STAT253/317 Winter 2013 Lecture 27

Yibi Huang

Mar 13, 2013

- Itô's Integral
- Itô's Formula

Adapted Processes

Let $\{B(t), t \geq 0\}$ be the standard Brownian motion process.

We say a stochastic process $\{X(t), t \geq 0\}$ is *adapted* to $\{B(t), t \geq 0\}$ if for each t, $X(t)$ is known given $\{B(u), 0 \leq u \leq t\}$, and $X(t)$ does not depend on what occurs in the future $\{B(u), u > t\}$.

Example. The following $X(t)$'s are all adapted to $\{B(t), t \geq 0\}$.

- $X(t) = f(t, B(t))$, where $f(t, x)$ is a non-random function
- $X(t) = \max_{0 \leq s \leq u} B(u)$
- $X(t) = B(t + a)$, if $a < 0$

However, if $a > 0$, $X(t) = B(t + a)$ is NOT adapted to $\{B(t), t \geq 0\}$ since $X(t)$ depends on the future $B(t + a)$.

Itô Integral

Let $B(t)$ be a standard Brownian Motion and $X(t)$ be a process adapted to $\{B(t), t \geq 0\}$ with the property that

$$\mathbb{E}\left[\int_0^T X^2(t)dt\right] \leq \infty \quad \text{for some } T$$

For $0 \leq a < b \leq T$, the integral $\int_a^b X(t)dB(t)$ is defined as

$$\int_a^b X(t)dB(t) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} X(t_j)[B(t_{j+1}) - B(t_j)]$$

where $\|\Pi\|$ is the mesh size $\max\limits_{0 \leq j \leq n-1} |t_{j+1} - t_j|$ of the partition

$$\Pi = \{a = t_0 < t_1 < \ldots < t_n = b\}.$$

We omit the proof to show that the integral is well-defined.

Properties of the Itô Integral

Let $X(t)$ be a process adapted to the filtration $\{\mathcal{F}_t, t \geq 0\}$ with

$$\mathbb{E}\left[\int_0^T X^2(t)dt\right] \leq \infty.$$

For $0 \leq t \leq T$, the process

$$I(t) = \int_0^t X(u)dB(u)$$

defined as in the previous page has the following properties

- **Continuity**: As a function of t, the paths of $I(t)$ are continuous
- **Adaptivity**: $\{I(t), t \geq 0\}$ is adapted to $\{B(t), t \geq 0\}$
- **Linearity**: If $\{X(t)\}$ and $\{Y(t)\}$ are both adapted to $\{B(t), t \geq 0\}$, then for all constants a and b

$$\int_0^t aX(u)+bY(t)dB(u) = a\int_0^t X(u)dB(u)+b\int_0^t Y(u)dB(u)$$

Proof. Omitted

Mean of Itô Integral

Let $I(t) = \int_0^t X(u)dB(u)$. Then $\mathbb{E}[I(t)] = 0$.

Proof.

We omit the justification of the interchangeability of taking limit and expectation. Since $\{X(t)\}$ is adapted to $\{B(t)\}$, $X(t_j)$ is independent of $B(t_{j+1}) - B(t_j)$.

$$\begin{aligned}
\mathbb{E}[I(t)] &= \lim_{\|\Pi\|\to 0} \mathbb{E}\Big[\sum_{j=0}^{n-1} X(t_j)[B(t_{j+1}) - B(t_j)]\Big] \\
&= \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} \mathbb{E}[X(t_j)]\underbrace{\mathbb{E}[B(t_{j+1}) - B(t_j)]}_{=0} \\
&= 0
\end{aligned}$$

Remark. This is not true for Stratonovich integral. For example, in Stratonovich sense,

$$\mathbb{E}\left[\int_0^t B(u)dB(u)\right] = \frac{1}{2}\mathbb{E}[B^2(t)] = \frac{t}{2} > 0.$$

Variance of Itô Integral

Let $I(t) = \int_0^t X(u)dB(u)$. Then $\mathrm{Var}(I(t)) = \int_0^t \mathbb{E}[X^2(u)]du$.

Proof. From that $\mathbb{E}[\sum_{j=0}^{n-1} X(t_j)[B(t_{j+1}) - B(t_j)]] = 0$, we know

$$\begin{aligned}
&\mathrm{Var}\left(\sum_{j=0}^{n-1} X(t_j)[B(t_{j+1}) - B(t_j)]\right) \\
&= \mathbb{E}\left\{\sum_{j=0}^{n-1} X(t_j)[B(t_{j+1}) - B(t_j)]\right\}^2 = I + II
\end{aligned}$$

where

$$I = \sum_{j=0}^{n-1} \mathbb{E}\{X^2(t_j)[B(t_{j+1}) - B(t_j)]^2\}$$

$$II = \sum_{0 \le i < j \le n-1} 2\mathbb{E}\{X(t_i)X(t_j)[B(t_{i+1}) - B(t_i)][B(t_{j+1}) - B(t_j)]\}$$

Since $\{X(t)\}$ is adapted to $\{B(t)\}$, $X(t_j)$ is independent of $B(t_{j+1}) - B(t_j)$, we have

$$I = \sum_{j=0}^{n-1} \mathbb{E}[X^2(t_j)]\mathbb{E}[B(t_{j+1}) - B(t_j)]^2 = \sum_{j=0}^{n-1} \mathbb{E}[X^2(t_j)](t_{j+1} - t_j)$$

which approaches $\int_0^t \mathbb{E}[X^2(u)]du$ as $\|\Pi\| \to 0$.

