# STAT253/317 Winter 2017 Lecture 25

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- Maximum of a Brownian Motion with drift $< 0$
- More Applications of Wald's Identities

# Maximum of a Brownian Motion with drift $< 0$

Let $\{B(t), t \geq 0\}$ be a Brownian Motion with drift coefficient $\mu < 0$ and variance parameter $\sigma^2$. Consider the maximum of the process

$$W = \max_{0 \leq t < \infty} B(t)$$

Then $W$ has the exponential distribution with rate $2|\mu|/\sigma^2$,

$$\mathrm{P}(W \geq w) = e^{-\frac{2|\mu|}{\sigma^2} w}, \quad w \geq 0.$$

*Proof.* Using the formula for $\mathrm{P}(B(T_{-a,b}) = b)$ we derived in the previous lecture, since $\mu < 0$, $\exp(2\mu a/\sigma^2) \to 0$, we have

$$\lim_{a \to \infty} \mathrm{P}(B(T_{-a,b}) = b) = \lim_{a \to \infty} \frac{e^{2\mu a/\sigma^2} - 1}{e^{2\mu a/\sigma^2} - e^{-2\mu b/\sigma^2}} = e^{2\mu b/\sigma^2}$$

The left-hand side becomes the probabilities that the process ever reaches $b$, that is, the probabilities that the maximum of the process ever exceeds $b$. Thus for $w = b$, we have

$$\mathrm{P}(W \geq w) = \exp(2\mu w/\sigma^2) = \exp(-2|\mu| w/\sigma^2).$$

# Hitting Times of Brownian Motion with Drift

Consider a Brownian motion process $\{B(t), t \geq 0\}$ with drift coefficient $\mu > 0$ and variance parameter $\sigma^2$ and consider the hitting time to the value $a > 0$

$$T_a = \min\{t : B(t) = a\}.$$

- Reflection principle doesn't apply to Brownian motion with drift.
- Alternative tool: Wald's identities
- $T_a$ is a stopping time: $\{T_a \leq t\} = \{\max_{0 \leq s \leq t} B(s) \geq a\}$ depends on $\{B(s), 0 \leq s \leq t\}$ only.
- $T_a$ is finite, but unbounded, can't apply Wald's identities directly. Try the truncated stopping time $T_a \wedge n = \min(T_a, n)$
- First Wald's identity $\Rightarrow$ $\mathbb{E}[B(T_a \wedge n)] = \mu\mathbb{E}[T_a \wedge n]$
- However, $|B(T \wedge n)|$ is not uniformly bounded. We cannot use the Dominated convergence theorem to show that

$$a = \mathbb{E}[B(T_a)] = \lim_{n\to\infty} \mathbb{E}[B(T_a \wedge n)] = \lim_{n\to\infty} \mu\mathbb{E}[T_a \wedge n] = \mu\mathbb{E}[T_a]$$

- Though the First Wald's identity doesn't help, the Third identity will. For $\theta > 0$, we have

$$\mathbb{E}[e^{\theta B(T_a \wedge n) - (\theta\mu + \frac{\theta^2\sigma^2}{2})(T_a \wedge n)}] = 1$$

- Since $\mu > 0$, $\mathrm{P}(T_a < \infty) = 1$, $(T_a \wedge n) \to T$ as $n \to \infty$
- By the continuity of Brownian motion path,

$$B(T_a \wedge n) \to B(T_a) = a \qquad \text{as } n \to \infty$$

- Since $B(T_a \wedge n) \leq a$ for all $n$, and that $\theta > 0$, $\mu > 0$, we have

$$e^{\theta B(T_a \wedge n) - (\theta\mu + \frac{\theta^2\sigma^2}{2})(T_a \wedge n)} \leq e^{\theta B(T_a \wedge n)} \leq e^{\theta a}$$

- By the Dominated Convergence Theorem,

$$\begin{aligned} 1 &= \lim_{n\to\infty} \mathbb{E}[e^{\theta B(T_a \wedge n) - (\theta\mu + \frac{\theta^2\sigma^2}{2})(T_a \wedge n)}] \\ &= \mathbb{E}[e^{\theta B(T_a) - (\theta\mu + \frac{\theta^2\sigma^2}{2})T_a}] = e^{\theta a}\mathbb{E}[e^{-(\theta\mu + \frac{\theta^2\sigma^2}{2})T_a}] \end{aligned}$$

