ELE 538B: Mathematics of High-Dimensional Data	Fall 2018
Homework 3	
Due date: Wednesday, Nov. 28, 2018 (at the beginning of class)	

You are allowed to drop 1 subproblem without penalty. In addition, up to 1 bonus point will be awarded to each subproblem for clean, well-organized, and elegant solutions.

1. Gaussian graphical models (*20 points*)

(a) Consider a p-dimensional Gaussian vector $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$. For any $1 \le u, v \le p$, show that

$$x_u \perp\!\!\!\perp x_v \mid \boldsymbol{x}_{\mathcal{V}\setminus\{u,v\}}$$

(namely, x_u and x_v are conditionally independent given all other variables) if and only if $\Theta_{u,v} = 0$. Here, $\boldsymbol{\Theta} = \boldsymbol{\Sigma}^{-1}$.

(b) In graphical lasso, the objective function includes a term $\log\det\boldsymbol{\Theta}$. Show that $g(\boldsymbol{\Theta}) := \log\det(\boldsymbol{\Theta})$ ($\boldsymbol{\Theta} \succ \boldsymbol{0}$) is a concave function.

Hint: A function $g(\boldsymbol{\Theta})$ is concave if $h(t) := g(\boldsymbol{\Theta} + t\boldsymbol{V})$ is concave for all t and $\boldsymbol{V}$ obeying $\boldsymbol{\Theta} + t\boldsymbol{V} \succ \boldsymbol{0}$.

2. Restricted isometry properties (*30 points*)

Recall that the restricted isometry constant $\delta_s \ge 0$ of $\boldsymbol{A}$ is the smallest constant such that

$$(1-\delta_s)\|\boldsymbol{x}\|_2^2 \le \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \le (1+\delta_s)\|\boldsymbol{x}\|_2^2 \tag{1}$$

holds for all s-sparse vector $\boldsymbol{x} \in \mathbb{R}^p$.

(a) Show that

$$|\langle \boldsymbol{A}\boldsymbol{x}_1, \boldsymbol{A}\boldsymbol{x}_2\rangle| \le \delta_{s_1+s_2}\|\boldsymbol{x}_1\|_2\|\boldsymbol{x}_2\|_2$$

for all pairs of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ that are supported on disjoint subsets $S_1, S_2 \subset \{1, \cdots, n\}$ with $|S_1| \le s_1$ and $|S_2| \le s_2$.

(b) For any $\boldsymbol{u}$ and $\boldsymbol{v}$, show that

$$|\langle \boldsymbol{u},\ (\boldsymbol{I} - \boldsymbol{A}^\top\boldsymbol{A})\boldsymbol{v}\rangle| \le \delta_s\|\boldsymbol{u}\|_2 \cdot \|\boldsymbol{v}\|_2,$$

where s is the cardinality of $\text{support}(\boldsymbol{u}) \cup \text{support}(\boldsymbol{v})$.

(c) Suppose that each column of $\boldsymbol{A}$ has unit norm. Show that $\delta_2 = \mu(\boldsymbol{A})$, where $\mu(\boldsymbol{A})$ is the mutual coherence of $\boldsymbol{A}$.

3. Statistical dimension (*10 points*)

Recall that for any convex cone $\mathcal{K}$, its statistical dimension and Gaussian width are defined respectively as

$$\text{stat-dim}(\mathcal{K}) := \mathbb{E}\big[\|\mathcal{P}_{\mathcal{K}}(\boldsymbol{g})\|_2^2\big]$$

and

$$w(\mathcal{K}) := \mathbb{E}\left[\sup_{\boldsymbol{z}\in\mathcal{K}, \|\boldsymbol{z}\|_2=1} \langle \boldsymbol{z}, \boldsymbol{g}\rangle\right],$$

where $\boldsymbol{g} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ and $\mathcal{P}_{\mathcal{K}}$ denotes the projection to $\mathcal{K}$ as

$$\mathcal{P}_{\mathcal{K}}(\boldsymbol{g}) = \arg\min_{\boldsymbol{z}\in\mathcal{K}} \|\boldsymbol{g} - \boldsymbol{z}\|_2.$$

(a) Prove that $w^2(\mathcal{K}) \leq \text{stat-dim}(\mathcal{K})$.

(b) (Optional (10 bonus points)) Prove the reverse inequality $\text{stat-dim}(\mathcal{K}) \leq w^2(\mathcal{K}) + 1$.
hint: Let $f(\cdot)$ be a function that is Lipschitz with respect to the Euclidean norm:

$$|f(\boldsymbol{u}) - f(\boldsymbol{v})| \leq M\|\boldsymbol{u} - \boldsymbol{v}\|_2 \qquad \forall \boldsymbol{u}, \boldsymbol{v}.$$

Then, $\text{Var}(f(\boldsymbol{g})) \leq M^2$.