

STAT253/317 Lecture 17

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7.4 Renewal Reward Processes

7.5.1 Alternating Renewal Processes

7.4 Renewal Reward Processes

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times $\{X_i, i \geq 1\}$. Let $R_i, i = 1, 2, \dots$ be i.i.d random variables. R_i may depend on the i th interarrival time X_i , but (X_i, R_i) are i.i.d. random variable pairs. The compound process

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

is called a *renewal reward process*. R_i may be considered as *reward* earned during the i th cycle, and $R(t)$ represents the total reward earned up to time t .

Proposition 7.3 If $\mathbb{E}[R_1] < \infty$ and $\mathbb{E}[X_1] < \infty$, then

- (a) $\lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}$ with probability 1
- (b) $\lim_{t \rightarrow \infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}$

Proof of Proposition 7.3(a)

We give the proof for (a) only. To prove this, write

$$\frac{R(t)}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \frac{N(t)}{t}$$

By the Strong Law of Large Numbers (SLLN) and that $\lim_{t \rightarrow \infty} N(t) = \infty$ w/ prob. 1, we know

$$\frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \rightarrow \mathbb{E}[R_1] \quad \text{as } t \rightarrow \infty \quad \text{w/ prob. 1.}$$

By Proposition 7.1

$$\frac{N(t)}{t} \rightarrow \frac{1}{\mathbb{E}[X_1]} \quad \text{as } t \rightarrow \infty.$$

The result thus follows.

Example 7.12 (A Car Buying Model)

- ▶ Mr. Brown buys a new car whenever his old one breaks down or reaches the age of T years
- ▶ Let Y_i be the lifetime of his i th car. Suppose Y_i 's are i.i.d with CDF

$$H(y) = P(Y \leq y), \quad \text{and density } h(y) = H'(y).$$

- ▶ Cost to buy a new car = C_1 ;
- ▶ If the car breaks down, an additional cost of C_2 is incurred.
- ▶ What is Mr. Brown's long run average cost (per unit of time, not per car)?

Example 7.12 (A Car Buying Model) Solutions

- ▶ An event occurs whenever Mr. Brown buys a new car
- ▶ Interarrival times: $X_i = \min(Y_i, T)$
- ▶ Cost incurred in the i th cycle: $R_i = C_1 + C_2 \mathbf{1}_{\{Y_i \leq T\}}$
- ▶ Are (X_i, R_i) , $i = 1, 2, \dots$ i.i.d?
- ▶ Total cost up to time t : $R(t) = \sum_{i=1}^{N(t)} R_i$

$$\mathbb{E}[X_i] = \int_0^\infty \min(y, T) h(y) dy = \int_0^T y h(y) dy + T(1 - H(T))$$

$$\mathbb{E}[R_i] = C_1 + C_2 P(Y_i \leq T) = C_1 + C_2 H(T)$$

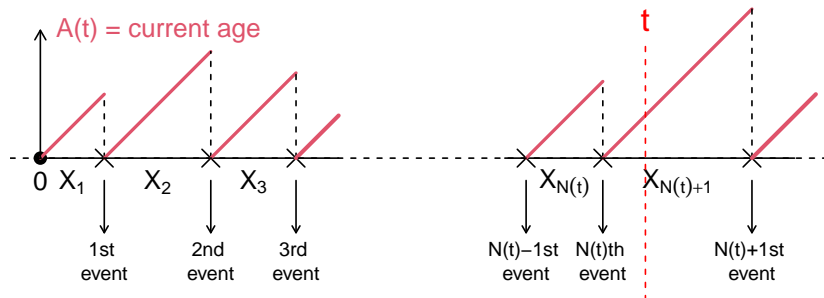
- ▶ average cost per car = $\mathbb{E}[R_i] = C_1 + C_2 H(T)$
- ▶ long-run average cost (per unit of time)

$$\lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{C_1 + C_2 H(T)}{\int_0^T y h(y) dy + T(1 - H(T))}$$

