

# Continuous-Time Markov Chains



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# Continuous-Time Markov Chains (CTMC)

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**Definitions.** A stochastic process  $\{X(t), t \geq 0\}$  with state space  $\mathcal{X}$  is called a *continuous-time Markov chain* if for any two states  $i, j \in \mathcal{X}$ ,

$$\begin{aligned} & \underbrace{\Pr(X(t+s) = j | X(s) = i, X(u) = x(u), \text{for } 0 \leq u < s)}_{\text{future}} \underbrace{\Pr(X(s) = i)}_{\text{present}} \underbrace{\Pr(X(u) = x(u) \text{ for } 0 \leq u < s)}_{\text{past}} \\ &= \underbrace{\Pr(X(t+s) = j | X(s) = i)}_{\text{future}} \underbrace{\Pr(X(s) = i)}_{\text{present}} \end{aligned}$$

If  $\Pr(X(t+s) = j | X(s) = i)$  does not depend on  $s$  for all  $i, j \in \mathcal{X}$ , then it is denoted as

$$P_{ij}(t) = \Pr(X(t+s) = j | X(s) = i),$$

and we say the CTMC is *homogeneous* in time.

In STAT253/317, we focus on homogeneous CTMC only.

# Exponential Waiting/Holding Time

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Let  $\{X(t), t \geq 0\}$  be a homogeneous continuous-time Markov chain. For  $i \in \mathcal{X}$ , let  $T_i$  denote the amount of time that  $X(t)$  stays in state  $i$  before making a transition into a different state.

**Claim:**  $T_i$  has the *memoryless property*.

$$\begin{aligned} & P(T_i \geq t + s | T_i \geq s) \\ &= P(X(u) = i, \text{ for } s \leq u \leq s + t | X(u) = i, \text{ for } 0 \leq u \leq s) \\ &= P(X(u) = i, \text{ for } s \leq u \leq s + t | X(s) = i) \quad (\text{Markov property}) \\ &= P(X(u) = i, \text{ for } 0 \leq u \leq t | X(0) = i) \quad (\text{time homogeneity}) \\ &= P(T_i \geq t) \quad \Rightarrow \quad \text{So } T_i \text{ is memoryless.} \end{aligned}$$

Recall that the exponential distribution is the only continuous distribution having the memoryless property.

Thus  $T_i \sim \text{Exp}(\nu_i)$  for some rate  $\nu_i$ .

# An Alternative Definition of CTMC

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A stochastic process  $\{X(t), t \geq 0\}$  with state space  $\mathcal{X}$  is a *continuous-time Markov chain* if

- (exponential waiting time) when the chain reaches a state  $i$ , the time it stays at state  $i \sim \text{Exp}(\nu_i)$ , where  $\nu_i$  is the transition rate at state  $i$
- (embedded with a discrete time Markov chain) when the process leaves state  $i$ , it enters another state  $j$  with probability  $P_{ij}$ , such that

$$P_{ii} = 0, \quad \sum_{j \in \mathcal{X}} P_{ij} = 1 \quad \text{for all } i, j \in \mathcal{X}.$$

**Remark:** The amount of time  $T_i$  the process spends in state  $i$ , and the next state visited, must be independent. For if the next state visited were dependent on  $T_i$ , then information as to how long the process has already been in state  $i$  would be relevant to the prediction of the next state—and this contradicts the Markovian assumption.

## Example: Poisson Process

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For a Poisson process with parameter  $\lambda$ , the holding-time parameters are constant:

$$q_i = \lambda, \quad i = 0, 1, 2, \dots$$

The process moves sequentially

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \dots$$

The transition matrix of the embedded chain is

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

## 6.3 Birth and Death Processes

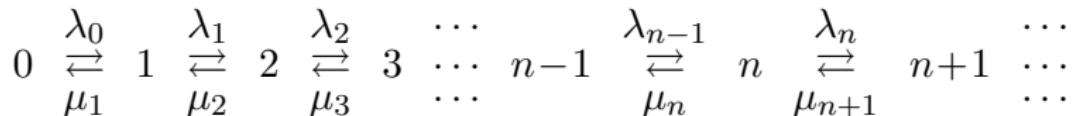
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Let  $X(t)$  = the number of people in the system at time  $t$ .

