STAT253/317 Lecture 15

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7.3. Limit Theorems

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Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d interarrival times X_i, $i = 1, 2, \ldots$ and $\mathbb{E}[X_i] = \mu$.

Explicit forms of $N(t)$ and $m(t) = \mathbb{E}[N(t)]$ are usually *unavailable*. However the limiting behavior of $N(t)$ and $m(t)$ is useful and intuitively makes sense.

As $t \to \infty$,

- $\dfrac{N(t)}{t} \to \dfrac{1}{\mu}$ with probability 1 **(Proposition 7.1)**
- $\dfrac{m(t)}{t} \to \dfrac{1}{\mu}$ **(Thm 7.1 Elementary Renewal Theorem)**

Remark.

- The number $1/\mu$ is called the **rate** of the renewal process
- Theorem 7.1 is not a simple consequence of Proposition. 7.1, since $X_n \to X$ w/ prob. 1 does not ensure $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

$X_n \to X$ Does Not Ensure $\mathbb{E}[X_n] \to \mathbb{E}[X]$

Example 7.8 Let U be a random variable which is uniformly distributed on (0, 1); and define the random variables X_n, $n \geq 1$, by

$$X_n = \begin{cases} 0 & \text{if } U > 1/n \\ n & \text{if } U \leq 1/n \end{cases}$$

Then $\mathrm{P}(X_n = 0) = P(U > 1/n) = 1 - 1/n \to 1$ as $n \to \infty$. So with probability 1

$$X_n \to X = 0.$$

However,

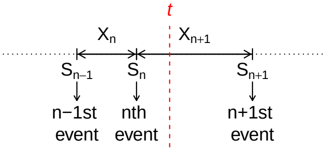
$$\mathbb{E}[X_n] = 0\mathrm{P}(X_n = 0) + n\mathrm{P}(X_n = n) = n \times \frac{1}{n} = 1 \quad \text{for all } n \geq 1.$$

and hence $\lim_{n\to\infty} \mathbb{E}[X_n] = 1 \neq \mathbb{E}[X] = \mathbb{E}[0] = 0.$

Proof of Proposition 7.1

Since $S_{N(t)} \leq t < S_{N(t)+1}$, we know

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{S_{N(t)+1}}{N(t)}.$$



By SLLN, $\frac{S_{N(t)}}{N(t)} = \frac{\sum_{i=1}^{N(t)} X_i}{N(t)} \to \mu$ as $N(t) \to \infty$, we obtain $\frac{S_{N(t)}}{N(t)} \to \mu$ as $t \to \infty$. Furthermore, writing

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \times \frac{N(t)+1}{N(t)}$$

we have that $S_{N(t)+1}/(N(t)+1) \to \mu$ by the same reasoning as before and

$$\frac{N(t)+1}{N(t)} \to 1 \text{ as } t \to \infty \quad \text{since } P(\lim_{t\to\infty} N(t) = \infty) = 1$$

Hence, $S_{N(t)+1}/N(t) \to \mu$.

Stopping Time

Definition. Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables. An integer-valued random variable $N > 0$ is said to be a *stopping time* w/ respect to $\{X_n : n \geq 1\}$ if the event $\{N = n\}$ is independent of $\{X_k : k \geq n+1\}$.

Example. (*Independent case.*)
If N is independent of $\{X_n : n \geq 1\}$, then N is a stopping time.

Example. (*Hitting Time* I.) For any set A, the first time X_n hits set A, $N_A = \min\{n : X_n \in A\}$, is a stopping time because

$$\{N_A = n\} = \{X_i \notin A \text{ for } i = 1, 2, \ldots, n-1, \text{ but } X_n \in A\}$$

is independent of $\{X_k : k \geq n+1\}$.

Example. (*Hitting Time* II.) For $n \geq 1$, let $S_n = \sum_{k=1}^{n} X_k$.
For any set A, $N_A = \min\{n : S_n \in A\}$, the first time S_n hits set A, is also a stopping time w/ respect to $\{X_n : n \geq 1\}$ because
$\{N_A = n\} = \{\sum_{k=1}^{i} X_k \notin A \text{ for } 1 \leq i \leq n-1, \text{ but } \sum_{k=1}^{n} X_k \in A\}$
is independent of $\{X_k : k \geq n+1\}$.

