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Integral Equations and Operator Theory

Limit Laws for R-diagonal Variables in a Tracial Probability Space

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Abstract. We study the weak convergence of sums of $*$-free, identically distributed tracial R-diagonal variables. The result parallels earlier results about free additive convolution on the real line. In particular, we determine under which conditions an infinitesimal array yields a sequence that converges to a given infinitely divisible tracial R-diagonal distribution. Thus, much of the work concerning sums of free (in the sense of Voiculescu) identically distributed positive random variables can be translated to the tracial R-diagonal context.

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1. Introduction

Given an R-diagonal random variable X in a tracial noncommutative W^*-probability space $(\mathcal{M},\tau)$, the $*$-distribution of X is entirely determined by the distribution μ_{X^*X} of X^*X. The latter distribution is a Borel probability measure on $\mathbb{R}_+ = [0,+\infty)$, uniquely determined by the requirement that

$$\int_{\mathbb{R}_+} u(t)d\mu_{X^*X}(t) = \tau(u(X^*X))$$

for every bounded Borel function u on $\mathbb{R}_+$. Therefore the addition of tracial $*$-free R-diagonal variables can be understood in terms of a convolution operation $\boxplus_{RD}$ defined on the set of Borel probability measures on $\mathbb{R}_+$, and satisfying the equation

$$\mu_{(X_1+X_2)^*(X_1+X_2)} = \mu_{X_1^*X_1} \boxplus_{RD} \mu_{X_2^*X_2} \tag{1.1}$$

if X_1 and X_2 are tracial R-diagonal, and $*$-free. In this context, the author [15] has proved an analog of Hinčin's theorem. Thus, given an infinitesimal array $\{\mu_{n,j} \ : \ n,j \in \mathbb{N}, \ j \le k_n\}$ of Borel probability measures on $\mathbb{R}_+$ (that is, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \min\{\mu_{n,j}((-\varepsilon,\varepsilon)) \ : \ 1 \le j \le k_n\} = 1)$$

such that the measures

$$\nu_n = \mu_{n,1} \boxplus_{RD} \mu_{n,2} \boxplus_{RD} \cdots \boxplus_{RD} \mu_{n,k_n}$$

have a weak limit ν, the measure ν must be infinitely divisible relative to the operation $\boxplus_{RD}$. We denote by $\mathcal{P}^+$ the family of all Borel probability measures defined on the positive half real line $\mathbb{R}_+$. The purpose of the present paper is to find conditions under which such an infinitesimal array does yield a sequence ν_n that converges weakly to a given $\boxplus_{RD}$-infinitely divisible measure ν. We restrict ourselves to the case in which each row $\{\mu_{n,j} \ : \ j = 1,\ldots,k_n\}$ consists of identically distributed measures and the result is closely related to earlier work [3]. More precisely,

Theorem 1.1. *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability measure on $[0,+\infty)$, let $k_1 < k_2 < \ldots$ be natural numbers, and set*

$$\nu_n = \underbrace{\mu_n \boxplus_{RD} \mu_n \boxplus_{RD} \cdots \boxplus_{RD} \mu_n}_{k_n \text{times}}, \qquad \nu_n' = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{k_n \text{times}}.$$

Then ν_n converges weakly to ν if and only if ν_n' converges weakly to ν'.

In fact, due to the free Lévy-Hinčin representation (Section 2), each $\boxplus_{RD}$-infinitely divisible measure or $\boxplus$-infinitely divisible measure can be uniquely determined by a free generating pair (γ,σ), where γ is a real constant, and σ is a finite Borel measure on $\mathbb{R}$. The above ν and ν' (appeared in the bijection in Theorem 1.1) share the same free generating pair, and could be denoted by $\nu = \nu_{\boxplus_{RD}}^{\gamma,\sigma}$ and $\nu' = \nu_{\boxplus}^{\gamma,\sigma}$. In the context of free generating pairs, there are more explicit conditions on μ_n that are equivalent to convergence of the array (Section 3). To be more accurate, set

$$\gamma_n = k_n \int \frac{x}{x^2+1} d\mu_n(x) \qquad \text{and} \qquad d\sigma_n(x) = \frac{k_n x^2}{x^2+1} d\mu_n(x),$$

then $\nu_n \longrightarrow \nu_{\boxplus_{RD}}^{\gamma,\sigma}$ weakly if and only if $\gamma_n \longrightarrow \gamma$ and $\sigma_n \longrightarrow \sigma$ weakly. Theorem 1.1 could also be proved using the connection between $\boxplus_{RD}$ and $\boxplus$ described in [6,7].

2. The Calculation of the $\boxplus_{RD}$ Convolution

As noted in the introduction, the convolution $\mu \boxplus_{RD} \nu$ of two measures $\mu,\nu \in \mathcal{P}^+$ is the distribution of $|X_1 + X_1|^2$, where X_1 and X_2 are $*$-free tracial R-diagonal random variables with distributions $\mu_{|X_1|^2} = \mu$ and $\mu_{|X_2|^2} = \nu$. We will not go through the detail of the definition for tracial R-diagonal variables, but refer readers to [9] for the bounded case and to [2,6,7] for the discussion of the unbounded case. We concentrate on analytic tools for the calculation of the $\boxplus_{RD}$ convolution in this section.

