Problems

February 10, 2017

1. Why is L2 norm a vector norm?

   Solution.

   We have to show that for any $\mathbf{v} \in \mathbb{R}^n$,

   $$\|\mathbf{v}\|_2 \geq 0, \tag{1}$$
   $$\|\alpha\mathbf{v}\|_2 = |\alpha|\|\mathbf{v}\|_2, \quad \forall \alpha \in \mathbb{R}, \tag{2}$$
   $$\|\mathbf{v}\|_2 = 0 \iff \mathbf{v} = \mathbf{0}, \tag{3}$$
   $$\|\mathbf{v}+\mathbf{u}\|_2 \leq \|\mathbf{v}\|_2 + \|\mathbf{u}\|_2. \tag{4}$$

   (1) is straightforward by the definition of L2 norm. (2) is followed as

   $$\|\alpha\mathbf{v}\|_2 = \|(\alpha v_1, \cdots, \alpha v_n)\|_2 = \sqrt{(\alpha v_1)^2 + \cdots + (\alpha v_n)^2} \tag{5}$$
   $$= \sqrt{\alpha^2}\sqrt{v_1^2 + \cdots + v_n^2} = |\alpha|\|\mathbf{v}\|_2. \tag{6}$$

   (3) holds, because

   $$\|\mathbf{v}\|_2 = \mathbf{0} \iff v_1^2 + \cdots + v_n^2 = 0 \iff v_1 = \cdots = v_n = 0 \iff \mathbf{v} = \mathbf{0}.$$

   The inequality (4), which is called as triangle inequality for vectors, is derived as

   $$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 =& (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})\\ =& \|\mathbf{u}\|_2^2 + 2(\mathbf{u}\cdot\mathbf{v}) + \|\mathbf{v}\|_2^2\\ \leq& \|\mathbf{u}\|_2^2 + 2|\mathbf{u}\cdot\mathbf{v}| + \|\mathbf{v}\|_2^2\\ \overset{(a)}{\leq}& \|\mathbf{u}\|_2^2 + 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|_2^2\\ =& (\|\mathbf{v}\|_2 + \|\mathbf{u}\|_2)^2,\end{aligned}$$

   where (a) holds by Cauchy-Schwarz inequality.

2. Verify that the Cauchy-Schwarz inequality holds for

   $$\mathbf{u} = (1, 3, 5, 2, 0, 1), \ \mathbf{v} = (0, 2, 4, 1, 3, 5).$$

Solution.

$$\begin{aligned}
\|\mathbf{u}\|_2 &= \sqrt{1^2+3^2+5^2+2^2+0^2+1^2} = \sqrt{40},\\
\|\mathbf{v}\|_2 &= \sqrt{0^2+2^2+4^2+1^2+3^2+5^2} = \sqrt{55},\\
\mathbf{u}\cdot\mathbf{v} &= 1\cdot 0+3\cdot 2+5\cdot 4+2\cdot 1+0\cdot 3+1\cdot 5 = 33,\\
(\mathbf{u}\cdot\mathbf{v})^2 &= 1089 \le \|\mathbf{u}\|_2^2\|\mathbf{v}\|_2^2 = 2200.
\end{aligned}$$

3. Let W_1 and W_2 be a subspaces of $\mathbb{R}^n$. Prove that the intersection $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

   Solution.

   Since every subspace includes the zero vector,

   $$\mathbf{0} \in W_1 \cap W_2,$$

   and therefore, we know that $W_1 \cap W_2$ is nonempty. Assume $\mathbf{x} \in W_1 \cap W_2$. Then, $\mathbf{x} \in W_1$ and $\mathbf{x} \in W_2$. By the definition of subspaces, for any $c \in \mathbb{R}$, $c\mathbf{x} \in W_1$ and $c\mathbf{x} \in W_2$. Thus, $c\mathbf{x} \in W_1 \cap W_2$, and $W_1 \cap W_2$ is closed under scalar multiplication. Also, by the definition of subspaces, for any $\mathbf{v}, \mathbf{u} \in W_1 \cap W_2$,

   $$\mathbf{v}+\mathbf{u} \in W_1,\ \ \mathbf{v}+\mathbf{u} \in W_2.$$

   Hence, $\mathbf{v}+\mathbf{u} \in W_1 \cap W_2$, which holds for any vectors. It was shown that $W_1 \cap W_2$ is closed under addition. Therefore, $W_1 \cap W_2$ is also a subspace.

4. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, then so is the set $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ for every nonzero scalar k.

   Solution.

   Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent,

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \implies c_1 = c_2 = c_3 = 0.$$

   Based on this, it is shown that $k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3$ are also linearly independent.

   $$\begin{aligned}
   c_1(k\mathbf{v}_1) + c_2(k\mathbf{v}_2) + c_3(k\mathbf{v}_3) = \mathbf{0} &\implies (c_1k)\mathbf{v}_1 + (c_2k)\mathbf{v}_2 + (c_3k)\mathbf{v}_3 = \mathbf{0}\\
   &\overset{(a)}{\implies} c_1k = c_2k = c_3k = 0 \overset{(b)}{\implies} c_1 = c_2 = c_3 = 0,
   \end{aligned}$$

   where (a) holds by the linear independence of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and (b) holds because k is nonzero.

5. Find all values of λ for which $\det(A) = 0$.

   $$A = \begin{bmatrix} \lambda-4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda-1 \end{bmatrix}$$

Solution. Cofactors and minors are denoted as C_{ij}, M_{ij}. Using cofactor expansion,

$$\begin{aligned}\det(A) =& a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ =&(\lambda - 4)M_{11} \\ =&(\lambda - 4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} \\ =&(\lambda - 4)(\lambda^2 - \lambda - 6) \\ =&(\lambda - 4)(\lambda + 2)(\lambda - 3).\end{aligned}$$

The values of λ satisfying $\det(A) = 0$ are $\lambda = 4, -2, 3$.

6. Find the determinant of the following matrix by inspection, not by calculation.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Solution.

Since the determinant is not affected by adding a multiple of row to another row, we can add $-1/4, -2/4, -3/4$ times the 4th row to 1st, 2nd, 3rd rows, respectively. Then, every element in the 4th column except for 4 becomes 0. In a similar way, every element except for those in the main diagonal becomes zero. Thus, the determinant of the given matrix is the same as that of

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix},$$

which is $\det(A) = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

7. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}.$$

Solution.

To find the eigenvalues we will solve the characteristic equation of A. Since

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix},$$

the characteristic equation $\det(\lambda I - A) = 0$ is

$$\begin{vmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{vmatrix} = 0.$$

Expanding and simplifying the determinant yields

$$\lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) = 0.$$

To find the eigenspaces corresponding to these eigenvalues we must solve the system

$$\begin{bmatrix} \lambda - 1 & -3 \\ -4 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

With $\lambda = -2$, it becomes

$$\begin{bmatrix} -3 & -3 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving this system yields

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

With $\lambda = -5$, it becomes

$$\begin{bmatrix} 4 & -3 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving this system yields

$$\begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

8. (a) Compute the operator norm of the following matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

Solution.
Since the operator norm of A is the same as $\sqrt{\lambda_{\max}(A^T A)}$, it suffices to figure out the eigenvalue of the following matrix:

$$A^T A = \begin{bmatrix} 35 & 44 \\ 44 & 56 \end{bmatrix}.$$

The characteristic equation is

$$\det(A^TA - \lambda I) = \lambda^2 - 91\lambda + 24 = 0$$

and we have eigenvalues

$$\lambda_1 \approx 90.7, \quad \lambda_2 \approx 0.265.$$

Thus, choosing the maximal one,

$$\|A^TA\| = \sqrt{90.7} \approx 9.53.$$

(b) What is the vector achieving the following maximum?

$$\max_{\|\mathbf{x}\| \neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

Solution.
The vector achieving the maximum is an eigenvector corresponding to the maximal eigenvalue of A^TA. Solving the following system,

$$A^TA\mathbf{x} = \lambda_1\mathbf{x}$$

we have

$$\mathbf{x} = t\begin{bmatrix} 0.62 \\ 0.785 \end{bmatrix}.$$

We can check that this achieves the maximum as

$$\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} = \left\| \begin{bmatrix} 2.19 \\ 5.00 \\ 7.81 \end{bmatrix} \right\| = 9.53$$

9. Show that the vectors

$$\mathbf{v}_1 = (1,0,0), \ \mathbf{v}_2 = (0,2,0),$$
$$\mathbf{v}_3 = (0,0,3), \ \mathbf{v}_4 = (1,1,1)$$

span $\mathbb{R}^3$ but do not form a basis for $\mathbb{R}^3$.

Solution.

We have to show first that

$$\text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_4\} = \mathbb{R}^3.$$

Since $\mathbf{v}_1, \cdots, \mathbf{v}_4 \in \mathbb{R}^3$, any linear combination of them is also in $\mathbb{R}^3$, and therefore,

$$\text{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_4\} \subset \mathbb{R}^3.$$

For any $(x, y, z) \in \mathbb{R}^3$,

$$(x, y, z) = x \cdot (1, 0, 0) + \frac{y}{2} \cdot (0, 2, 0) + \frac{z}{3} \cdot (0, 0, 3) = x\mathbf{v}_1 + \frac{y}{2}\mathbf{v}_2 + \frac{z}{3}\mathbf{v}_3,$$

and $(x, y, z) \in \operatorname{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}$. Thus,

$$\mathbb{R}^3 \subset \operatorname{span}\{\mathbf{v}_1, \cdots, \mathbf{v}_4\}.$$

Even though the vectors span $\mathbb{R}^3$, they are not a basis, because they are linearly dependent. In $\mathbb{R}^3$, the maximum number of elements in a linearly independent set is 3, so the vectors cannot be linearly independent.

