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Hinčin's Theorem for Additive Free Convolutions of Tracial R-Diagonal $*$-Distributions

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Abstract

Hinčin proved that any limit law associated with a triangular array of uniformly infinitesimal random variables is infinitely divisible. Analogous results for the additive and multiplicative free convolution were proved by Bercovici, Belinschi and Pata. We prove an analogous result for the $\boxplus_{RD}$ convolution of measures defined on the positive half-line. This is the convolution arising from the addition of $*$-free R-diagonal elements of a tracial, noncommutative probability space.

Keywords Free probability · R-diagonal · Infinitely divisible

1 Introduction

A measure μ on the real line $\mathbb{R}$ is said to be infinitely divisible relative to the classical convolution $*$ if for every natural number n, there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n * \mu_n * \cdots * \mu_n}_{n \text{ times}}.$$

Hinčin [1] characterized $*$-infinitely divisible measures as all the possible weak limit of sequences of the form $\delta_{c_n} * \mu_{n1} * \mu_{n2} * \cdots * \mu_{nk_n}$, where δ_{c_n} is the Dirac mass at $c_n \in \mathbb{R}$, and probability measures μ_{nj} are an infinitesimal array, in the sense that for

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every $\varepsilon > 0$,

$$\lim_{n\to\infty} \min \{ \mu_{nj}((-\varepsilon, \varepsilon)) : 1 \leq j \leq k_n \} = 1.$$

Analogous results to Hinčin's theorem were proved in [2,3], where the classical convolution $*$ is replaced by the additive free convolution $\boxplus$ and the multiplicative free convolution $\boxtimes$, respectively.

One can also define the $\boxplus_{RD}$ convolution [4,5] $\mu \boxplus_{RD} \nu$ of two measures μ, ν defined on the positive half-line. This operation $\boxplus_{RD}$ arises from the addition of $*$-free R-diagonal elements of a tracial noncommutative probability space. A bounded random variable in a tracial noncommutative probability space called R-diagonal if its R-transform has a special diagonal form [6]. The definition of R-diagonality was extended by Haagerup and Schultz in [5] to unbounded operators affiliated with a finite von Neumann algebra. Two such random variables T_1 and T_2 are $*$-free if the sets $\{T_1, T_1^*\}$ and $\{T_2, T_2^*\}$ are free in the sense of Voiculescu [7]. It was first observed in [4,5] that the measure $\mu \boxplus_{RD} \nu$ is the probability distribution of $|T_1 + T_2|^2$ where T_1 and T_2 are two $*$-free tracial R-diagonal variables such that $\mu_{|T_1|^2} = \mu$ and $\mu_{|T_2|^2} = \nu$. Note that the sum of $*$-free tracial R-diagonal variables is still tracial R-diagonal [8]. Moreover, the $*$-distribution of every R-diagonal variable T is uniquely determined by the distribution of positive variable $T^*T = |T|^2$.

It is our purpose to show that the analogue of Hinčin's theorem holds for this operator $\boxplus_{RD}$. As observed above, this result can be rephrased at the level of tracial R-diagonality. In other words, every weak limit of sums of uniformly infinitesimal $*$-free tracial R-diagonal variables (generally unbounded) is necessarily $\boxplus$-infinitely divisible. The appropriate notion of an infinitesimal array is defined in Sect. 2.

Even though our approach in this paper is similar to Hinčin's theorem for additive free convolutions [2], there are technical differences between the calculation of the $\boxplus_{RD}$ convolution and the calculation of the additive free convolution.

2 Basic Ingredients

The analogue of the Fourier transform for the $\boxplus_{RD}$ convolution was described in [9]. We denote by $\mathbb{C}$ the complex plane, by $\mathbb{C}^+$ the upper half-plane, by $\mathbb{C}^-$ the lower half-plane, and by $\mathcal{P}^+$ the family of all Borel probability measures defined on the positive half-line $\mathbb{R}_+ = [0, +\infty)$. For two numbers $\alpha, \beta > 0$ we introduce the domains

$$\Lambda_\alpha = \mathbb{C}\backslash\{ z = x + iy \in \mathbb{C} \ : \ x > 0, \ \alpha|y| < |x| \}$$

and

$$\Lambda_{\alpha,\beta} = \{ z = x + iy \in \mathbb{C} \ : \ |z| > \beta \} \cap \Lambda_\alpha.$$

