

STAT253/317 Winter 2022 Lecture 2

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4.2 Chapman-Kolmogorov Equation

Joint Distribution of Random Variables in a Markov Chain

Suppose $\{X_n : n = 0, 1, 2, \dots\}$ is a stationary Markov chain with

- ▶ state space \mathfrak{X} and
- ▶ transition probabilities $\{P_{ij} : i, j \in \mathfrak{X}\}$.

Define $\pi_0(i) = P(X_0 = i)$, $i \in \mathfrak{X}$ to be the distribution of X_0 .

What is the joint distribution of X_0, X_1, X_2 ?

$$\begin{aligned} &P(X_0 = i_0, X_1 = i_1, X_2 = i_2) \\ &= P(X_0 = i_0)P(X_1 = i_1|X_0 = i_0)P(X_2 = i_2|X_1 = i_1, X_0 = i_0) \\ &= P(X_0 = i_0)P(X_1 = i_1|X_0 = i_0)P(X_2 = i_2|X_1 = i_1) \quad (\text{Markov}) \\ &= \pi_0(i_0)P_{i_0i_1}P_{i_1i_2} \end{aligned}$$

In general,

$$\begin{aligned} &P(X_0 = i_0, X_1 = i_1, X_2 = i_2, \dots, X_{n-1} = i_{n-1}, X_n = i_n) \\ &= \pi_0(i_0)P_{i_0i_1}P_{i_1i_2} \cdots P_{i_{n-1}i_n} \end{aligned}$$

n -Step Transition Probabilities

Suppose $\{X_n\}$ is a stationary Markov chain with state space \mathfrak{X} . Define the n -step transition probabilities

$$P_{ij}^{(n)} = \mathbb{P}(X_{n+k} = j | X_k = i) \quad \text{for } i, j \in \mathfrak{X} \text{ and } n, k = 0, 1, 2, \dots$$

How to calculate $P_{ij}^{(n)}$?

Example: Ehrenfest Model, 4 Balls

$$\mathbb{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \left(\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right) \end{matrix}$$

Q1 Find $P_{4,2}^{(2)} = P(X_2 = 2 | X_0 = 4)$.

Only one possible path: $4 \rightarrow 3 \rightarrow 2$,
so $P_{4,2}^{(2)} = P_{4,3}P_{3,2} = 1 \cdot (3/4) = 3/4$.

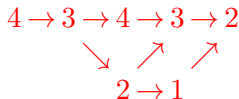
Q2 Find $P_{4,2}^{(3)} = P(X_3 = 2 | X_0 = 4)$.

Impossible to go from 4 to 2 in odd number of steps,
so $P_{4,2}^{(3)} = 0$.

$$\mathbb{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Q3 Find $P_{4,2}^{(4)} = P(X_4 = 2 | X_0 = 4)$.

Possible paths:



$$\begin{aligned} P_{4,2}^{(4)} &= P_{4,3}P_{3,4}P_{4,3}P_{3,2} + P_{4,3}P_{3,2}P_{2,3}P_{3,2} + P_{4,3}P_{3,2}P_{2,1}P_{1,2} \\ &= 1 \cdot \frac{1}{4} \cdot 1 \cdot \frac{3}{4} + 1 \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{3}{4} + 1 \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{3}{4} = \frac{3}{4} \end{aligned}$$

Q4 Find $P_{4,2}^{(10)} = P(X_{10} = 2 | X_0 = 4)$.

Too many paths to list, likely to miss a few.

Chapman-Kolmogorov Equations

Suppose $\{X_n\}$ is a stationary Markov chain with state space \mathfrak{X} . Define the n -step transition probabilities

$$P_{ij}^{(n)} = P(X_{n+k} = j | X_k = i) \quad \text{for } i, j \in \mathfrak{X} \text{ and } n, k = 0, 1, 2, \dots$$

Then for all $m, n \geq 1$,

$$P_{ij}^{(m+n)} = \sum_{k \in \mathfrak{X}} P_{ik}^{(m)} P_{kj}^{(n)}$$

Proof.