We first recall that there is an analogue of the Fourier transform that linearizes the usual free convolution of measures on $\mathcal{P}^+$. To do so, we denote by $\mathbb{C}$ the complex plane and by $\mathbb{R}$ the real line. For two numbers $\alpha, \beta > 0$ we introduce the domains

$$\Lambda_\alpha = \mathbb{C} \setminus \{\, z = x + iy \in \mathbb{C} \;:\; x > 0,\ \alpha|y| < |x| \,\}$$

and

$$\Lambda_{\alpha,\beta} = \{\, z = x + iy \in \mathbb{C} \;:\; |z| > \beta \,\} \cap \Lambda_\alpha.$$

Given a probability measure $\mu \in \mathcal{P}^+$, one defines the Cauchy transform

$$G_\mu(z) = \int_{[0,+\infty)} \frac{1}{z-t}\, d\mu(t), \qquad z \in \mathbb{C}\backslash\mathbb{R}_+.$$

The reciprocal $F_\mu(z) = 1/G_\mu(z)$ has a right inverse F_μ^{-1} defined in some truncated cone $\Lambda_{\alpha,\beta}$. The Voiculescu transform $\phi_\mu(z) = F_\mu^{-1}(z) - z$ of $\mu \in \mathcal{P}^+$ is defined in $\Lambda_{\alpha,\beta}$. It is known that $\lim_{n\to\infty} \phi_\mu(z)/z = 0$ as $|z| \to \infty$ and $z \in \Lambda_\alpha$. More importantly, the transform ϕ_μ linearizes $\boxplus$ the usual free convolution. In the following we only state this for measures in $\mathcal{P}^+$.

Theorem 2.1. *If $\mu, \nu \in \mathcal{P}^+$, then $\phi_{\mu\boxplus\nu} = \phi_\mu + \phi_\nu$ in any truncated cone $\Lambda_{\alpha,\beta}$ where all three functions involved are defined.*

This remarkable result (Theorem 2.1) was essentially discovered by Voiculescu [13] for compactly supported measures on the real line and his work was extended in [8] to measures with finite variance and in [5] to the general case (see also [14]).

We seek a function that could linearize the $\boxplus_{RD}$ convolution. Fortunately, one of the basic propositions [10, Proposition 15.6 on page 241] from the combinatorial theory of R-diagonal elements can be understood as presenting to us what the linearizing transform should be. The combinatorial setting is precisely introduced at [1, Section 2]. We state the analytic setting as "ϕ^{RD}-transform" in Theorem 2.2 below, after first reviewing the needed definition.

Following the notation in [2], one defines

$$\widetilde{G}_\mu(z) = \frac{1}{z}(1 + \phi_\mu(z)/z) \qquad \text{on } \Lambda_{\alpha,\beta}$$

and

$$\widetilde{F}_\mu(z) = 1/\widetilde{G}_\mu(z) = \frac{z}{1 + \phi_\mu(z)/z} \qquad \text{on } \Lambda_{\alpha,\beta}.$$

From [2], we know $\widetilde{F}_\mu(z)/z$ tends to 1 as $|z| \to \infty$ and $z \in \Lambda_\alpha$. Using this property and the similar argument in [5] one can show that for every $\alpha > 0$ there exists $\beta > 0$ such that $\widetilde{F}_\mu^{-1}$ is defined on $\Lambda_{\alpha,\beta}$. The function

$$\phi_\mu^{RD}(z) = \widetilde{F}_\mu^{-1}(z) - z \qquad \text{on } \Lambda_{\alpha,\beta} \tag{2.1}$$

has the property $\lim \phi_\mu^{RD}(z)/z = 0$ as $|z| \to \infty$ and $z \in \Lambda_{\alpha,\beta}$. The following nice result from [2] shows that the transform ϕ^{RD} linearizes the $\boxplus_{RD}$ convolution.

Theorem 2.2. *If $\mu, \nu \in \mathcal{P}^+$, then $\phi^{RD}_{\mu \boxplus_{RD} \nu}(z) = \phi^{RD}_\mu(z) + \phi^{RD}_\nu(z)$ in any truncated cone $\Lambda_{\alpha,\beta}$ where all three functions involved are defined.*

Arguments similar to those used in [3, Lemma 2.2] shows that

Lemma 2.3. *Let $\mu \in \mathcal{P}^+$, n a natural number, and set*

$$\mu_n = \underbrace{\mu \boxplus_{RD} \mu \boxplus_{RD} \cdots \boxplus_{RD} \mu}_{n \text{ times}}.$$

Assume that $\widetilde{F}^{-1}_{\mu_n}$ exists in a truncated cone $\Lambda_{\alpha,\beta}$. Then $\widetilde{F}^{-1}_\mu$ also exists in that same truncated cone.

If ν_n and ν are elements of $\mathcal{P}^+$, then we say that the sequence ν_n *converges weakly* to ν, if

$$\lim_{n\to\infty} \int_{[0,+\infty)} f(t)\, d\nu_n(t) = \int_{[0,+\infty)} f(t)\, d\nu(t),$$

for every bounded continuous function f on $\mathbb{R}_+$. We say that a sequence of tracial R-diagonal variables $\{X_n\}$ (generally, unbounded) *converges weakly*, or *converges in distribution* or *converges in law* to a tracial R-diagonal variable X, if the sequence of measures $\mu_{|X_n|^2} \in \mathcal{P}^+$ converges weakly to the measure $\mu_{|X|^2} \in \mathcal{P}^+$. Therefore, the weak convergence of tracial R-diagonal $*$-distributions can be rephrased in terms of the weak convergence of probability measures in $\mathcal{P}^+$.

