STAT253/317 Winter 2022 Lecture 21

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Section 8.7 The Model G/M/1

8.7 The Model G/M/1

The $G/M/1$ model assumes

- i.i.d times between successive arrivals with an arbitrary distribution G
- i.i.d service times $\sim \text{Exp}(\mu)$
- a single server; and
- first come, first serve

Just like $M/G/1$ system, there is also a discrete-time Markov chain embedded in an $G/M/1$ system. Let

$$
\begin{aligned}
Y_n &= \#\text{ of customers in the system seen by the } n\text{th arrival}, \quad n \geq 1 \\
D_n &= \#\text{ of customers the server can possibly serve} \\
&\qquad \text{between the } (n-1)\text{st and the } n\text{th arrival}, \quad n \geq 1
\end{aligned}
$$

Observed that $\{Y_n, n \geq 0\}$ and $\{D_n, n \geq 1\}$ are related as follows

$$
Y_{n+1} = \begin{cases} Y_n + 1 - D_{n+1} & \text{if } Y_n + 1 \geq D_{n+1} \\ 0 & \text{if } Y_n + 1 < D_{n+1} \end{cases}, \quad n \geq 1
$$

A Markov Chain embedded in $G/M/1$ (Cont'd)

- By the memoryless property of the exponential service time, the remaining service time of the customer being served at an arrival is also $\sim \text{Exp}(\mu)$.
- Thus starting from the $(n-1)$st arrival, the events of completion of servicing a customer constitute a Poisson process of rate μ.
- Let G_n be the time elapsed between the $(n-1)$st and the nth arrival.
- Then given G_n, D_n is Poisson with mean μG_n.
- As G_n's are i.i.d $\sim G$, we can conclude that $D_1, D_2, \ldots$ are i.i.d. with distribution

$$\begin{aligned}
\delta_k = \mathrm{P}(D_n = k) &= \int_0^\infty \mathrm{P}(D_n = k | G_n = y) G(dy) \\
&= \int_0^\infty \frac{(\mu y)^k}{k!} e^{-\mu y} G(dy)
\end{aligned}$$

A Markov Chain embedded in $G/M/1$ (Cont'd)

The transition probabilities P_{ij} for the Markov chain $\{Y_n, n \geq 0\}$ are thus:

$$P_{ij} = \mathrm{P}(Y_{n+1} = j | Y_n = i)$$
$$= \begin{cases} \mathrm{P}(D_{n+1} \geq i+1) = \sum_{k=i+1}^{\infty} \delta_k & \text{if } j = 0 \\ \mathrm{P}(D_{n+1} = i+1-j) = \delta_{i+1-j}, & \text{if } j \geq 1, i+1 \geq j \\ 0 & \text{if } i+1 < j \end{cases}$$

i.e., the transition probability matrix is

$$\mathbb{P} = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} & \begin{pmatrix} \sum_{k=1}^{\infty} \delta_k & \delta_0 & 0 & 0 & 0 & \cdots \\ \sum_{k=2}^{\infty} \delta_k & \delta_1 & \delta_0 & 0 & 0 & \cdots \\ \sum_{k=3}^{\infty} \delta_k & \delta_2 & \delta_1 & \delta_0 & 0 & \cdots \\ \sum_{k=4}^{\infty} \delta_k & \delta_3 & \delta_2 & \delta_1 & \delta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

A Markov Chain embedded in $G/M/1$ (Cont'd)

To find the stationary distribution $\pi_i = \lim_{n\to\infty} \mathrm{P}(Y_n = i)$, $i = 0, 1, 2, \ldots$, we have to solve the equations

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} = \sum_{i=j-1}^{\infty} \pi_i \delta_{i+1-j},\ j \geq 1 \quad \text{and} \quad \sum_{j=0}^{\infty} \pi_j = 1$$

Let us try a solution of the form $\pi_j = c\beta^j$, $j \geq 0$. Substituting into the equation above leads to

$$c\beta^j = \sum\nolimits_{i=j-1}^{\infty} c\beta^i \delta_{i+1-j} \quad \text{(Divide both sides by } c\beta^{j-1}\text{)}$$

$$\Rightarrow \quad \beta = \sum\nolimits_{i=j-1}^{\infty} \beta^{i+1-j} \delta_{i+1-j} = \sum\nolimits_{i=0}^{\infty} \beta^i \delta_i$$

Observe that $\sum_{i=0}^{\infty} \beta^i \delta_i$ is exactly the generating function of D_n $g(s) = \mathbb{E}[s^{D_n}]$ taking value at $s = \beta$.
Thus if we can find $0 < \beta < 1$ such that $\beta = g(\beta)$, then

$$\pi_j = (1-\beta)\beta^j, \quad j \geq 0$$

is a stationary distribution of $\{Y_n\}$.