Suppose that whenever there are  $n$  people in the system, then

- (i) new arrivals enter the system at an exponential rate  $\lambda_n$ , and
- (ii) people leave the system at an exponential rate  $\mu_n$ .

Such an  $\{X(t), t \geq 0\}$  is called a *birth and death process*.



Suppose the process is at state  $i > 0$  at time  $t$ . Then

$$B_i = \text{waiting time until the next birth} \sim \text{Exp}(\lambda_i)$$

$$D_i = \text{waiting time until the next death} \sim \text{Exp}(\mu_i)$$

Hence, the waiting time until the next transition out of state  $i$  is  $\min(B_i, D_i) \sim \text{Exp}(\lambda_i + \mu_i)$ , from which we can get

$$\nu_i = \lambda_i + \mu_i, \text{ for } i > 0$$

## 6.3 Birth and Death Processes (Cont'd)

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Moreover, given the process is at state  $i > 0$  at time  $t$ , the probability that the next transition is a birth rather than a death is

$$P_{i,i+1} = P(B_i < D_i) = \frac{\lambda_i}{\lambda_i + \mu_i},$$

which implies  $P_{i,i-1} = P(D_i < B_i) = \frac{\mu_i}{\lambda_i + \mu_i}$ , for  $i > 0$ .

As only birth is possible at state 0, we know  $\nu_0 = \lambda_0$  and  $P_{01} = 1$ .  
To sum up, a birth and death process is a CTMC with state space  $\mathcal{X} = \{0, 1, 2, \dots\}$  such that

$$\nu_i = \lambda_i + \mu_i, i > 0, \quad \nu_0 = \lambda_0,$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, i > 0$$

$$P_{01} = 1, \quad P_{i,j} = 0 \quad \text{if } |i - j| > 1$$

The parameters  $\{\lambda_n\}_{n=0}^{\infty}$  and  $\{\mu_n\}_{n=1}^{\infty}$  are called, respectively, the arrival (or birth) and departure (or death) rates.

## Examples of Birth and Death Processes

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- Poisson Processes:  $\mu_n = 0$ ,  $\lambda_n = \lambda$  for all  $n \geq 0$
- Pure Birth Process:

$$\mu_n = 0 \quad \Rightarrow \quad \nu_i = \lambda_i, \quad P_{i,i+1} = 1, \quad P_{i,i-1} = 0$$

- Yule Processes (Pure Birth Process with Linear Growth rate): If there are  $n$  people and each independently gives birth at an exponential rate  $\lambda$ , then the total rate at which births occur is  $n\lambda$ .

$$\mu_n = 0, \quad \lambda_n = n\lambda$$

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$$\mu_n = 0, \quad \lambda_n = n\lambda$$

*Reason:* Let

$$B_i = \text{time until the } i\text{th individual give birth} \sim \text{Exp}(\lambda), \quad i = 1, \dots, n$$

So the time until the next (first) birth when there are  $n$  individuals in the population is

$$\min(B_1, B_2, \dots, B_n) \sim \text{Exp}(\lambda + \lambda + \dots + \lambda) = \text{Exp}(n\lambda)$$

## Example: Linear Growth Model with Immigration

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- each individual independently gives birth at an exponential rate  $\lambda$
- each individual independently die at at an exponential rate  $\mu$
- new immigrants come in at an exponential rate  $\theta$

Such a process is a birth-death process with birth and death rates

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Reason: Let

$B_i$  = time until the  $i$ th individual give birth  $\sim \text{Exp}(\lambda)$ ,  $i = 1, \dots, n$

$T$  = time until the next new immigrant comes in  $\sim \text{Exp}(\theta)$

So the time until the population size increase from  $n$  to  $n+1$  is

$$\min(B_1, \dots, B_n, T) \sim \text{Exp}(\lambda + \dots + \lambda + \theta) = \text{Exp}(n\lambda + \theta)$$

So the rate until the next birth is  $\lambda_n = n\lambda + \theta$ .

