STAT253/317 Lecture 3: 4.3 Classification of States

Introduce state diagram.

Definition. Consider a Markov chain $\{X_n, n \geq 0\}$ with state space $\mathfrak{X}$. For two states $i, j \in \mathfrak{X}$, we say state j is **accessible** from state i if $P_{ij}^{(n)} > 0$ for some n, and we denote it as

$$i \to j.$$

Note that **accessibility is transitive**: for $i, j, k \in \mathfrak{X}$,

if $i \to j$ and $j \to k$, then $i \to k$.

Proof.

$$\begin{aligned} i \to j \quad &\Rightarrow \quad P_{ij}^{(m)} > 0 \text{ for some } m \\ j \to k \quad &\Rightarrow \quad P_{jk}^{(n)} > 0 \text{ for some } n \end{aligned}$$

By Chapman-Kolmogorov Equation:

$$P_{ik}^{(m+n)} = \sum\nolimits_{l \in \mathfrak{X}} P_{il}^{(m)} P_{lk}^{(n)} \geq P_{ij}^{(m)} P_{jk}^{(n)} > 0,$$

which shows $i \to k$.

Communicability

Definition. Consider a Markov chain $\{X_n, n \geq 0\}$ chain with state space $\mathfrak{X}$. Two states $i, j \in \mathfrak{X}$ are said to **communicate** if $i \to j$, and $j \to i$. We denote it as

$$i \longleftrightarrow j.$$

Fact. Communicability is also transitive, meaning that

$$\text{if } i \longleftrightarrow j \text{ and } j \longleftrightarrow k, \text{ then } i \longleftrightarrow k.$$

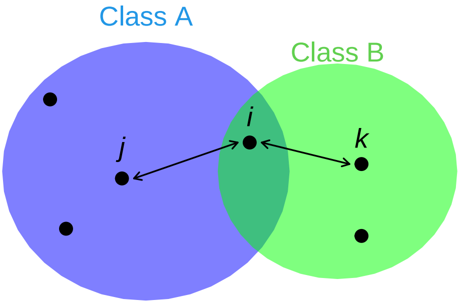
The proof is straight forward from the transitivity of accessibility.

Communicative Class

Definition. Two states that communicate with each other are in the same **class**. A state that communicates with no other states itself is a class.

Fact. Two classes are either identical or disjoint.

Proof. If two classes A and B have one state i in common, then all states in A communicate with i and all states in B do too. Consequently, all states with A can communicate with states in B (through state i). Class A and Class B must be identical.



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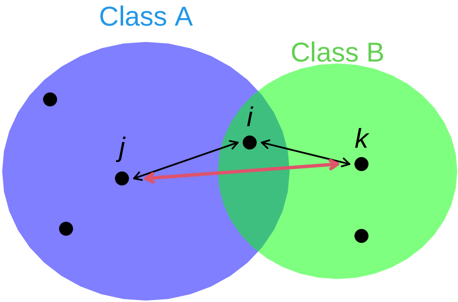

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Example 1. Specify the classes of the following Markov chains.

$$\mathbb{P}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0 & 0 & 0.2 & 0.8 \\ 0 & 0 & 0.9 & 0.1 \end{pmatrix} \end{array} \qquad \mathbb{P}_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Example 2. How many classes does the Ehrenfest diffusion model with K balls have?

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For $\mathbb{P}_1$, $1 \leftrightarrow 2 \to 3 \leftrightarrow 4$. Classes: $\{$1,2$\}$, $\{$3,4$\}$.

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For $\mathbb{P}_1$, $1 \leftrightarrow 2 \rightarrow 3 \leftrightarrow 4$. Classes: {1,2}, {3,4}.

For $\mathbb{P}_2$,
$$\begin{tikzcd} 1 \arrow[r, leftrightarrow] & 2 \\ 4 \arrow[loop below] & 3 \arrow[l] \arrow[u] \arrow[ul] \arrow[loop below] \end{tikzcd}$$
. Classes: {1,2}, {3}, {4}.

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For $\mathbb{P}_1$, $1 \leftrightarrow 2 \to 3 \leftrightarrow 4$. Classes: {1,2}, {3,4}.

For $\mathbb{P}_2$, $\begin{array}{ccc} 1 & \longleftrightarrow & 2 \\ & \nwarrow & \uparrow \\ 4 & \leftarrow & 3 \\ \circlearrowleft & & \circlearrowleft \end{array}$. Classes: {1,2}, {3}, {4}.

Example 2. How many classes does the Ehrenfest diffusion model with K balls have?

All states communicate. Only one class.

Irreducibility

A Markov chain is said to be **irreducible** if it has only 1 class.

Recurrence & Transience

Consider a Markov chain $\{X_n, n \geq 0\}$ chain with state space $\mathfrak{X}$.
For $i \in \mathfrak{X}$, define

$$f_{ii}^{(n)} = \mathrm{P}(X_n = i, X_v \neq i \text{ for } v = 1, 2, \ldots, n-1 \mid X_0 = i)$$

If $\sum_{n\geq 1} f_{ii}^{(n)} = 1$, we say state i is **recurrent**
If $\sum_{n\geq 1} f_{ii}^{(n)} < 1$, we say state i is **transient**

- It's generally difficult to compute $\sum_{n\geq 1} f_{ii}^{(n)}$ directly.
  We need other tools to determine whether a state is recurrent or transient.

