slope) indicates a deviation from the proportional-hazards assumption.

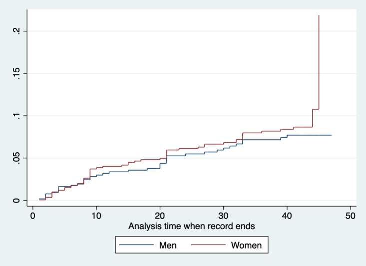
Test of proportional-hazards assumption

Time function: Analysis time

	rho	chi2	df	Prob>chi2
female	0.03383	0.12	1	0.7254
grade	0.00149	0.00	1	0.9869
part_time	-0.04947	0.30	1	0.5813
lag	-0.08136	0.71	1	0.3997
stm	0.04949	0.32	1	0.5727
married	-0.05574	0.33	1	0.5647
Global test		1.54	6	0.9568

Stratified Cox Model

- If some variable does not fulfill the PH assumption, we can use it as a strata (group) variable.
- In the stratified Cox model, we relax the assumption that everyone faces the same baseline hazard.
- The baseline hazards are allowed to differ by group, while the coefficients β are constrained to be the same across groups. Ex: $h_g(t|x) = h_{0g}(t)e^{x\beta}$, where g indicates the gender groups.



- The cost of this model is that the effect of `female` is not identified.

Time-Varying Covariates

Extended Cox Model

- There are cases that require time-varying covariates: e.g., when one is repeatedly unemployed, the macroeconomic conditions change.
- Extended Cox model:

$$h(t|x) = h_0(t)e^{x_t \beta}$$

- For two individuals i and j,

$$\frac{h(t|x_{it})}{h(t|x_{jt})} = \frac{h_0(t)e^{x_{it}\beta}}{h_0(t)e^{x_{jt}\beta}} = e^{(x_{it}-x_{jt})\beta}$$

  This hazard ratio between two individuals is a function of t, the PH assumption no longer holds.
- Estimation: in the likelihood, x_i is replaced by $x_i(t_j)$ ...

Parametric Models

- Proportional hazard specification: $h(t|x) = h_0(t)e^{x\beta} \rightarrow$ flexible hazard functions

Semi-parametric model:

$$\text{Cox PH:} \quad h(t|x) = \underbrace{h_0(t)}_{\text{unparameterized}} e^{x\beta}$$

Parametric models:

$$\text{Weibull:} \quad h(t|x) = h_0(t, \alpha, \gamma) \cdot e^{x\beta} = \alpha t^{\alpha-1} e^{\gamma} \cdot e^{x\beta} \rightarrow (\alpha, \gamma, \beta)$$

$$\text{Exponential:} \quad h(t|x) = h_0(t, \alpha) \cdot e^{x\beta} = e^{\alpha} \cdot e^{x\beta} \rightarrow (\alpha, \beta)$$

Notice that there is no constant term in vector x.

- The estimates from the parametric PH model should be roughly similar to that from the Cox model. Otherwise there is evidence of a misparameterized underlying baseline hazard.

Parametric Models Comparison

	(1) Cox	(2) Exponen~l	(3) Weibull	(4) Loglogit	(5) Lognormal
main					
female	0.106	0.141	0.139	-0.155	-0.148
	(0.202)	(0.213)	(0.209)	(0.238)	(0.240)
grade	0.289***	0.300***	0.293**	-0.315***	-0.308**
	(0.0846)	(0.0911)	(0.0896)	(0.0942)	(0.0953)
part_time	1.210***	1.323***	1.289***	-1.524***	-1.555***
	(0.268)	(0.287)	(0.278)	(0.364)	(0.341)
lag	-0.0138	-0.0152	-0.0147	0.0105	0.00809
	(0.00981)	(0.0103)	(0.0102)	(0.0125)	(0.0117)
stm	0.106***	0.108***	0.102***	-0.106***	-0.104***
	(0.0205)	(0.0173)	(0.0178)	(0.0208)	(0.0198)
married	0.995***	1.050***	1.030***	-1.263***	-1.247***
	(0.267)	(0.294)	(0.289)	(0.300)	(0.315)
_cons		-6.231***	-5.914***	5.973***	6.007***
		(0.307)	(0.422)	(0.362)	(0.367)
ll	-535.6	-290.0	-289.6	-287.5	-286.5
aic	1083.2	593.9	595.2	591.1	589.0
bic	1104.7	619.0	623.8	619.7	617.6
N	265	265	265	265	265

Standard errors in parentheses
* p<0.05, ** p<0.01, *** p<0.001

- Better model fit but counterintuitive signs of coef. for some models?
- Can be more precise on coef.; Low robustness to distribution misspecification.

Hazards from Various Models

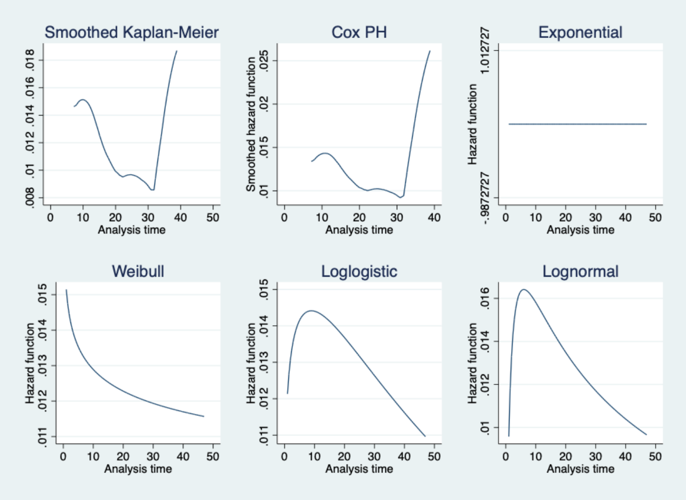


Figure 1: Hazard rates from various models, evaluated at the mean of the regressors

- Exponential: constant hazard; Weibull: monotonic hazard; Loglogistic and Lognormal: inverted U-shaped hazard; Cox PH: flexible hazard.