Similarly, one can show that the death rate is  $\mu_n = n\mu$ .

## Example: $M/M/s$ Queueing Model

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- $s$  servers
- Poisson arrival of customers, rate =  $\lambda$
- Exponential service time, rate =  $\mu$

⇒ a birth and death process with constant birth rate  $\lambda_n = \lambda$ , and  
death (departure)rate  $\mu_n = \min(n, s)\mu$ .

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⇒ a birth and death process with constant birth rate  $\lambda_n = \lambda$ , and death (departure)rate  $\mu_n = \min(n, s)\mu$ .

*Reason:* Suppose, there are  $n$  customer in the system at time  $t$ . At most  $\min(n, s)$  of them are being served. Let  $S_i$  be remaining service time of the  $i$ th server  $\sim \text{Exp}(\mu)$ . Then, the waiting time until the next departure is

$$\min(S_1, \dots, S_{\min(s,n)}) \sim \text{Exp}(\min(s, n)\mu).$$

## 6.4 The Transition Probability Function $P_{ij}(t)$

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Recall the transition probability function  $P_{ij}(t)$  of a CTMC  $\{X(t), t \geq 0\}$  is

$$P_{ij}(t) = \text{P}(X(t+s) = j | X(s) = i)$$

**Example.** (Poisson Processes with rate  $\lambda$ )

$$P_{ij}(t) = \text{P}(N(t+s) = j | N(s) = i)$$

$$= \text{P}(N(t+s) - N(s) = j - i) = \begin{cases} e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases}$$

### Properties of Transition Probability Functions

- $P_{ij}(t) \geq 0$  for all  $i, j \in \mathcal{X}$  and  $t \geq 0$
- (Row sums are 1)  $\sum_j P_{ij}(t) = 1$  for all  $i \in \mathcal{X}$  and  $t \geq 0$

## Lemma 6.3 Chapman-Kolmogorov Equation

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For all  $i, j \in \mathcal{X}$  and  $t \geq 0$ ,

$$P_{ij}(t+s) = \sum_{k \in \mathcal{X}} P_{ik}(t)P_{kj}(s)$$

*Proof.*

$$\begin{aligned} & P_{ij}(t+s) \\ &= \text{P}(X(t+s) = j | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} \text{P}(X(t+s) = j, X(t) = k | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} \text{P}(X(t+s) = j | X(t) = k, X(0) = i) \text{P}(X(t) = k | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} \text{P}(X(t+s) = j | X(t) = k) \text{P}(X(t) = k | X(0) = i) \text{ (Markov Property)} \\ &= \sum_{k \in \mathcal{X}} P_{kj}(s)P_{ik}(t) \end{aligned}$$

## The matrix notation

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Let  $\mathbf{P}(t) = [P_{ij}(t)]$  be the transition matrix at time  $t$ .  
We have  $\mathbf{P}(0) = \mathbf{I}$ . And C-K equations read

$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s)$$

One way to specify a CTMC is through  $\{\mathbf{P}(t)\}_{t \geq 0}$ . But this requires an infinite number of matrices. Can we simplify it?

Key: use derivatives  $\mathbf{P}'(t)$

# Transition rate matrix / infinitesimal generator Q

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Assume that

$$\mathbf{P}'(0) = \lim_{h \rightarrow 0} \frac{\mathbf{P}(h) - \mathbf{P}(0)}{h} \quad \text{exists.}$$

In other words, for each  $i, j$ ,

$$P'_{ij}(0) = \lim_{h \rightarrow 0} \frac{P_{ij}(h) - P_{ij}(0)}{h} \quad \text{exists.}$$

We will denote such limit as  $\mathbf{Q} = [q_{ij}]$ , the transition rate matrix.  
What does this tell us about  $\mathbf{P}'(t)$  for  $t > 0$ ?