Given a probability measure $\mu \in \mathcal{P}^+$, one defines the Cauchy transform

$$G_\mu(z) = \int_{[0,+\infty)} \frac{1}{z-t}\, d\mu(t), \qquad z \in \mathbb{C} \setminus \mathbb{R}_+.$$

The reciprocal $F_\mu(z) = 1/G_\mu(z)$ has a right inverse F_μ^{-1} defined in some truncated cone $\Lambda_{\alpha,\beta}$. The Voiculescu transform $\phi_\mu(z) = F_\mu^{-1}(z) - z$ is defined in $\Lambda_{\alpha,\beta}$. One can also define

$$\widetilde{G}_\mu(z) = \frac{1}{z}(1 + \phi_\mu(z)/z), \qquad \widetilde{F}_\mu(z) = 1/\widetilde{G}_\mu(z) = \frac{z}{1 + \phi_\mu(z)/z} \qquad \text{on } \Lambda_{\alpha,\beta}.$$

As observed in [9], $\widetilde{F}_\mu$ has a right inverse $\widetilde{F}_\mu^{-1}$ defined in some truncated cone $\Lambda_{\alpha',\beta'}$. Moreover, the function

$$\phi_\mu^{RD}(z) = \widetilde{F}_\mu^{-1}(z) - z \qquad \text{on } \Lambda_{\alpha',\beta'} \tag{1}$$

has the property

$$\lim_{|z|\to\infty} \phi_\mu^{RD}(z)/z = 0 \qquad \text{for } z \in \Lambda_{\alpha',\beta'}.$$

The fundamental result proved in [9] is that $\phi^{RD}_{\mu \boxplus_{RD} \nu}(z) = \phi_\mu^{RD}(z) + \phi_\nu^{RD}(z)$ in any truncated cone $\Lambda_{\alpha',\beta'}$ where all three functions involved are defined.

Weak convergence of probability measures on $\mathbb{R}_+$ can be translated [9] to convergence properties of the function ϕ^{RD} for the $\boxplus_{RD}$ convolution. First of all, we say that a sequence μ_n of probability measures on $\mathbb{R}_+$ converges weakly to a measure $\mu \in \mathcal{P}^+$, if $\lim_{n\to\infty} \int_{[0,\infty)} f(t) d\mu_n(t) = \int_{[0,\infty)} f(t) d\mu(t)$ for every bounded continuous function f on $\mathbb{R}_+$. Then the translation works in the following way. Given probability measures μ and μ_n, $n \geq 1$ on $\mathbb{R}_+$, the sequence μ_n converges weakly to μ if and only if $\phi_{\mu_n}^{RD} \to \phi_\mu^{RD}$ uniformly on the compact subsets of $\Lambda_{\alpha,\beta}$ for some $\alpha, \beta > 0$, and $\phi_{\mu_n}^{RD}(z)/z \to 0$ uniformly in n as $|z| \to \infty$, $z \in \Lambda_{\alpha,\beta}$.

The actual definition of the function ϕ^{RD} in Eq. (1) involves inverting an analytic function. To avoid calculating such inverse, we introduce the following approximation of ϕ^{RD}, first observed by [9]. Such an observation enables us to present weak limit theorem directly in terms of Voiculescu transform and Cauchy transform.

Proposition 2.1 *Let μ belong to a tight family $\mathcal{F} \subset P^+$.*

(i) *For every $\alpha > 0$ there exists $\beta > 0$ such that $\widetilde{F}_\mu^{-1}$ (and hence ϕ_μ^{RD}) is defined in $\Lambda_{\alpha,\beta}$ for every $\mu \in \mathcal{F}$.*

(ii) *Let α and β be such that ϕ_μ^{RD} is defined in $\Lambda_{\alpha,\beta}$ for every $\mu \in \mathcal{F}$, and write*

$$\phi_\mu^{RD}(z) = z^2 \left[G_\mu(z) - 1/z\right] \left(1 + w_\mu(z)\right) \qquad z \in \Lambda_{\alpha,\beta},\ \mu \in \mathcal{F}.$$

Then $\sup_{\mu \in \mathcal{F}} |w_\mu(z)| = 0$ *as* $|z| \to \infty$ *and* $z \in \Lambda_{\alpha,\beta}$.