In the study of limit laws, it is important to translate weak convergence of probability measures into convergence properties of the corresponding ϕ^{RD} transforms. This is achieved by the following result [2].

Proposition 2.4. *Let $\mu_n \in \mathcal{P}^+$ be a sequence of probability measures. The following assertions are equivalent:*

(i) *there exists a probability measure μ on $\mathbb{R}_+$ such that μ_n converges weakly to μ as $n \to \infty$;*

(ii) *there exist α, $\beta > 0$ such that the sequence $\phi^{RD}_{\mu_n}$ converges uniformly to φ on the compact subsets of $\Lambda_{\alpha,\beta}$, and $\lim \phi^{RD}_{\mu_n}(z)/z = 0$ uniformly in n as $|z| \to \infty$, $z \in \Lambda_{\alpha,\beta}$. In particular, $\phi^{RD}_{\mu_n}(x) = o(|x|)$ uniformly in n for $x \to -\infty$ and $x \in (-\infty, 0)$;*

(iii) *there exist α', $\beta' > 0$ such that functions $\phi^{RD}_{\mu_n}$ are defined on $\Lambda_{\alpha',\beta'}$ for every n, $\lim_{n\to\infty} \phi^{RD}_{\mu_n}(iy)$ exists for every $y > \beta'$, and $\lim \phi^{RD}_{\mu_n}(iy)/y = 0$ uniformly in n as $y \to \infty$.*

Moreover, if (i) *and* (ii) *are satisfied, then $\varphi = \phi^{RD}_\mu$ in $\Lambda_{\alpha,\beta}$.*

The actual definition of the transform ϕ^{RD} in equation (2.1) requires inverting an analytic function. In order to avoid calculating such inverse, we will use an approximation for ϕ^{RD} which enables us to present weak limit theorems directly in terms of Voiculescu transform and Cauchy transform. We start with a nice observation from [2].

Proposition 2.5. *Let μ belong to a tight family $\mathcal{F} \subset P^+$.*

(i) *For every* $\alpha > 0$ *there exists* $\beta > 0$ *such that* $\widetilde{F}_{\mu}^{-1}$ *(and hence* ϕ_{μ}^{RD}*) is defined in* $\Lambda_{\alpha,\beta}$ *for every* $\mu \in \mathcal{F}$.

(ii) *Let* α *and* β *be such that* ϕ_{μ}^{RD} *is defined in* $\Lambda_{\alpha,\beta}$ *for every* $\mu \in \mathcal{F}$*, and write*

$$\phi_{\mu}^{RD}(z) = z^2 \left[G_{\mu}(z) - 1/z\right](1 + w_{\mu}(z)) \qquad z \in \Lambda_{\alpha,\beta},\ \mu \in \mathcal{F}.$$

Then $\sup_{\mu \in \mathcal{F}} |w_{\mu}(z)| = 0$ *as* $|z| \to \infty$ *and* $z \in \Lambda_{\alpha,\beta}$.

For a sequence of measures converges weakly to the Dirac mass at zero, one has the following property [15].

Proposition 2.6. *Let* $\mu_n \in \mathcal{P}^+$ *be a sequence converging weakly to* δ_0*. Then there exist* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$\phi_{\mu_n}^{RD}(z) = z^2 \left[G_{\mu_n}(z) - 1/z\right](1 + w_n(z)) \qquad z \in \Lambda_{\alpha,\beta},$$

and

$$\lim_{n \to \infty} w_n(z) = 0, \qquad z \in \Lambda_{\alpha,\beta}.$$

3. The Lévy–Hinčin Formula for the $\boxplus_{RD}$ Convolution

An important class of measures connects with the study of limit laws of tracial R-diagonal variables in free probability. An R-diagonal $*$-distribution ν will be said to be $\boxplus$-*infinitely divisible* if for every natural number n, there exists an R-diagonal $*$-distribution μ_n such that

$$\nu = \underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ times}},$$

where $\boxplus$ denotes the usual additive free convolution of $*$-distributions. Analogously, a measure $\nu \in \mathcal{P}^+$ will be said to be $\boxplus_{RD}$-*infinitely divisible* if for every natural number n, there exists a measure $\mu_n \in \mathcal{P}^+$ such that

$$\nu = \underbrace{\mu_n \boxplus_{RD} \mu_n \boxplus_{RD} \cdots \boxplus_{RD} \mu_n}_{n \text{ times}}.$$

In fact, the usual free convolution $\boxplus$ of tracial R-diagonal distributions can be rephrased in terms of the $\boxplus_{RD}$ convolution of probability measures on $\mathbb{R}_+$ by the definition in Eq. (1.1). It follows that the $*$-distribution of a tracial R-diagonal variable X (generally, unbounded) is $\boxplus$-infinitely divisible if and only if $\mu_{|X|^2}$ is $\boxplus_{RD}$-infinitely divisible.