# Kolmogorov's equations

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By definition, one has

$$\begin{aligned}\mathbf{P}'(t) &= \lim_{h \rightarrow 0} \frac{\mathbf{P}(t+h) - \mathbf{P}(t)}{h} = \lim_{h \rightarrow 0} \frac{\mathbf{P}(t)\mathbf{P}(h) - \mathbf{P}(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\mathbf{P}(t)(\mathbf{P}(h) - \mathbf{I})}{h} = \mathbf{P}(t)\mathbf{Q}.\end{aligned}$$

This is the so-called Kolmogorov's forward equations.

Similarly you can prove backward equations

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t).$$

These imply  $\mathbf{P}(t) = \exp(t\mathbf{Q})$ .

# Transition rate matrix

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How to compute  $\mathbf{Q}$ ?

## Lemma 6.2a

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For any  $i, j \in \mathcal{X}$ , we have

$$q_{ii} = \lim_{h \rightarrow 0} \frac{P_{ii}(h) - 1}{h} = -\nu_i$$

*Proof.* Let  $T_i$  be the amount of time the process stays in state  $i$  before moving to other states.

$$\begin{aligned} P_{ii}(h) &= \text{P}(X(h) = i | X(0) = i) \\ &= \text{P}(X(h) = i, \text{no transition in } (0, h] | X(0) = i) \\ &\quad + \text{P}(X(h) = i, 2 \text{ or more transitions in } (0, h] | X(0) = i) \\ &= \text{P}(T_i > h) + o(h) \\ &= e^{-\nu_i h} + o(h) \\ &= 1 - \nu_i h + o(h) \end{aligned}$$

## Lemma 6.2b

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For any  $i \neq j \in \mathcal{X}$ , we have

$$q_{ij} = \lim_{h \rightarrow 0} \frac{P_{ij}(h)}{h} = \nu_i P_{ij}$$

*Proof.*

$$\begin{aligned} P_{ij}(h) &= \text{P}(X(h) = j | X(0) = i) \\ &= \text{P}(X(h) = j, 1 \text{ transition in } (0, h] | X(0) = i) \\ &\quad + \text{P}(X(h) = j, 2 \text{ or more transitions in } (0, h] | X(0) = i) \\ &= \text{P}(T_i < h) P_{ij} + o(h) \\ &= (1 - e^{-\nu_i h}) P_{ij} + o(h) \\ &= \nu_i P_{ij} h + o(h) \end{aligned}$$

For finite state space case  $\mathcal{X} = \{1, 2, \dots, m\}$ , define the matrices

$$\mathbf{P}(t) = \begin{bmatrix} P_{11}(t) & \cdots & P_{1m}(t) \\ \vdots & & \vdots \\ P_{m1}(t) & \cdots & P_{mm}(t) \end{bmatrix}, \quad \mathbf{P}'(t) = \begin{bmatrix} P'_{11}(t) & \cdots & P'_{1m}(t) \\ \vdots & & \vdots \\ P'_{m1}(t) & \cdots & P'_{mm}(t) \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & & \vdots \\ q_{m1} & \cdots & q_{mm} \end{bmatrix} = \begin{bmatrix} -\nu_1 & \nu_1 P_{12} & \cdots & \nu_1 P_{1m} \\ \nu_2 P_{21} & -\nu_2 & \cdots & \nu_2 P_{2m} \\ \vdots & \vdots & & \vdots \\ \nu_m P_{m1} & \nu_m P_{m2} & \cdots & -\nu_m \end{bmatrix}$$