The preceding statement can be made over a sequence of measures converges weakly to the Dirac mass at zero.

Proposition 2.2 *Let $\mu_n \in \mathcal{P}^+$ be a sequence converging weakly to δ_0. Then there exist $\alpha > 0$ and $\beta > 0$ such that*

$$\phi^{RD}_{\mu_n}(z) = z^2 \left[G_{\mu_n}(z) - 1/z\right](1 + w_n(z)) \qquad z \in \Lambda_{\alpha,\beta},$$

and $\lim_{n\to\infty} w_n(z) = 0,\ z \in \Lambda_{\alpha,\beta}$.

Proof As in Proposition 2.1, it is possible to find a truncated cone $\Lambda_{\alpha,\beta}$ such that ϕ_{μ_n} has the above representation for $\Lambda_{\alpha,\beta}$. Therefore, we only need to prove the last assertion.

We know that $\lim_{n\to\infty} \phi^{RD}_{\mu_n} = 0$ uniformly on the compact subsets of $\Lambda_{\alpha,\beta}$ for some $\alpha, \beta \in (0, +\infty)$. In addition, we know that $\phi^{RD}_{\mu_n}(z) = o(|z|)$ uniformly in n for $|z| \to \infty$ and $z \in \Lambda_{\alpha,\beta}$. Therefore, for a fixed $\varepsilon > 0$, there exist $\alpha' < \alpha$ and $\beta' > \beta$ such that for every $w \in \Lambda_{\alpha',\beta'}$ and n big enough, $|\phi^{RD}_{\mu_n}(w)| < \varepsilon|w|$. This implies for some fixed $k > 0$ (independent of ε) the derivative

$$\left|\left(\phi^{RD}_{\mu_n}\right)'(w)\right| < k\varepsilon \qquad \text{for } n \text{ big enough.}$$

Moreover, since the sequence μ_n converges weakly to δ_0, it is also clear that $\widetilde{F}_{\mu_n}(z) \to z$ as $n \to \infty$. Therefore, the domain joining z and $\widetilde{F}_{\mu_n}(z)$ is entirely contained in $\Lambda_{\alpha',\beta'}$ for n big enough. Proceeding as in the proof of [10, Proposition 2.5], we have from (1)

$$\begin{aligned}
\phi^{RD}_{\mu_n}(z) &= \widetilde{F}^{-1}_{\mu_n}(z) - z \\
&= \widetilde{F}^{-1}_{\mu_n}(z) - \widetilde{F}^{-1}_{\mu_n}(\widetilde{F}_{\mu_n}(z)) \\
&= \int_{\widetilde{F}_{\mu_n}(z)}^{z} \left(\widetilde{F}^{-1}_{\mu_n}\right)'(\zeta)\, d\zeta \\
&= \int_{\widetilde{F}_{\mu_n}(z)}^{z} 1 + \left(\phi^{RD}_{\mu_n}\right)'(\zeta)\, d\zeta \\
&= z - \widetilde{F}_{\mu_n}(z) + \int_{\widetilde{F}_{\mu_n}(z)}^{z} \left(\phi^{RD}_{\mu_n}\right)'(\zeta)\, d\zeta.
\end{aligned}$$

We observe that

$$\left|\int_{\widetilde{F}_{\mu_n}(z)}^{z} \left(\phi^{RD}_{\mu_n}\right)'(\zeta)\, d\zeta\right| \leq \left|z - \widetilde{F}_{\mu_n}(z)\right| k\varepsilon.$$

Meanwhile, $z - \widetilde{F}_{\mu_n}(z) = z - z/(1 + \phi_{\mu_n}/z) = \phi_{\mu_n}(z)(1 - \phi_{\mu_n}/(z + \phi_{\mu_n}))$ and $\lim_{n\to\infty} \phi_{\mu_n}(z) \to 0$ for all $z \in \Lambda_{\alpha',\beta'}$. The proof is completed by applying [10, Proposition 2.7] to $\phi_{\mu_n}(z)$. ◻

We now state the basic ingredient for the main theorem.