Example 7.18 Current Age

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times $\{X_i, i \geq 1\}$. Consider the **current age** of the item in use at time t

$$A(t) = t - S_{N(t)}.$$

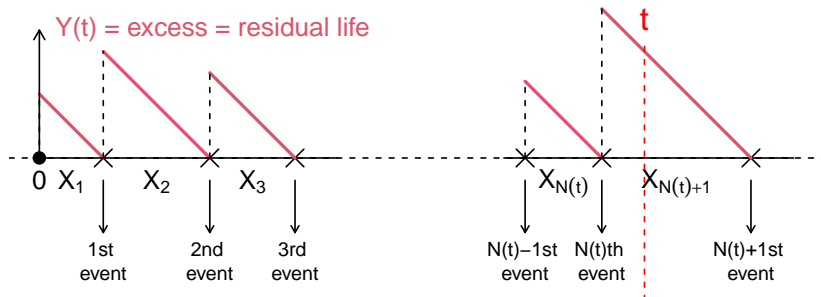


What is the long-run average of age $\lim_{t \rightarrow \infty} \frac{\int_0^t A(s) ds}{t}$?

Example 7.19 Residual Life of a Renewal Process

Consider the **residual life** or **excess** of the item in use at time t

$$Y(t) = S_{N(t)+1} - t.$$

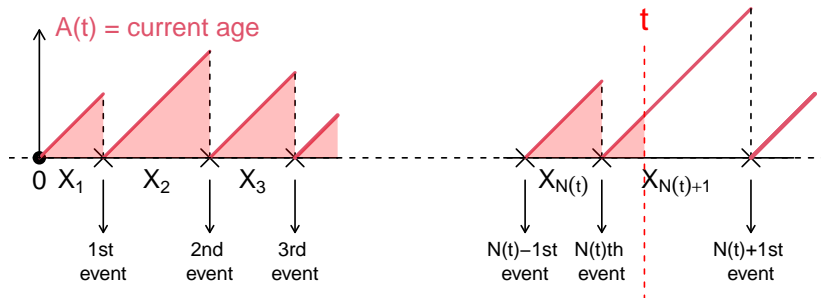


What is the long-run average of residual life

$$\lim_{t \rightarrow \infty} \frac{\int_0^t Y(s) ds}{t}?$$

Example 7.18 Age of a Reward Renewal Process (Cont'd)

Observe that $\int_0^t A(s)ds$ is the area of the shaded regions below.



$$\sum_{i=1}^{N(t)} \frac{X_i^2}{2} \leq \int_0^t A(s)ds < \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2}$$

Observe that $\sum_{i=1}^{N(t)} \frac{X_i^2}{2}$ is a renewal reward process

$R(t) = \sum_{i=1}^{N(t)} R_i$ with reward $R_i = X_i^2/2$.

Example 7.18 Current Age (Cont'd)

Since

$$R(t) \leq \int_0^t A(s)ds < R(t) + \frac{X_{N(t)+1}^2}{2},$$

and

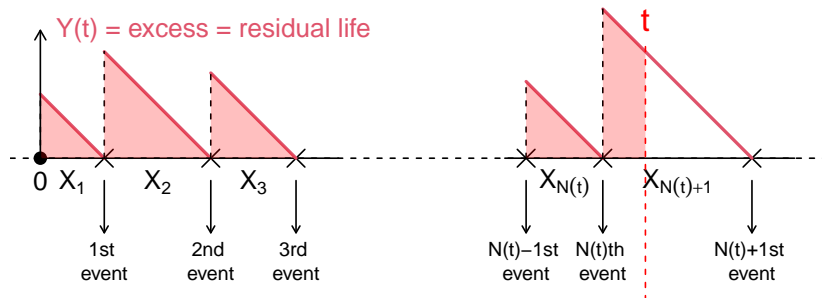
$$\frac{X_{N(t)+1}^2}{2t} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

by Proposition 7.3, the long-run average age of the item in use is

$$\frac{\int_0^t A(s)ds}{t} = \lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}.$$

