STAT253/317 Lecture 10

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5.3 The Poisson Processes

Properties of Poisson Processes

Outline:

- Interarrival times of events are i.i.d Exponential with rate λ
- Conditional Distribution of the Arrival Times
- Superposition & Thinning ....(Lecture 11)
- "Converse" of Superposition & Thinning .... (Lecture 11)

Arrival & Interarrival Times of Poisson Processes

Let

$$
\begin{aligned}
S_n &= \text{Arrival time of the } n\text{-th event}, n = 1, 2, \ldots \\
T_1 &= S_1 = \text{Time until the 1st event occurs} \\
T_n &= S_n - S_{n-1} \\
&= \text{time elapsed between the } (n-1)\text{st and } n\text{-th event,} \\
&\quad\ n = 2, 3, \ldots
\end{aligned}
$$

Proposition 5.1

The interarrival times $T_1, T_2, \ldots, T_k, \ldots,$ are i.i.d $\sim Exp(\lambda)$.

Consequently, as the distribution of the sum of n i.i.d Exp(λ) is $Gamma(n, \lambda)$, the arrival time of the nth event is

$$S_n = \sum_{i=1}^{n} T_i \sim Gamma(n, \lambda)$$

Proof of Proposition 5.1

$$
\begin{aligned}
&\mathrm{P}(T_{n+1} > t | T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
&= \mathrm{P}(0 \text{ event in } (s_n, s_n + t] | T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{where } s_n = t_1 + t_2 + \cdots + t_n) \\
&= \mathrm{P}(0 \text{ event in } (s_n, s_n + t]) \quad (\text{by indep increment}) \\
&= \mathrm{P}(N(s_n + t) - N(s_n) = 0) \\
&= e^{-\lambda t}
\end{aligned}
$$

where the last step comes from the fact that

- $N(s_n + t) - N(s_n) \sim$ Poisson(λt) and
- $P(N = k) = e^{-\mu} \mu^k / k!$ if $N \sim$ Poisson(μ), $k = 0, 1, 2 \ldots$

This shows that T_{n+1} is $\sim Exp(\lambda)$, and is independent of $T_1, T_2, \ldots, T_n$.

Definition 3 of the Poisson Process

A continuous-time stochastic process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

(i) $N(0) = 0$,

(ii) $N(t)$ counts the number of events that have occurred up to time t (i.e., it is a counting process).

(iii) The times between events are independent and identically distributed with an Exp(λ) distribution.

We have seen how Definition 5.1 implies (i), (ii) and (iii) in Definition 3. The proof of the converse is omitted.

5.3.5 Conditional Distribution of Arrival Times is Uniform

Given $N(t) = 1$, then T_1, the arrival time of the first event $\sim$ Uniform$(0, t)$

Proof. For $s < t$,

$$\begin{aligned}
\mathrm{P}(T_1 \leq s | N(t) = 1) &= \frac{\mathrm{P}(T_1 \leq s, N(t) = 1)}{\mathrm{P}(N(t) = 1)} \\
&= \frac{\mathrm{P}(\text{1 event in } (0, s], \text{ no events in } (s, t])}{\mathrm{P}(N(t) = 1)} \\
&= \frac{\mathrm{P}(N(s) = 1)\mathrm{P}(N(t) - N(s) = 0)}{\mathrm{P}(N(t) = 1)} \text{ by indep. increment} \\
&=^* \frac{(\lambda s e^{-\lambda s})(e^{-\lambda(t-s)})}{\lambda t e^{-\lambda t}} = \frac{s}{t}, \quad s < t.
\end{aligned}$$

where the step $=^*$ comes from the fact that

- $N(s) \sim$ Poisson(λs), $N(t) - N(s) \sim$ Poisson$(\lambda(t - s))$, and $N(t) \sim$ Poisson(λt)
- $P(N = k) = e^{-\mu}\mu^k/k!$ if $N \sim$ Poisson(μ), $k = 0, 1, 2 \ldots$