Lemma 2.3 *For every $\alpha, \beta > 0$ there exists $\varepsilon > 0$ such that the following is true: If $\mu \in \mathcal{P}^+$ is a measure such that $\mu([0, \varepsilon)) > 1 - \varepsilon$, then ϕ_μ^{RD} is defined in $\Lambda_{\alpha,\beta}$.*

Proof From [2, Lemma 5], given every $\alpha, \beta > 0$, there exists $\varepsilon > 0$ such that F_μ, F_μ^{-1} and ϕ_μ are defined in $\Lambda_{\alpha,\beta}$. Hence $\widetilde{F}_\mu$ is defined in $\Lambda_{\alpha,\beta}$. (The original argument applies to functions defined on $\Lambda_{\alpha,\beta} \cap \mathbb{C}^+$ but it is not hard to extend to functions defined on $\Lambda_{\alpha,\beta}$.) Fix such $\varepsilon > 0$ and $z = x + iy \in \Lambda_{\alpha,\beta}$. By the similar argument in [2, Lemma 5] we can make the following estimate

$$\left|\frac{F_\mu(z)}{z} - 1\right| \leq 2\varepsilon \left(\frac{1}{\beta/\sqrt{1+\alpha^2}} + \sqrt{1+\alpha^2}\right).$$

It is known that $\phi'_\mu(z) = o(1)$ as $z \to \infty,\ z \in \Lambda_{\alpha,\beta}$. This implies that $(F_\mu^{-1})'(z) = 1 + o(1)$ as $z \to \infty$. Therefore, $(F_\mu^{-1})'(z)$ has a uniform bounded $M_{\alpha,\beta}$ in the domain $\Lambda_{\alpha,\beta}$. For any $z \in \Lambda_{\alpha,\beta}$ we have the following estimate:

$$\begin{aligned}
|1 - z/\widetilde{F}_\mu(z)| &= \left|1 - z\left(\frac{1}{z} + \frac{1}{z^2}\phi_\mu(z)\right)\right| \\
&= \left|\frac{F_\mu^{-1}(z) - z}{z}\right| \\
&= \left|\frac{F_\mu^{-1}(z) - F_\mu^{-1}(F_\mu(z))}{z}\right| \\
&= \left|\frac{\int_{F_\mu(z)}^{z} (F_\mu^{-1}(\zeta))' d\zeta}{z}\right| \\
&\leq M_{\alpha,\beta}\left|\frac{z - F_\mu(z)}{z}\right| \\
&\leq 2\varepsilon M_{\alpha,\beta}\left[\frac{1}{\beta/\sqrt{1+\alpha^2}} + \sqrt{1+\alpha^2}\right].
\end{aligned}$$

Now following the same argument from [2, Lemma 5], it is clear that $\widetilde{F}_\mu(z)$ has a right inverse defined in $\Lambda_{\alpha,\beta}$. □

Infinitely divisible related to the $\boxplus_{RD}$ convolution is characterized in terms of the function ϕ^{RD} as follows.

Theorem 2.4 *A measure $\mu \in \mathcal{P}^+$ is $\boxplus_{RD}$-infinitely divisible if and only if the function ϕ_μ^{RD} can be extended to an analytic function $\psi : \mathbb{C}\backslash[0, \infty) \to \mathbb{C}$ such that*

(A) $\psi(\overline{z}) = \overline{\psi(z)}$ *and* $\mathfrak{Im}\ \psi(z) \leq 0$ *if* $\mathfrak{Im}\ z > 0$,
(B) $\psi(x) \geq 0$ *if* $x \in (-\infty, 0)$, *and*
(C) $\lim_{|z|\to\infty, z\in\Lambda_\alpha} \psi(z)/z = 0$.

