# STAT253/317 Lecture 14

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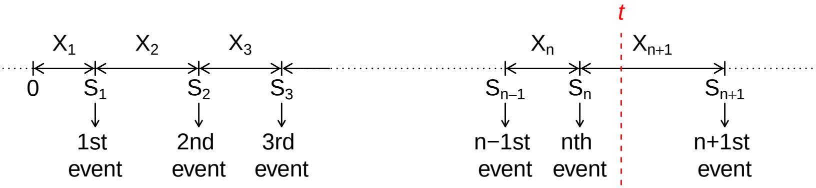
Chapter 7 Renewal Processes

# Chapter 7 Renewal Processes

Recall the interarrival times of a Poisson process are i.i.d exponential random variables.

A **renewal process** is a counting process of which the interarrival times are i.i.d., but may not have an exponential distribution.

# Definition of a Renewal Process




Let $X_1$, $X_2, \ldots$ be i.i.d random variables with $\mathbb{E}[X_i] < \infty$, and $\mathrm{P}(X_i = 0) < 1$. Let

$$S_0 = 0, \quad S_n = X_1 + \ldots + X_n, \ n \geq 1.$$

Define

$$N(t) = \max\{n : S_n \leq t\}.$$

Then $\{N(t), t \geq 0\}$ is called a *renewal process*.

- Events are called "*renewals*". The interarrival times between events $X_1$, $X_2, \ldots$ are also called "renewals"
- A more general definition allows the first renewal $X_1$ to be of a different distribution, called a **delayed renewal process**

# Renewal Processes Are Well-Defined

Renewal processes are well-defined in the sense that

$$\mathrm{P}(\max\{n : S_n \le t\} < \infty) = 1 \quad \text{for all } t > 0.$$

$$\begin{aligned}
\text{By SLLN} &\Rightarrow \mathrm{P}\left(\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1]\right) = 1 \\
&\Rightarrow \ \mathrm{P}\left(\lim_{n\to\infty} S_n = \infty\right) = 1 \\
&\Rightarrow \ \text{For any } t\text{, w/ prob. 1 } S_n < t \quad \text{for only finitely many } n \\
&\Rightarrow \ \mathrm{P}(\max\{n : S_n \le t\} < \infty) = 1 \quad \text{for all } t > 0
\end{aligned}$$

# Examples of Renewal Processes

- Replacement of light bulbs: $N(t) = \#$ of replaced light bulbs by time $t$, is a renewal process
- Consider a homogeneous, irreducible, positive recurrent, discrete time Markov chain, started from a state $i$. Let

$$N_i(t) = \text{number of visits to state } i \text{ by time } t.$$

Then $\{N_i(t), t \geq 0\}$ is a renewal process.

# Basic Properties of Renewal Processes

- $\mathrm{P}(\lim_{t\to\infty} N(t) = \infty) = 1$

  Reason: $\lim_{t\to\infty} N(t) < \infty$ can happen only when $X_i = \infty$ for some $i$.

  $$\left\{\lim_{t\to\infty} N(t) < \infty\right\} \subseteq \bigcup\nolimits_{i=1}^{\infty} \{X_i = \infty\}$$

  However, as the interarrival times of a renewal process are required to have finite means $\mathbb{E}[X_i] < \infty$, which implies $\mathrm{P}(X_i = \infty) = 0$, we must have

  $$\mathrm{P}\left(\lim_{t\to\infty} N(t) < \infty\right) \le \mathrm{P}\left(\bigcup_{i=1}^{\infty} \{X_i = \infty\}\right) \le \sum_{i=1}^{\infty} \mathrm{P}(X_i = \infty) = 0.$$

- Not memoryless in general

  $\Rightarrow$ No independent or stationary increments in general
  $\mathrm{P}(N(t+h) - N(t) = 1)$ depends on the current lifetime
  $A(t) = t - S_{N(t)}$

# Things of Interest

- Distribution of $N(t)$:

$$\mathrm{P}(N(t) = n), \quad n = 0, 1, 2, \ldots$$

- Renewal function:

$$m(t) = \mathbb{E}[N(t)]$$

- Residual life (a.k.a. excess life, overshoot, excess over the boundary):

$$B(t) = S_{N(t)+1} - t$$

- Current age (a.k.a. current life, undershoot):

$$A(t) = t - S_{N(t)}$$

- Total life: $C(t) = A(t) + B(t)$
- Inspection paradox: $C(t)$ and the interarrival time $X_i$ have different distributions.

## 7.2. Distribution of $N(t)$

Let

$$F_n(t) = \mathrm{P}(S_n \leq t)$$

be the CDF of the arrival time $S_n = X_1 + \cdots + X_n$ of the $n$th event. Observe that

$$\{N(t) \geq n\} \quad \Leftrightarrow \quad \{S_n \leq t\}$$

Thus
$$\begin{aligned} \mathrm{P}(N(t) = n) &= \mathrm{P}(N(t) \geq n) - \mathrm{P}(N(t) \geq n+1) \\ &= \mathrm{P}(S_n \leq t) - \mathrm{P}(S_{n+1} \leq t) \\ &= F_n(t) - F_{n+1}(t) \end{aligned}$$

This formula looks simple but is generally <u>USELESS</u> in practice since $F_n(t)$ is often intractable.