Before overviewing the characterization of $\boxplus_{RD}$-infinitely divisible measure, we need to recall the Nevanlinna representation [11, 5.3] of $\boxplus$-infinitely divisible measures. A characterization of $\boxplus$-infinitely divisible measures $\nu \in \mathcal{P}^+$ is given by the following free analogue of the Lévy-Hinčin formula (Theorem 3.1). The original formula in [4,5] is developed for measures on $\mathbb{R}$. It can be connected to the Lévy-Hinčin formula in classical probability (see the deep analogy between the Gaussian measure and the semicircle law [12]). However, here in Theorem 3.1 we state it in a different domain because we are interested in measures on $\mathbb{R}_+$ related to tracial R-diagonal variables.

Theorem 3.1. *A measure $\nu \in \mathcal{P}^+$ is $\boxplus$-infinitely divisible if and only if there exist a finite positive Borel measure σ on $\mathbb{R}_+$ and a real number γ such that the Voiculescu transform has an analytic continuation in the following form*

$$\phi_\nu(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma(t) \qquad z \in \mathbb{C} \setminus \mathbb{R}_+.$$

In the rest of this section we will introduce a characterization of $\boxplus_{RD}$-infinitely divisible measure $\nu \in \mathcal{P}^+$ which is given by the $\boxplus_{RD}$ analogue of the Lévy-Hinčin formula. The following result is first observed in [1].

Theorem 3.2. *A measure $\nu \in \mathcal{P}^+$ is $\boxplus_{RD}$-infinitely divisible if and only if there exist a finite positive Borel measure σ on $\mathbb{R}_+$ and a real number γ such that the ϕ_ν^{RD} transform has an analytic continuation in the following form*

$$\phi_\nu^{RD}(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma(t) \qquad z \in \mathbb{C} \setminus \mathbb{R}_+.$$

Example. The connection between Theorem 3.1 and Theorem 3.2 can be seen more clearly from examples. For instance, if $\gamma = 1$ and $\sigma = 0$, then Theorem 3.1 yields $\nu = \delta_1$ (the Dirac mass at 1), meanwhile Theorem 3.2 gives $\nu = \Pi_{1;1}$ (the Marchenko-Pastur distribution of parameter (1,1)). Such measure $\Pi_{1;1}$ in Theorem 3.2 can be further rephrased at the level of tracial R-diagonal elements. Indeed, let c be the standard circular element (a type of tracial R-diagonal variables) in a C^*-probability space. Then the distribution of $|c|^2$ is $\Pi_{1;1}$. Moreover, the work [1] shows that the measure $\Pi_{1;1} = \mu_{|c|^2}$ actually corresponds to the central limit for the $\boxplus_{RD}$ convolution.

Before giving another illustration of Theorem 3.2, we need two more definitions. We denote $\boxtimes$ the usual multiplicative free convolution. Measures $\mu \in \mathcal{P}^+$ have *free additive convolutions powers* with real exponent $t \in [1, +\infty)$. More precisely, for $\mu \in \mathcal{P}^+$ and $t \in [1, +\infty)$, there exists a unique measure $\nu \in \mathcal{P}^+$, such that $\phi_\nu = t\phi_\mu$, denoted by $\nu = \mu^{\boxplus t}$. When t is an integer, $\mu^{\boxplus t}$ is the t-fold convolution $\mu \boxplus \cdots \boxplus \mu$. We refer the reader to [10, pp. 228-231] for the detail about compactly supported measures on $\mathbb{R}_+$.

Example. Another special case of the analogy between Theorems 3.1 and 3.2 is the following. Suppose $\gamma = \sqrt{p-1}$ with $p > 2$ and $\sigma = \delta_q$ with $q = 1/\sqrt{p-1}$. Then, on one hand, the formula in Theorem 3.1 yields $\nu = \Pi_{p;q}$ the free Poisson distribution of rate p and jump size q (see e.g. in [10, Proposition 12.11]). On the other hand, the one in Theorem 3.2 gives $\nu = (\Pi_{pq;1} \boxtimes \Pi_{q;1})^{\boxplus 1/q}$ corresponding to the "Possion" R-diagonal variable b such that the distribution of $|b|^2$ is ν (by the work [1]). It is known that the determining series of the tracial R-diagonal variable b is $p\sum_{n=1}^{\infty} q^n z^n$.

We begin with a proof of Theorem 3.2.

Proof of Theorem 3.2. Part (1) Assume that ν is a $\boxplus_{RD}$-infinitely divisible measure in $\mathcal{P}^+$. We set the measure $\nu_n \in \mathcal{P}^+$ to satisfy the condition:

$$\nu = \underbrace{\nu_n \boxplus \cdots \boxplus \nu_n}_{n \text{ times}}$$

for natural numbers n. Fix a truncated angle $\Lambda_{\alpha,\beta}$ where ϕ_{ν}^{RD} and $\phi_{\nu_n}^{RD}$ exist. By the $\boxplus_{RD}$-infinitely divisible property, we have $\phi_{\nu_n}^{RD}(z) = \phi_{\nu}^{RD}(z)/n \to 0$ as $n \to \infty$, for all $z \in \Lambda_{\alpha,\beta}$. Hence, $\nu_n \to \delta_0$ as $n \to \infty$. From Proposition 2.6 we know for all $z \in \Lambda_{\alpha,\beta}$

$$\phi_{\nu}^{RD}(z) = n\phi_{\nu_n}^{RD}(z) = n\left[z^2\left(G_{\nu_n}(z) - \frac{1}{z}\right)\right](1 + w_n(z)), \tag{3.1}$$