  That is,

$$\mathbb{E}[e^{-(\theta\mu + \frac{\theta^2\sigma^2}{2})T_a}] = e^{-\theta a} \qquad \text{for all } \theta > 0$$

$$\mathbb{E}[e^{-(\theta\mu + \frac{\theta^2\sigma^2}{2})T_a}] = e^{-\theta a}$$

Make a change of variable by letting $\lambda = -(\theta\mu + \frac{\theta^2\sigma^2}{2})$. Then by solving the quadratic equation

$$\sigma^2\theta^2 + 2\mu\theta + 2\lambda = 0$$

for $\theta$, we will get

$$\theta = \frac{-\mu \pm \sqrt{\mu^2 - 2\lambda\sigma^2}}{\sigma^2}.$$

Since $\theta > 0$, we know that

$$\theta = \frac{-\mu + \sqrt{\mu^2 - 2\lambda\sigma^2}}{\sigma^2} > 0 \quad \text{for } \lambda \leq \frac{\mu^2}{2\sigma^2}$$

Hence, we can get the MGF for $T_a$,

$$M(\lambda) = \mathbb{E}[e^{\lambda T_a}] = \exp\left(\frac{\mu - \sqrt{\mu^2 - 2\lambda\sigma^2}}{\sigma^2}a\right), \quad \lambda \leq \frac{\mu^2}{2\sigma^2}$$

MGF for $T_a$ is

$$M(\lambda) = \mathbb{E}[e^{\lambda T_a}] = \exp\left(\frac{\mu - \sqrt{\mu^2 - 2\lambda\sigma^2}}{\sigma^2} a\right), \quad \lambda \le \frac{\mu^2}{2\sigma^2}$$

The cumulant generating function

$$K(\lambda) = \log(M(\lambda)) = \frac{a}{\sigma^2}(\mu - \sqrt{\mu^2 - 2\lambda\sigma^2}), \text{ for } \lambda \le \frac{\mu^2}{2\sigma^2}.$$

from which we can get that

$$K'(\lambda) = a(\mu^2 - 2\lambda\sigma^2)^{-1/2}, \quad K''(\lambda) = a\sigma^2(\mu^2 - 2\lambda\sigma^2)^{-3/2},$$

and that $\mathbb{E}[T_a] = K'(0) = \frac{a}{\mu}$, and $\mathrm{Var}(T_a) = K''(0) = \frac{a\sigma^2}{\mu^3}$.

It is possible to find the distribution with the MGF given above. The probability density of $T_a$ is

$$f(t) = \frac{a}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{(a-\mu t)^2}{2\sigma^2 t}\right), \quad t > 0$$

**Corollary:** For a Brownian motion process $\{B(t), t \geq 0\}$ with drift coefficient $\mu > 0$ and variance parameter $\sigma^2$, the distribution of $\max_{0 \leq s \leq t} B(s)$ is

$$\begin{aligned} \mathrm{P}\left(\max_{0 \leq s \leq t} B(s) \geq a\right) &= \mathrm{P}(T_a \leq t) \\ &= \int_0^t \frac{a}{\sigma\sqrt{2\pi u^3}} \exp\left(-\frac{(a-\mu u)^2}{2\sigma^2 u}\right) du \end{aligned}$$

# Example: $T_{-a,b}$ for SBM

For constants $a, b > 0$, recall in Lecture 24 we let $T = T_{-a,b}$ be the first time $t$ the standard Brownian Motion process hits $-a$ or $b$

$$T_{-a,b} = \min\{t : B(t) = -a, \text{ or } B(t) = b\}$$

and used the first and second Wald's identity to show that

$$\mathrm{P}(B(T) = -a) = \frac{b}{a+b}, \quad \mathrm{P}(B(T) = b) = \frac{a}{a+b}$$

and that

$$\mathbb{E}[T] = ab.$$

Today we will use the third Wald's identity to find the distribution of $T$.

