

$$(1) \vdash (\neg A \rightarrow A) \rightarrow A$$

证明:

$$(1) \neg A \rightarrow A, A \vdash A \quad (\in)$$

$$(2) \neg A \rightarrow A, \neg A \vdash \neg A \quad (\in)$$

$$(3) \neg A \rightarrow A, \neg A \vdash \neg A \rightarrow A \quad (\in)$$

$$(4) \neg A \rightarrow A, \neg A \vdash A \quad (2)(3)(\rightarrow -)$$

$$(5) \neg A \rightarrow A \vdash A \quad (1)(4)(-)$$

$$(6) \vdash (\neg A \rightarrow A) \rightarrow A \quad (5)(\rightarrow +)$$

$$(2) \vdash (A \vee B \rightarrow C) \leftrightarrow (A \rightarrow C) \wedge (B \rightarrow C)$$

证明:

$$(1) A \vee B \rightarrow C, A \vdash A \quad (\in)$$

$$(2) A \vee B \rightarrow C, A \vdash A \vee B \quad (1)(\vee +)$$

$$(3) A \vee B \rightarrow C, A \vdash A \vee B \rightarrow C \quad (\in)$$

$$(4) A \vee B \rightarrow C, A \vdash C \quad (2)(3)(\rightarrow -)$$

$$(5) A \vee B \rightarrow C \vdash A \rightarrow C \quad (4)(\rightarrow +)$$

$$(6) A \vee B \rightarrow C, B \vdash B \quad (\in)$$

$$(7) A \vee B \rightarrow C, B \vdash A \vee B \quad (6)(\vee +)$$

$$(8) A \vee B \rightarrow C, B \vdash A \vee B \rightarrow C \quad (\in)$$

$$(9) A \vee B \rightarrow C, B \vdash C \quad (7)(8)(\rightarrow -)$$

$$(10) A \vee B \rightarrow C \vdash B \rightarrow C \quad (9)(\rightarrow +)$$

$$(11) A \vee B \rightarrow C \vdash (A \rightarrow C) \wedge (B \rightarrow C) \quad (5)(10)(\wedge +)$$

$$(12) \vdash (A \vee B \rightarrow C) \rightarrow (A \rightarrow C) \wedge (B \rightarrow C) \quad (11)(\rightarrow +)$$

$$(13) (A \rightarrow C) \wedge (B \rightarrow C), A \vee B, A \vdash A \quad (\in)$$

- (14) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B, A \vdash (A \rightarrow C) \wedge (B \rightarrow C) \quad (\in)$
- (15) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B, A \vdash A \rightarrow C \quad (14)(\wedge-)$
- (16) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B, A \vdash C \quad (13)(15)(\rightarrow-)$
- (17) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B, B \vdash B \quad (\in)$
- (18) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B, B \vdash (A \rightarrow C) \wedge (B \rightarrow C) \quad (\in)$
- (19) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B, B \vdash B \rightarrow C \quad (18)(\wedge-)$
- (20) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B, B \vdash C \quad (17)(19)(\rightarrow-)$
- (21) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B \vdash A \vee B \quad (\in)$
- (22) $(A \rightarrow C) \wedge (B \rightarrow C), A \vee B \vdash C \quad (16)(20)(21)(\vee-)$
- (23) $(A \rightarrow C) \wedge (B \rightarrow C) \vdash A \vee B \rightarrow C \quad (22)(\rightarrow+)$
- (24) $\vdash (A \rightarrow C) \wedge (B \rightarrow C) \rightarrow (A \vee B \rightarrow C) \quad (23)(\rightarrow+)$
- (25) $\vdash (A \vee B \rightarrow C) \leftrightarrow (A \rightarrow C) \wedge (B \rightarrow C) \quad (12)(24)(\leftrightarrow+)$

(5) $\vdash \neg(A \rightarrow B) \leftrightarrow A \wedge \neg B$

证明:

