

STAT253/317 Lecture 7

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- Using the Iterative Relationship
- 4.5.3 Random Walk w/ Reflective Boundary at 0
- 4.7 Branching Processes

Using the Iterative Relationship

Many Markov chains $\{X_n\}$ have some iterative relationships between consecutive terms, e.g.,

$$X_{n+1} = g(X_n, \xi_{n+1}) \quad \text{for all } n$$

where $\{\xi_n, n = 0, 1, 2, \dots\}$ are some i.i.d. random variables and X_n is independent of $\{\xi_k : k > n\}$.

In many cases, we can use the iterative relationship to find $\mathbb{E}[X_n]$ and $\text{Var}[X_n]$ without knowing the distribution of X_n .

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] \\ \text{Var}(X_{n+1}) &= \mathbb{E}[\text{Var}(X_{n+1}|X_n)] + \text{Var}(\mathbb{E}[X_{n+1}|X_n])\end{aligned}$$

Example 1: Simple Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } q = 1 - p \end{cases}$$

So

$$\begin{aligned}\mathbb{E}[X_{n+1}|X_n] &= p(X_n + 1) + q(X_n - 1) = X_n + p - q \\ \text{Var}[X_{n+1}|X_n] &= 4pq\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = \mathbb{E}[X_n] + p - q \\ \text{Var}(X_{n+1}) &= \mathbb{E}[\text{Var}(X_{n+1}|X_n)] + \text{Var}(\mathbb{E}[X_{n+1}|X_n]) \\ &= \mathbb{E}[4pq] + \text{Var}(X_n + p - q) = 4pq + \text{Var}(X_n)\end{aligned}$$

So

$$\mathbb{E}[X_n] = n(p - q) + \mathbb{E}[X_0], \quad \text{Var}(X_n) = 4npq + \text{Var}(X_0)$$

Example 2: Ehrenfest Urn Model with M Balls

Recall that

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{M-X_n}{M} \\ X_n - 1 & \text{with probability } \frac{X_n}{M} \end{cases}$$

We have

$$\mathbb{E}[X_{n+1}|X_n] = (X_n+1) \times \frac{M-X_n}{M} + (X_n-1) \times \frac{X_n}{M} = 1 + \left(1 - \frac{2}{M}\right) X_n.$$

Thus

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = 1 + \left(1 - \frac{2}{M}\right) \mathbb{E}[X_n]$$

Subtracting $M/2$ from both sides of the equation above, we get

$$\mathbb{E}[X_{n+1}] - \frac{M}{2} = \left(1 - \frac{2}{M}\right) \left(\mathbb{E}[X_n] - \frac{M}{2}\right)$$

Thus

$$\mathbb{E}[X_n] - \frac{M}{2} = \left(1 - \frac{2}{M}\right)^n \left(\mathbb{E}[X_0] - \frac{M}{2}\right)$$

Example 3: Branching Processes (Section 4.7)

Consider a population of individuals.

- ▶ All individuals have the same lifetime
- ▶ Each individual will produce a random number of offsprings at the end of its life

Let X_n = size of the n -th generation, $n = 0, 1, 2, \dots$

If $X_{n-1} = k$, the k individuals in the $(n-1)$ -th generation will independently produce $Z_{n,1}, Z_{n,2}, \dots, Z_{n,k}$ new offsprings, and $Z_{n,1}, Z_{n,2}, \dots, Z_{n,X_{n-1}}$ are i.i.d such that

$$P(Z_{n,i} = j) = P_j, \quad j \geq 0.$$

We suppose that $P_j < 1$ for all $j \geq 0$.

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i} \tag{1}$$

$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \dots\}$.

Mean of a Branching Process

Let $\mu = \mathbb{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$ be the mean # of offsprings produced by an individual. Since $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$ and $Z_{n,i}$'s are i.i.d., we have

$$\mathbb{E}[X_n|X_{n-1}] = \mathbb{E}\left[\sum_{i=1}^{X_{n-1}} Z_{n,i} \middle| X_{n-1}\right] = X_{n-1}\mathbb{E}[Z_{n,i}] = X_{n-1}\mu$$

So

$$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n|X_{n-1}]] = \mathbb{E}[X_{n-1}\mu] = \mu\mathbb{E}[X_{n-1}]$$

Then

$$\mathbb{E}[X_n] = \mu\mathbb{E}[X_{n-1}] = \mu^2\mathbb{E}[X_{n-2}] = \dots = \mu^n\mathbb{E}[X_0]$$

- ▶ If $\mu < 1 \Rightarrow \mathbb{E}[X_n] \rightarrow 0$ as $n \rightarrow \infty \Rightarrow \lim_{n \rightarrow \infty} P(X_n \geq 1) = 0$
the branching processes will eventually die out.
- ▶ What if $\mu = 1$ or $\mu > 1$?