Example of Non-Stopping Times

- (*Last visit time*) The last time that X_n visit a set A

  $$N_A = \max\{n : X_n \in A\}$$

  is NOT a stopping time.
  Clearly we need to know whether A will be visited again in the future to determine such a time.
- The time X_n reaches its maximum,

  $$N = \min\{n : X_n = \max_{k \geq 1} X_k\},$$

  is NOT a stopping time since

  $$\{N = n\} = \{X_n > X_k \text{ for } 1 \leq k < n \text{ and } k \geq n+1\}$$

  depends on $\{X_k : k \geq n+1\}$.

Renewal Processes and Stopping Times

Consider a renewal process $N(t)$. With respect to its interarrival times X_1, $X_2, \ldots$,

- $N(t)$ is NOT a stopping time.

  $$N(t) = n \Leftrightarrow X_1 + \cdots + X_n \leq t \text{ and } X_1 + \cdots + X_{n+1} > t,$$

  depends on X_{n+1}.
- But $N(t) + 1$ is a stopping time, since

  $$\begin{aligned} N(t) + 1 = n &\Leftrightarrow N(t) = n - 1 \\ &\Leftrightarrow X_1 + \cdots + X_{n-1} \leq t \text{ and } X_1 + \cdots + X_n > t, \end{aligned}$$

  is independent of X_{n+1}, $X_{n+2}, \ldots$.

Wald's Equation

If $X_1, X_2 \ldots$ are i.i.d. with $\mathbb{E}[X_i] < \infty$, and if N is a stopping time for this sequence with $\mathbb{E}[N] < \infty$, then

$$\mathbb{E}\left[\sum_{j=1}^{N} X_j\right] = \mathbb{E}[N]\mathbb{E}[X_1]$$

Proof. Let us define the indicator variable

$$I_j = \begin{cases} 1 & \text{if } j \leq N \\ 0 & \text{if } j > N. \end{cases}$$

We have

$$\sum_{j=1}^{N} X_j = \sum_{j=1}^{\infty} X_j I_j$$

Hence

$$\mathbb{E}\left[\sum_{j=1}^{N} X_j\right] = \mathbb{E}\left[\sum_{j=1}^{\infty} X_j I_j\right] = \sum_{j=1}^{\infty} \mathbb{E}[X_j I_j] \qquad (1)$$

Proof of Wald's Equation (Cont'd)

Note I_j and X_j are independent because

$$I_j = 0 \quad \Leftrightarrow \quad N < j \quad \Leftrightarrow \quad N \leq j - 1$$

and the event $\{N \leq j-1\}$ depends on $X_1, \ldots, X_{j-1}$ only, but not X_j. From (1), we have

$$\begin{aligned} \mathbb{E}\left[\sum_{j=1}^{N} X_j\right] &= \sum_{j=1}^{\infty} \mathbb{E}[X_j I_j] = \sum_{j=1}^{\infty} \mathbb{E}[X_j]\mathbb{E}[I_j] \\ &= \mathbb{E}[X_1] \sum_{j=1}^{\infty} \mathbb{E}[I_j] = \mathbb{E}[X_1] \sum_{j=1}^{\infty} \mathrm{P}(N \geq j) \\ &= \mathbb{E}[X_1]\mathbb{E}[N] \end{aligned}$$

Here we use the alternative formula $\mathbb{E}[N] = \sum_{j=1}^{\infty} \mathrm{P}(N \geq j)$ to find expected values of non-negative integer valued random variables.

