Refined analysis of local convergence: implicit regularization

Cong Ma

University of Chicago, Autumn 2021

A natural least-squares formulation

$$\text{given:} \qquad y_k \;=\; (\boldsymbol{a}_k^\top \boldsymbol{x}^\star)^2, \quad 1 \le k \le m$$

$$\Downarrow$$

$$\text{minimize}_{\boldsymbol{x}\in\mathbb{R}^n} \quad f(\boldsymbol{x}) = \frac{1}{4m}\sum_{k=1}^{m}\Big[\big(\boldsymbol{a}_k^\top \boldsymbol{x}\big)^2 - y_k\Big]^2$$

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- **pros:** often exact as long as sample size is sufficiently large
- **cons:** $f(\cdot)$ is highly nonconvex
  $\longrightarrow$ *computationally challenging!*

Wirtinger flow (Candès, Li, Soltanolkotabi '14)

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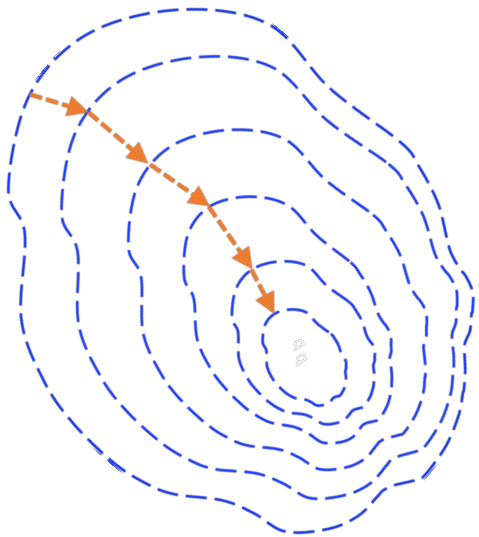

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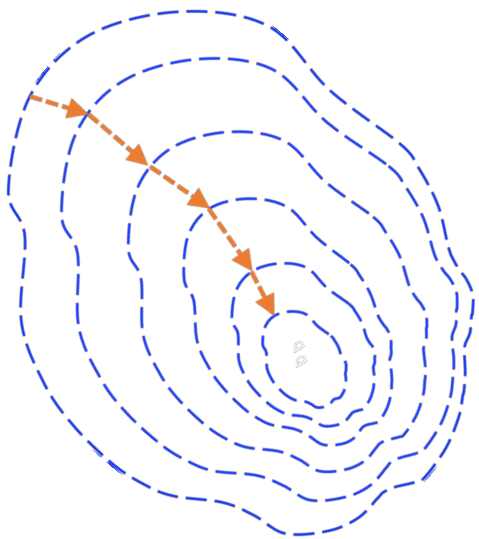

- **spectral initialization:** $\boldsymbol{x}^0 \leftarrow$ leading eigenvector of certain data matrix
- **gradient descent:**

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \eta\,\nabla f(\boldsymbol{x}^t), \qquad t = 0, 1, \cdots$$

First theory of WF

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) := \min\{\|\boldsymbol{x}^t \pm \boldsymbol{x}^\star\|_2\}$$

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Under i.i.d. Gaussian design, WF with spectral initialization achieves

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \lesssim \left(1 - \frac{\eta}{4}\right)^{t/2} \|\boldsymbol{x}^\star\|_2,$$

with high prob., provided that step size $\eta \lesssim 1/n$ *and sample size:* $m \gtrsim n \log n$.

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with high prob., provided that step size and sample size: .

- Iteration complexity: $O(n \log \frac{1}{\epsilon})$
- Sample complexity: $O(n \log n)$
- Derived based on (worst-case) local geometry

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Gaussian designs: $\boldsymbol{a}_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n), \quad 1 \le k \le m$

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Consequence (Candès et al '14): WF attains ε-accuracy within $O\big(n \log \frac{1}{\varepsilon}\big)$ iterations if $m \asymp n \log n$

Generic optimization theory gives pessimistic bounds

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Improved theory of WF

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) := \min\{\|\boldsymbol{x}^t \pm \boldsymbol{x}^\star\|_2\}$$

Theorem 9.2 (Ma, Wang, Chi, Chen '17)

Under i.i.d. Gaussian design, WF with spectral initialization achieves

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- Iteration complexity: $O(n \log \frac{1}{\epsilon}) \searrow O(\log n \log \frac{1}{\epsilon})$
- Sample complexity: $O(n \log n)$
- Derived based on finer analysis of GD trajectory