Variance of a Branching Process

Let $\sigma^2 = \text{Var}[Z_{n,i}] = \sum_{j=0}^{\infty} (j - \mu)^2 P_j$. $\text{Var}(X_n)$ may be obtained using the conditional variance formula

$$\text{Var}(X_n) = \mathbb{E}[\text{Var}(X_n|X_{n-1})] + \text{Var}(\mathbb{E}[X_n|X_{n-1}]).$$

Again from that $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathbb{E}[X_n|X_{n-1}] = X_{n-1}\mu, \quad \text{Var}(X_n|X_{n-1}) = X_{n-1}\sigma^2$$

and hence

$$\text{Var}(\mathbb{E}[X_n|X_{n-1}]) = \text{Var}(X_{n-1}\mu) = \mu^2 \text{Var}(X_{n-1})$$

$$\mathbb{E}[\text{Var}(X_n|X_{n-1})] = \sigma^2 \mathbb{E}[X_{n-1}] = \sigma^2 \mu^{n-1} \mathbb{E}[X_0].$$

Variance of a Branching Process

So

$$\begin{aligned}\mathrm{Var}(X_n) &= \sigma^2 \mu^{n-1} \mathbb{E}[X_0] + \mu^2 \mathrm{Var}(X_{n-1}) \\&= \sigma^2 \mu^{n-1} \mathbb{E}[X_0] + \mu^2 (\sigma^2 \mu^{n-2} \mathbb{E}[X_0] + \mu^2 \mathrm{Var}(X_{n-2})) \\&= \sigma^2 (\mu^{n-1} + \mu^n) \mathbb{E}[X_0] + \mu^4 \mathrm{Var}(X_{n-2}) \\&= \sigma^2 (\mu^{n-1} + \mu^n) \mathbb{E}[X_0] + \mu^4 (\sigma^2 \mu^{n-3} \mathbb{E}[X_0] + \mu^2 \mathrm{Var}(X_{n-3})) \\&= \sigma^2 (\mu^{n-1} + \mu^n + \mu^{n+1}) \mathbb{E}[X_0] + \mu^6 \mathrm{Var}(X_{n-3}) \\&\quad \vdots \\&= \sigma^2 (\mu^{n-1} + \mu^n + \dots + \mu^{2n-2}) \mathbb{E}[X_0] + \mu^{2n} \mathrm{Var}(X_0) \\&= \begin{cases} \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu} \right) \mathbb{E}[X_0] + \mu^{2n} \mathrm{Var}(X_0) & \text{if } \mu \neq 1 \\ n\sigma^2 \mathbb{E}[X_0] + \mathrm{Var}(X_0) & \text{if } \mu = 1 \end{cases}\end{aligned}$$

4.5.1 The Gambler's Ruin Problem

- ▶ A gambler repeatedly plays a game until he goes bankrupt or his fortune reaches N .
- ▶ In each game, he can win \$1 with probability p or lose \$1 with probability $q = 1 - p$.
- ▶ Outcomes of different games are independent
- ▶ Define X_n = the gambler's fortune after the n th game.
- ▶ $\{X_n\}$ is a simple random walk w/ absorbing boundaries at 0 and N .

$$P_{00} = P_{NN} = 1, P_{i,i+1} = p, P_{i,i-1} = q, i = 1, 2, \dots, N - 1$$

- ▶ Two recurrent classes: $\{0\}$ and $\{N\}$
one transient class $\{1, 2, \dots, N - 1\}$
- ▶ Regardless of the initial fortune X_0 , eventually $\lim_{n \rightarrow \infty} X_n = 0$ or N as all states are transient except 0 or N .

4.5.1 The Gambler's Ruin Problem

Denote A as the event that the gambler's fortune reaches N before reaches 0. Then

$$P_i = P(A|X_0 = i).$$

Conditioning on the outcome of the first game,

$$\begin{aligned} P_i &= P(A|X_0 = i, \text{he wins the 1st game}) \underbrace{P(\text{he wins the 1st game})}_{=p} \\ &\quad + P(A|X_0 = i, \text{he loses the 1st game}) \underbrace{P(\text{he loses the 1st game})}_{=q} \\ &= P(A|X_0 = i, X_1 = i+1)p + P(A|X_0 = i, X_1 = i-1)q \\ &= \underbrace{P(A|X_1 = i+1)}_{=P_{i+1}} p + \underbrace{P(A|X_1 = i-1)}_{=P_{i-1}} q \quad \because \text{Markov} \\ &\hspace{15em} \text{because} \end{aligned}$$