In matrix notation,

Forward equation:  $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

Backward equation:  $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$

# Limiting Probabilities

## 6.5. Limiting Probabilities

**Definition.** Just like discrete-time Markov chains, if the probability that a continuous-time Markov chain will be in state  $j$  at time  $t$ ,  $P_{ij}(t)$ , converges to a limiting value  $P_j$  independent of the initial state  $i$ , for all  $i \in \mathcal{X}$

$$P_j = \lim_{t \rightarrow \infty} P_{ij}(t) > 0$$

then we say  $P_j$  is the *limiting probability* of state  $j$ . If  $P_j$  exists for all  $j \in \mathcal{X}$ , we say  $\{P_j\}_{j \in \mathcal{X}}$  is the *limiting distribution* of the process.

**Remark.** If  $\lim_{t \rightarrow \infty} P_{ij}(t)$  exists, we must have

$$\lim_{t \rightarrow \infty} P'_{ij}(t) = 0.$$

# Balanced equations

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Recall the forward equations  $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

If you set  $t \rightarrow \infty$ , you have

$$0 = p^\top \mathbf{Q},$$

where  $p = (P_1, P_2, \dots)^\top$

This is the same as saying that

$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} \quad \text{for all } j \in \mathcal{X}$$

# Interpretation of the Balanced Equations

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$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$$

$\nu_j P_j$  = rate at which the process **leaves** state  $j$

$\sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$  = rate at which the process **enters** state  $j$

Balanced equations means that the rates at which the process enters and leaves state  $j$  are equal.

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The limiting distribution  $\{P_j\}_{j \in \mathcal{X}}$  can be obtained by solving the balanced equations along with the equation  $\sum_{j \in \mathcal{X}} P_j = 1$ .

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**Remarks.** Just like discrete-time Markov chains, a sufficient condition for the existence of a limiting distribution is that the chain is irreducible and positive recurrent.

# Examples

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- **Poisson processes:**  $\mu_n = 0$ ,  $\lambda_n = \lambda$  for all  $n \geq 0$

$$\nu_i = \lambda, P_{i,i+1} = 1, q_{i,i+1} = \nu_i P_{i,i+1} = \lambda$$

Balanced equations:

$$\nu_j P_j = P_{j-1} q_{j-1,j} \Rightarrow \lambda P_j = \lambda P_{j-1} \Rightarrow P_j = P_{j-1}$$

No limiting distribution exists. The chain is not irreducible. All states are transient.

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Balanced equations:

$$\nu_j P_j = P_{j-1} q_{j-1,j} \Rightarrow \lambda P_j = \lambda P_{j-1} \Rightarrow P_j = P_{j-1}$$

No limiting distribution exists. The chain is not irreducible. All states are transient.

- **Pure birth processes with  $\lambda_n > 0$  for all  $n$**

No limiting distribution exists. All states are transient.

# Birth and Death Processes

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For a birth and death process,

$$\nu_0 = \lambda_0,$$

$$\nu_i = \lambda_i + \mu_i, i > 0$$

$$P_{01} = 1,$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad i > 0$$

$$P_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}, \quad i > 0$$

$$P_{i,j} = 0 \quad \text{if } |i - j| > 1$$

$$\Rightarrow q_{i,i+1} = \nu_i P_{i,i+1} = \lambda_i, \quad i \geq 0$$

$$q_{i,i-1} = \nu_i P_{i,i-1} = \mu_i, \quad i \geq 1$$

# Balanced Equations for Birth and Death Processes

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The balanced equations  $\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$  for a birth and death process are

$$\lambda_0 P_0 = \mu_1 P_1$$

$$(\mu_1 + \lambda_1) P_1 = \lambda_0 P_0 + \mu_2 P_2,$$

$$(\mu_2 + \lambda_2) P_2 = \lambda_1 P_1 + \mu_3 P_3,$$

$$\vdots$$

$$(\mu_{n-1} + \lambda_{n-1}) P_{n-1} = \lambda_{n-2} P_{n-2} + \mu_n P_n$$

$$(\mu_n + \lambda_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$$

Adding up all the equations above and eliminating the common terms on both sides, we get

$$\lambda_n P_n = \mu_{n+1} P_{n+1} \quad n \geq 0,$$

We hence just need to solve  $\lambda_n P_n = \mu_{n+1} P_{n+1}$  for the limiting distribution.