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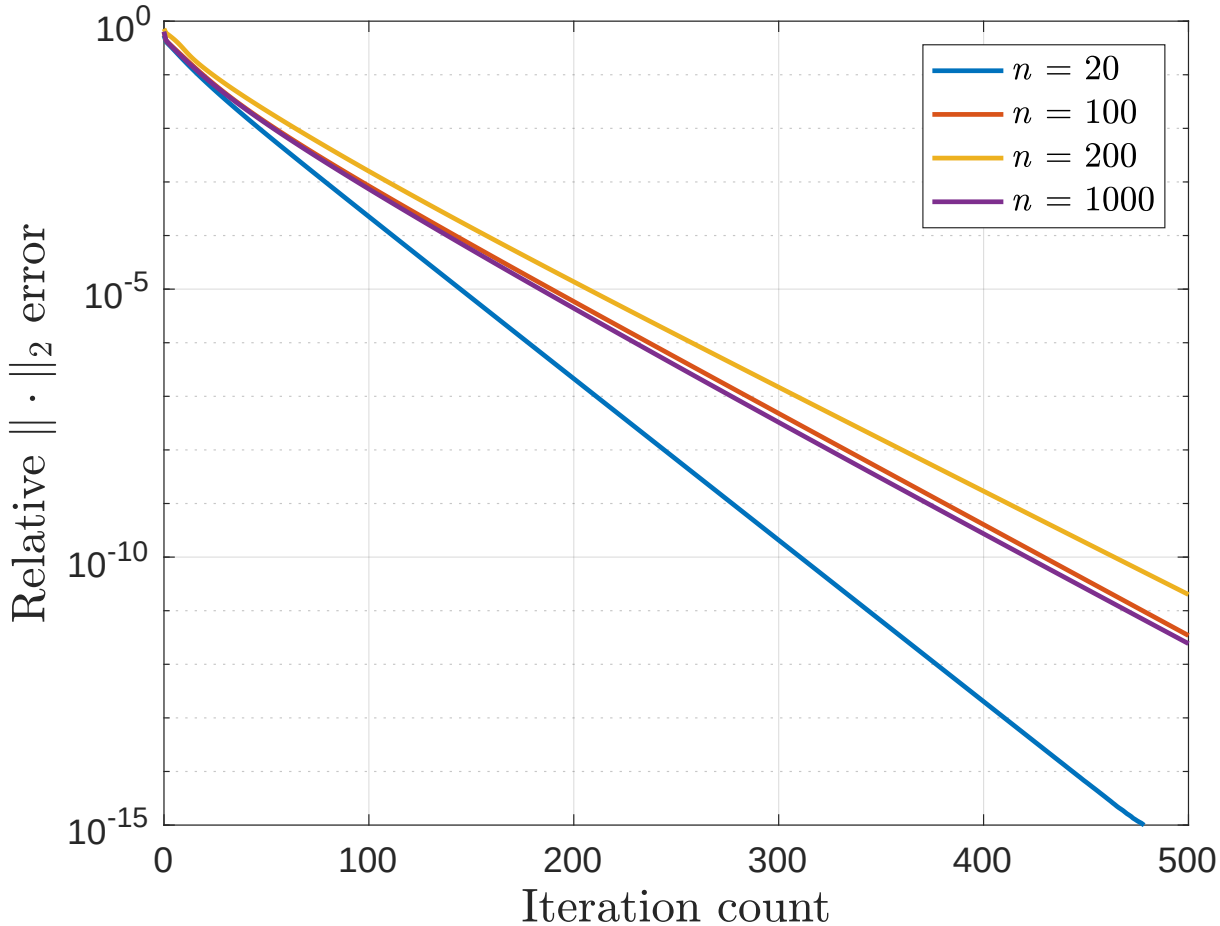
Numerical efficiency with $\eta_t = 0.1$



Vanilla GD (WF) converges fast for a constant step size!

A second look at gradient descent theory

Which local region enjoys both strong convexity and smoothness?

$$\nabla^2 f(\boldsymbol{x}) = \frac{1}{m}\sum_{k=1}^{m}\left[3\left(\boldsymbol{a}_k^\top \boldsymbol{x}\right)^2 - \left(\boldsymbol{a}_k^\top \boldsymbol{x}^\star\right)^2\right]\boldsymbol{a}_k\boldsymbol{a}_k^\top$$

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- Not sufficiently smooth if $\boldsymbol{x}$ and $\boldsymbol{a}_k$ are too close (coherent)