We get a set of equations

$$\begin{aligned} P_i &= pP_{i+1} + qP_{i-1} \quad \text{for } i = 1, 2, \dots, N-1. \\ P_0 &= 0, \quad P_N = 1 \end{aligned}$$

Solving the equations $P_i = pP_{i+1} + qP_{i-1}$

$$\begin{aligned}(p+q)P_i &= pP_{i+1} + qP_{i-1} && \text{since } p+q=1 \\ \Leftrightarrow q(P_i - P_{i-1}) &= p(P_{i+1} - P_i) \\ \Leftrightarrow P_{i+1} - P_i &= (q/p)(P_i - P_{i-1})\end{aligned}$$

As $P_0 = 0$,

~~$$P_2 - P_1 = (q/p)(P_1 - P_0) = (q/p)P_1$$~~

~~$$P_3 - P_2 = (q/p)(P_2 - P_1) = (q/p)^2 P_1$$~~

~~$$\vdots$$~~

~~$$+) P_i - P_{i-1} = (q/p)(P_{i-1} - P_{i-2}) = (q/p)(q/p)^{i-2} P_1 = (q/p)^{i-1} P_1$$~~

Adding up the equations above we get

$$P_i - P_1 = [q/p + (q/p)^2 + \cdots + (q/p)^{i-1}] P_1$$

Solving the equations $P_i = pP_{i+1} + qP_{i-1}$

$$\begin{aligned}(p+q)P_i &= pP_{i+1} + qP_{i-1} && \text{since } p+q=1 \\ \Leftrightarrow q(P_i - P_{i-1}) &= p(P_{i+1} - P_i) \\ \Leftrightarrow P_{i+1} - P_i &= (q/p)(P_i - P_{i-1})\end{aligned}$$

As $P_0 = 0$,

$$P_2 - P_1 = (q/p)(P_1 - P_0) = (q/p)P_1$$

$$P_3 - P_2 = (q/p)(P_2 - P_1) = (q/p)^2 P_1$$

$$\vdots$$

$$P_i - P_{i-1} = (q/p)(P_{i-1} - P_{i-2}) = (q/p)(q/p)^{i-2} P_1 = (q/p)^{i-1} P_1$$

Adding up the equations above we get

$$P_i - P_1 = [q/p + (q/p)^2 + \cdots + (q/p)^{i-1}] P_1$$

From

$$P_i - P_1 = [q/p + (q/p)^2 + \cdots + (q/p)^{i-1}] P_1$$

we get

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1 & \text{if } p \neq q \\ iP_1 & \text{if } p = q \end{cases}$$

Two gamblers Aiden, and Baker, with initial fortune A, the other with initial fortune B.

As $P_N = 1$, we get

$$P_1 = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N} & \text{if } p \neq 0.5 \\ 1/N & \text{if } p = 0.5 \end{cases}$$

They continue playing until one of them is out of money.

So

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 0.5 \\ i/N & \text{if } p = 0.5 \end{cases}$$

What's the prob that Aiden wins all the money at the end?

$X_0=A, N=A+B$

If the gambler will never quit with whatever fortune he has ($N = \infty$), then

P(Aiden eventually wins)

$$\lim_{N \rightarrow \infty} P_i = \begin{cases} 1 - (q/p)^i & \text{if } p > 0.5 \\ 0 & \text{if } p \leq 0.5 \end{cases}$$

4.5.3 Random Walk w/ Reflective Boundary at 0

- ▶ State Space = $\{0, 1, 2, \dots\}$
- ▶ $P_{01} = 1, P_{i,i+1} = p, P_{i,i-1} = 1 - p = q$, for $i = 1, 2, 3 \dots$
- ▶ Only one class, irreducible
- ▶ For $i < j$, define

$$N_{ij} = \min\{m > 0 : X_m = j | X_0 = i\}$$

= first time to reach state j when starting from state i

- ▶ Observe that $N_{0n} = N_{01} + N_{12} + \dots + N_{n-1,n}$
By the Markov property, $N_{01}, N_{12}, \dots, N_{n-1,n}$ are indep.
- ▶ Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i+1 \\ 1 + N_{i-1,i}^* + N_{i,i+1}^* & \text{if } X_1 = i-1 \end{cases} \quad (2)$$

Observe that $N_{i,i+1}^* \sim N_{i,i+1}$, and $N_{i,i+1}^*$ is indep of $N_{i-1,i}^*$.