# The Renewal Function $m(t)$

Recall that if a random variable $X$ takes non-negative integer values $\{0, 1, 2, \ldots\}$, then $\mathbb{E}[X] = \sum_{n=1}^{\infty} \mathrm{P}(X \geq n)$. Therefore the renewal function can be written as

$$\begin{aligned} m(t) = \mathbb{E}[N(t)] &= \sum_{n=1}^{\infty} \mathrm{P}(N(t) \geq n) \\ &= \sum_{n=1}^{\infty} \mathrm{P}(S_n \leq t) = \sum_{n=1}^{\infty} F_n(t) \end{aligned}$$

- It can be shown that the renewal function $m(t)$ can uniquely determine the interarrival distribution $F$. So the only renewal process with linear renewal function $m(t) = \lambda t$ is the Poisson process with rate $\lambda$.
- The formula $m(t) = \sum_{n=1}^{\infty} F_n(t)$ is again generally <u>useless</u> since $F_n(t)$ often times has no closed form expression. We need more tools.

# The Renewal Equation

Conditioning on $X_1 = x$, observe that

$$(N(t)|X_1 = x) = \begin{cases} 1 + N(t-x) & \text{if } x \le t \\ 0 & \text{if } x > t \end{cases}$$

Assuming that the interarrival distribution $F$ is continuous with density function $f$. Then

$$\begin{aligned} m(t) = \mathbb{E}[N(t)] &= \int_0^\infty \mathbb{E}[N(t)|X_1 = x] f(x) dx \\ &= \int_0^t (1 + \mathbb{E}[N(t-x)]) f(x) dx + \int_t^\infty 0 f(x) dx \\ &= \int_0^t (1 + m(t-x)) f(x) dx = F(t) + \int_0^t m(t-x) f(x) dx \end{aligned}$$

The equation

$$m(t) = F(t) + \int_0^t m(t-x) f(x) dx$$

is called the *renewal equation*.

# Example 7.3

Suppose the interarrival times $X_i$ are i.i.d. uniform on $(0, 1)$. The density and CDF of $X_i$'s are respectively

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } x > 1. \end{cases}$$

For $0 \le t \le 1$, the renewal equation is

$$m(t) = t + \int_0^t m(t-x)dx = t + \int_0^t m(x)dx$$

Differentiating the equation with respect to $t$ yields

$$m'(t) = 1 + m(t) \Rightarrow \frac{d}{dt}(1 + m(t)) = 1 + m(t) \Rightarrow 1 + m(t) = Ke^t.$$

or $m(t) = Ke^t - 1$. Since $m(0) = 0$, we can see that $K = 1$ and obtain that $m(t) = e^t - 1$ for $0 \le t \le 1$.

What if $1 \le t \le 2$?

For $1 \le t \le 2$, $F(t) = 1$, the renewal equation is

$$m(t) = 1 + \int_0^1 m(t-x)dx = 1 + \int_{t-1}^t m(x)dx$$

Differentiating the preceding equation yields

$$m'(t) = m(t) - m(t-1) = m(t) - [e^{t-1} - 1] = m(t) + 1 - e^{t-1}$$

Multiplying both side by $e^{-t}$, we get

$$\underbrace{e^{-t}(m'(t) - m(t))}_{\frac{d}{dt}[e^{-t}m(t)]} = e^{-t} - e^{-1}$$

Integrating over $t$ from 1 to $t$, we get

$$\begin{aligned}
e^{-t}m(t) &= e^{-1}m(1) + e^{-1}\int_1^t e^{-(s-1)} - 1ds \\
&= e^{-1}m(1) + e^{-1}[1 - e^{-(t-1)} - (t-1)] \\
\Rightarrow m(t) &= e^{t-1}m(1) + e^{t-1} - 1 - e^{t-1}(t-1) \\
&= e^t + e^{t-1} - 1 - te^{t-1} \qquad \text{(Note } m(1) = e - 1\text{)}
\end{aligned}$$

In general for $n \leq t \leq n+1$, the renewal equation is

$$m(t) = 1 + \int_{t-1}^{t} m(x)dx \quad \Rightarrow \quad m'(t) = m(t) - m(t-1)$$

Multiplying both side by $e^{-t}$, we get

$$\frac{d}{dt}(e^{-t}m(t)) = e^{-t}(m'(t) - m(t)) = -e^{-t}m(t-1)$$

Integrating over $t$ from 1 to $t$, we get

$$e^{-t}m(t) = e^{-n}m(n) - \int_{n}^{t} e^{-s}m(s-1)ds$$

Thus we can find $m(t)$ iteratively.