Variance of Itô Integral (Cont'd)

For II, as $\{X(t)\}$ is adapted, for $t_i < t_j$, $X(t_j)$, $X(t_i)$, and $B(t_{i+1}) - B(t_i)$ depend only on those happen by time t_j, and hence are independent of $B(t_{j+1}) - B(t_j)$. Thus we have

$$\begin{aligned} &\mathbb{E}\{X(t_j)X(t_i)[B(t_{i+1}) - B(t_i)][B(t_{j+1}) - B(t_j)]\} \\ &= \mathbb{E}\{X(t_j)X(t_i)[B(t_{i+1}) - B(t_i)]\} \underbrace{\mathbb{E}[B(t_{j+1}) - B(t_j)]}_{=0} = 0 \end{aligned}$$

As all the terms in II are 0, we know $II = 0$. We can see that

$$\begin{aligned} \mathrm{Var}(I(t)) = \mathbb{E}[I^2(t)] &= \lim_{\|\Pi\| \to 0} \mathbb{E}\left\{\sum_{j=0}^{n-1} X(t_j)[B(t_{j+1}) - B(t_j)]\right\}^2 \\ &= \lim_{\|\Pi\| \to 0} I + II = \int_0^t \mathbb{E}[X^2(u)]du. \end{aligned}$$

Again we omit the justification of the interchangeability of taking limit and expectation.

Quadratic Variation of Itô Integral

Let $I(t) = \int_0^t X(u)dB(u)$. The quadratic variation of $I(t)$ up to time t is

$$[I, I](t) = \int_0^t X^2(u)du.$$

Informal Proof.

$$\begin{aligned}\sum_{j=0}^{n-1}[I(t_{j+1}) - I(t_j)]^2 &= \sum_{j=0}^{n-1}\left(\int_{t_j}^{t_{j+1}} X(u)dB(u)\right)^2 \\ &\approx \sum_{j=0}^{n-1} X^2(t_j)[B(t_{j+1}) - B(t_j)]^2 \\ &\approx \sum_{j=0}^{n-1} X^2(t_j)(t_{j+1} - t_j)\end{aligned}$$

As $\|\Pi\| \to 0$, the quantity on the left hand side approaches the quadratic variation of $I(t)$ up to time t, $[I, I](t)$, and the last quantity on the right hand side approaches $\int_0^t X^2(u)du$.

Remark. The two approximations above all need rigorous justifications.

Itô's Formula

Let $u(x,t)$ be a function of $x \in \mathbb{R}$ and $t \geq 0$ that is twice continuously differentiable in x and once continuously differentiable in t, and let $\{B(t)\}$ be a Brownian motion process. Denote by u_t, u_x, and u_{xx} the first and second partial derivatives of u with respect to the variables t and x. Then

$$\begin{aligned} u(B(t),t) - u(0,0) = &\int_0^t u_x(B(s),s)dB(s) \\ &+ \int_0^t u_t(B(s),s)ds + \frac{1}{2}\int_0^t u_{xx}(B(s),s)ds. \end{aligned}$$

Example 1. Let $u(t,x) = x^2/2$, then $u_t = 0$, $u_x = x$, $u_{xx} = 1$. By Itô's Formula, we have

$$\frac{1}{2}B^2(t) - \frac{1}{2}\underbrace{B^2(0)}_{=0} = \int_0^t B(s)dB(s) + \frac{1}{2}\underbrace{\int_0^t 1ds}_{=t}$$

consistent with the result we got in Lecture 26, that

$$\int_0^t B(s)dB(s) = [B^2(t) - t]/2.$$

Example 2

Let $u(x,t) = f(t)x$, where $f(t)$ is a differentiable function. Then $u_t = f'(t)x$, $u_x = f(t)$, $u_{xx} = 0$. By Itô's Formula, we have

$$f(t)B(t) - f(0)B(0) = \int_0^t f(s)dB(s) + \int_0^t f'(s)B(s)ds$$

Then we get the "integration-by-part formula" for non-random integrant

$$\int_0^t f(s)dB(s) = f(t)B(t) - f(0)B(0) - \int_0^t f'(s)B(s)ds.$$

It is not hard to show that when $X(t) = f(t)$ is non-random, the Itô integral

$$I(t) = \int_0^t X(u)dB(u), \quad t \geq 0$$

is also a Gaussian process. However, when $X(t)$ is random, $\{I(t)\}$ is usually no longer Gaussian.