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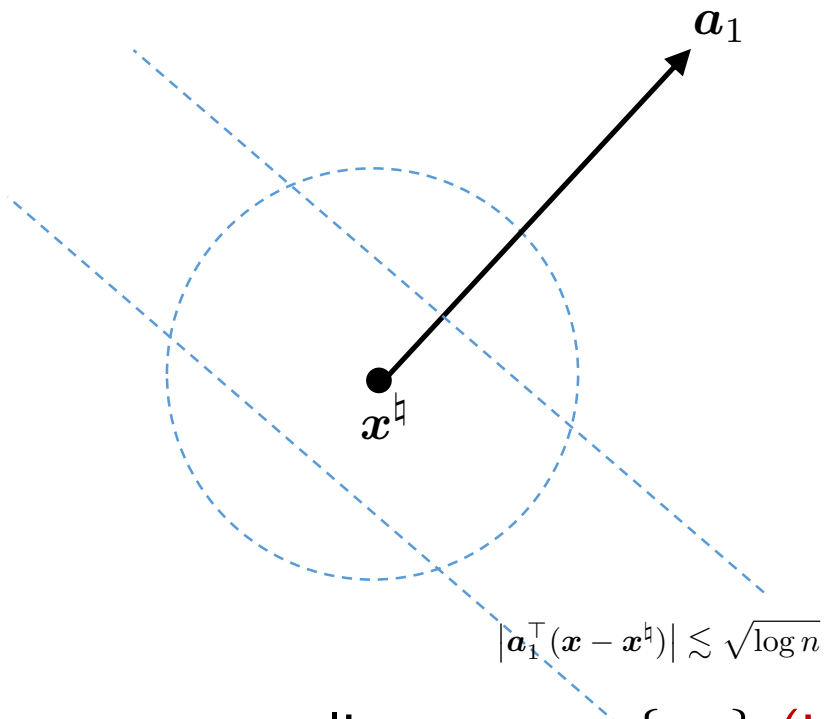

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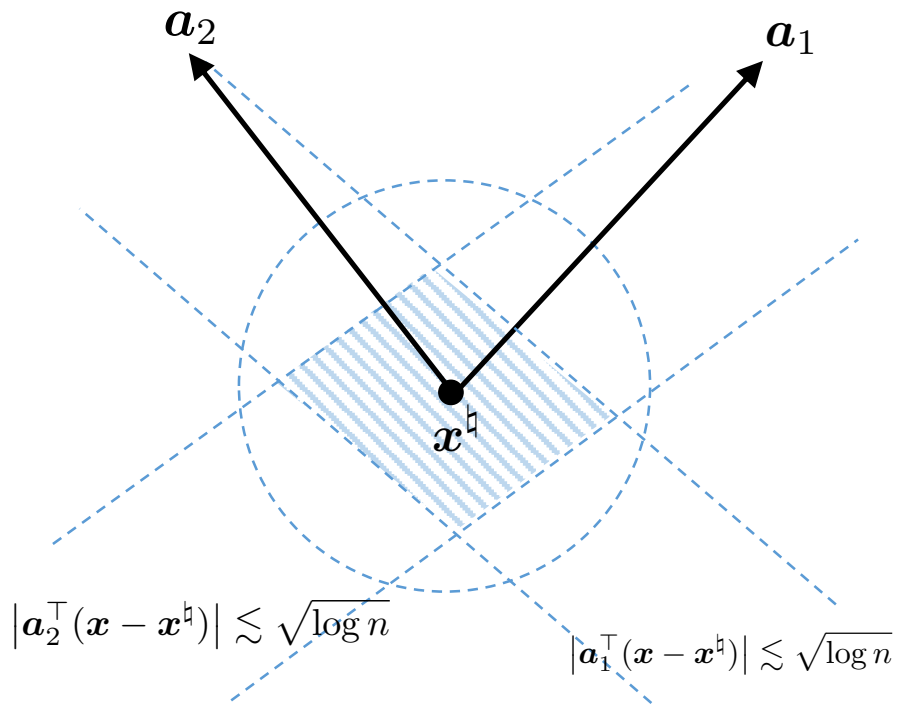

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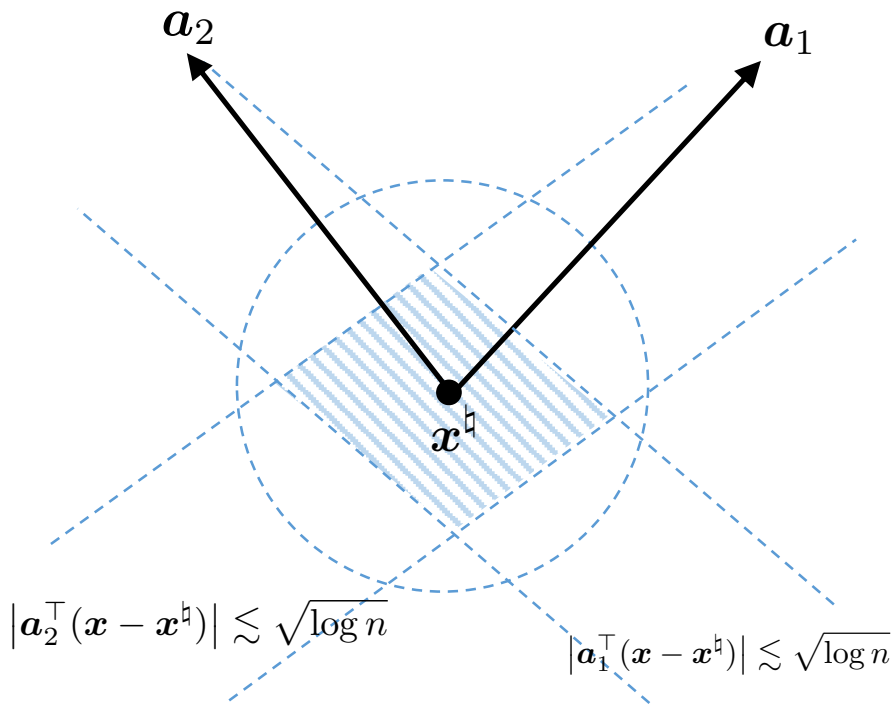