## 6.6. Time Reversibility

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**Definition.** A continuous-time Markov chain with state space  $\mathcal{X}$  is *time reversible* if

$$P_i q_{ij} = P_j q_{ji}, \quad \text{for all } i, j \in \mathcal{X} \quad (\text{detailed balanced equation})$$

If a distribution  $\{P_j\}$  on  $\mathcal{X}$  satisfies the detailed balanced equation, then it is a stationary distribution for the process.

**Example.** We have just shown that for Birth and Death processes, the balanced equations would lead to the detailed balanced equations, which are

$$\lambda_n P_n = \mu_{n+1} P_{n+1}, \quad n \geq 0$$

# Limiting Dist'n for Birth and Death Processes

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Solving  $\lambda_n P_n = \mu_{n+1} P_{n+1}$ ,  $n \geq 0$  for the limiting distribution, we get

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1}}{\mu_n} \frac{\lambda_{n-2}}{\mu_{n-1}} P_{n-2} = \dots = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} P_0$$

To meet the requirement  $\sum_{n=0}^{\infty} P_n = 1$ , we need

$$\sum_{n=0}^{\infty} P_n = P_0 + P_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} = 1$$

In other words, to have a limiting distribution, it is necessary that

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$

# Limiting Dist'n for Birth and Death Processes (Cont'd)

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If  $\sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}$  is finite, the limiting distribution is

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}}$$

and

$$P_k = \frac{\lambda_0\lambda_1\cdots\lambda_{k-1}/(\mu_1\mu_2\cdots\mu_k)}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0\lambda_1\cdots\lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}}, \quad k \geq 1$$

**Lemma: (Ratio Test)** If  $a_n \geq 0$  for all  $n$ , then

$$\sum_{n=1}^{\infty} a_n \begin{cases} < \infty & \text{if } \lim_{n \rightarrow \infty} a_n/a_{n-1} < 1 \\ = \infty & \text{if } \lim_{n \rightarrow \infty} a_n/a_{n-1} > 1 \end{cases}$$

For  $a_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$ ,  $a_n/a_{n-1} = \lambda_{n-1}/\mu_n$ . By the ratio test, if

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\mu_n} < 1,$$

then  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$ , the limiting distribution exists.

#### Example 6.4 Linear Growth Model with Immigration

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Using the Ratio Test,

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\mu_n} = \lim_{n \rightarrow \infty} \frac{(n-1)\lambda + \theta}{n\mu} = \frac{\lambda}{\mu}$$

So the linear growth model has a limiting distribution if  $\lambda < \mu$ .

## Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

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- single-server service station. Service times are i.i.d.  $\sim Exp(\mu)$
- Poisson arrival of customers with rate  $\lambda$

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- Upon arrival, a customer would
  - go into service if the server is free (queue length = 0)
  - join the queue if 1 to  $N - 1$  customers in the station, or
  - **walk away** if  $N$  or more customers in the station

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**Q:** What fraction of potential customers are lost?

## Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

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- single-server service station. Service times are i.i.d.  $\sim \text{Exp}(\mu)$
- Poisson arrival of customers with rate  $\lambda$
- Upon arrival, a customer would
  - go into service if the server is free (queue length = 0)
  - join the queue if 1 to  $N - 1$  customers in the station, or
  - **walk away** if  $N$  or more customers in the station

**Q:** What fraction of potential customers are lost?

Let  $X(t)$  be the number of customers in the station at time  $t$ .