$$\begin{aligned} P_{ij}^{(m+n)} &= P(X_{m+n} = j | X_0 = i) \\ &= \sum_{k \in \mathfrak{X}} P(X_{m+n} = j, X_m = k | X_0 = i) \\ &= \sum_{k \in \mathfrak{X}} P(X_m = k | X_0 = i) P(X_{m+n} = j | X_m = k, X_0 = i) \\ &= \sum_{k \in \mathfrak{X}} P(X_m = k | X_0 = i) P(X_{m+n} = j | X_m = k) \quad (\text{Markov}) \\ &= \sum_{k \in \mathfrak{X}} P_{ik}^{(m)} P_{kj}^{(n)} \end{aligned}$$

Chapman-Kolmogorov Equation in Matrix Notation

For $n = 1, 2, 3, \dots$, let

$$\mathbb{P}^{(n)} = \begin{pmatrix} P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots & P_{0j}^{(n)} & \cdots \\ P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \cdots & P_{1j}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i0}^{(n)} & P_{i1}^{(n)} & P_{i2}^{(n)} & \cdots & P_{ij}^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

be the n -step transition probability matrix.

The Chapman-Kolmogorov equation just asserts that

$$\mathbb{P}^{(m+n)} = \mathbb{P}^{(m)} \times \mathbb{P}^{(n)}$$

Note $\mathbb{P}^{(1)} = \mathbb{P}$, $\Rightarrow \mathbb{P}^{(2)} = \mathbb{P}^{(1)} \times \mathbb{P}^{(1)} = \mathbb{P} \times \mathbb{P} = \mathbb{P}^2$.

By induction,

$$\mathbb{P}^{(n)} = \mathbb{P}^{(n-1)} \times \mathbb{P}^{(1)} = \mathbb{P}^{n-1} \times \mathbb{P} = \mathbb{P}^n$$

Define $\pi_n(i) = P(X_n = i)$, $i \in \mathfrak{X}$ to be the marginal distribution of X_n , $n = 1, 2, \dots$. Then again by the law of total probabilities,

$$\begin{aligned}\pi_n(j) &= P(X_n = j) \\ &= \sum_{k \in \mathfrak{X}} P(X_0 = k) P(X_n = j | X_0 = k) \\ &= \sum_{k \in \mathfrak{X}} \pi_0(k) P_{kj}^{(n)}\end{aligned}\tag{1}$$

Suppose the state space \mathfrak{X} is $\{0, 1, 2, \dots\}$.

If we write the marginal distribution of X_n as a row vector

$$\pi_n = (\pi_n(0), \pi_n(1), \pi_n(2), \dots),$$

then the equation (1) is

$$\pi_n = \pi_0 \mathbb{P}^{(n)} = \pi_0 \mathbb{P}^n$$

Example: Ehrenfest Model, 4 Balls

$$\mathbb{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \left(\begin{array}{ccccc} 0 & 4/4 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 4/4 & 0 \end{array} \right) \end{matrix}$$

Q3 Find $P_{4,2}^{(4)} = P(X_4 = 2 | X_0 = 4)$.

Q4 Find $P_{4,2}^{(10)} = P(X_{10} = 2 | X_0 = 4)$.

Q5 Given $P(X_0 = i) = 1/5$ for $i = 0, 1, 2, 3, 4$, find $P(X_4 = 2)$

Q6 Find $P(X_{10} = 2, X_k \geq 2, \text{ for } 1 \leq k \leq 9 | X_0 = 4)$

$$\mathbb{P}^2 = \mathbb{P} \times \mathbb{P} = \begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \left(\begin{array}{ccccc} 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 5/8 & 0 & 3/8 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 3/8 & 0 & 5/8 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \end{array} \right) \end{array}$$

$$\mathbb{P}^3 = \mathbb{P} \times \mathbb{P}^2 = \begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \left(\begin{array}{ccccc} 0 & 5/8 & 0 & 3/8 & 0 \\ 5/32 & 0 & 3/4 & 0 & 3/32 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 3/32 & 0 & 3/4 & 0 & 5/32 \\ 0 & 3/8 & 0 & 5/8 & 0 \end{array} \right) \end{array}$$

$$\mathbb{P}^4 = \mathbb{P}^2 \times \mathbb{P}^2 = \begin{array}{c} \begin{array}{ccccc} & 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \left(\begin{array}{ccccc} 5/32 & 0 & 3/4 & 0 & 3/32 \\ 0 & 17/32 & 0 & 15/32 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 15/32 & 0 & 5/32 & 0 \\ 3/32 & 0 & 3/4 & 0 & 5/32 \end{array} \right) \end{array}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

$$\mathbb{P}^4 = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 5/32 & 0 & 3/4 & 0 & 3/32 \\ 0 & 17/32 & 0 & 15/32 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 15/32 & 0 & 5/32 & 0 \\ 3/32 & 0 & \boxed{3/4} & 0 & 5/32 \end{pmatrix} \end{matrix}$$

For Q3, $P(X_4 = 2 | X_0 = 4) = P_{42}^{(4)} = 3/4$.
which agrees with our previous calculation.