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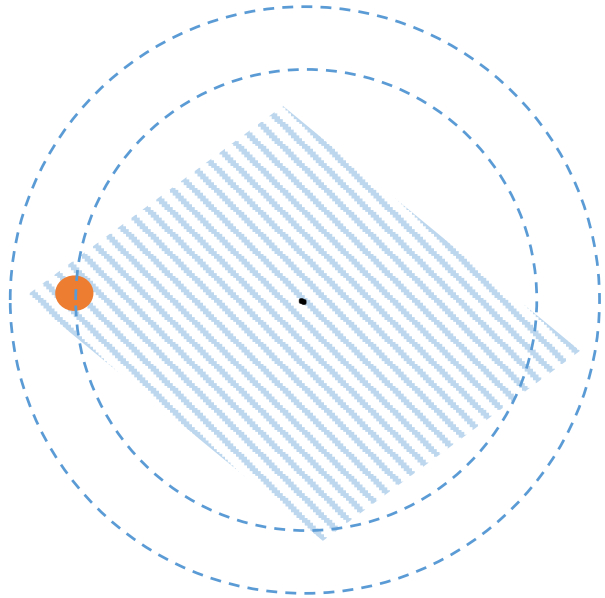
Prior works suggest enforcing regularization (e.g. truncation, projection, regularized loss) to promote incoherence

Encouraging message: GD is implicitly regularized

region of local strong convexity + smoothness



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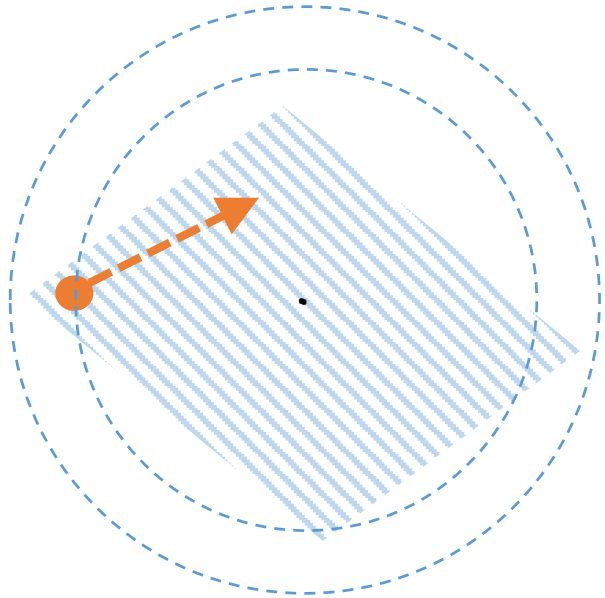