An equivalent characterization

Proposition 4.1: State i is

$$\begin{cases} \text{recurrent if} & \sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty \\ \text{transient if} & \sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty \end{cases}$$

Proof: Suppose that $X_0 = i$, and consider the random variable $N(i) = \sum_{n=1}^{\infty} 1\{X_n = i\}$

We will use two way to calculate the expectation of $N(i)$. First, by definition we have

$$\mathbb{E}[N(i)] = \mathbb{E}[\sum_{n=1}^{\infty} 1\{X_n = i\}] = \sum_{n=1}^{\infty} \mathbb{E}[1\{X_n = i\}]$$

$$= \sum_{n=1}^{\infty} \mathrm{P}\{X_n = i\}] = \sum_{n=1}^{\infty} P_{ii}^{(n)}$$

In addition, we have

$$\mathbb{E}[N(i)] = \sum_{k=0}^{\infty} \mathrm{P}(N(i) \geq k) = \sum_{k=0}^{\infty} (\sum_{n \geq 1} f_{ii}^{(n)})^k$$

Corollary 4.2

If $i \longleftrightarrow j$, and i is recurrent, then j is also recurrent.

Proof.

$$\begin{aligned} i \to j \quad &\Rightarrow \quad P_{ij}^{(k)} > 0 \text{ for some } k \\ j \to i \quad &\Rightarrow \quad P_{ji}^{(l)} > 0 \text{ for some } l \end{aligned}$$

By Chapman-Kolmogorov Equation:

$$P_{jj}^{(l+n+k)} \geq P_{ji}^{(l)} P_{ii}^{(n)} P_{ij}^{(k)}, \text{ for all } k = 0, 1, 2, \ldots$$

Thus

$$\sum_{n=1}^{\infty} P_{jj}^{(n)} \geq \sum_{n=1}^{\infty} P_{jj}^{(l+n+k)} \geq \underbrace{P_{ji}^{(l)}}_{>0} \underbrace{\sum_{n=1}^{\infty} P_{ii}^{(n)}}_{=\infty} \underbrace{P_{ij}^{(k)}}_{>0} = \infty$$

Corollary 4.2 implies that all states of a finite irreducible Markov chain are recurrent.

Finite irreducible MC

Theorem All states of a finite irreducible Markov chain are recurrent.

Example: One-Dimensional Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob. } p \\ X_n - 1 & \text{with prob. } 1-p \end{cases}$$

- State space $\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
- All states communicate

$$\cdots \longleftrightarrow -2 \longleftrightarrow -1 \longleftrightarrow 0 \longleftrightarrow 1 \longleftrightarrow 2 \longleftrightarrow \cdots$$

Only one class $\Rightarrow$ Irreducible

$\Rightarrow$ States are all transient or all recurrent.

It suffices to check whether 0 is recurrent or transient, i.e., whether

$$\sum_{n=1}^{\infty} P_{00}^{(n)} = \infty \text{ or } < \infty$$

Example: One-Dimensional Random Walk (Cont'd)

$$P_{00}^{(2n+1)} = 0 \quad \text{(Why?)}$$

$$\begin{aligned}
P_{00}^{(2n)} &= \binom{2n}{n} p^n (1-p)^n \\
&= \frac{(2n)!}{n!\,n!} p^n (1-p)^n \\
&\approx \frac{(2n)^{2n+0.5} e^{-2n} \sqrt{2\pi}}{(n^{n+0.5} e^{-n} \sqrt{2\pi})^2} p^n (1-p)^n \\
&= \frac{1}{\sqrt{\pi n}} [4p(1-p)]^n
\end{aligned}$$

$$\boxed{\text{Stirling's Formula: } n! \approx n^{n+0.5} e^{-n} \sqrt{2\pi}}$$

Thus

$$\sum_{n=1}^{\infty} P_{ii}^{2n} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} [4p(1-p)]^n \begin{cases} < \infty & \text{if } p \neq 1/2 \\ = \infty & \text{if } p = 1/2 \end{cases}$$

Conclusion: One-dimensional random walk is recurrent if $p = 1/2$, and transient otherwise.

Example: Two-Dimensional Symmetric Random Walk

Irreducible. Just check if 0 is recurrent.

$$
\begin{aligned}
P_{00}^{(2n)} &= \sum_{i=0}^{n} \frac{(2n)!}{i!i!(n-i)!(n-i)!} \left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n} \underbrace{\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i}}_{=\binom{2n}{n}} \left(\frac{1}{4}\right)^{2n} \\
&= \binom{2n}{n}^2 \left(\frac{1}{4}\right)^{2n} \approx \frac{1}{\pi n} \quad \text{by Stirling's Formula}
\end{aligned}
$$

Thus $\sum_{n=1}^{\infty} P_{00}^{(2n)} = \infty$.

Two-dimensional symmetric random walk is **recurrent**.

Example: d-Dimensional Symmetric Random Walk

In general, for a d-dimensional symmetric random walk, it can be shown that

$$P_{00}^{(2n)} \approx (1/2)^{d-1} \left(\frac{d}{n\pi} \right)^{d/2}$$

Thus

$$\sum_{n=1}^{\infty} P_{00}^{(2n)} \begin{cases} = \infty & \text{for } d = 1 \text{ or } 2 \\ < \infty & \text{for } d \geq 3 \end{cases} .$$

"A drunken man will find his way home.
A drunken bird might be lost forever."