Example 7.19 Residual Life (Cont'd)

Similarly, for the residual life, $\int_0^t Y(s)ds$ is the area of the shaded regions below.



$$\sum_{i=1}^{N(t)} \frac{X_i^2}{2} \leq \int_0^t Y(s)ds < \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2}$$

By the same argument, the long-run average of residual life of the item in use is

$$\frac{\int_0^t Y(s)ds}{t} = \lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}.$$

7.5.1 Alternating Renewal Processes

Considers a system that can be in one of two states: **ON** or **OFF**. Initially it is ON, and remains ON for a time Z_1 ; it then goes OFF and remains OFF for a time Y_1 . It then goes ON for a time Z_2 ; then OFF for a time Y_2 ; then on, and so on. Suppose

- ▶ (Z_k, Y_k) are i.i.d random vectors, though Z_k and Y_k might depend on each other
- ▶ Y_k, Z_k are non-negative with finite means.

Then a renewal process $\{N(t), t \geq 0\}$ with interarrival times

$$X_k = Z_k + Y_k, \quad k \geq 1$$

is called an *alternating renewal process*. Let

$$U(t) = \begin{cases} 1 & \text{if the system is ON at time } t \\ 0 & \text{otherwise} \end{cases}$$

Q: What is the long-run proportion of time that the system is ON?

$$\lim_{t \rightarrow \infty} \frac{\int_0^t U(s) ds}{t} ?$$

Alternating Renewal Processes (Cont'd)

An alternating renewal process can be regarded as a renewal reward process with reward $R_i = Z_i$,

$$R(t) = \sum_{i=1}^{N(t)} Z_i$$

Then

$$R(t) \leq \int_0^t U(s)ds < R(t) + Z_{N(t)+1}$$

By Proposition 7.3, with probability 1,

$$\lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]}$$

and hence

$$\lim_{t \rightarrow \infty} \frac{\int_0^t U(s)ds}{t} = \lim_{t \rightarrow \infty} \frac{R(t)}{t} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]} = \frac{\mathbb{E}[\text{ON}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]}.$$

Definition: Lattice Distribution

A random variable X is said to have a **lattice** distribution if there is an $h > 0$ for which

$$\sum_{k=-\infty}^{\infty} P(X = kh) = 1,$$

i.e., X is lattice if it only takes on integral multiples of some nonnegative number h . The largest h having this property is called the *period* of X .

Examples.

- ▶ Continuous distributions, mixtures of discrete and continuous distributions are both non-lattice.
- ▶ Integer-valued random variables are lattice, e.g., Poisson, binomial
- ▶ A lattice distribution must be discrete, but a discrete distribution may not be lattice, e.g., if

$$P(X = 1/n) = 1/2^n, \quad n = 1, 2, 3, \dots$$

then X is discrete but non-lattice because we cannot find an $h > 0$ such that all $1/n$'s are all multiples of h .

Theorem

If the interarrival distribution is non-lattice, then

$$\lim_{t \rightarrow \infty} P(\text{ON at time } t) = \lim_{t \rightarrow \infty} P(U(t) = 1) = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]}$$

Remark. If interarrival distribution is lattice, $\lim_{t \rightarrow \infty} P(U(t) = 1)$ may not exist.

Exercise 7.39

- ▶ Two machines work independently, each functions for an exponential time with rate λ and then fails
- ▶ A single repairmen. All repair times are independent with distribution function G
- ▶ If the repairmen is free when a machine fails, he will begin repairing that machine immediately; Otherwise, that machine must wait until the other machine has been repaired.
- ▶ Once repaired, a machine is as good as a new one.
- ▶ What proportion of time is the repairmen idle?

Solution.

