Generic analysis of local convergence

Cong Ma

University of Chicago, Autumn 2021

Outline

- Low-rank matrix sensing
- Phase retrieval
- Low-rank matrix completion

Low-rank matrix sensing

- Groundtruth: rank-r matrix $\boldsymbol{M}^\star \in \mathbb{R}^{n_1 \times n_2}$
- Observations:

$$y_i = \langle \boldsymbol{A}_i, \boldsymbol{M}^\star \rangle, \qquad \text{for } 1 \le i \le m$$

- Goal: recover $\boldsymbol{M}^\star$ based on linear measurements $\{\boldsymbol{A}_i, y_i\}_{1 \le i \le m}$

How many measurements are needed

- $m \geq n_1 n_2$ "generic" measurements suffice given theory of solving linear equations
- But $\boldsymbol{M}^\star$ only has $O((n_1 + n_2)r)$ degrees of freedom. Ideally, one hope for using only $O((n_1 + n_2)r)$ measurements

Recovery is possible if $\{A_i\}$'s satisfy restricted isometry property

Restricted isometry property (RIP)

Define linear operator $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \mapsto \mathbb{R}^m$ t obe

$$\mathcal{A}(\boldsymbol{M}) = [m^{-1/2} \langle \boldsymbol{A}_i, \boldsymbol{M} \rangle]_{1 \le i \le m}$$

Definition 8.1

The operator $\mathcal{A}$ is said to satisfy r-RIP with RIP constant $\delta_r < 1$ if

$$(1 - \delta_r) \|\boldsymbol{M}\|_{\mathsf{F}}^2 \le \|\mathcal{A}(\boldsymbol{M})\|_2^2 \le (1 + \delta_r) \|\boldsymbol{M}\|_{\mathsf{F}}^2$$

holds simultaneously for all $\boldsymbol{M}$ of rank at most r.

- Many random designs satisfy RIP with high probability
- For instance, when $\boldsymbol{A}_i$ is composed of i.i.d. $\mathcal{N}(0, 1)$ entries, $\mathcal{A}$ obeys r-RIP with constant δ_r as soon as $m \gtrsim (n_1 + n_2) r / \delta_r^2$

An optimization-based method

Consider the simple case when $\boldsymbol{M}^\star$ is psd and rank 1, i.e.,

$$\boldsymbol{M}^\star = \boldsymbol{x}^\star \boldsymbol{x}^{\star\top}$$

Then least-squares estimation yields

$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \qquad f(\boldsymbol{x}) = \frac{1}{4m}\sum_{i=1}^{m}\Big(\langle \boldsymbol{A}_i, \boldsymbol{x}\boldsymbol{x}^\top\rangle - y_i\Big)^2$$

Gradient descent

Starting from $\boldsymbol{x}^0$, one proceeds by

$$\begin{aligned}\boldsymbol{x}^{t+1} &= \boldsymbol{x}^t - \eta \nabla f(\boldsymbol{x}^t) \\ &= \boldsymbol{x}^t - \frac{\eta}{m}\sum_{i=1}^m \Big(\langle \boldsymbol{A}_i, \boldsymbol{x}^t \boldsymbol{x}^{t\top} \rangle - y_i \Big) \boldsymbol{A}_i \boldsymbol{x}^t\end{aligned}$$

Here we made simplifying assumption that A_i is symmetric

- Under random design, when $m \to \infty$, this mirrors PCA problem with loss $\frac{1}{4}\|\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star \boldsymbol{x}^{\star\top}\|_\mathrm{F}^2$; GD works locally
- How about finite-sample case?

RIP helps

Local convergence of gradient descent

Theorem 8.2

Suppose that $\mathcal{A}$ obeys 4-RIP with constant $\delta_4 \leq 1/44$. If $\|\boldsymbol{x}^0 - \boldsymbol{x}^\star\|_2 \leq \|\boldsymbol{x}^\star\|_2/12$, then GD with $\eta = 1/(3\|\boldsymbol{x}^\star\|_2^2)$ obeys

$$\|\boldsymbol{x}^t - \boldsymbol{x}^\star\|_2 \leq \left(\tfrac{11}{12}\right)^t \|\boldsymbol{x}^0 - \boldsymbol{x}^\star\|_2, \qquad \textit{for } t = 0, 1, 2, \ldots$$

- local linear convergence within basin of attraction $\{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \leq \|\boldsymbol{x}^\star\|_2/12\}$

Proof of Theorem 8.2

In view of theory of gradient descent for locally strongly convex and smooth functions, it suffices to prove that

$$0.25\|\boldsymbol{x}^\star\|_2^2 \boldsymbol{I}_n \preceq \nabla^2 f(\boldsymbol{x}) \preceq 3\|\boldsymbol{x}^\star\|_2^2 \boldsymbol{I}_n$$

holds for all

$$\{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \le \|\boldsymbol{x}^\star\|_2/12\}$$

To analyzing spectral properties of $\nabla^2 f(\boldsymbol{x})$, we focus on quadratic forms

$$\boldsymbol{z}^\top \nabla^2 f(\boldsymbol{x}) \boldsymbol{z}$$

Proof of Theorem 8.2 (cont.)