Proposition 7.2

$$\mathbb{E}[S_{N(t)+1}] = \mu(m(t)+1)$$

Proof: Since $N(t)+1$ is a stopping time, by Wald's equation, we have

$$\mathbb{E}[S_{N(t)+1}] = \mathbb{E}\left[\sum_{j=1}^{N(t)+1} X_j\right] = \mathbb{E}[N(t)+1]\mathbb{E}[X_1] = (m(t)+1)\mu$$

Since $S_{N(t)+1} = t + Y(t)$, where $Y(t)$ is the residual life at t, taking expectations and using the result above yields

$$\mathbb{E}[S_{N(t)+1}] = \mu(m(t)+1) = t + \mathbb{E}[Y(t)].$$

So far we have proved Proposition 7.2 and can deduce that

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{\mathbb{E}[Y(t)]}{t\mu}.$$

Proof of the Elementary Renewal Theorem

First from Proposition 7.2, we have

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{\mathbb{E}[Y(t)]}{t\mu} \geq \frac{1}{\mu} - \frac{1}{t} \quad \Rightarrow \quad \lim_{t\to\infty} \frac{m(t)}{t} \geq \frac{1}{\mu}.$$

It remains to show that $\lim_{t\to\infty} \dfrac{m(t)}{t} \leq \dfrac{1}{\mu}$.

If the interarrival times $X_1, X_2, \ldots$ are bounded by a constant M, then the residual life $Y(t)$ is also bounded by M. Hence,

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \lim_{t\to\infty} \frac{1}{\mu} - \frac{1}{t} + \frac{M}{t\mu} = \frac{1}{\mu}$$

The Elementary Renewal Theorem for renewal process with **bounded interarrival times** is proved.

Proof of the Elementary Renewal Theorem (Cont'd)

In general, if the interarrival times $X_1, X_2, \dots$ are not bounded, we fix a constant M and define a new renewal process $N_M(t)$ with the truncated interarrival times

$$\min(X_1, M), \min(X_2, M), \dots, \min(X_n, M), \dots.$$

Because $\min(X_i, M) \le X_i$ for all i, it follows that $N_M(t) \ge N(t)$ for all t.

$$\lim_{t\to\infty} \frac{m(t)}{t} = \lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} \le \lim_{t\to\infty} \frac{\mathbb{E}[N_M(t)]}{t} = \frac{1}{\mathbb{E}[\min(X_1, M)]}$$

by the Elementary Renewal Theorem with bounded interarrival times. Note the inequality above is valid for all $M > 0$. Letting $M \to \infty$ yields

$$\lim_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\mu}.$$

Here we use the fact that $\mathbb{E}[\min(X_1, M)] \to \mathbb{E}[X_1] = \mu$ as $M \to \infty$.

Example 7.6 (M/G/1 with no Queue)

- Single-server bank
- Potential customers arrive at a Poisson rate λ
- Customers enter the bank only if the server is free
- Service times are i.i.d. with mean μ_G, indep. of the arrival
- Let $N(t) =$ number of customers entry the bank by time t and those who arrive finding the server busy and walk away don't count. Is $\{N(t) : t \geq 0\}$ a (delayed) renewal process?

Ans. An interarrival time $T_i = G_i + W_i$ where

$G_i =$ service time, i.i.d., w/ mean μ_G

$W_i =$ waiting time until the next customer arrives after the previous one i

As potential customers arrive following a Poisson process, by the memoryless property, W_i's are i.i.d. Exp(λ).

The interarrival times $\{T_i\} = \{G_i + W_i\}$ are i.i.d. The events of customers entering constitutes a renewal process

Example 7.6 (M/G/1 with no Queue)

Q: What is the rate at which customers enter the bank?

- As $\mathbb{E}[T_i] = \mathbb{E}[G_i] + \mathbb{E}[W_i] = \mu_G + \dfrac{1}{\lambda}$, by the Elementary Renewal Theorem, the rate is

$$\frac{1}{\mathbb{E}[T_i]} = \frac{1}{\mu_G + \frac{1}{\lambda}} = \frac{\lambda}{\lambda\mu_G + 1}$$

Q: What is the proportion of potential customers that are lost?

- As potential customers arrive at rate λ, and customers enter at the rate $\dfrac{\lambda}{\lambda\mu_G + 1}$, the proportion that actually enter the bank is

$$\frac{\lambda/(\lambda\mu_G + 1)}{\lambda} = \frac{1}{\lambda\mu_G + 1}$$

So the proportion that is lost is $1 - \dfrac{1}{\lambda\mu_G + 1} = \dfrac{\lambda\mu_G}{\lambda\mu_G + 1}$.