Proof Assume that a measure $\mu \in \mathcal{P}^+$ is $\boxplus_{RD}$-infinitely divisible. Then for every natural number n, there exists $\mu_n \in \mathcal{P}^+$ such that

$$\mu = \underbrace{\mu_n \boxplus_{RD} \mu_n \boxplus_{RD} \cdots \boxplus_{RD} \mu_n}_{n \text{ times}}.$$

This implies $\phi_\mu^{RD} = n\phi_{\mu_n}^{RD}$ in some domain $\Lambda_{\alpha,\beta}$ where all functions involved are defined. Since $\phi_{\mu_n}^{RD}(z) = \phi_\mu^{RD}(z)/n$ uniformly converges to zero on the compact subsets of $\Lambda_{\alpha,\beta}$ as $n \to \infty$, the sequence μ_n converges weakly to δ_0 as $n \to \infty$. For the point mass δ_0, functions G_{δ_0}, F_{δ_0}, $F_{\delta_0}^{-1}$, ϕ_{δ_0}, $\widetilde{F}_{\delta_0}$, $\widetilde{F}_{\delta_0}^{-1}$ and $\phi_{\delta_0}^{RD}$ are defined on $\mathbb{C}\backslash\mathbb{R}_+$. Thus, when n large enough, $\phi_{\mu_n}^{RD}(z)$ is defined on the domain $\Lambda_{\alpha,\beta}$ with arbitrary large α and arbitrary small β. Hence, $\phi_\mu^{RD}(z) = n\phi_{\mu_n}^{RD}(z)$ can be extended to an analytic function ψ on $\mathbb{C}\backslash\mathbb{R}_+$, with $\psi(\overline{z}) = \overline{\psi(z)}$. In addition, by Proposition 2.2, one has

$$\phi_{\mu_n}^{RD}(z) = z^2[G_{\mu_n}(z) - 1/z](1 + w_n(z))$$

where $\lim_{n\to\infty} w_n(z) = 0$. Hence

$$\mathfrak{Im}\phi_\mu^{RD}(z) = n\mathfrak{Im}\left[z^2(G_{\mu_n}(z) - 1/z)\right]\mathfrak{Re}(1 + w_n(z)) + n\mathfrak{Re}\left[z^2(G_{\mu_n}(z) - 1/z)\right]\mathfrak{Im}w_n(z).$$

We denote $\mathrm{I}_n(z) = n\mathfrak{Im}\left[z^2(G_{\mu_n}(z) - 1/z)\right]\mathfrak{Re}(1 + w_n(z))$ and $\mathrm{II}_n(z) = n\mathfrak{Re}\left[z^2(G_{\mu_n}(z) - 1/z)\right]\mathfrak{Im}w_n(z)$. In the second $\mathrm{II}_n(z)$ term, one can choose n large enough such that $|w_n(z)| < 1/2$, and hence

$$\left|n\left[z^2(G_{\mu_n}(z) - 1/z)\right]\right| = \left|\frac{\phi_\mu^{RD}(z)}{1 + w_n(z)}\right| < 2|\phi_\mu^{RD}(z)|, \qquad z \in \Lambda_{\alpha,\beta}.$$

Therefore, $n\mathfrak{Re}\left[z^2(G_{\mu_n}(z) - 1/z)\right]$ stays bounded. Since $\lim_{n\to\infty}\mathfrak{Im}w_n(z) = 0$, $\lim_{n\to\infty}\mathrm{II}_n(z) = 0$ for every $z \in \Lambda_{\alpha,\beta}$. In the first $\mathrm{I}_n(z)$ term, $\mathfrak{Re}(1 + w_n(z)) > 0$ as n big enough. When $\mathfrak{Im}z > 0$ and $z = x + iy \in \Lambda_{\alpha,\beta}$,

$$n\mathfrak{Im}\left[z^2(G_{\mu_n}(z) - 1/z)\right] = n\int_{[0,+\infty)} \frac{-yt^2}{y^2 + (x-t)^2} d\mu_n(t) \le 0.$$

Therefore, $\mathrm{I}_n(z) \le 0$ as $\mathfrak{Im}z > 0$. Combining the observation of the above properties for terms I_n and II_n, we have $\mathfrak{Im}\phi_\mu^{RD}(z) \le 0$, if $\mathfrak{Im}z > 0$, $z \in \Lambda_{\alpha,\beta}$. Next, for $x < 0$, $G_{\mu_n}(x) \ge 1/x$ and $w_n(x)$ only take real values. This implies that for such x, $\phi_{\mu_n}^{RD}(x) = x^2[G_{\mu_n}(x) - 1/x](1 + w_n(x)) \ge 0$ as n big enough, and hence $\phi_\mu^{RD}(x) = n\phi_{\mu_n}^{RD}(x) \ge 0$ for $x > 0$. Finally, observing the way we extended $\phi_\mu^{RD}(z)$ to $\psi(z)$, we have $\mathfrak{Im}\psi(z) \le 0$ if $\mathfrak{Im}z > 0$, $\psi(x) \ge 0$ if $x < 0$, and $\lim_{|z|\to\infty, z\in\Lambda_\alpha}\psi(z)/z = \lim_{|z|\to\infty, z\in\Lambda_\alpha}\phi_\mu^{RD}(z)/z = 0$.