Example: Ehrenfest Model, 4 Balls (Cont'd)

To find $P_{4,2}^{(10)}$ for Q4, it's awful lots of work to compute $\mathbb{P}^{10} \dots$

There are ways to save some work. By the C-K equation,

$$\mathbb{P}_{4,2}^{(10)} = \underbrace{\mathbb{P}_{4,0}^{(5)}\mathbb{P}_{0,2}^{(5)}}_{=0} + \mathbb{P}_{4,1}^{(5)}\mathbb{P}_{1,2}^{(5)} + \underbrace{\mathbb{P}_{4,2}^{(5)}\mathbb{P}_{2,2}^{(5)}}_{=0} + \mathbb{P}_{4,3}^{(5)}\mathbb{P}_{3,2}^{(5)} + \underbrace{\mathbb{P}_{4,4}^{(5)}\mathbb{P}_{4,2}^{(5)}}_{=0}$$

because it's impossible to move between even states in odd number of moves.

We just need to find $\mathbb{P}_{4,1}^{(5)}$, $\mathbb{P}_{4,3}^{(5)}$, $\mathbb{P}_{1,2}^{(5)}$, and $\mathbb{P}_{3,2}^{(5)}$.

Example: Ehrenfest Model, 4 Balls (Cont'd)

$$\mathbb{P}^5 = \mathbb{P}^2 \times \mathbb{P}^3$$

$$\begin{aligned}
 &= \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 5/8 & 0 & 3/8 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 3/8 & 0 & 5/8 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \end{pmatrix} \end{matrix} \times \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 5/8 & 0 & 3/8 & 0 \\ 5/32 & 0 & 3/4 & 0 & 3/32 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 3/32 & 0 & 3/4 & 0 & 5/32 \\ 0 & 3/8 & 0 & 5/8 & 0 \end{pmatrix} \end{matrix} \\
 &= \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} & & 0 & & \\ & & 3/4 & & \\ & & 0 & & \\ & & 3/4 & & \\ 0 & 15/32 & 0 & 17/32 & 0 \end{pmatrix} \end{matrix}
 \end{aligned}$$

So

$$\mathbb{P}_{4,2}^{(10)} = \mathbb{P}_{4,1}^{(5)} \mathbb{P}_{1,2}^{(5)} + \mathbb{P}_{4,3}^{(5)} \mathbb{P}_{3,2}^{(5)} = \frac{15}{32} \times \frac{3}{4} + \frac{17}{32} \times \frac{3}{4} = \frac{3}{4}.$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

Q5: Given $P(X_0 = i) = 1/5$ for $i = 0, 1, 2, 3, 4$, find $P(X_4 = 2)$.

$$\pi_0 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

$$\pi_4 = \pi_0 \mathbb{P}^4 = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \begin{pmatrix} 5/32 & 0 & 3/4 & 0 & 3/32 \\ 0 & 17/32 & 0 & 15/32 & 0 \\ 1/8 & 0 & 3/4 & 0 & 1/8 \\ 0 & 15/32 & 0 & 17/32 & 0 \\ 3/32 & 0 & 3/4 & 0 & 5/32 \end{pmatrix}$$

$$\begin{aligned} \pi_4(2) &= \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right) \begin{pmatrix} 3/4 \\ 0 \\ 3/4 \\ 0 \\ 3/4 \end{pmatrix} \\ &= \frac{1}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot \frac{3}{4} + \frac{1}{5} \cdot 0 + \frac{1}{5} \cdot \frac{3}{4} = \frac{9}{20} \end{aligned}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

Q6: Find $P(X_{10} = 2, X_k \geq 2, \text{ for } 1 \leq k \leq 9 | X_0 = 4)$.