4.5.3 Random Walk w/ Reflective Boundary at 0 (Cont'd)

Let $m_i = \mathbb{E}(N_{i,i+1})$. Taking expected value on Equation (2), we get

$$m_i = \mathbb{E}[N_{i,i+1}] = 1 + q\mathbb{E}[N_{i-1,i}^*] + q\mathbb{E}[N_{i,i+1}^*] = 1 + q(m_{i-1} + m_i)$$

Rearrange terms we get $pm_i = 1 + qm_{i-1}$ or

$$\begin{aligned} m_i &= \frac{1}{p} + \frac{q}{p}m_{i-1} \\ &= \frac{1}{p} + \frac{q}{p}\left(\frac{1}{p} + \frac{q}{p}m_{i-2}\right) \\ &= \frac{1}{p} \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \dots + \left(\frac{q}{p}\right)^{i-1} \right] + \left(\frac{q}{p}\right)^i m_0 \end{aligned}$$

Since $N_{01} = 1$, which implies $m_0 = 1$.

$$m_i = \begin{cases} \frac{1-(q/p)^i}{p-q} + \left(\frac{q}{p}\right)^i & \text{if } p \neq 0.5 \\ 2i + 1 & \text{if } p = 0.5 \end{cases}$$

Mean of $N_{0,n}$

Recall that $N_{0n} = N_{01} + N_{12} + \dots + N_{n-1,n}$

$$\begin{aligned}\mathbb{E}[N_{0n}] &= m_0 + m_1 + \dots + m_{n-1} \\ &= \begin{cases} \frac{n}{p-q} - \frac{2pq}{(p-q)^2} [1 - (\frac{q}{p})^n] & \text{if } p \neq 0.5 \\ n^2 & \text{if } p = 0.5 \end{cases}\end{aligned}$$

When

$$\begin{array}{lll} p > 0.5 & \mathbb{E}[N_{0n}] \approx \frac{n}{p-q} - \frac{2pq}{(p-q)^2} & \text{linear in } n \\ p = 0.5 & \mathbb{E}[N_{0n}] = n^2 & \text{quadratic in } n \\ p < 0.5 & \mathbb{E}[N_{0n}] = O(\frac{2pq}{(p-q)^2} (\frac{q}{p})^n) & \text{exponential in } n \end{array}$$

Exercise 4.50 on p.284

A Markov chain has transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{matrix}$$

Communicating classes:

$$\begin{array}{ccc} \{1, 2\} & \{3, 4\} & \{5, 6\} \\ \uparrow & \uparrow & \uparrow \\ \text{transient} & \text{recurrent} & \text{recurrent} \end{array}$$

Find $\lim_{n \rightarrow \infty} P^{(n)}$.

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n \rightarrow \infty} P_{ij}^{(n)} = 0$ if j is transient, hence,

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n \rightarrow \infty} P_{ij}^{(n)} = 0$ if j is NOT accessible from i

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

The two classes $\{3,4\}$ and $\{5,6\}$ do not communicate and hence the transition probabilities in between are all 0.

Exercise 4.50 on p.284 (Cont'd)

Since the Markov chain restricted to the closed class $\{3,4\}$ is also

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a Markov chain with the transition matrix $\begin{matrix} & \begin{matrix} 3 & 4 \end{matrix} \\ \begin{matrix} 3 \\ 4 \end{matrix} & \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \end{matrix}$ and the

limiting distribution of a two-state Markov chain with the transition matrix $\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ is $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$, we get

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{matrix}$$

Exercise 4.50 on p.284 (Cont'd)

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{matrix}$$

For the same reason,

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{matrix}$$

Exercise 4.50 on p.284 (Cont'd)

It remains to find

$$\pi_{ij} = \lim_{n \rightarrow \infty} P_{ij}^{(n)}$$

from a transient state $i = 1, 2$
to a recurrent state $j = 3, 4, 5$, or 6 .

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{matrix}$$

By the Chapman-Kolmogorov Equation,

$$\begin{aligned} P_{13}^{(n+1)} &= P_{11}P_{13}^{(n)} + P_{12}P_{23}^{(n)} + P_{13}P_{33}^{(n)} + P_{14}P_{43}^{(n)} + P_{15}P_{53}^{(n)} + P_{16}P_{63}^{(n)} \\ &= 0.2P_{13}^{(n)} + 0.4P_{23}^{(n)} + 0 + 0.3P_{43}^{(n)} + 0 + 0.1 \underbrace{P_{63}^{(n)}}_{=0} \end{aligned}$$

where $P_{63}^{(n)} = 0$ since state 3 and 6 do not communicate.