Conversely, it is known that if an analytic function ψ satisfies conditions (A), (B) and (C), then it has a Nevalinna representation

$$\psi(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t} d\sigma(t), \qquad \text{on } z \in \mathbb{C} \setminus \mathbb{R}_+.$$

Note that σ is a probability measure on $\mathbb{R}_+$ for which $\int_{[0,+\infty)} d\sigma(t)$ is finite, and γ is real. It is known that every probability measure σ on $\mathbb{R}_+$ can be approximated by σ_N, $\lim_{N\to\infty} \sigma_N = \sigma$, where $\sigma_N = \sum_{n=1}^N \lambda_n \delta_{t_n}$ for $t_n \geq 0$, $\lambda_n > 0$, $N \in \mathbb{N}$. If we denote by $\Pi_{1;\gamma}$ the Marchenko-Pastur distribution with rate 1 and jump size γ, then $\phi^{RD}_{\Pi_{1;\gamma}}(z) = \gamma$. If we choose $\theta_n = (\Pi_{t_n\lambda_n;1} \boxtimes \Pi_{t_n;1})^{\boxplus 1/t_n}$, then $\phi^{RD}_{\theta_n}(z) = \lambda_n t_n z/(z-t_n)$. By the method in [11], one can find a measure $\omega_n \in \mathcal{P}^+$ such that $\phi^{RD}_{\omega_n}(z) = \lambda_n/(z-t_n)$. It can be verified that if we set

$$\mu_N = \Pi_{1;\gamma} \boxplus_{RD} \theta_1 \boxplus_{RD} \omega_1 \boxplus_{RD} \cdots \boxplus_{RD} \theta_N \boxplus_{RD} \omega_N,$$

then

$$\phi^{RD}_{\mu_N}(z) = \gamma + \int_{[0,+\infty)} \frac{1+tz}{z-t}\, d\sigma_N(t), \qquad \text{on } \mathbb{C} \setminus \mathbb{R}_+.$$

Since $\lim_{N\to\infty} \phi^{RD}_{\mu_N}(z) = \psi(z)$ uniformly on the compact subsets of their domain and $\phi^{RD}_{\mu_N}(z) = o(|z|)$ uniformly in N as $|z| \to \infty$, the sequence μ_N converges weakly to the measure μ with $\psi = \phi^{RD}_{\mu}$. The same argument can be applied to ψ/k for $k = 2, 3, \ldots$. This process shows that there exists a probability measure $\mu_k \in \mathcal{P}^+$ such that $\phi^{RD}_{\mu_k} = \psi/k$. This is because ψ/k also satisfies all conditions in the theorem for $k = 2, 3, \ldots$. Therefore, $\phi^{RD}_{\mu}(z) = k \cdot (\psi(z)/k) = k \cdot \phi^{RD}_{\mu_k}(z)$, which implies that μ equals to the k-fold convolution $\mu_k \boxplus_{RD} \cdots \boxplus_{RD} \mu_k$. Hence, μ is $\boxplus_{RD}$-infinitely divisible. □

3 The Main Theorem

Theorem 3.1 *Let $\{T_{ij}, i \geq 1, 1 \leq j \leq k_i\}$ be an array of tracial R-diagonal random variables, and let the measure $\mu_{ij} \in \mathcal{P}^+$ represent the distribution of $|T_{ij}|^2$, such that*

(i) *T_{i1}, T_{i2}, ..., T_{ik_i} are $*$-free for every i;*
(ii) *$\lim_{i\to\infty} \max_{1\leq j\leq k_i} \mu_{ij}(\{t : t > \varepsilon\}) = 0$ for every $\varepsilon > 0$ (the array $\{T_{ij}\}$ is uniformly infinitesimal); and*
(iii) *the distribution μ_i of $|\sum_{i=1}^{k_i} T_{ij}|^2$ converges weakly to a limit $\mu \in \mathcal{P}^+$.*

Then the measure μ must be $\boxplus_{RD}$-infinitely divisible.