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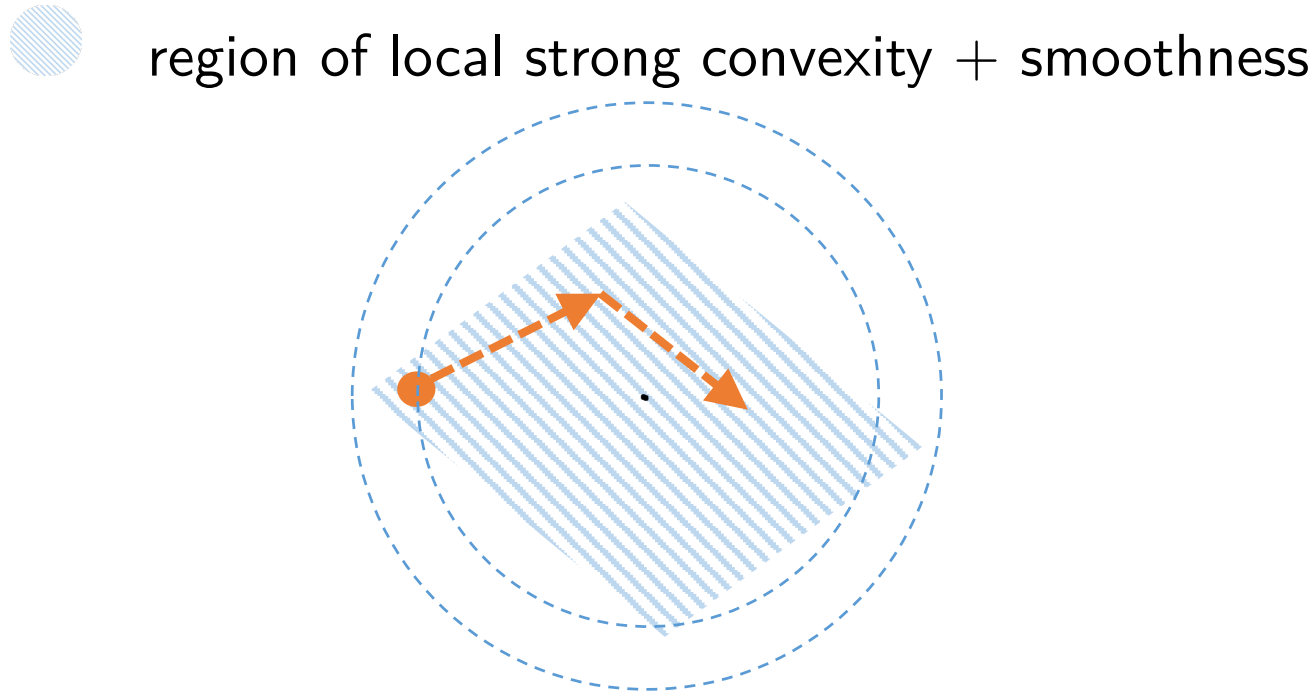

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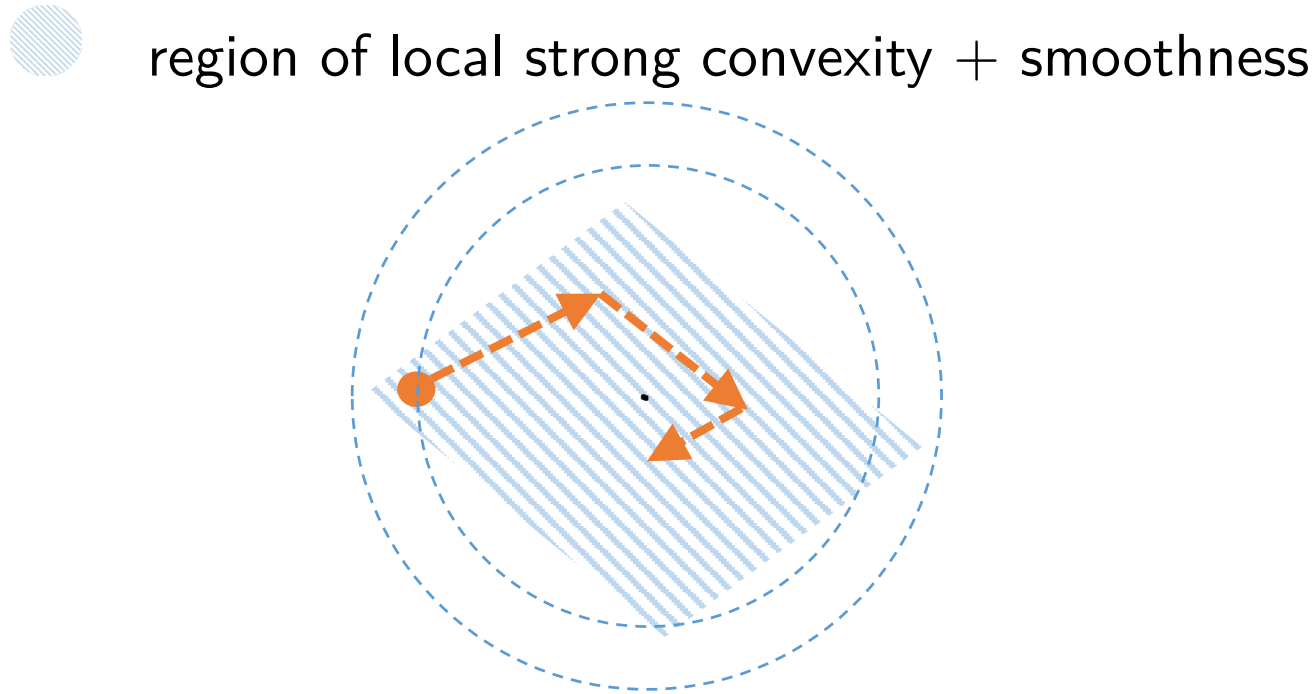