Simple calculations show

$$\boldsymbol{z}^\top \nabla^2 f(\boldsymbol{x})\boldsymbol{z} = \frac{1}{m}\sum_{i=1}^{m}\langle \boldsymbol{A}_i, \boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top}\rangle(\boldsymbol{z}^\top\boldsymbol{A}_i\boldsymbol{z}) + 2(\boldsymbol{z}^\top\boldsymbol{A}_i\boldsymbol{x})^2,$$

which admits a more "compact" expression

$$\begin{aligned}\boldsymbol{z}^\top \nabla^2 f(\boldsymbol{x})\boldsymbol{z} &= \langle \mathcal{A}(\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top}), \mathcal{A}(\boldsymbol{z}\boldsymbol{z}^\top)\rangle \\ &\quad + \frac{1}{2}\langle \mathcal{A}(\boldsymbol{z}\boldsymbol{x}^\top + \boldsymbol{x}\boldsymbol{z}^\top), \mathcal{A}(\boldsymbol{z}\boldsymbol{x}^\top + \boldsymbol{x}\boldsymbol{z}^\top)\rangle\end{aligned}$$

RIP preserves inner product

Lemma 8.3

Suppose that $\mathcal{A}$ satisfies $2r$-RIP with constant $\delta_{2r} < 1$, then

$$|\langle \mathcal{A}(\boldsymbol{X}), \mathcal{A}(\boldsymbol{Y}) \rangle - \langle \boldsymbol{X}, \boldsymbol{Y} \rangle| \le \delta_{2r} \|\boldsymbol{X}\|_{\mathrm{F}} \|\boldsymbol{X}\|_{\mathrm{F}}$$

Proof of Theorem 8.2 (cont.)

Apply Lemma 8.3 to obtain

$$\begin{aligned}
&\Big|\langle \mathcal{A}(\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top}), \mathcal{A}(\boldsymbol{z}\boldsymbol{z}^\top)\rangle - \langle \boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top}, \boldsymbol{z}\boldsymbol{z}^\top\rangle\Big| \\
&\quad \le \delta_4 \|\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top}\|_{\mathrm{F}} \|\boldsymbol{z}\boldsymbol{z}^\top\|_{\mathrm{F}} \le 3\delta_4 \|\boldsymbol{x}^\star\|_2^2 \|\boldsymbol{z}\|_2^2,
\end{aligned}$$

while last relation uses $\|\boldsymbol{x} - \boldsymbol{x}^\star\|_2 \le \|\boldsymbol{x}^\star\|_2$. Similarly, one has

$$\begin{aligned}
&\Big|\langle \mathcal{A}(\boldsymbol{z}\boldsymbol{x}^\top + \boldsymbol{x}\boldsymbol{z}^\top), \mathcal{A}(\boldsymbol{z}\boldsymbol{x}^\top + \boldsymbol{x}\boldsymbol{z}^\top)\rangle - \|\boldsymbol{z}\boldsymbol{x}^\top + \boldsymbol{x}\boldsymbol{z}^\top\|_{\mathrm{F}}^2\Big| \\
&\quad \le \delta_4 \|\boldsymbol{z}\boldsymbol{x}^\top + \boldsymbol{x}\boldsymbol{z}^\top\|_{\mathrm{F}}^2 \le 4\delta_4 \|\boldsymbol{x}\|_2^2 \|\boldsymbol{z}\|_2^2 \le 16\delta_4 \|\boldsymbol{x}\|_2^2 \|\boldsymbol{z}\|_2^2
\end{aligned}$$

Proof of Theorem 8.2 (cont.)

Define

$$g(\boldsymbol{x},\boldsymbol{z}) := \langle \boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{x}^\star\boldsymbol{x}^{\star\top}, \boldsymbol{z}\boldsymbol{z}^\top \rangle + \frac{1}{2}\|\boldsymbol{z}\boldsymbol{x}^\top + \boldsymbol{x}\boldsymbol{z}^\top\|_{\mathrm{F}}^2$$

Key conclusion so far: when $\|\boldsymbol{x}-\boldsymbol{x}^\star\|_2 \leq \|\boldsymbol{x}^\star\|_2$, $\boldsymbol{z}^\top \nabla^2 f(\boldsymbol{x})\boldsymbol{z}$ is close to $g(\boldsymbol{x},\boldsymbol{z})$

It boils down to upper and lower bounding $g(\boldsymbol{x},\boldsymbol{z})$—a much easier task

Spectral initialization

Construct

$$\boldsymbol{M} = \frac{1}{m}\sum_{i=1}^{m} y_i \boldsymbol{A}_i$$