where $\lim_{n\to\infty} w_n(z) = 0$. We define

$$\gamma_n = n\int_{[0,+\infty)} \frac{t}{t^2+1}d\nu_n(t), \qquad d\rho_n(t) = \frac{nt^2}{t^2+1}d\nu_n(t),$$

and rewrite

$$n\left[z^2\left(G_{\nu_n}(z) - \frac{1}{z}\right)\right] = n\int_{[0,+\infty)} \frac{t}{t^2+1}d\nu_n(t) + \int_{[0,+\infty)} \frac{1+tz}{z-t}\frac{nt^2}{t^2+1}d\nu_n(t)$$

$$= \gamma_n + \int_{[0,+\infty)} \frac{1+tz}{z-t}d\rho_n(t). \tag{3.2}$$

First of all, we can show the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is bounded. Indeed, from equations (3.1) and (3.2), for $z \in \Lambda_{\alpha,\beta}$

$$\begin{aligned}\phi_{\nu}^{RD}(z) &= \lim_{n\to\infty} \frac{\phi_{\nu}^{RD}(z)}{1+w_n(z)}\\ &= \lim_{n\to\infty} n\left[z^2\left(G_{\nu_n}(z) - \frac{1}{z}\right)\right]\\ &= \lim_{n\to\infty} n\int_{[0,+\infty)} \frac{zt}{z-t}d\nu_n(t).\end{aligned}$$

Taking $z = i$ and taking the imaginary part of the above equation, we obtain

$$-\mathfrak{Im}\,\phi_{\nu}^{RD}(i) = \lim_{n\to\infty}\int_{[0,+\infty)} \frac{nt^2}{t^2+1}\,d\nu_n(t).$$

Thus, the family $\{\rho_n\}_{n\in\mathbb{N}}$ is bounded. Secondly we can prove $\{\rho_n\}_{n\in\mathbb{N}}$ is a tight family. Indeed, for $z = iy \in \Lambda_{\alpha,\beta}$, we observe from equation (3.2) that

$$\begin{aligned}\frac{1}{2}\rho_n(\{t : t \geq y\}) &= \int_{t\geq y} \frac{t^2+y^2}{2(t^2+y^2)}d\rho_n(t)\\ &\leq \int_{[0,+\infty)} \frac{t^2+t^2}{2(t^2+y^2)}d\rho_n(t)\\ &= \int_{[0,+\infty)} \frac{t^2}{t^2+y^2}d\rho_n(t)\\ &\leq \int_{[0,+\infty)} \frac{t^2+1}{t^2+y^2}d\rho_n(t)\\ &= -\frac{n}{y}\mathfrak{Im}\left((iy)^2\left[G_{\nu_n}(iy) - \frac{1}{iy}\right]\right).\end{aligned} \tag{3.3}$$

Using Proposition 2.5, we observe that the function $w_n(z)$ in equation (3.1) satisfy $w_n(z) = o(1)$ as $|z| \to \infty, z \in \Lambda_{\alpha,\beta}$. As a result, by decreasing the cone we may assume that $|w_n(iy)| < 1/2$, and we deduce that

$$\left| n(iy)^2 \left[G_{\nu_n}(iy) - \frac{1}{iy} \right] \right| \leq \frac{|\phi_\nu^{RD}(iy)|}{|1 + w_n(iy)|} \leq 2|\phi_\nu^{RD}(iy)|. \tag{3.4}$$

Since $\lim_{y\to\infty} \phi_\nu^{RD}(iy)/y = 0$, inequalities (3.3) and (3.4) imply that the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is tight. Hence, it has weakly convergent subsequences. If a subsequence ρ_{n_j} converges weakly to a measure ρ', then from equations (3.1) and (3.2) for $z = iy$ we have

$$\begin{aligned}
\int_{[0,+\infty)} \frac{y}{y^2+t^2}(1+t^2)d\rho'(t) &= \lim_{j\to\infty} \int_{[0,+\infty)} \frac{y}{y^2+t^2}(1+t^2)d\rho_{n_j}(t) \\
&= \lim_{j\to\infty} -n_j \mathfrak{Im}\left((iy)^2 \left[G_{\nu_{n_j}}(iy) - \frac{1}{iy} \right] \right) \\
&= \lim_{j\to\infty} \frac{-\mathfrak{Im}\phi_\nu^{RD}(iy)}{(1 + w_{n_j}(iy))} \\
&= -\mathfrak{Im}\phi_\nu^{RD}(iy)
\end{aligned}$$

