STAT253/317 Lecture 9

Cong Ma

Chapter 5 Poisson Processes

5.2 Exponential Distribution

Let X follow exponential distribution with rate λ: $X \sim Exp(\lambda)$.

- Density: $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- CDF: $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$
- $\mathbb{E}(X) = 1/\lambda$, $\mathbb{V}ar(X) = 1/\lambda^2$
- If $X_1, \ldots, X_n$ are i.i.d $Exp(\lambda)$, then $S_n = X_1 + \cdots + X_n \sim Gamma(n, \lambda)$, with density

$$f_{S_n}(x) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

The Exponential Distribution is Memoryless ($\star\star\star\star\star$)

Lemma: for all $s, t \geq 0$

$$\mathrm{P}(X > t + s \mid X > t) = \mathrm{P}(X > s)$$

Proof.

$$\begin{aligned}
\mathrm{P}(X > t + s | X > t) &= \frac{\mathrm{P}(X > t + s \text{ and } X > t)}{\mathrm{P}(X > t)} \\
&= \frac{\mathrm{P}(X > t + s)}{\mathrm{P}(X > t)} \\
&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = \mathrm{P}(X > s)
\end{aligned}$$

Implication. If the lifetime of batteries has an Exponential distribution, then *a used battery is as good as a new one*, as long as it's not dead!

Another Important Property of the Exponential

If $X_1, \dots, X_n$ are independent, $X_i, \sim Exp(\lambda_i)$ for $i = 1, \dots, n$ then

(i) $\min(X_1, \dots, X_n) \sim Exp(\lambda_1 + \dots + \lambda_n)$, and

(ii) $\mathrm{P}\big(X_j = \min(X_1, \dots, X_n)\big) = \dfrac{\lambda_j}{\lambda_1 + \dots + \lambda_n}$

Proof of (i)

$$\begin{aligned} &\mathrm{P}(\min(X_1, \dots, X_n) > t) = \mathrm{P}(X_1 > t, \dots, X_n > t) \\ &= \mathrm{P}(X_1 > t) \dots \mathrm{P}(X_n > t) = e^{-\lambda_1 t} \cdots e^{-\lambda_n t} \\ &= e^{-(\lambda_1 + \dots + \lambda_n)t}. \end{aligned}$$

Proof of (ii)

$$\begin{aligned}
& \mathrm{P}\big(X_j = \min(X_1, \dots, X_n)\big) \\
&= \mathrm{P}(X_j < X_i \text{ for } i = 1, \dots, n, i \neq j) \\
&= \int_0^\infty \mathrm{P}(X_j < X_i \text{ for } i \neq j | X_j = t) \lambda_j e^{-\lambda_j t} dt \\
&= \int_0^\infty \mathrm{P}(t < X_i \text{ for } i \neq j) \lambda_j e^{-\lambda_j t} dt \\
&= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} \mathrm{P}(X_i > t) dt \\
&= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} e^{-\lambda_i t} dt \\
&= \lambda_j \int_0^\infty e^{-(\lambda_1 + \cdots + \lambda_n) t} dt \\
&= \frac{\lambda_j}{\lambda_1 + \cdots + \lambda_n}
\end{aligned}$$

Example 5.8: Post Office

- A post office has two clerks.
- Service times for clerk $i \sim Exp(\lambda_i)$, $i = 1$, 2
- When you arrive, both clerks are busy but no one else waiting. You will enter service when either clerk becomes free.
- Find $\mathbb{E}[T]$, where $T =$ the amount of time you spend in the post office.

Solution. Let $R_i =$ remaining service time of the customer with clerk i, $i = 1$, 2.