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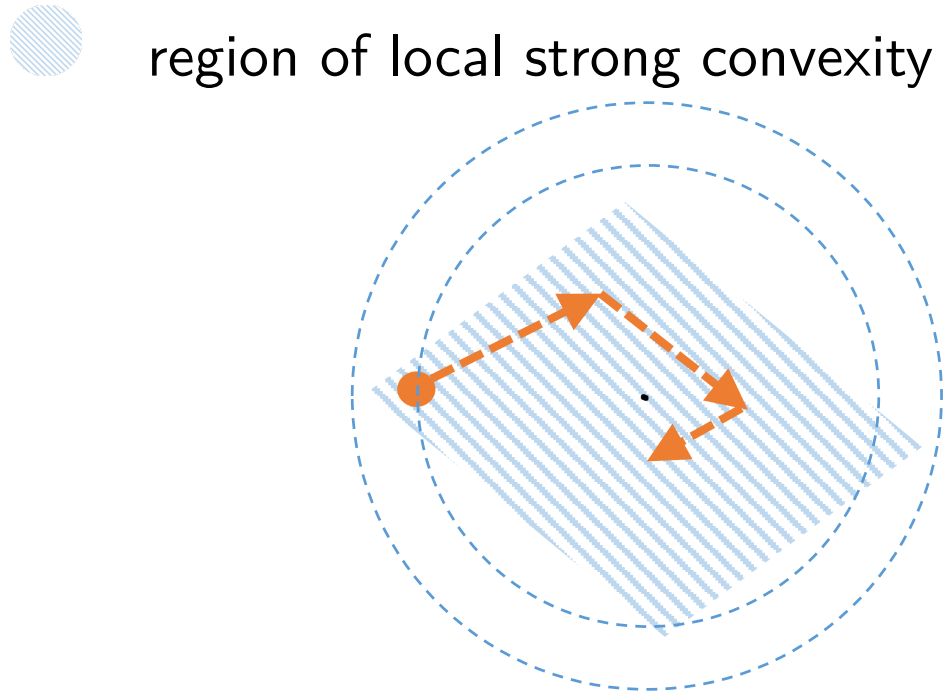

GD implicitly forces iterates to remain incoherent with $\{\boldsymbol{a}_k\}$

$$\max_k \left|\boldsymbol{a}_k^\top (\boldsymbol{x}^t - \boldsymbol{x}^\star)\right| \lesssim \sqrt{\log n}\, \|\boldsymbol{x}^\star\|_2, \quad \forall t$$

— cannot be derived from generic optimization theory; relies on finer statistical analysis for entire trajectory of GD

Theoretical guarantees for local refinement stage

Theorem 9.3 (Ma, Wang, Chi, Chen '17)

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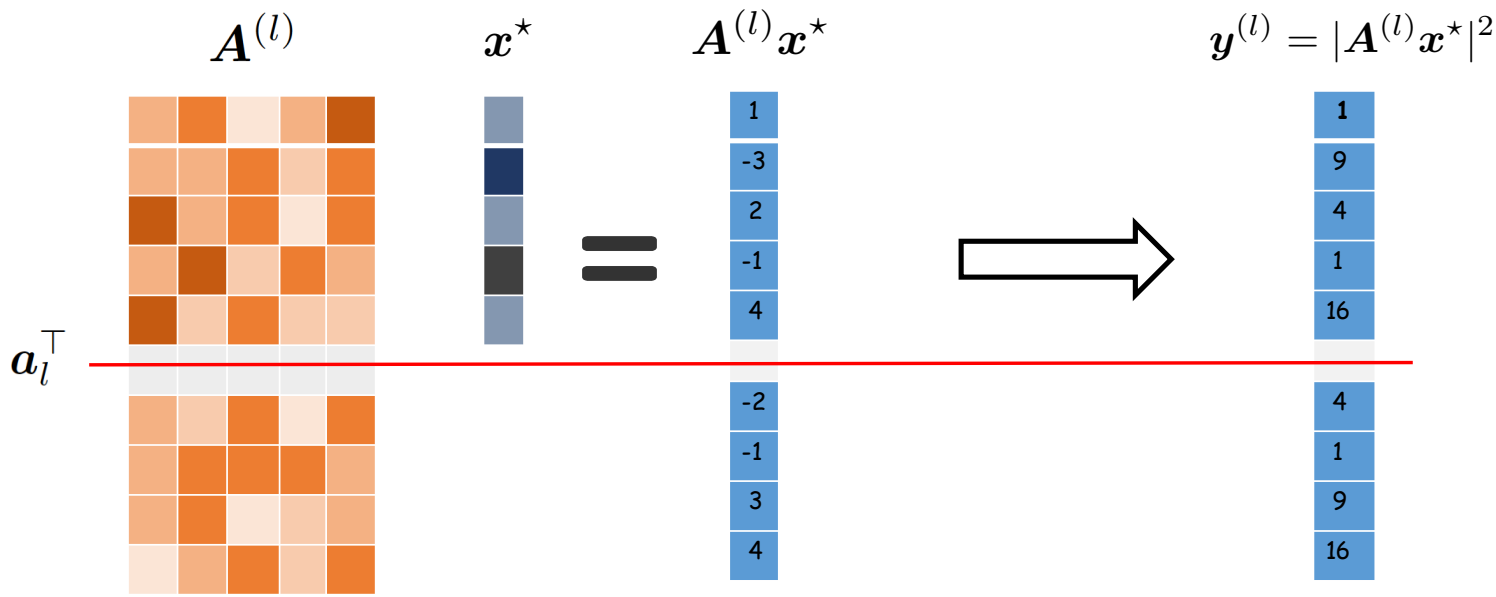
- $\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \lesssim \left(1-\frac{\eta}{2}\right)^t \|\boldsymbol{x}^\star\|_2$ *(linear convergence)*