$\{X(t), t \geq 0\}$  is a birth-death process with the birth and death rates below

$$\mu_n = \begin{cases} 0 & \text{if } n = 0 \\ \mu & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad \lambda_n = \begin{cases} \lambda & \text{if } 0 \leq n < N \\ 0 & \text{if } n \geq N \end{cases}$$

## Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

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Solving  $\lambda_n P_n = \mu_{n+1} P_{n+1}$  for the limiting distribution

$$P_1 = (\lambda/\mu)P_0$$

$$P_2 = (\lambda/\mu)P_1 = (\lambda/\mu)^2 P_0$$

⋮

$$P_i = (\lambda/\mu)^i P_0, \quad i = 1, 2, \dots, N$$

Plugging  $P_i = (\lambda/\mu)^i P_0$  into  $\sum_{i=0}^N P_i = 1$ , one can solve for  $P_0$  and get

$$P_i = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^i$$

Answer: The fraction of customers lost is  $P_N = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^N$

# Duration Times for Birth and Death Processes

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Let

$$T_i = \text{time to move from state } i \text{ to state } i+1, \quad i = 0, 1, \dots$$

Suppose at some moment  $X(t) = i$ . Let

$$B_i = \text{time until the next birth} \sim \text{Exp}(\lambda_i)$$

$$D_i = \text{time until the next death} \sim \text{Exp}(\mu_i)$$

Then

$$\begin{aligned} T_i &= \begin{cases} B_i & \text{if the next step is } i \rightarrow i+1, \text{ i.e., } B_i < D_i \\ D_i + T_{i-1} + T_i^* & \text{if the next step is } i \rightarrow i-1, \text{ i.e., } D_i < B_i \end{cases} \\ &= \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i, \text{ occur w/ prob. } \frac{\lambda_i}{\lambda_i + \mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i, \text{ occur w/ prob. } \frac{\mu_i}{\lambda_i + \mu_i} \end{cases} \end{aligned}$$

Note

- $T_i^*$  has the same distribution as  $T_i$
- $T_{i-1}$  and  $T_i^*$  are indep. of  $B_i$  and  $D_i$  because it's Markov

# Duration Times for Birth and Death Processes

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Taking expected value of

$$T_i = \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i, \text{ occur w/ prob. } \frac{\lambda_i}{\lambda_i + \mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i, \text{ occur w/ prob. } \frac{\mu_i}{\lambda_i + \mu_i} \end{cases}$$

we get

$$\begin{aligned}\mathbb{E}[T_i] &= \mathbb{E}[\min(B_i, D_i)] + (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i]) \frac{\mu_i}{\lambda_i + \mu_i} \\ &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i])\end{aligned}$$

We obtain the recursive formula

$$\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$$

# Duration Times for Birth and Death Processes (Cont'd)

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Since  $T_0 \sim \text{Exp}(\lambda_0)$ ,  $\mathbb{E}[T_0] = 1/\lambda_0$ .

Using the recursive formula  $\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$ , we have

$$\mathbb{E}[T_0] = \frac{1}{\lambda_0}$$

$$\mathbb{E}[T_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} \mathbb{E}[T_0] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0}$$

$$\mathbb{E}[T_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} \left( \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0} \right) = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2 \lambda_1} + \frac{\mu_2 \mu_1}{\lambda_2 \lambda_1 \lambda_0}$$

⋮

$$\begin{aligned}\mathbb{E}[T_i] &= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} \mathbb{E}[T_{i-1}] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \cdots + \frac{\mu_i \mu_{i-1} \cdots \mu_2 \mu_1}{\lambda_i \lambda_{i-1} \cdots \lambda_2 \lambda_1 \lambda_0} \\ &= \frac{1}{\lambda_i} \left( 1 + \sum_{k=1}^i \prod_{j=1}^k \frac{\mu_{i-j+1}}{\lambda_{i-j}} \right)\end{aligned}$$