10. Prove that if $\mathbf{u}$ is a nonzero $n \times 1$ column vector, then the outer product $\mathbf{u}\mathbf{u}^T$ is a symmetric matrix of rank 1.

Solution.

$$\mathbf{u}\mathbf{u}^T = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2^2 & \cdots & u_2u_n \\ \vdots & \vdots & & \vdots \\ u_nu_1 & u_nu_2 & \cdots & u_n^2 \end{bmatrix}.$$

It is symmetric, because

$$\left(\mathbf{u}\mathbf{u}^T\right)_{ij} = (\mathbf{u})_i \left(\mathbf{u}^T\right)_j = u_iu_j,$$
$$\left(\mathbf{u}\mathbf{u}^T\right)_{ji} = (\mathbf{u})_j \left(\mathbf{u}^T\right)_i = u_iu_j$$

for any $1 \le i, j \le n$. The m-th row of $\mathbf{u}\mathbf{u}^T$ is

$$\mathbf{r}_m \left(\mathbf{u}\mathbf{u}^T\right) = u_m \cdot \mathbf{u}^T.$$

Thus, every row is a scalar multiple of $\mathbf{u}^T$, which implies that the row space has dimension 1.

11. Prove that if $V = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, and if $W = \{\mathbf{w}_1, \cdots, \mathbf{w}_{n-k}\}$ is a basis for the null space of the matrix A that has the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ as its successive rows, then $V \cup W$ is a basis for $\mathbb{R}^n$.

Solution.

Since $V \cup W$ contains n vectors, it suffices to show that $V \cup W$ is linearly independent. Assume

$$\sum_{i=1}^{k} c_i\mathbf{v}_i + \sum_{j=1}^{n-k} d_j\mathbf{w}_j = 0.$$

Equivalently,

$$\sum_{i=1}^{k} c_i \mathbf{v}_i = \sum_{j=1}^{n-k} (-d_j) \mathbf{w}_j.$$

The left-hand side is a linear combination of elements of V and is included in the row space of A. The right-hand side is a linear combination of elements of W and is included in the null space of A. The only vector that is in the row space and the null space of a matrix is the zero vector,

$$\sum_{i=1}^{k} c_i \mathbf{v}_i = \sum_{j=1}^{n-k} (-d_j) \mathbf{w}_j = \mathbf{0}.$$

Since V and W is linearly independent sets, $c_i = d_j = 0$ for all i, j. Thus, $V \cup W$ is a linearly independent set.

12. Find the least squares solution of $A\mathbf{x} = \mathbf{b}$ by solving the associated normal system, and show that the least square error vector is orthogonal to the column space of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}.$$

Solution. The normal system for the equation $A\mathbf{x} = \mathbf{b}$ is $A^T A\mathbf{x} = A^T \mathbf{b}$, or,

$$\begin{bmatrix} 21 & 25 \\ 25 & 35 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}.$$

The solution of this is

$$\mathbf{x} = \frac{1}{11} \begin{bmatrix} 20 \\ -8 \end{bmatrix},$$

which is the least squares solution of $A\mathbf{x} = \mathbf{b}$. The least square error vector is

$$\mathbf{b} - A\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \frac{1}{11} \begin{bmatrix} 28 \\ 16 \\ 40 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} -6 \\ -27 \\ 15 \end{bmatrix}.$$

This is orthogonal to the column space of A, because

$$\frac{1}{11} \begin{bmatrix} -6 \\ -27 \\ 15 \end{bmatrix}^T \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 0,$$

$$\frac{1}{11} \begin{bmatrix} -6 \\ -27 \\ 15 \end{bmatrix}^T \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = 0.$$

13. Find the eigenvalue decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Solution.

Following the method from Problem 7, eigenvalues and corresponding eigenvectors are

$$\begin{aligned} &\lambda = 0,\ \mathbf{x}_1 = (1,0,-1), \mathbf{x}_2(0,1,-1), \\ &\lambda = 3,\ \mathbf{x}_3 = (1,1,1). \end{aligned}$$

