STAT253/317 Lecture 1

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4.1 Introduction to Markov Chains

Stochastic Processes

A stochastic process is a family of random variables $\{X_t : t \in \mathcal{T}\}$ such that

- For each $t \in \mathcal{T}$, X_t is a random variable

The index set $\mathcal{T}$ can be discrete or continuous

- $\mathcal{T} = \{0, 1, 2, 3, 4\}$
- $\mathcal{T} = \mathbb{R}, \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}^3$

Examples:

- Discrete Time Markov Chains .... Chapter 4
- Poisson Processes, Counting Processes .... Chapter 5
- Continuous Time Markov Chains .... Chapter 6
- Renewal Theory .... Chapter 7
- Queuing Theory .... Chapter 8
- Brownian Motion .... Chapter 10

4.1 Introduction to Markov Chain

Consider a stochastic process $\{X_n : n = 0, 1, 2, \dots\}$ taking values in a finite or countable set $\mathfrak{X}$.

- $\mathfrak{X}$ is called the **state space**
- If $X_n = i$, $i \in \mathfrak{X}$, we say the process is in state i at time n
- Since $\mathfrak{X}$ is countable, there is a 1-1 map from $\mathfrak{X}$ to the set of non-negative integers $\{0, 1, 2, 3, \dots\}$
  From now on, we assume $\mathfrak{X} = \{0, 1, 2, 3, \dots\}$

Definition

A stochastic process $\{X_n : \ n = 0, 1, 2, \dots\}$ is called a **Markov chain** if it has the following property:

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_2 = i_2, X_1 = i_1, X_0 = i_0)$$
$$= P(X_{n+1} = j \mid X_n = i)$$

for all states i_0, i_1, $i_2, \dots, i_{n-1}$, i, $j \in \mathfrak{X}$ and $n \geq 0$.

Transition Probability Matrix

If $P(X_{n+1} = j | X_n = i) = P_{ij}$ does not depend on n, then the process $\{X_n : n = 0, 1, 2, \ldots\}$ is called a **stationary Markov chain**. From now on, we consider stationary Markov chain only.

$\{P_{ij}\}$ is called the **transition probabilities**.
The matrix

$$\mathbb{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i0} & P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

is called the **transition probability matrix**.
Naturally, the transition probabilities $\{P_{ij}\}$ satisfy the following

- $P_{ij} \geq 0$ for all i, j
- Rows sums are 1: $\sum_j P_{ij} = 1$ for all i.

In other words, $\mathbb{P}\mathbf{1} = \mathbf{1}$, where $\mathbf{1} = (1, 1, \ldots, 1, \ldots)^\top$

Example 1: construct Markov Chain from i.i.d. sequence

Let $\{Y_n\}_{n\geq 0}$ be an i.i.d. sequence. The following two stochastic processes $\{X_n\}_{n\geq 0}$ are Markov chains

- $X_n = Y_n$
- $X_n = \sum_{k=0}^{n} Y_n$

Example 2: Random Walk

Consider the following random walk on integers

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } 1-p \end{cases}$$

This is a Markov chain because given $X_n, X_{n-1}, X_{n-2}, \ldots$, the distribution of X_{n+1} depends only on X_n but not $X_{n-1}, X_{n-2}, \ldots$.
The state space is

$$\mathfrak{X} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} = \mathbb{Z} = \text{all integers}$$

The transition probability is

$$P_{ij} = \begin{cases} p & \text{if } j = i+1 \\ 1-p & \text{if } j = i-1 \\ 0 & \text{otherwise} \end{cases}$$

Example 2: Random Walk (Cont'd)

$$
\mathbb{P} = \begin{array}{c c}
 & \begin{array}{ccccccccc} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \end{array} \\
\begin{array}{c} \vdots \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} &
\begin{pmatrix}
\ddots & \ddots & & & & & & & \\
\ddots & 0 & p & & & & & & \\
 & 1-p & 0 & p & & & & & \\
 & & 1-p & 0 & p & & & & \\
 & & & 1-p & 0 & p & & & \\
 & & & & 1-p & 0 & p & & \\
 & & & & & 1-p & 0 & p & \\
 & & & & & & 1-p & 0 & \ddots \\
 & & & & & & & \ddots & \ddots
\end{pmatrix}
\end{array}
$$

Example 3: Ehrenfest Diffusion Model

Two containers A and B, containing a sum of K balls. At each stage, a ball is selected at random from the totality of K balls, and move to the other container. Let

$X_0 = \#$ of balls in container A in the beginning

$X_n = \#$ of balls in container A after n movements, $n = 1, 2, 3, \ldots$

$$\mathfrak{X} = \{0, 1, 2, \ldots, K\}$$

$$P_{ij} = \begin{cases} \dfrac{i}{K} & \text{if } j = i - 1 \\[2ex] \dfrac{K-i}{K} & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

Joint Distribution of Random Variables in a Markov Chain

Suppose $\{X_n : n = 0, 1, 2, \ldots\}$ is a stationary Markov chain with

- state space $\mathfrak{X}$ and
- transition probabilities $\{P_{ij} : i, j \in \mathfrak{X}\}$.

Define $\pi_0(i) = \mathrm{P}(X_0 = i)$, $i \in \mathfrak{X}$ to be the distribution of X_0.

What is the joint distribution of X_0, X_1, X_2?

$$
\begin{aligned}
&\mathrm{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2) \\
&= \mathrm{P}(X_0 = i_0)\mathrm{P}(X_1 = i_1 | X_0 = i_0)\mathrm{P}(X_2 = i_2 | X_1 = i_1, X_0 = i_0) \\
&= \mathrm{P}(X_0 = i_0)\mathrm{P}(X_1 = i_1 | X_0 = i_0)\mathrm{P}(X_2 = i_2 | X_1 = i_1) \quad \text{(Markov)} \\
&= \pi_0(i_0) P_{i_0 i_1} P_{i_1 i_2}
\end{aligned}
$$

In general,

$$
\begin{aligned}
&\mathrm{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}, X_n = i_n) \\
&= \pi_0(i_0) P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}
\end{aligned}
$$