provided that step size $\eta \asymp 1/\log n$ *and sample size* $m \gtrsim n \log n$.

- Attains ε accuracy within $O\big(\log n \log \frac{1}{\varepsilon}\big)$ iterations

Key proof idea: leave-one-out analysis

For each $1 \le l \le m$, introduce leave-one-out iterates $\boldsymbol{x}^{t,(l)}$ by dropping lth measurement



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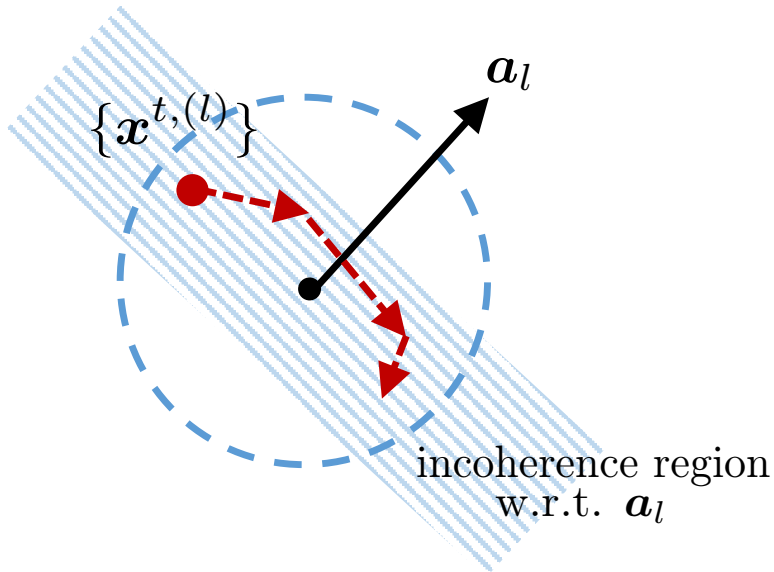

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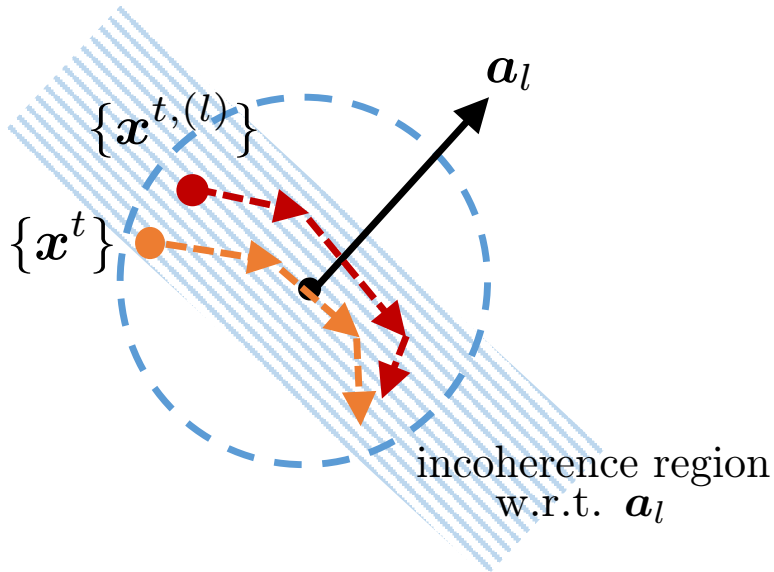