Proof Given $\alpha, \beta > 0$, we will show that the function ϕ^{RD}_{μ} can be continued analytically to $\Lambda_{\alpha,\beta}$, and the continuation maps $\mathbb{C}^+ \cap \Lambda_{\alpha,\beta}$ to $\mathbb{C}^- \cup \mathbb{R}$ and maps $(-\infty, 0)$

to $\mathbb{R}_+$. Indeed, choose $\varepsilon > 0$, so that the conclusion of Lemma 2.3 holds. For $\delta > 0$ independent from α, β, ε, we set

$$U_\delta = \{z : \Im\mathfrak{m} z > \delta\}.$$

Assumption (i) of the theorem implies that the function $\phi^{RD}_{\mu_i} = \sum_{j=1}^{k_i} \phi^{RD}_{\mu_{ij}}$ is defined in $\Lambda_{\alpha,\beta}$ for i sufficiently large. Assumption (ii) of the theorem implies that $\phi^{RD}_{\mu_{ij}}(z) = z^2[G_{\mu_{ij}}(z) - 1/z](1 + w_{ij}(z))$ with $\lim_{i\to\infty} \sup_{1\le j\le k_i} |w_{ij}(z)| = 0$ for all $z \in \Lambda_{\alpha,\beta}$. Since $z^2[G_{\mu_{ij}}(z) - 1/z]$ maps $\Lambda_{\alpha,\beta} \cap U_\delta$ to $\mathbb{C}^- \cup \mathbb{R}$ and takes no real values unless μ_{ij} equals a Dirac measure, the functions $\phi^{RD}_{\mu_{ij}}$ and $\phi^{RD}_{\mu_i}$ map $\Lambda_{\alpha,\beta} \cap U_\delta$ to $\mathbb{C}^- \cup \mathbb{R}$ when i is big enough. Since $\mathbb{C}^-$ is conformally equivalent to a disk, the family $\phi^{RD}_{\mu_i}$ is normal on $\Lambda_{\alpha,\beta} \cap U_\delta$. Hence, by applying the Vitali-Montel theorem, there exists a subsequence $\phi^{RD}_{\mu_{i_n}}$ which converges uniformly on the compact subsets of $\Lambda_{\alpha,\beta}$ to an analytic function φ, or to infinity. Note now that the case of an infinite limit can be excluded, because $\phi^{RD}_{\mu_{i_n}}$ converges to ϕ^{RD}_μ for z in some open subset of $\Lambda_{\alpha,\beta}$ by assumption (iii). Therefore, $\varphi(z) = \phi^{RD}_\mu(z)$ for such z. As a result, φ on $\Lambda_{\alpha,\beta}$ is an analytic continuation of ϕ^{RD}_μ. It follows that the range of φ on the intersection of $\Lambda_{\alpha,\beta}$ with $\bigcup_{\delta>0} U_\delta = \mathbb{C}^+$ is contained in the $\mathbb{C}^- \cup \mathbb{R}$. In addition, since $x^2[G_{\mu_{ij}}(x) - 1/x] \ge 0$ for $x < 0$, using the similar argument one can show $\varphi(x) \ge 0$ for $x < 0$. Thus, ϕ^{RD}_μ can be continued analytically to $\bigcup_{\alpha,\beta>0} \Lambda_{\alpha,\beta} = \mathbb{C}\backslash\mathbb{R}_+$ with desired properties (A) (B) and (C) of Proposition 2.2, and therefore μ is $\boxplus_{RD}$-infinitely divisible. $\square$

Conversely, every $\boxplus_{RD}$-infinitely divisible measure μ occurs as a week limit of this type. Indeed, if $\mu = \underbrace{\mu_n \boxplus_{RD} \cdots \boxplus_{RD} \mu_n}_{n \text{ times}}$, one can choose $k_i = i$ and $|T_{i1}|^2, |T_{i2}|^2, \ldots, |T_{ik_i}|^2$ with distribution μ_n.

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Data Availability All data generated or analysed during this study are included in this published article.

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