Problems

February 16, 2017

1. Let $X_1, \cdots, X_n$ be a sequence of i.i.d. random variables, of which cdf is $F_X(\cdot)$.
   (a) What is CDF of $X_{\max} = \max_i X_i$?
   (b) What is CDF of $X_{\min} = \min_i X_i$?
2. Show how the Chebyshev inequality can be derived from Markov inequality.
3. If X is a continuous random variable having CDF F_X, show that the random variable $Y = F_X(X)$ is uniformly distributed in $(0, 1)$.
4. Suppose you can generate a random variable U uniformly distributed in $(0, 1)$. How would you use it to simulate a continuous random variable X having a arbitrary distribution function $F(\cdot)$?
5. Suppose you have access to two independent random variables U_1 and U_2, both uniformly distributed in $[0, 1]$. How would you use them to simulate two continuous random variables X_1 and X_2 with a given joint distribution $F(\cdot, \cdot)$?
6. Prove the weak law of large number using Chebyshev's inequality.
7. Let $W \sim \mathcal{N}(0, 1)$ and $Q(x) = \mathbb{P}[W > x]$.
   (a) Show that
   $$Q(x) < \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right).$$
   (b) Show that
   $$Q(x) > \frac{1}{\sqrt{2\pi}x} \left(1 - \frac{1}{x^2}\right) \exp\left(-\frac{x^2}{2}\right) \quad \forall\ x > 1.$$
8. Prove Cauchy-Schwarz inequality.
   $$\mathbb{E}[XY]^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2].$$

9. The amount of weight, W, that a bridge can withstand without damage, is a Gaussian random variable with mean μ_W and variance σ_W^2. Suppose the weight of cars $X_1, X_2, \cdots, X_n$ are i.i.d. random variables with mean μ_X and variance σ_X^2. How many cars would have to be on the bridge for the probability of damage to exceed 0.1 ?

10. Show that if X and Y are independent and

$$\begin{aligned} &X \sim \mathcal{N}(\mu_X, \sigma_X^2), \\ &Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2), \\ &Z = X + Y, \end{aligned}$$

then

$$Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$