

6.5. Limiting Probabilities

Definition. Just like discrete-time Markov chains, if the probability that a continuous-time Markov chain will be in state j at time t , $P_{ij}(t)$, converges to a limiting value P_j independent of the initial state i , for all $i \in \mathcal{X}$

$$P_j = \lim_{t \rightarrow \infty} P_{ij}(t) > 0$$

then we say P_j is the *limiting probability* of state j . If P_j exists for all $j \in \mathcal{X}$, we say $\{P_j\}_{j \in \mathcal{X}}$ is the *limiting distribution* of the process.

Remark. If $\lim_{t \rightarrow \infty} P_{ij}(t)$ exists, we must have

$$\lim_{t \rightarrow \infty} P'_{ij}(t) = 0.$$

Balanced equations

Recall the forward equations $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

If you set $t \rightarrow \infty$, you have

$$0 = p^\top \mathbf{Q},$$

where $p = (P_1, P_2, \dots)^\top$

This is the same as saying that

$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} \quad \text{for all } j \in \mathcal{X}$$

Interpretation of the Balanced Equations

$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$$

$\nu_j P_j$ = rate at which the process **leaves** state j

$\sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$ = rate at which the process **enters** state j

Balanced equations means that the rates at which the process enters and leaves state j are equal.

The limiting distribution $\{P_j\}_{j \in \mathcal{X}}$ can be obtained by solving the balanced equations along with the equation $\sum_{j \in \mathcal{X}} P_j = 1$.

Remarks. Just like discrete-time Markov chains, a sufficient condition for the existence of a limiting distribution is that the chain is irreducible and positive recurrent.

Examples

- **Poisson processes:** $\mu_n = 0$, $\lambda_n = \lambda$ for all $n \geq 0$

$$\nu_i = \lambda, \quad P_{i,i+1} = 1, \quad q_{i,i+1} = \nu_i P_{i,i+1} = \lambda$$

Balanced equations:

$$\nu_j P_j = P_{j-1} q_{j-1,j} \quad \Rightarrow \quad \lambda P_j = \lambda P_{j-1} \quad \Rightarrow \quad P_j = P_{j-1}$$

No limiting distribution exists. The chain is not irreducible.
All states are transient.

- **Pure birth processes with $\lambda_n > 0$ for all n**

No limiting distribution exists. All states are transient.

Birth and Death Processes

For a birth and death process,

$$\begin{aligned} \nu_0 &= \lambda_0, \\ \nu_i &= \lambda_i + \mu_i, \quad i > 0 \\ P_{01} &= 1, \\ P_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i}, \quad i > 0 \\ P_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i}, \quad i > 0 \\ P_{i,j} &= 0 \quad \text{if } |i - j| > 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} q_{i,i+1} &= \nu_i P_{i,i+1} = \lambda_i, \quad i \geq 0 \\ q_{i,i-1} &= \nu_i P_{i,i-1} = \mu_i, \quad i \geq 1 \end{aligned}$$

Balanced Equations for Birth and Death Processes

The balanced equations $\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$ for a birth and death process are

$$\begin{aligned}\lambda_0 P_0 &= \mu_1 P_1 \\ (\mu_1 + \lambda_1) P_1 &= \lambda_0 P_0 + \mu_2 P_2, \\ (\mu_2 + \lambda_2) P_2 &= \lambda_1 P_1 + \mu_3 P_3, \\ &\vdots \\ (\mu_{n-1} + \lambda_{n-1}) P_{n-1} &= \lambda_{n-2} P_{n-2} + \mu_n P_n \\ (\mu_n + \lambda_n) P_n &= \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}\end{aligned}$$

Adding up all the equations above and eliminating the common terms on both sides, we get

$$\lambda_n P_n = \mu_{n+1} P_{n+1} \quad n \geq 0,$$

We hence just need to solve $\lambda_n P_n = \mu_{n+1} P_{n+1}$ for the limiting distribution.

6.6. Time Reversibility

Definition. A continuous-time Markov chain with state space \mathcal{X} is *time reversible* if

$$P_i q_{ij} = P_j q_{ji}, \quad \text{for all } i, j \in \mathcal{X} \quad (\text{detailed balanced equation})$$

If a distribution $\{P_j\}$ on \mathcal{X} satisfies the detailed balanced equation, then it is a stationary distribution for the process.

Example. We have just shown that for Birth and Death processes, the balanced equations would lead to the detailed balanced equations, which are

$$\lambda_n P_n = \mu_{n+1} P_{n+1}, \quad n \geq 0$$

Limiting Dist'n for Birth and Death Processes

Solving $\lambda_n P_n = \mu_{n+1} P_{n+1}$, $n \geq 0$ for the limiting distribution, we get

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1}}{\mu_n} \frac{\lambda_{n-2}}{\mu_{n-1}} P_{n-2} = \dots = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} P_0$$

To meet the requirement $\sum_{n=0}^{\infty} P_n = 1$, we need

$$\sum_{n=0}^{\infty} P_n = P_0 + P_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} = 1$$

In other words, to have a limiting distribution, it is necessary that

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$

Limiting Dist'n for Birth and Death Processes (Cont'd)

If $\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$ is finite, the limiting distribution is

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}}$$

and

$$P_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1} / (\mu_1 \mu_2 \cdots \mu_k)}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}}, \quad k \geq 1$$

Lemma: (Ratio Test) If $a_n \geq 0$ for all n , then

$$\sum_{n=1}^{\infty} a_n \begin{cases} < \infty & \text{if } \lim_{n \rightarrow \infty} a_n/a_{n-1} < 1 \\ = \infty & \text{if } \lim_{n \rightarrow \infty} a_n/a_{n-1} > 1 \end{cases}$$

For $a_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$, $a_n/a_{n-1} = \lambda_{n-1}/\mu_n$. By the ratio test, if

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\mu_n} < 1,$$

then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$, the limiting distribution exists.