Let $n \rightarrow \infty$ and recall we've shown earlier that $\lim_{n \rightarrow \infty} P_{43}^{(n)} = 6/13$. We get the equation

$$\pi_{13} = 0.2\pi_{13} + 0.4\pi_{23} + 0.3 \times \frac{6}{13}.$$

Exercise 4.50 on p.284 (Cont'd)

Similarly,

$$\begin{aligned}P_{23}^{(n+1)} &= P_{21}P_{13}^{(n)} + P_{22}P_{23}^{(n)} + P_{23}P_{33}^{(n)} + P_{24}P_{43}^{(n)} + P_{25}P_{53}^{(n)} + P_{26}P_{63}^{(n)} \\&= 0.1P_{13}^{(n)} + 0.3P_{23}^{(n)} + 0 + 0.4P_{43}^{(n)} + 0 + 0.2 \underbrace{P_{63}^{(n)}}_{=0}\end{aligned}$$

where $P_{63}^{(n)} = 0$ since state 3 and 6 do not communicate. Let $n \rightarrow \infty$ and recall we've shown earlier that $\lim_{n \rightarrow \infty} P_{43}^{(n)} = 6/13$. We get the equation

$$\pi_{23} = 0.1\pi_{13} + 0.3\pi_{23} + 0.4 \times \frac{6}{13}.$$

Along with the equation $\pi_{13} = 0.2\pi_{13} + 0.4\pi_{23} + 0.3 \times \frac{6}{13}$ obtained on the previous page, we get

$$\pi_{13} = \frac{37}{52} \times \frac{6}{13} = \frac{111}{338}, \quad \pi_{23} = \frac{35}{52} \times \frac{6}{13} = \frac{105}{338}$$

Exercise 4.50 on p.284 (Cont'd)

Similarly

$$\begin{aligned}P_{15}^{(n+1)} &= P_{11}P_{15}^{(n)} + P_{12}P_{25}^{(n)} + P_{13}P_{35}^{(n)} + P_{14}P_{45}^{(n)} + P_{15}P_{55}^{(n)} + P_{16}P_{65}^{(n)} \\&= 0.2P_{15}^{(n)} + 0.4P_{25}^{(n)} + 0 + \underbrace{0.3P_{45}^{(n)}}_{=0} + 0 + 0.1P_{65}^{(n)}\end{aligned}$$

$$\begin{aligned}P_{25}^{(n+1)} &= P_{21}P_{15}^{(n)} + P_{22}P_{25}^{(n)} + P_{23}P_{35}^{(n)} + P_{24}P_{45}^{(n)} + P_{25}P_{55}^{(n)} + P_{26}P_{65}^{(n)} \\&= 0.1P_{15}^{(n)} + 0.3P_{25}^{(n)} + 0 + \underbrace{0.4P_{45}^{(n)}}_{=0} + 0 + 0.2P_{65}^{(n)}\end{aligned}$$

where $P_{45}^{(n)} = 0$ since state 4 and 5 do not communicate. Letting $n \rightarrow \infty$ and since $\lim_{n \rightarrow \infty} P_{65}^{(n)} = 2/7$, we get the equations

$$\pi_{15} = 0.2\pi_{15} + 0.4\pi_{25} + 0.1(2/7)$$

$$\pi_{25} = 0.1\pi_{15} + 0.3\pi_{25} + 0.2(2/7)$$

and can find the solutions

$$\pi_{15} = \frac{15}{52} \times \frac{2}{7} = \frac{15}{182}, \quad \pi_{25} = \frac{17}{52} \times \frac{2}{7} = \frac{17}{182}.$$

Exercise 4.50 on p.284 (Cont'd)

One can use the same method to find that

$$\begin{aligned}\pi_{14} &= \frac{37}{52} \times \frac{7}{13}, & \pi_{24} &= \frac{35}{52} \times \frac{7}{13} \\ \pi_{16} &= \frac{15}{52} \times \frac{5}{7}, & \pi_{26} &= \frac{17}{52} \times \frac{5}{7}\end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & \frac{37}{52} \times \frac{6}{13} & \frac{37}{52} \times \frac{7}{13} & \frac{15}{52} \times \frac{2}{7} & \frac{15}{52} \times \frac{5}{7} \\ 0 & 0 & \frac{35}{52} \times \frac{6}{13} & \frac{35}{52} \times \frac{7}{13} & \frac{17}{52} \times \frac{2}{7} & \frac{17}{52} \times \frac{5}{7} \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{matrix}$$