Review of Order Statistics

Suppose X_1, X_2,. . . , X_n are i.i.d. random variables with a common density $f(x)$. Their joint density would be the product of the marginal density

$$f(x_1, x_2, \ldots, x_n) = f(x_1)f(x_2)\ldots f(x_n).$$

Let $X_{(i)}$ be the ith smallest number among $X_1, X_2, \ldots, X_n$. $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ is called the order statistics of $X_1, X_2, \ldots, X_n$

- $X_{(1)}$ is the minimum
- $X_{(n)}$ is the maximum
- $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$

The joint density of $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is

$$h(x_1, x_2, \ldots, x_n) = \begin{cases} n!f(x_1)f(x_2)\ldots f(x_n), & \text{if } x_1 \leq x_2 \leq \ldots \leq x_n. \\ 0 & \text{otherwise} \end{cases}$$

Example

If $U_1, U_2, \ldots, U_n$ are indep. Uniform$(0, t)$, their common density is

$$f(u) = \begin{cases} 1/t, & \text{for } 0 < u < t. \\ 0 & \text{otherwise} \end{cases}$$

The joint density of their order statistics $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ is

$$h(u_1, u_2, \ldots, u_n) = n! f(u_1) f(u_2) \ldots f(u_n) = n!(1/t)^n$$

for $0 \leq u_1 \leq u_2 \leq \ldots \leq u_n < t$ and 0 elsewhere.

Theorem 5.2

Given $N(t) = n$,

$$(S_1, S_2, \ldots, S_n) \sim (U_{(1)}, U_{(2)}, \ldots, U_{(n)})$$

where $(U_{(1)}, \ldots, U_{(n)})$ are the order statistics of $(U_1, \ldots, U_n) \sim$ i.i.d Uniform $(0, t)$, i.e., the joint conditional density of $S_1, S_2, \ldots$, S_n is

$$f(s_1, s_2, \ldots, s_n | N(t) = n) = n!/t^n, \; 0 < s_1 < s_2 < \ldots < s_n$$

Proof. The event that $S_1 = s_1$, $S_2 = s_2, \ldots$, $S_n = s_n$, $N(t) = n$ is equivalent to the event $T_1 = s_1$, $T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$. Hence, by Proposition 5.1, we have the conditional joint density of $S_1, \ldots, S_n$ given $N(t) = n$ as follows:

$$f(s_1, \ldots, s_n | N(t) = n) = \frac{f(s_1, \ldots, s_n, N(t) = n)}{\mathrm{P}(N(t) = n)}$$

$$= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \ldots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{e^{-\lambda t}(\lambda t)^n / n!}$$

$$= n! t^{-n}, \quad 0 < s_1 < \ldots < s_n < t$$

Example 5.21. Insurance claims comes according to a Poisson process $\{N(t)\}$ with rate λ. Let

- S_i = the time of the ith claims
- C_i = amount of the ith claims, i.i.d with mean μ, indep. of $\{N(t)\}$

Then the total discounted cost by time t at discount rate α is given by

$$D(t) = \sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}.$$

Then

$$\begin{aligned}
\mathbb{E}[D(t)|N(t)] &= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}\middle| N(t)\right] \overset{(5.2)}{=} \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_{(i)}}\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_i}\right] = \sum_{i=1}^{N(t)} \mathbb{E}[C_i]\mathbb{E}\left[e^{-\alpha U_i}\right] \\
&= N(t)\mu \int_0^t \frac{1}{t} e^{-\alpha x} dx = N(t)\frac{\mu}{\alpha t}(1 - e^{-\alpha t})
\end{aligned}$$

Thus $\mathbb{E}[D(t)] = \mathbb{E}[N(t)]\frac{\mu}{\alpha t}(1 - e^{\alpha t}) = \frac{\lambda\mu}{\alpha}(1 - e^{-\alpha t})$