Example 6.4 Linear Growth Model with Immigration

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Using the Ratio Test,

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\mu_n} = \lim_{n \rightarrow \infty} \frac{(n-1)\lambda + \theta}{n\mu} = \frac{\lambda}{\mu}$$

So the linear growth model has a limiting distribution if $\lambda < \mu$.

Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

- ▶ single-server service station. Service times are i.i.d. $\sim \text{Exp}(\mu)$
- ▶ Poisson arrival of customers with rate λ
- ▶ Upon arrival, a customer would
 - ▶ go into service if the server is free (queue length = 0)
 - ▶ join the queue if 1 to $N - 1$ customers in the station, or
 - ▶ **walk away** if N or more customers in the station

Q: What fraction of potential customers are lost?

Let $X(t)$ be the number of customers in the station at time t .

$\{X(t), t \geq 0\}$ is a birth-death process with the birth and death rates below

$$\mu_n = \begin{cases} 0 & \text{if } n = 0 \\ \mu & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad \lambda_n = \begin{cases} \lambda & \text{if } 0 \leq n < N \\ 0 & \text{if } n \geq N \end{cases}$$

Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

Solving $\lambda_n P_n = \mu_{n+1} P_{n+1}$ for the limiting distribution

$$P_1 = (\lambda/\mu)P_0$$

$$P_2 = (\lambda/\mu)P_1 = (\lambda/\mu)^2 P_0$$

$$\vdots$$

$$P_i = (\lambda/\mu)^i P_0, \quad i = 1, 2, \dots, N$$

Plugging $P_i = (\lambda/\mu)^i P_0$ into $\sum_{i=0}^N P_i = 1$, one can solve for P_0 and get

$$P_i = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^i$$

Answer: The fraction of customers lost is $P_N = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^N$

Duration Times for Birth and Death Processes

Let

T_i = time to move from state i to state $i + 1$, $i = 0, 1, \dots$

Suppose at some moment $X(t) = i$. Let

B_i = time until the next birth $\sim \text{Exp}(\lambda_i)$

D_i = time until the next death $\sim \text{Exp}(\mu_i)$

Then

$$\begin{aligned} T_i &= \begin{cases} B_i & \text{if the next step is } i \rightarrow i + 1, \text{ i.e., } B_i < D_i \\ D_i + T_{i-1} + T_i^* & \text{if the next step is } i \rightarrow i - 1, \text{ i.e., } D_i < B_i \end{cases} \\ &= \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i, \text{ occur w/ prob. } \frac{\lambda_i}{\lambda_i + \mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i, \text{ occur w/ prob. } \frac{\mu_i}{\lambda_i + \mu_i} \end{cases} \end{aligned}$$

Note

- ▶ T_i^* has the same distribution as T_i
- ▶ T_{i-1} and T_i^* are indep. of B_i and D_i because it's Markov

Duration Times for Birth and Death Processes

Taking expected value of

$$T_i = \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i, \text{ occur w/ prob. } \frac{\lambda_i}{\lambda_i + \mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i, \text{ occur w/ prob. } \frac{\mu_i}{\lambda_i + \mu_i} \end{cases}$$

we get

$$\begin{aligned} \mathbb{E}[T_i] &= \mathbb{E}[\min(B_i, D_i)] + (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i]) \frac{\mu_i}{\lambda_i + \mu_i} \\ &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i]) \end{aligned}$$

We obtain the recursive formula

$$\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$$

Duration Times for Birth and Death Processes (Cont'd)

Since $T_0 \sim \text{Exp}(\lambda_0)$, $\mathbb{E}[T_0] = 1/\lambda_0$.

Using the recursive formula $\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$, we have

$$\mathbb{E}[T_0] = \frac{1}{\lambda_0}$$

$$\mathbb{E}[T_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} \mathbb{E}[T_0] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0}$$

$$\mathbb{E}[T_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} \left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0} \right) = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2 \lambda_1} + \frac{\mu_2 \mu_1}{\lambda_2 \lambda_1 \lambda_0}$$

\vdots

$$\mathbb{E}[T_i] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} \mathbb{E}[T_{i-1}] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \cdots + \frac{\mu_i \mu_{i-1} \cdots \mu_2 \mu_1}{\lambda_i \lambda_{i-1} \cdots \lambda_2 \lambda_1 \lambda_0}$$

$$= \frac{1}{\lambda_i} \left(1 + \sum_{k=1}^i \prod_{j=1}^k \frac{\mu_{i-j+1}}{\lambda_{i-j}} \right)$$