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|---|--|
| (1) $\neg(A \rightarrow B), \neg A, A \vdash A \quad (\in)$ | (11) $\neg(A \rightarrow B) \vdash \neg B \quad (\neg+)$ |
| (2) $\neg(A \rightarrow B), \neg A, A \vdash \neg A \quad (\in)$ | (12) $\neg(A \rightarrow B) \vdash A \wedge \neg B \quad (\wedge+)$ |
| (3) $\neg(A \rightarrow B), \neg A, A \vdash B \quad (12)(\neg-)$ | (13) $\vdash \neg(A \rightarrow B) \rightarrow A \wedge \neg B \quad (\rightarrow+)$ |
| (4) $\neg(A \rightarrow B), \neg A \vdash A \rightarrow B \quad (\rightarrow+)$ | (14) $A \wedge \neg B, A \rightarrow B \vdash A \rightarrow B \quad (\in)$ |
| (5) $\neg(A \rightarrow B), \neg A \vdash \neg(A \rightarrow B) \quad (\in)$ | (15) $A \wedge \neg B, A \rightarrow B \vdash A \wedge \neg B \quad (\in)$ |
| (6) $\neg(A \rightarrow B) \vdash \neg\neg A \quad (\neg+)$ | (16) $A \wedge \neg B, A \rightarrow B \vdash A \quad (\wedge-)$ |
| (7) $\neg(A \rightarrow B) \vdash A \quad (\neg\neg-)$ | (17) $A \wedge \neg B, A \rightarrow B \vdash \neg B \quad (\wedge-)$ |
| (8) $\neg(A \rightarrow B), B, A \vdash B \quad (\in)$ | (18) $A \wedge \neg B, A \rightarrow B \vdash B \quad (\rightarrow-)$ |
| (9) $\neg(A \rightarrow B), B \vdash A \rightarrow B \quad (\rightarrow+)$ | (19) $A \wedge \neg B \vdash \neg(A \rightarrow B) \quad (\neg+)$ |
| (10) $\neg(A \rightarrow B), B \vdash \neg(A \rightarrow B) \quad (\in)$ | (20) $\vdash A \wedge \neg B \rightarrow \neg(A \rightarrow B) \quad (\rightarrow+)$ |
| | (21) $\vdash \neg(A \rightarrow B) \leftrightarrow A \wedge \neg B \quad (\leftrightarrow+)$ |

$$(7) \vdash A \wedge B \leftrightarrow A \wedge (\neg A \vee B)$$

证明:

- 1 $A \wedge B \vdash A \wedge B \quad (\in)$
- 2 $A \wedge B \vdash A \quad (1)(\wedge-)$
- 3 $A \wedge B \vdash B \quad (1)(\wedge-)$
- 4 $A \wedge B \vdash \neg A \vee B \quad (3)(\vee+)$
- 5 $A \wedge B \vdash A \wedge (\neg A \vee B) \quad (2)(4)(\wedge+)$
- 6 $\vdash A \wedge B \rightarrow A \wedge (\neg A \vee B) \quad (5)(\rightarrow +)$
- 7 $A \wedge (\neg A \vee B) \vdash A \wedge (\neg A \vee B) \quad (\in)$
- 8 $A \wedge (\neg A \vee B) \vdash A \quad (7)(\wedge-)$
- 9 $A \wedge (\neg A \vee B) \vdash \neg A \vee B \quad (7)(\wedge-)$
- 10 $A \wedge (\neg A \vee B), B \vdash B \quad (\in)$
- 11 $A \wedge (\neg A \vee B), \neg A \vdash \neg A \quad (\in)$
- 12 $A \wedge (\neg A \vee B), \neg A \vdash A \wedge (\neg A \vee B) \quad (\in)$
- 13 $A \wedge (\neg A \vee B), \neg A \vdash A \quad (12)(\wedge-)$
- 14 $A \wedge (\neg A \vee B), \neg A \vdash B \quad (13)(11)(\neg-)$
- 15 $A \wedge (\neg A \vee B) \vdash B \quad (14)(10)(9)(\vee-)$
- 16 $A \wedge (\neg A \vee B) \vdash A \wedge B \quad (8)(15)(\wedge+)$
- 17 $\vdash A \wedge (\neg A \vee B) \rightarrow A \wedge B \quad (16)(\rightarrow +)$
- 18 $\vdash A \wedge B \leftrightarrow A \wedge (\neg A \vee B) \quad (6)(17)(\leftrightarrow +)$