This implies that if another subsequence ρ_{n_k} converges weakly to a measure σ, then $(1+t^2)d\rho'(t)$ and $(1+t^2)d\sigma(t)$ must have the same Poisson integral (in a open subset of imagine line so everywhere in $\mathbb{C}\backslash\mathbb{R}_+$), and this can only happen if $\rho' = \sigma$. Hence the sequence ρ_n converges weakly to a unique σ. Next, we show the limit of γ_n exists. Indeed, since the continuous function $(1+tz)/(z-t)$ stays bounded for $z \in \mathbb{C}\backslash\mathbb{R}_+$,

$$\lim_{n\to\infty} \int_{[0,+\infty)} \frac{1+tz}{z-t} d\rho_n(t)$$

exists and equals to $\int_{[0,+\infty)} (1+tz)/(z-t)d\sigma(t)$. Consequently, equations (3.1) and (3.2) imply that

$$\begin{aligned}
\lim_{n\to\infty} \gamma_n &= \lim_{n\to\infty} \left[n \left[z^2 \left(G_{\nu_n}(z) - \frac{1}{z} \right) \right] - \int_{[0,+\infty)} \frac{1+tz}{z-t} d\rho_n(t) \right] \\
&= \phi_\nu^{RD}(z) - \lim_{n\to\infty} \int_{[0,+\infty)} \frac{1+tz}{z-t} d\rho_n(t)
\end{aligned}$$

also exists. Denote by γ the limit of γ_n. Finally, we have the result that ϕ_ν^{RD} has an analytic continuation (still denoted ϕ_ν^{RD}) such that

$$\phi_\nu^{RD}(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t} d\sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}_+.$$

Part (2): Conversely, suppose that $\phi_\nu^{RD}(z)$ on some truncated cone has a analytic continuation $\psi(z)$ such that

$$\psi(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t} d\sigma(t), \qquad \text{on } z \in \mathbb{C} \setminus \mathbb{R}_+.$$

Then the conclusion follows from [15, Theorem 1.4]. □

4. Limit Laws for Tracial *R*-diagonal Variables

The purpose of this section is to prove the result announced in the introduction on partial domains of attraction. We begin by showing that the analogy between Theorem 3.1 and Theorem 3.2 extends to all infinitely divisible measures in $\mathcal{P}^+$.

In order to simplify the statement of our result, we introduce some additional notation. Fix a finite positive Borel measure σ on $\mathbb{R}_+$ and a real number γ. We denote by $\nu_{\boxplus}^{\gamma,\sigma}$ the $\boxplus$-infinitely divisible such that

$$\phi_{\nu_{\boxplus}^{\gamma,\sigma}}(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma(t), \qquad z \in \mathbb{C} \setminus \mathbb{R}_+,$$

and we denote by $\nu_{RD}^{\gamma,\sigma}$ the $\boxplus_{RD}$-infinitely divisible measure determined by the formula

$$\phi^{RD}_{\nu_{RD}^{\gamma,\sigma}}(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma(t), \qquad z \in \mathbb{C} \setminus \mathbb{R}_+.$$

The following result is essentially contained in [3]. Here we are interested in the version for measures on $\mathbb{R}_+$. From the original proof [3], it is not hard to obtain Theorem 4.1 for measures in $\mathcal{P}^+$ by writing Cauchy and Voiculescu transforms on a different truncated cone.

Theorem 4.1. *Let $\nu_{\boxplus}^{\gamma,\sigma}$ be a $\boxplus$-infinitely divisible measure, let $\mu_n \in \mathcal{P}^+$ be a sequence, and let $k_1 < k_2 < \dots$ be natural numbers. The following assertions are equivalent:*

(i) *the sequence* $\underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{k_n \text{ times}}$ *converges weakly to* $\nu_{\boxplus}^{\gamma,\sigma}$*;*

(ii) *the measures*

$$d\sigma_n(x) = k_n \frac{x^2}{x^2+1}\, d\mu_n(x)$$

converge weakly to σ and

$$\lim_{n\to\infty} k_n \int_{[0,+\infty)} \frac{x}{x^2+1}\, d\mu_n(x) = \gamma.$$

We are now ready to state the first main result of this section.

Theorem 4.2. *Fix a finite positive Borel measure σ on $\mathbb{R}_+$, a real number γ, a sequence $\mu_n \in \mathcal{P}^+$, and a sequence of positive integers $k_1 < k_2 < \dots$. The following assertions are equivalent:*

(i) *the sequence* $\underbrace{\mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n}_{k_n \text{ times}}$ *converges weakly to* $\nu_{\boxplus}^{\gamma,\sigma}$*;*

(ii) *the sequence* $\underbrace{\mu_n \boxplus_{RD} \mu_n \boxplus_{RD} \cdots \boxplus_{RD} \mu_n}_{k_n \text{ times}}$ *converges weakly to* $\nu_{RD}^{\gamma,\sigma}$*;*

(iii) *the measures*

$$d\sigma_n(x) = k_n \frac{x^2}{x^2+1}\, d\mu_n(x)$$

converge weakly to σ *and*

$$\lim_{n\to\infty} k_n \int_{[0,+\infty)} \frac{x}{x^2+1}\, d\mu_n(x) = \gamma.$$

Proof. The equivalence of (i) and (iii) is just a restatement of Theorem 4.1. We will prove the equivalence of (ii) and (iii). Assume first that (ii) holds. For simplicity, we denote by

$$\nu_n = \underbrace{\mu_n \boxplus_{RD} \mu_n \boxplus_{RD} \cdots \boxplus_{RD} \mu_n}_{k_n \text{ times}}.$$