$$\mathbb{E}[e^{\theta B(T \wedge n) - \theta^2 (T \wedge n)/2}] = 1$$

## Example: $T_{-a,b}$ for SBM (Cont'd)

- Since $\mathrm{P}(T < \infty) = 1$, $(T \wedge n) \to T$ as $n \to \infty$ w/ prob. 1.
- Since $-a \leq B(T \wedge n) \leq b \Rightarrow |B(T \wedge n)| \leq a + b$, for all $n$,

$$e^{\theta B(T\wedge n) - \theta^2 (T \wedge n)/2} \leq e^{\theta(a+b)}.$$

- By the Dominated Convergence Theorem,

$$\begin{aligned}
1 &= \lim_{n\to\infty} \mathbb{E}[e^{\theta B(T\wedge n) - \theta^2 (T\wedge n)/2}] = \mathbb{E}[e^{\theta B(T) - \theta^2 T/2}] \\
&= e^{-\theta a}\mathbb{E}[e^{-\theta^2 T/2} | B(T) = -a]\mathrm{P}(B(T) = -a) \\
&\quad + e^{\theta b}\mathbb{E}[e^{-\theta^2 T/2} | B(T) = b]\mathrm{P}(B(T) = b) \\
&= \frac{be^{-\theta a}}{a+b}\mathbb{E}[e^{-\theta^2 T/2} | B(T) = -a] + \frac{ae^{\theta b}}{a+b}\mathbb{E}[e^{-\theta^2 T/2} | B(T) = b]
\end{aligned}$$

# Example: $T_{-a,b}$ for SBM (Cont'd)

$$a+b = be^{-\theta a}\mathbb{E}[e^{-\theta^2 T/2}|B(T)=-a] + ae^{\theta b}\mathbb{E}[e^{-\theta^2 T/2}|B(T)=b].$$

This equation is valid if we change $\theta$ to $-\theta$:

$$a+b = be^{\theta a}\mathbb{E}[e^{-\theta^2 T/2}|B(T)=-a] + ae^{-\theta b}\mathbb{E}[e^{-\theta^2 T/2}|B(T)=b].$$

Now we have 2 equations with 2 unknowns
$x = \mathbb{E}[e^{-\theta^2 T/2}|B(T)=-a]$ and $y = \mathbb{E}[e^{-\theta^2 T/2}|B(T)=b]$.

$$\begin{cases} be^{-\theta a}x + ae^{\theta b}y = a+b \\ be^{\theta a}x + ae^{-\theta b}y = a+b \end{cases}$$

Solving the equations we can get that

$$x = \frac{a+b}{b}\frac{e^{b\theta}-e^{-b\theta}}{e^{(a+b)\theta}-e^{-(a+b)\theta}}, \quad y = \frac{a+b}{a}\frac{e^{a\theta}-e^{-a\theta}}{e^{(a+b)\theta}-e^{-(a+b)\theta}}$$

# Example: $T_{-a,b}$ for SBM (Cont'd)

Thus

$$\begin{aligned}
\mathbb{E}[e^{-\theta^2 T/2}] &= \mathbb{E}[e^{-\theta^2 T/2} | B(T) = -a] \mathrm{P}(B(T) = -a) \\
&\quad + \mathbb{E}[e^{-\theta^2 T/2} | B(T) = -a] \mathrm{P}(B(T) = b) \\
&= \frac{b}{a+b} x + \frac{a}{a+b} y = \frac{e^{b\theta} - e^{-b\theta} + e^{a\theta} - e^{-a\theta}}{e^{(a+b)\theta} - e^{-(a+b)\theta}} \\
&= \frac{e^{-a\theta} + e^{-b\theta} - e^{-(a+b)\theta}(e^{-a\theta} + e^{-b\theta})}{1 - e^{-2(a+b)\theta}} \\
&= \frac{e^{-a\theta} + e^{-b\theta}}{1 + e^{-(a+b)\theta}}
\end{aligned}$$

Let $\lambda = \theta^2/2$, we get that $\theta = \sqrt{2\lambda}$ and the moment generating function of $T$ that

$$\mathbb{E}[e^{-\lambda T}] = \frac{e^{-a\sqrt{2\lambda}} + e^{-b\sqrt{2\lambda}}}{1 + e^{-(a+b)\sqrt{2\lambda}}}$$