Example 3. Stock Price Model

First consider a discrete-time model that S_n is the price of a certain stock at period n. Assume that the return in one period is a constant μ plus a noise ϵ, that is

$$\frac{S_{n+1} - S_n}{S_n} = \mu + \epsilon_{n+1}.$$

The noise terms ϵ_i's are usually assume to be i.i.d. with mean 0 and variance σ^2. As we shrink the length of one time period, the mean return μ should be shrinked proportionally, and the variability of the noise should be decreased, too.

For these reason, in continuous-time, stock price is often modeled as

$$\frac{S(t+\Delta t) - S(t)}{S(t)} = \mu \Delta t + \sigma \underbrace{[B(t+\Delta t) - B(t)]}_{\sim N(0,\Delta t)} \qquad (1)$$

or is written as

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t). \qquad (2)$$

Example 3. Stock Price Model

The formal meaning of equation (2) is the integral equation,

$$S(t) - S(0) = \int_0^t \mu S(t)dt + \int_0^t \sigma S(t)dB(t).$$

From the expression (1) above, we can see that the last term $\int_0^t \sigma S(t)dB(t)$ is an Itô integral rather than a Stratonovich integral.

Now we will use Itô's formula to find a solution to the integral equation above. Let $u(x,t) = \exp(ax + bt)$, then $u_t = bu(x,t)$, $u_x = au(x,t)$, $u_{xx} = a^2 u(x,t)$. By Itô's Formula, we have

$$u(B(s),s) - 1 = \int_0^t a\, u(B(s),s)dB(s) + \int_0^t (b + \frac{a^2}{2})u(B(s),s)ds.$$

We can see that if let $a = \sigma$, $b = \mu - \sigma^2/2$, then

$$S(t) = u(B(s),s) = \exp(\sigma B(t) + (\mu - \sigma^2/2)t)$$

is a solution to the integral equation above.

Sketch of Proof of Itô's Formula

Let $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a *partition* of $[0, t]$.
Taking a Taylor expansion of $u(x, t)$ with respect to the point $(B(t_j), t_j)$, we have

$$
\begin{aligned}
u(x,t) - u(B(t_j), t_j) = {} & u_x(B(t_j), t_j)(x - B(t_j)) \\
& + u_t(B(t_j), t_j)(t - t_j) \\
& + \frac{1}{2} u_{xx}(B(t_j), t_j)(x - B(t_j))^2 \\
& + u_{tx}(B(t_j), t_j)[x - B(t_j)](t - t_j) \\
& + \frac{1}{2} u_{tt}(B(t_j), t_j)(t - t_j)^2 \\
& + \text{higher-order terms}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
u(B(t), t) - u(B(0), 0) &= \sum_{j=0}^{n-1} u(B(t_{j+1}), t_{j+1}) - u(B(t_j), t_j) \\
&= I + II + III + IV + V
\end{aligned}
$$

where

$$
\begin{aligned}
I &= \sum_{j=0}^{n-1} u_x(B(t_j), t_j)(B(t_{j+1}) - B(t_j)) \longrightarrow \int_0^t u_x(B(t), t)dB(t) \\
II &= \sum_{j=0}^{n-1} u_t(B(t_j), t_j)(t_{j+1} - t_j) \qquad\qquad\quad \longrightarrow \int_0^t u_t(B(t), t)dt \\
III &= \frac{1}{2}\sum_{j=0}^{n-1} u_{xx}(B(t_j), t_j)(B(t_{j+1}) - B(t_j))^2 \\
&\approx \frac{1}{2}\sum_{j=0}^{n-1} u_{xx}(B(t_j), t_j)(t_{j+1} - t_j) \qquad \longrightarrow \int_0^t u_{xx}(B(t), t)dt \\
IV &= \sum_{j=0}^{n-1} u_{tx}(B(t_j), t_j)[B(t_{j+1}) - B(t_j)](t_{j+1} - t_j) \longrightarrow 0 \\
V &= \frac{1}{2}\sum_{j=0}^{n-1} u_{tt}(B(t_j), t_j)(t_{j+1} - t_j)^2 \qquad\qquad\qquad \longrightarrow 0
\end{aligned}
$$

as $\|\Pi\| \to 0$.

The limit of I and II are straightforward from the definition of Itô's integral and Riemann integral.
The approximation of III must be done by showing

$$\sum_{j=0}^{n-1} u_{xx}(B(t_j), t_j)[(B(t_{j+1}) - B(t_j))^2 - (t_{j+1} - t_j)] \to 0$$

as $\|\Pi\| \to 0$.