Unobserved Heterogeneity

- In duration analysis, the unobserved heterogeneity will lead to inconsistent estimates even if it's not correlated with the explanatory variables[1]. Consider for example that there are groups of unemployed people that differ by the unobserved skill level, which will affect their hazard function.

$$\begin{aligned} h_i(t) &= h_0(t)\alpha_i e^{x_i\beta}, \ \ \alpha_i > 0 \\ &= h_0(t)e^{x_i\beta+\nu_i}, \ \ \nu_i = \ln \alpha_i \end{aligned}$$

  The unobserved heterogeneity enters multiplicatively on the hazard function as a group-level *frailty*: α_g. The log frailty ν_g is analogous to random effects[2] in panel data.

[1] Unlike in linear models, where the estimates will be consistent if the unobserved heterogeneity is not correlated with the regressors.

[2] In this shared frailty model, the effects α_i (can also be group effects) is assumed to be random and is governed by a Gamma distribution with mean 1. We will also talk about the fixed effects case later.

Unobserved Heterogeneity

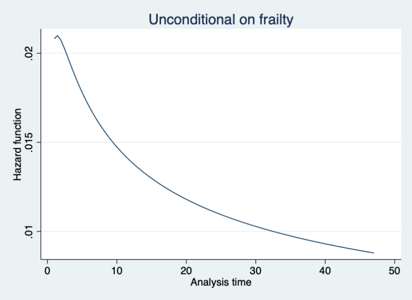
```
. streg, dist(weibull) frailty(invgau) vce(robust) nolog
nohr
```

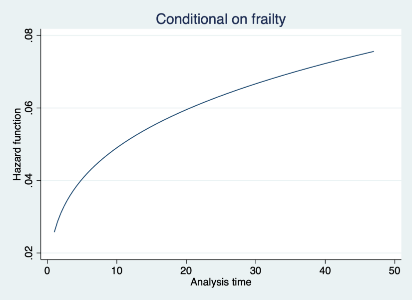
```
Weibull PH regression
Inverse-Gaussian frailty

No. of subjects =    265                          Number of obs =     265
No. of failures =    107
Time at risk    =  8,087
                                                  Wald chi2(0)  =       .
Log pseudolikelihood = -318.11508                 Prob > chi2   =       .

                                 (Std. err. adjusted for 265 clusters in id)
```

_t	Coefficient	Robust std. err.	z	P>|z|	[95% conf.	interval]
_cons	-3.906223	.2703001	-14.45	0.000	-4.436002	-3.376445
/ln_p	.2467046	.0768851	3.21	0.001	.0960125	.3973967
/lntheta	2.575637	.2272414	11.33	0.000	2.130253	3.021022
p	1.279801	.0983977			1.100773	1.487946
1/p	.7813715	.0600759			.6720673	.9084527
theta	13.13969	2.985881			8.416992	20.51225

- The log likelihood increases from -535.6177 (Cox PH with 6 regressors) to -318.1151.

Unobserved Heterogeneity





Weibull Hazard

Discrete Duration

Discrete-time hazards

- The T periods indexed by $t = 1, \ldots, T$ are grouped into A intervals indexed by $a = 1, \ldots, A$, unequally spaced intervals are allowed.

$$h(t_a|x) = \mathbb{P}(t_{a-1} \leq T < t_a | T \geq t_{a-1}, x(t_{a-1}))$$

- Why discrete durations is a problem: we need to consider three indexes i, t, a in the derivation.
  - PH model of continuous durations:

$$h(t|x) = h_0(t)e^{x\beta}$$

  - PH model of discrete durations associated with the continuous model:

$$h(t|x) = h_0(t)e^{x(t_{a-1})\beta}$$

    The regressors are constant within the interval (a) but can vary across intervals, and $h_0(t)$ can vary within the interval (a).

Discrete-time hazards

- Two solutions:
  1. Use index a, group $h_0(t)$ (more common)
     - Consider a binary choice model for transitions:

$$d = \begin{cases} 1, & \text{if the spell ends} \\ 0, & \text{otherwise} \end{cases}$$

     - And we fit a simple (stacked) Logit model on it:

$$\mathbb{P}(t_{a-1} \leq T < t_a | T \geq t_{a-1}, x) = F(\lambda_a + x(t_{a-1})\beta)$$

       where β is restricted to be constant over time, and the intercept λ_a is allowed to vary across intervals.
  2. Use index t, add group indicators for each a (dummies for each interval a are included as regressors)
     - Complementary log-log: equivalent to a Cox PH, also called a grouped Cox PH.

Discrete-time hazards

	(1) logit	(2) cloglog
y		
female	-0.351	-0.335
	(0.213)	(0.192)
grade	-0.0559	-0.0543
	(0.136)	(0.134)
part_time	1.137***	1.091***
	(0.292)	(0.252)
stm	-0.321***	-0.328***
	(0.0684)	(0.0641)
married	1.027***	1.000***
	(0.248)	(0.263)
ll	-587.2	-588.3
aic	1212.5	1214.6
bic	1345.4	1347.5
N	8085	8085

Standard errors in parentheses
* p<0.05, ** p<0.01, *** p<0.001

Appendix

References

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