A Markov Chain embedded in $G/M/1$ (Cont'd)

Claim:
The equation

$$\beta = g(\beta)$$

has a solution between 0 and 1 iff $g'(1) = E[D_n] = \mu\mathbb{E}[G_n] > 1$.

This condition is intuitive since if

$$\begin{aligned} &\text{the average service time } 1/\mu \\ > &\text{ the average interarrival time of customers } \mathbb{E}[G_n], \end{aligned}$$

the queue will become longer and longer and the system will ultimately explode.

PASTA Principle Does Not Apply to G/M/1

With the stationary distribution $\{\pi_j, j \geq 0\}$, one might attempt to calculate L, the average number of customers in the system as

$$\mathbb{E}[Y_n] = \sum_{k=0}^{\infty} k\pi_k = \sum_{k=0}^{\infty} k(1-\beta)\beta^k = \frac{\beta}{1-\beta}.$$

However, the PASTA principle does not apply as the arrival process is not Poisson. Recall

$$\begin{aligned} a_k = \pi_k &= \text{proportion of arrivals see } k \text{ in the system} \\ P_k &= \text{proportion of time having } k \text{ customers in the system,} \end{aligned}$$

W of G/M/1

Though we cannot use $\{\pi_j\}$ to find L, we can use it to find W. Let W_n be the waiting time of nth customer in the system. If he/she sees k customers at arrival, then W_n is the total service time of $k+1$ customers. That is,

$$\begin{aligned}\mathbb{E}[W_n | Y_n = k] &= \mathbb{E}[\text{sum of } k+1 \text{ i.i.d. Exp}(\mu) \text{ service times}] \\ &= \frac{k+1}{\mu}.\end{aligned}$$

Thus

$$\begin{aligned}W &= \sum_{k=0}^{\infty} \mathbb{E}[W_n | Y_n = k] \mathrm{P}(Y_n = k) = \sum_{k=0}^{\infty} \mathbb{E}[W_n | Y_n = k] \pi_k \\ &= \sum_{k=0}^{\infty} \frac{k+1}{\mu}(1-\beta)\beta^k = \frac{1}{\mu(1-\beta)}\end{aligned}$$

Here we use the identity $\displaystyle\sum_{k=0}^{\infty}(k+1)x^k = \frac{1}{(1-x)^2}$.

L, W_Q, L_Q of G/M/1

By the Little's Formula, we know $L = \lambda W$, in which λ is the arrival rate of customers, which is the reciprocal of the mean interarrival time $\mathbb{E}[G_n]$

$$\lambda = \frac{1}{\mathbb{E}[G_n]}$$

Thus

$$L = \lambda W = \frac{1}{\mathbb{E}[G_n]} \frac{1}{\mu(1-\beta)} = \frac{1}{\mu \mathbb{E}[G_n](1-\beta)}$$

Moreover,

$$W_Q = W - \mathbb{E}[\text{Service Time}] = W - \frac{1}{\mu} = \frac{\beta}{\mu(1-\beta)}$$

$$L_Q = \lambda W_Q = \frac{\beta}{\mu \mathbb{E}[G_n](1-\beta)}$$

8.9.3 $G/M/k$

Just like $G/M/1$ system, $G/M/k$ system can also be analyzed as a Markov Chain. Let

$$\begin{aligned} Y_n &= \# \text{ of customers in the system seen by the } n\text{th arrival}, \ n \geq 1 \\ D_n &= \# \text{ of customers the } k \text{ servers can possibly serve} \\ &\quad\ \text{between the } (n-1)\text{st and the } n\text{th arrival}, \ n \geq 1 \end{aligned}$$

Observed again that $\{Y_n, n \geq 0\}$ and $\{D_n, n \geq 1\}$ are related as follows

$$Y_{n+1} = \begin{cases} Y_n + 1 - D_{n+1} & \text{if } Y_n + 1 \geq D_{n+1} \\ 0 & \text{if } Y_n + 1 < D_{n+1} \end{cases}, \quad n \geq 1$$

One can show that the distribution of D_{n+1} depends on Y_n but not $Y_{n-1}, Y_{n-2}, \ldots$ and hence $\{Y_n\}$ is a Markov chain. The transition probabilities are given in p.544-545 (p.565-566 in 10ed)

8.9.4 $M/G/k$

Unlike $G/M/k$, the method to analyze $M/G/1$ cannot be used to analyze $M/G/k$. If we follow the lines as we do in $M/G/1$

$$\begin{aligned} Y_n &= \# \text{ of customers in the system} \\ &\qquad \text{leaving behind at the } n\text{th departure}, \ n \geq 1 \\ D_n &= \# \text{ of customers entered the system} \\ &\qquad \text{during the service time of the } n\text{th customer}, \ n \geq 1 \end{aligned}$$

As there are more than one server, the service times are not disjoint, and hence D_n's are not independent.

In fact, there is NO known exact formula for L, W, L_Q, W_Q of an $M/G/k$ system.