- Note R_i's are indep. $\sim Exp(\lambda_i)$, $i = 1$, 2 by the memoryless property
- Observe $T = \min(R_1, R_2) + S$ where S is your service time
- Using the property of exponential distributions,

$$\min(R_1, R_2) \sim Exp(\lambda_1 + \lambda_2) \quad \Rightarrow \quad \mathbb{E}[\min(R_1, R_2)] = \frac{1}{\lambda_1 + \lambda_2}$$

Example 5.8: Post Office (Cont'd)

As for your service time S, observe that

$$S \sim \begin{cases} Exp(\lambda_1) & \text{if } R_1 < R_2 \\ Exp(\lambda_2) & \text{if } R_2 < R_1 \end{cases} \Rightarrow \begin{array}{l} \mathbb{E}[S|R_1 < R_2] = 1/\lambda_1 \\ \mathbb{E}[S|R_2 < R_1] = 1/\lambda_2 \end{array}$$

Recall that $\mathrm{P}(R_1 < R_2) = \lambda_1/(\lambda_1 + \lambda_2)$ So

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[S|R_1 < R_2]\mathrm{P}(R_1 < R_2) + \mathbb{E}[S|R_2 < R_1]\mathrm{P}(R_2 < R_1) \\ &= \frac{1}{\lambda_1} \times \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2} \times \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{2}{\lambda_1 + \lambda_2} \end{aligned}$$

Hence the expected amount of time you spend in the post office is

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}[\min(R_1, R_2)] + \mathbb{E}[S] \\ &= \frac{1}{\lambda_1 + \lambda_2} + \frac{2}{\lambda_1 + \lambda_2} = \frac{3}{\lambda_1 + \lambda_2}. \end{aligned}$$

5.3.1. Counting Processes

A counting process $\{N(t)\}$ is a cumulative count of number of events happened up to time t.

Definition.

A stochastic processes $\{N(t), t \geq 0\}$ is a *counting process* satisfying

(i) $N(t) = 0, 1, \ldots$ (integer valued),

(ii) If $s < t$, then $N(s) \leq N(t)$.

(iii) For $s < t$, $N(t) - N(s) =$ number of events that occur in the interval $(s, t]$.

Definition.

A process $\{X(t), t \geq 0\}$ is said to have *stationary increments* if for any $t > s$, the distribution of $X(t) - X(s)$ depends on s and t only through the difference $t - s$, for all $s < t$.
That is, $X(t+a) - X(s+a)$ has the same distribution as $X(t) - X(s)$ for any constant a.

Definition.

A process $\{X(t), t \geq 0\}$ is said to have *independent increments* if for any $s_1 < t_1 \leq s_2 < t_2 \leq \ldots \leq s_k < t_k$, the random variable $X(t_1) - X(s_1)$, $X(t_2) - X(s_2), \ldots, X(t_k) - X(s_k)$ are independent, i.e. the numbers of events that occur in **disjoint** time intervals are **independent**.

Example. Modified simple random walk $\{X_n, n \geq 0\}$ is a process with independent and stationary increment, since $X_n = \sum_{k=0}^{n} \xi_k$ where ξ_k's are i.i.d with $\mathrm{P}(\xi_k = 1) = p$ and $\mathrm{P}(\xi_k = 0) = 1 - p$.

Definition 5.1 of Poisson Processes

A Poisson process with rate $\lambda > 0$ $\{N(t), t \geq 0\}$ is a counting process satisfying

(i) $N(0) = 0$,

(ii) For $s < t$, $N(t) - N(s)$ is independent of $N(s)$ (independent increment)

(iii) For $s < t$, $N(t) - N(s) \sim Poi(\lambda(t-s))$, i.e.,

$$\mathrm{P}(N(t) - N(s) = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$$

Remark: In (iii), the distribution of $N(t) - N(s)$ depends on $t - s$ only, not s, which implies $N(t)$ has stationary increment.

Definition 5.3 of Poisson Processes

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ, $\lambda > 0$, if

(i) $N(0) = 0$.

(ii) The process has stationary and independent increments.

(iii) $\mathrm{P}(N(h) = 1) = \lambda h + o(h)$.

(iv) $\mathrm{P}(N(h) \geq 2) = o(h)$.

Theorem 5.1 Definitions 5.1 and 5.3 are equivalent.
[Proof of Definitions 5.1 ⇒ Definition 5.3]
From Definitions 5.1, $N(h) \sim Poi(h)$. Thus

$$\begin{aligned}\mathrm{P}(N(h) = 1) &= \lambda h e^{-\lambda h} = \lambda h + o(h) \\ \mathrm{P}(N(h) \geq 2) &= 1 - \mathrm{P}(N(h) = 0) - \mathrm{P}(N(h) = 1) \\ &= 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} = o(h)\end{aligned}$$

Proof of Definitions 5.3 ⇒ Definition 5.1:
See textbook.