In order to find the orthonormal basis of eigenspace, we have to find the orthonormal basis of $\mathbf{x}_1, \mathbf{x}_2$. First, normalize $\mathbf{x}_1, \mathbf{x}_2$ into $\frac{1}{\sqrt{2}}(1,0,-1)$, $\frac{1}{\sqrt{2}}(0,1,-1)$. Subtract projection of the second one onto the first one as

$$\frac{1}{\sqrt{2}}(0,1,-1) - \left[\frac{1}{\sqrt{2}}(0,1,-1) \cdot \frac{1}{\sqrt{2}}(1,0,-1)\right] \frac{1}{\sqrt{2}}(1,0,-1) = \frac{1}{\sqrt{2}}\left(-\frac{1}{2}, 1, -\frac{1}{2}\right).$$

This vector is normalized into $\frac{1}{\sqrt{6}}(1,-2,1)$. So, an orthonormal basis for $\operatorname{span}(\mathbf{x}_1, \mathbf{x}_2)$ is

$$\left\{\frac{1}{\sqrt{2}}(1,0,-1),\ \frac{1}{\sqrt{6}}(1,-2,1)\right\}.$$

Since $\mathbf{x}_3$ is orthogonal to the other two eigenvectors, it suffices to normalize it as

$$\frac{\mathbf{x}_3}{\|\mathbf{x}_3\|} = \frac{1}{\sqrt{3}}(1,1,1).$$

Based on these results, A is decomposed as

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}^T.$$

14. Find a singular value decomposition of A.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Solution.

(SVD is not unique. The following solution shows one of possible options.)

$$A^T A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I.$$

The eigenvalue is 2 and every nonzero vector is an eigenvector corresponding to this eigenvalue. We can choose any orthonormal basis of $\mathbb{R}^2$, one of which is

$$\{(0,1),\ (1,0)\}.$$

Using these as column vectors, construct an orthogonal matrix V, which orthogonally diagonalizes A^TA, as

$$V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Another orthogonal matrix for SVD is made as

$$A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_1 = \frac{A\mathbf{v}_1}{\|A\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$A\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{u}_2 = \frac{A\mathbf{v}_1}{\|A\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since a singular value of A is 1,

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

and the decomposition is

$$A = U\Sigma V^T = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T.$$

15. Suppose that A has the singular value decomposition

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 24 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

(a) Find orthonormal bases for the four fundamental spaces of A.

Solution.

We can use the columns of U, V in the decomposition to construct bases of the fundamental spaces. Since the rank A is 2, the first two columns of U form an orthonormal basis for $\mathrm{col}(A)$:

$$\left\{\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)\right\}.$$

The last two columns of U form an orthonormal basis for null(A^T):

$$\left\{\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\right\}.$$

The first two rows of V form an orthonormal basis for row(A):

$$\left\{\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right),\left(\frac{2}{3},\frac{2}{3},-\frac{1}{3}\right)\right\}.$$

The last row of V forms an orthonormal basis for null(A):

$$\left\{\left(-\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)\right\}.$$

(b) Find the reduced singular value decomposition of A.
Solution.

$$A=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}24 & 0\\ 0 & 12\end{bmatrix}\begin{bmatrix}\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}.$$

16. Prove that nuclear norm has the following property:

$$\|A\|_* \triangleq \sum_{i=1}^{\min(m,n)} \sigma_i = \operatorname{tr}(\sqrt{A^TA}),$$

for $A\in\mathbb{R}^{m\times n}$, where $\sqrt{A^TA}$ if a matrix B such that $B^2=A^TA$.

Solution.

$\sqrt{A^TA}$ is well-defined, because it is positive semidefinite. Since A^TA is symmetric, it can be decomposed as

$$A^TA=VDV^T,$$

where D is nonnegative and diagonal, and V is orthogonal, and the elements of D are eigenvalues of A^TA, which are the squares of singular values of A. Let $B=V\sqrt{D}V^T$. Then,

$$B^2=V\sqrt{D}V^TV\sqrt{D}V^T=V\sqrt{D}\sqrt{D}V^T=VDV^T=A^TA.$$

However,

$$\begin{aligned}\operatorname{tr}(B)&=\sum_{i=1}^n B_{ii}=\sum_{i=1}^n\sum_{j=1}^n\left(V\sqrt{D}\right)_{ij}\left(V^T\right)_{ji}=\sum_{i=1}^n\sum_{j=1}^n\left(V\sqrt{D}\right)_{ij}v_{ij}\\ &\overset{(a)}{=}\sum_{i=1}^n\sum_{j=1}^n v_{ij}\sigma_j v_{ij}=\sum_{j=1}^n\sigma_j\sum_{i=1}^n v_{ij}^2\overset{(b)}{=}\sum_{j=1}^n\sigma_j\end{aligned}$$

where (a) holds because D is diagonal, and (b) holds since every column of V has unit norm. When $m < n$, $\sigma_j = 0$ for $j > \min(m, n)$. Thus, it becomes

$$\mathrm{tr}(B) = \sum_{i=1}^{\min(m,n)} \sigma_i.$$