The limit $\lim_{n\to\infty} \nu_n$ exists and is $\boxplus_{RD}$-infinitely divisible (by [15]). Therefore, the notation $\nu_{RD}^{\gamma,\sigma}$ is reasonable. By Lemma 2.3, there exists a truncated cone $\Lambda_{\alpha,\beta}$ such that both $\phi_{\mu_n}^{RD}(z)$ and $\phi_{\nu_n}^{RD}(z)$ are defined,

$$\lim_{n\to\infty} k_n \phi_{\mu_n}^{RD}(z) = \phi_{\nu_{RD}^{\gamma,\sigma}}^{RD}(z) \qquad z \in \Lambda_{\alpha,\beta},$$

and $\left|k_n \phi_{\mu_n}^{RD}(z)\right| \le v(z)$ with $\lim_{|z|\to\infty, z\in\Lambda_\alpha} v(z)/z = 0$ (by Proposition 2.4). Then we have

$$\left|\phi_{\mu_n}^{RD}(z)\right| \le \frac{v(z)}{k_n} \underset{n\to\infty}{\longrightarrow} 0.$$

Therefore, the sequence μ_n converges weakly to δ_0 by Proposition 2.4 (indeed $\phi_{\delta_0}^{RD} = 0$). From Proposition 2.5, we get

$$\phi_{\mu_n}^{RD}(z) = z^2 \left[G_{\mu_n}(z) - \frac{1}{z}\right](1 + w_n(z))$$

where $|w_n(z)| \le w(z)$ and $\lim_{|z|\to\infty, z\in\Lambda_\alpha} w(z) = 0$ that is

$$\sup_{n\in\mathbb{N}} |w_n(z)| = o(1), \qquad \text{as } |z| \to \infty,\ z \in \Lambda_\alpha.$$

Using Proposition 2.5, we conclude that

$$k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z}\right] = \frac{k_n \phi_{\mu_n}^{RD}(z)}{(1 + w_n(z))} \underset{n\to\infty}{\longrightarrow} \phi_{\nu_{RD}^{\gamma,\sigma}}^{RD}(z), \quad z \in \Lambda_{\alpha,\beta}. \tag{4.1}$$

By decreasing the cone we may assume $w(z) < 1/2$, and thus

$$\left|k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z}\right]\right| \le \frac{v(z)}{|1 - w(z)|} \le 2v(z), \quad z \in \Lambda_{\alpha,\beta}. \tag{4.2}$$

Observe that for $z \in \mathbb{C} \setminus \mathbb{R}_+$

$$\begin{aligned} k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z}\right] &= \int_{[0,+\infty)} \frac{k_n t z}{z - t}\, d\mu_n(t) \\ &= k_n \int_{[0,\infty)} \frac{t}{t^2+1}\, d\mu_n(t) + \int_{[0,\infty)} \frac{1 + tz}{z - t} \frac{k_n t^2}{t^2+1}\, d\mu_n(t). \end{aligned} \tag{4.3}$$

We denote

$$\gamma_n = k_n \int_{[0,+\infty)} \frac{t}{t^2+1}\, d\mu_n(t)$$

$$d\sigma_n = \frac{k_n t^2}{t^2+1}\, d\mu_n(t)$$

From inequality (4.2) for $z = iy, y \in \mathbb{R}_+$, we see that

$$\begin{aligned}
\frac{1}{2}\sigma_n(\{t : t \geq y\}) &= \frac{1}{2}\int_{[y,+\infty)} \frac{y^2+t^2}{y^2+t^2}\, d\sigma_n(t) \\
&\leq \int_{[0,+\infty)} \frac{1+t^2}{y^2+t^2}\, d\sigma_n(t) \\
&= -\frac{k_n}{y}\mathfrak{Im}\, \left(z^2\left[G_{\mu_n}(z) - \frac{1}{z}\right]\right) \\
&\leq \frac{2v(iy)}{y} \\
&= \frac{2iv(iy)}{iy}.
\end{aligned}$$

The sequence $\{\sigma_n\}$ is uniformly bounded. In addition, $\lim_{y\to\infty} v(iy)/(iy) = 0$ (see the earlier discussion). We have that $\{\sigma_n\}$ is a tight sequence. From equation (4.3) and inequality (4.2) for $z = i$, we observe

$$\left|\int_{[0,+\infty)} d\sigma_n(t)\right| = \left|-k_n\mathfrak{Im}\, \left(i^2\left[G_{\mu_n}(i) - \frac{1}{i}\right]\right)\right| \leq 2v(i).$$

Hence, there exists a weakly convergent subsequence, say $\{\sigma_{n_j}\}$. Let σ' be free weak limit of $\{\sigma_{n_j}\}$. As in [3, Theorem 3.4], taking $z = x + iy \in \Lambda_{\alpha,\beta}$, then equation (4.1) and Theorem 3.2 imply

$$\begin{aligned}
\int_{[0,+\infty)} \frac{y}{y^2+(x-t)^2}(1+t^2)\, d\sigma'(t) &= \lim_{j\to\infty}\int_{[0,+\infty)} \frac{y(1+t^2)}{y^2+(x-t)^2}\, d\sigma_{n_j}(t) \\
&= -\lim_{j\to\infty} k_{n_j}\mathfrak{Im}\, \left(z^2\left[G_{\mu_n}(z) - \frac{1}{z}\right]\right) \\
&= -\mathfrak{Im}\, \phi^{RD}_{\nu^{\gamma,\sigma}_{RD}}(z) \\
&= -\mathfrak{Im}\, \int_{[0,+\infty)} \frac{1+t(x+iy)}{(x-t)+iy}\, d\sigma(t) \\
&= \int_{[0,+\infty)} \frac{y}{y^2+(x-t)^2}(t^2+1)\, d\sigma(t).
\end{aligned}$$