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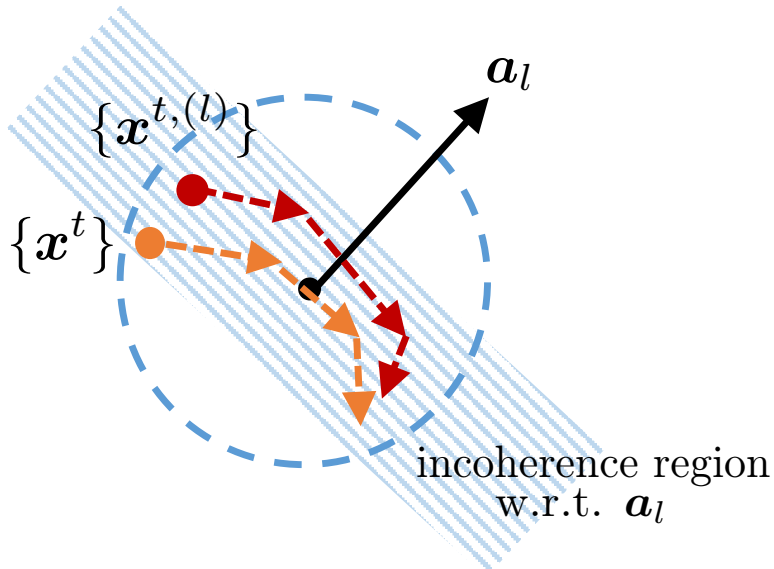

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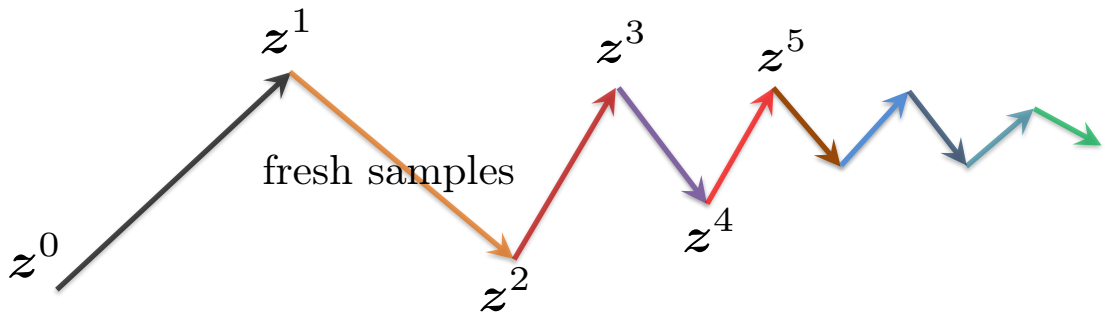
$$\Longrightarrow \quad \boldsymbol{x}^t \text{ is } \underbrace{\text{nearly independent of}}_{\text{nearly orthogonal to}} \boldsymbol{a}_l$$

No need of sample splitting

- Several prior works use sample-splitting: require fresh samples at each iteration; not practical but helps analysis



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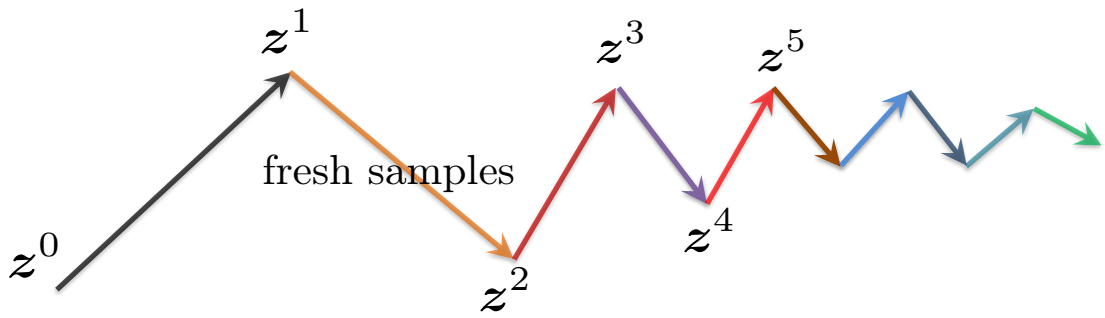
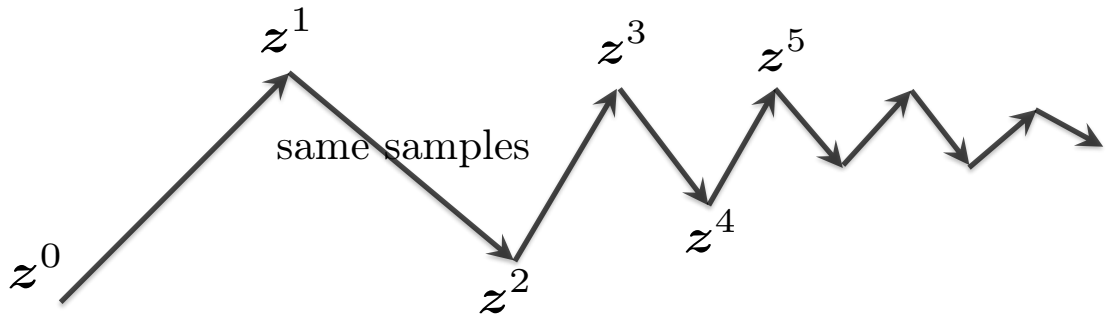


- **This tutorial:** reuses all samples in all iterations



Low-rank matrix completion