Tip: Create another process $\{W_n, n = 0, 1, 2, \dots\}$ with an absorbing state A

$$W_n = \begin{cases} X_n & \text{if } X_k \geq 2 \text{ for all } k = 0, 1, 2, \dots, n \\ A & \text{if } X_k < 2 \text{ for some } k \leq n \end{cases}$$

What is the state space of $\{W_n\}$? $\{A, 2, 3, 4\}$

Is $\{W_n\}$ a Markov chain?

$$W_{n+1} = \begin{cases} A & \text{if } W_n = A \\ W_n + 1 & \text{with prob. } \frac{4-W_n}{4} \text{ if } W_n \neq A \\ W_n - 1 & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 3 \text{ or } 4 \\ A & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 2 \end{cases}$$

Yes, $\{W_n\}$ is a Markov chain.

Example: Ehrenfest Model, 4 Balls (Cont'd)

What is the transition probability of $\{W_n\}$?

$$\mathbb{P}_W = \begin{matrix} & A & 2 & 3 & 4 \\ \begin{matrix} A \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2/4 & 0 & 2/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Observe that $\mathbb{P}_{W,i,j}$ equals the transition prob. of the original process $\mathbb{P}_{i,j}$ for $i, j \neq A$.

$$\mathbb{P} = \begin{matrix} & 0 & 1 & 2 & 3 & 4 \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 4/4 & 0 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 & 0 \\ 0 & 2/4 & 0 & 2/4 & 0 \\ 0 & 0 & 3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{matrix}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

How does $\{W_n\}$ helps us to solve Q6?

$$\begin{aligned} \text{Observe that } P(X_{10} = 2, X_k \geq 2, \text{ for } 1 \leq k \leq 9 | X_0 = 4) \\ = P(W_{10} = 2 | W_0 = 4) = P_{W,4,2}^{(10)} \end{aligned}$$

It's still an awful lot of work to compute $P_{W,4,2}^{(10)}$.

By the same way we calculate $P_{4,2}^{(10)}$, using C-K equation, we know

$$\mathbb{P}_{W,4,2}^{(10)} = \mathbb{P}_{W,4,A}^{(5)} \underbrace{\mathbb{P}_{W,A,2}^{(5)}}_{=0} + \underbrace{\mathbb{P}_{W,4,2}^{(5)} \mathbb{P}_{W,2,2}^{(5)}}_{=0} + \mathbb{P}_{W,4,3}^{(5)} \mathbb{P}_{W,3,2}^{(5)} + \underbrace{\mathbb{P}_{W,4,4}^{(5)} \mathbb{P}_{W,4,2}^{(5)}}_{=0}$$

in which

- ▶ $\mathbb{P}_{W,A,2}^{(5)} = 0$ because $\{W_n\}$ will never leave A .
- ▶ $\mathbb{P}_{W,4,2}^{(5)} = \mathbb{P}_{W,4,4}^{(5)} = 0$ because $\{W_n\}$ can never get from 4 to an even numbered state in odd numbers of steps.

Just need to find $\mathbb{P}_{W,4,3}^{(5)}$ and $\mathbb{P}_{W,3,2}^{(5)}$.

Example: Ehrenfest Model, 4 Balls (Cont'd)

$$\mathbb{P}_W^{(2)} = \begin{matrix} & \begin{matrix} A & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} A \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 3/8 & 0 & 1/8 \\ 3/8 & 0 & 5/8 & 0 \\ 0 & 3/4 & 0 & 1/4 \end{pmatrix} \end{matrix}, \quad \mathbb{P}_W^{(3)} = \begin{matrix} & \begin{matrix} A & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} A \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 11/16 & 0 & 5/16 & 0 \\ 3/8 & 15/32 & 0 & 5/32 \\ 3/8 & 0 & 5/8 & 0 \end{pmatrix} \end{matrix}$$

$$\mathbb{P}_W^{(5)} = \mathbb{P}_W^{(2)} \times \mathbb{P}_W^{(3)} = \begin{matrix} & \begin{matrix} A & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} A \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 75/256 & & & \\ 0 & 25/64 & & \end{pmatrix} \end{matrix}$$

So

$$\mathbb{P}_{W,4,2}^{(10)} = \mathbb{P}_{W,4,3}^{(5)} \mathbb{P}_{W,3,2}^{(5)} = \frac{25}{64} \times \frac{75}{256} = \frac{1875}{16384}.$$

For a generalization of Q6, see the discussion starting from the bottom of p.202 to Example 4.14 on p.203 of the 12th edition of the textbook (or p.192-193 of the 11th edition).