- ▶ ON when the repairmen is idle, OFF when busy
- ▶ length of ON (idle) time: $Z \sim \text{Exp}(2\lambda)$, $\mathbb{E}[Z] = 1/(2\lambda)$
- ▶ length of OFF (busy) time Y ; want to find $\mathbb{E}[Y]$

Exercise 7.39 Solutions

- ▶ T = length of time to repair the first failing machine $\sim G$
- ▶ U = the time the working machine can function after the first machine failed. By the memoryless property, $U \sim \text{Exp}(\lambda)$
- ▶ Note that

$$Y = \begin{cases} T & \text{if } U > T \\ T + Y' & \text{if } U < T \end{cases}$$
$$= T + Y' \mathbf{1}_{\{U < T\}}$$

where Y' is the time the repairmen remains busy after the first failing machine is fixed. Note Y' is independent of T and U , and has the same distribution as Y . Thus

$$\mathbb{E}[Y] = \mathbb{E}[T] + \mathbb{E}[Y]P(T > U) \Rightarrow \mathbb{E}[Y] = \frac{\mathbb{E}[T]}{P(T < U)}$$

- ▶ long-run proportion of ON (idle) time

$$\frac{\mathbb{E}[Z]}{\mathbb{E}[Z] + \mathbb{E}[Y]} = \frac{1/(2\lambda)}{1/(2\lambda) + \mathbb{E}[Y]}$$

Example 7.23 & 7.24

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times $X_i, i = 1, 2, \dots$, where $\mu = \mathbb{E}[X_i]$ and $F(x) = P(X_i \leq x)$. Consider the **current age** of the item in use at time t

$$A(t) = t - S_{N(t)},$$

and the **residual life** of the item in use at time t

$$Y(t) = S_{N(t)+1} - t.$$

Proposition. The long-run proportion of time that $A(t) \leq x$ is the same as the long-run proportion of time that $Y(t) \leq x$, and is equal to

$$F_e(x) = \frac{1}{\mu} \int_0^x (1 - F(u)) du.$$

Furthermore, if F is non-lattice, then

$$\lim_{t \rightarrow \infty} P(A(t) \leq x) = \lim_{t \rightarrow \infty} P(Y(t) \leq x) = F_e(x).$$

Example 7.23 Current Age(Con'd)



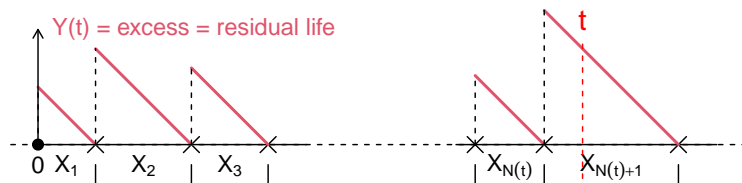
- ▶ let's say the system is ON at time t if $A(t) \leq x$
- ▶ length of ON time $Y_i = \min(X_i, x)$

$$\begin{aligned}\mathbb{E}[Y_i] &= \mathbb{E}[\min(X_i, x)] = \int_0^\infty \mathbb{P}(\min(X_i, x) > u) du \\ &= \int_0^x (1 - F(u)) du\end{aligned}$$

- ▶ length of a cycle = X_i , $\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_i] = \mu$
- ▶ long-run proportion of time that $A(t) \leq x$ is

$$\frac{\mathbb{E}[\text{ON}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{1}{\mu} \int_0^x (1 - F(u)) du$$

Example 7.24 Residual Life (Con'd)



- ▶ let's say the system is OFF at time t if $Y(t) \leq x$
- ▶ length of OFF time $Z_i = \min(X_i, x)$

$$\mathbb{E}[Z_i] = \mathbb{E}[\min(X_i, x)] = \int_0^x (1 - F(u)) du$$

- ▶ length of a cycle $= X_i$, $\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_i] = \mu$
- ▶ long-run proportion of time that $Y(t) \leq x$ is

$$\frac{\mathbb{E}[\text{OFF}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{1}{\mu} \int_0^x (1 - F(u)) du$$

Remark: The ON time in Example 7.23 is not the same as the ON time in Example 7.24