Hence, $(1+t^2)\, d\sigma'(t)$ and $(1+t^2)\, d\sigma(t)$ have the same Poisson integral (in an open set and hence everywhere of $\mathbb{C}\backslash\mathbb{R}_+$), and this can only happen when $\sigma' = \sigma$. This means that all weakly convergent subsequences of $\{\sigma_n\}$ have the same limit. Thus, we conclude that the sequence $\{\sigma_n\}$ converges weakly to σ. Finally,

$$\begin{aligned}\gamma &= \phi^{RD}_{\nu^{\gamma,\sigma}_{RD}}(z) - \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma(t)\\ &= \lim_{n\to\infty} \left\{ \phi^{RD}_{\nu^{\gamma,\sigma}_{RD}} - \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma_n(t) \right\}\\ &= \lim_{n\to\infty} \left\{ \gamma_n + \phi^{RD}_{\nu^{\gamma,\sigma}_{RD}}(z) - k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z} \right] \right\}.\end{aligned}$$

Since

$$\phi^{RD}_{\nu^{\gamma,\sigma}_{RD}}(z) - k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z} \right] \underset{n\to\infty}{\longrightarrow} 0,$$

we see that $\lim_{n\to\infty} \gamma_n$ exists and equals γ.

Conversely, assume that (iii) holds. Since

$$\mu_n(\{t : |t| > \varepsilon\}) \le \frac{1+\varepsilon^2}{\varepsilon^2} \int_{[0,+\infty)} \frac{t^2}{1+t^2} d\mu_n(t) \le \frac{1+\varepsilon^2}{\varepsilon^2} \frac{1}{k_n} \sigma_n(\mathbb{R}_+)$$

for every $\varepsilon > 0$, we deduce that the sequence μ_n converges weakly to δ_0. By Proposition 2.5 and 2.6 there exists a truncated cone $\Lambda_{\alpha,\beta}$ such that

$$\phi^{RD}_{\mu_n}(z) = z^2 \left[G_{\mu_n}(z) - \frac{1}{z} \right] (1 + w_n(z)),$$

where $\lim_{n\to\infty} w_n(z) = 0$, $|w_n(z)| \le w(z)$, and $\lim_{z\to\infty, z\in\Lambda_\alpha} w(z) = 0$. Using (4.3), we have

$$\begin{aligned}k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z} \right] &= \gamma_n + \int_{[0,\infty)} \frac{1+tz}{z-t}\, d\sigma_n(t)\\ &\underset{n\to\infty}{\longrightarrow} \gamma + \int_{[0,\infty)} \frac{1+tz}{z-t}\, d\sigma(t)\end{aligned}$$

for $z \in \Lambda_{\alpha,\beta}$ via condition (iii). This immediately implies that

$$\lim_{n\to\infty} k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z} \right] = \phi^{RD}_{\nu^{\gamma,\sigma}_{RD}}(z).$$

Hence for all $z \in \Lambda_{\alpha,\beta}$,

$$\phi^{RD}_{\nu_n}(z) = k_n \phi^{RD}_{\mu_n}(z) = k_n z^2 \left[G_{\mu_n}(z) - \frac{1}{z} \right] (1 + w_n(z)) \underset{n\to\infty}{\longrightarrow} \phi^{RD}_{\nu^{\gamma,\sigma}_{RD}}(z),$$

that is, $\lim_{n\to\infty} \phi^{RD}_{\nu_n}(z)$ exists. From Proposition 2.4 (iii), to conclude the proof it suffices to show that

$$k_n (iy)^2 \left[G_{\mu_n}(iy) - \frac{1}{iy} \right] = o(y), \quad \text{as } y \to +\infty,$$

uniformly in n. Since γ_n is a bounded sequence, we have $\lim_{y\to+\infty} \gamma_n/y = 0$ uniformly in n. This leaves us to show that

$$\int_{[0,+\infty)} \frac{1+ity}{iy-t}\, d\sigma_n(t) = o(y), \quad \text{as } y \to +\infty,$$

uniformly in n. It is not hard to verify that for $y > 1$,

$$|(1+ity)(-iy-t)| \leq \sqrt{1+t^2y^2}\sqrt{y^2+t^2} \leq y(t^2+y^2),$$

$$\left|\frac{1+ity}{iy-t}\right| \leq y, \qquad \text{and} \qquad \left|\frac{1}{iy-t}\right| \leq \frac{\sqrt{t^2+y^2}}{t^2+y^2} \leq \sqrt{2}\frac{1}{y+t}.$$

These inequalities immediately imply that if $M > 0$ and $y > 1$, then

$$\left|\int_{[0,+\infty)} \frac{1+ity}{iy-t}\, \sigma_n(t)\right| \leq \int_{[0,M)} \sqrt{2}\frac{1+ty}{y+t}\, d\sigma_n(t) + y\sigma_n(\{t:\ t \geq M\}).$$

Since the function $(1+ty)/(y+t)$ is monotone increasing for any constant $y > 1$, we have

$$\left|\int_{[0,+\infty)} \frac{1+ity}{iy-t}\, \sigma_n(t)\right| \leq \sqrt{2}\frac{1+My}{y+M} + y\sigma_n(\{t:\ t \geq M\}).$$

The desired conclusion follows from the assumption that σ_n is a tight sequence. □

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