Problems

February 9, 2017

1. Why is L2 norm a vector norm?

2. Verify that the Cauchy-Schwarz inequality holds for
$$\mathbf{u} = (1, 3, 5, 2, 0, 1), \ \ \mathbf{v} = (0, 2, 4, 1, 3, 5).$$

3. Let W_1 and W_2 be a subspaces of $\mathbb{R}^n$. Prove that the intersection $W_1 \cap W_2$ is a subspace of $\mathbb{R}^n$.

4. Prove that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set, then so is the set $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ for every nonzero scalar k.

5. Find all values of λ for which $\det(A) = 0$.
$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

6. Find the determinant of the following matrix by inspection, not by calculation.
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

7. Find the eigenvalues and corresponding eigenvectors of the matrix
$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}.$$

8. (a) Compute the operator norm of the following matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}.$$

(b) What is the vector achieving the following maximum?

$$\max_{\|\mathbf{x}\| \neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}$$

9. Show that the vectors

$$\mathbf{v}_1 = (1, 0, 0),\ \mathbf{v}_2 = (0, 2, 0),$$
$$\mathbf{v}_3 = (0, 0, 3),\ \mathbf{v}_4 = (1, 1, 1)$$

span $\mathbb{R}^3$ but do not form a basis for $\mathbb{R}^3$.

10. Prove that if $\mathbf{u}$ is a nonzero $n \times 1$ column vector, then the outer product $\mathbf{u}\mathbf{u}^T$ is a symmetric matrix of rank 1.

11. Prove that if $V = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, and if $W = \{\mathbf{w}_1, \cdots, \mathbf{w}_{n-k}\}$ is a basis for the null space of the matrix A that has the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ as its successive rows, then $V \cup W$ is a basis for $\mathbb{R}^n$.

12. Find the least squares solution of $A\mathbf{x} = \mathbf{b}$ by solving the associated normal system, and show that the least square error vector is orthogonal to the column space of A.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}.$$

13. Find the eigenvalue decomposition of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

14. Find a singular value decomposition of A.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

15. Suppose that A has the singular value decomposition

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 24 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

(a) Find orthonormal bases for the four fundamental spaces of A.

(b) Find the reduced singular value decomposition of A.

16. Prove that nuclear norm has the following property:

$$\|A\|_* \triangleq \sum_{i=1}^{\min(m,n)} \sigma_i = \operatorname{tr}(\sqrt{A^T A}),$$

for $A \in \mathbb{R}^{m\times n}$, where $\sqrt{A^T A}$ if a matrix B such that $B^2 = A^T A$.