

Continuous-Time Markov Chains



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University of Chicago, Winter 2026

Continuous-Time Markov Chains (CTMC)

Definitions. A stochastic process $\{X(t), t \geq 0\}$ with state space \mathcal{X} is called a *continuous-time Markov chain* if for any two states $i, j \in \mathcal{X}$,

$$\begin{aligned} & \underbrace{P(X(t+s) = j)}_{\text{future}} \mid \underbrace{X(s) = i}_{\text{present}}, \underbrace{X(u) = x(u), \text{ for } 0 \leq u < s}_{\text{past}} \\ &= P(\underbrace{X(t+s) = j}_{\text{future}} \mid \underbrace{X(s) = i}_{\text{present}}) \end{aligned}$$

If $P(X(t+s) = j \mid X(s) = i)$ does not depend on s for all $i, j \in \mathcal{X}$, then it is denoted as

$$P_{ij}(t) = P(X(t+s) = j \mid X(s) = i),$$

and we say the CTMC is *homogeneous* in time.

In STAT253/317, we focus on homogeneous CTMC only.

Exponential Waiting/Holding Time

Let $\{X(t), t \geq 0\}$ be a homogeneous continuous-time Markov chain. For $i \in \mathcal{X}$, let T_i denote the amount of time that $X(t)$ stays in state i before making a transition into a different state.

Claim: T_i has the *memoryless property*.

$$\begin{aligned} & P(T_i \geq t + s | T_i \geq s) \\ &= P(X(u) = i, \text{ for } s \leq u \leq s + t | X(u) = i, \text{ for } 0 \leq u \leq s) \\ &= P(X(u) = i, \text{ for } s \leq u \leq s + t | X(s) = i) \quad (\text{Markov property}) \\ &= P(X(u) = i, \text{ for } 0 \leq u \leq t | X(0) = i) \quad (\text{time homogeneity}) \\ &= P(T_i \geq t) \quad \Rightarrow \quad \text{So } T_i \text{ is memoryless.} \end{aligned}$$

Recall that the exponential distribution is the only continuous distribution having the memoryless property.

Thus $T_i \sim \text{Exp}(\nu_i)$ for some rate ν_i .

An Alternative Definition of CTMC

A stochastic process $\{X(t), t \geq 0\}$ with state space \mathcal{X} is a *continuous-time Markov chain* if

- (exponential waiting time) when the chain reaches a state i , the time it stays at state $i \sim \text{Exp}(\nu_i)$, where ν_i is the transition rate at state i
- (embedded with a discrete time Markov chain) when the process leaves state i , it enters another state j with probability P_{ij} , such that

$$P_{ii} = 0, \quad \sum_{j \in \mathcal{X}} P_{ij} = 1 \quad \text{for all } i, j \in \mathcal{X}.$$

Remark: The amount of time T_i the process spends in state i , and the next state visited, must be independent. For if the next state visited were dependent on T_i , then information as to how long the process has already been in state i would be relevant to the prediction of the next state—and this contradicts the Markovian assumption.

Example: Poisson Process

For a Poisson process with parameter λ , the holding-time parameters are constant:

$$q_i = \lambda, \quad i = 0, 1, 2, \dots$$

The process moves sequentially

$$0 \rightarrow 1 \rightarrow 2 \rightarrow \dots$$

The transition matrix of the embedded chain is

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

6.3 Birth and Death Processes

Let $X(t)$ = the number of people in the system at time t .

Suppose that whenever there are n people in the system, then

- (i) new arrivals enter the system at an exponential rate λ_n , and
- (ii) people leave the system at an exponential rate μ_n .

Such an $\{X(t), t \geq 0\}$ is called a *birth and death process*.

$$\begin{array}{ccccccccccccccc} & \lambda_0 & & \lambda_1 & & \lambda_2 & & \cdots & & \lambda_{n-1} & & \lambda_n & & \cdots \\ 0 & \rightleftharpoons & 1 & \rightleftharpoons & 2 & \rightleftharpoons & 3 & \cdots & n-1 & \rightleftharpoons & n & \rightleftharpoons & n+1 & \cdots \\ & \mu_1 & & \mu_2 & & \mu_3 & & \cdots & & \mu_n & & \mu_{n+1} & & \cdots \end{array}$$

Suppose the process is at state $i > 0$ at time t . Then

B_i = waiting time until the next birth $\sim \text{Exp}(\lambda_i)$

D_i = waiting time until the next death $\sim \text{Exp}(\mu_i)$

Hence, the waiting time until the next transition out of state i is $\min(B_i, D_i) \sim \text{Exp}(\lambda_i + \mu_i)$, from which we can get

$$\nu_i = \lambda_i + \mu_i, \text{ for } i > 0$$

6.3 Birth and Death Processes (Cont'd)

Moreover, given the process is at state $i > 0$ at time t , the probability that the next transition is a birth rather than a death is

$$P_{i,i+1} = P(B_i < D_i) = \frac{\lambda_i}{\lambda_i + \mu_i},$$

which implies $P_{i,i-1} = P(D_i < B_i) = \frac{\mu_i}{\lambda_i + \mu_i}$, for $i > 0$.

As only birth is possible at state 0, we know $\nu_0 = \lambda_0$ and $P_{01} = 1$.

To sum up, a birth and death process is a CTMC with state space $\mathcal{X} = \{0, 1, 2, \dots\}$ such that

$$\begin{aligned}\nu_i &= \lambda_i + \mu_i, i > 0, & \nu_0 &= \lambda_0, \\ P_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i}, & P_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i}, i > 0 \\ P_{01} &= 1, & P_{i,j} &= 0 \quad \text{if } |i - j| > 1\end{aligned}$$

The parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ are called, respectively, the arrival (or birth) and departure (or death) rates.

Examples of Birth and Death Processes

- Poisson Processes: $\mu_n = 0$, $\lambda_n = \lambda$ for all $n \geq 0$
- Pure Birth Process:

$$\mu_n = 0 \quad \Rightarrow \quad \nu_i = \lambda_i, \quad P_{i,i+1} = 1, \quad P_{i,i-1} = 0$$

- Yule Processes (Pure Birth Process with Linear Growth rate): If there are n people and each independently gives birth at an exponential rate λ , then the total rate at which births occur is $n\lambda$.

$$\mu_n = 0, \quad \lambda_n = n\lambda$$

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$$\mu_n = 0, \quad \lambda_n = n\lambda$$

Reason: Let

B_i = time until the i th individual give birth $\sim \text{Exp}(\lambda)$, $i = 1, \dots, n$

So the time until the next (first) birth when there are n individuals in the population is

$$\min(B_1, B_2, \dots, B_n) \sim \text{Exp}(\lambda + \lambda + \dots + \lambda) = \text{Exp}(n\lambda)$$

So the rate until the next birth is $\lambda_n = n\lambda$.

Example: Linear Growth Model with Immigration

- each individual independently gives birth at an exponential rate λ
- each individual independently die at at an exponential rate μ
- new immigrants come in at an exponential rate θ

Such a process is a birth-death process with birth and death rates

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Reason: Let

B_i = time until the i th individual give birth $\sim \text{Exp}(\lambda)$, $i = 1, \dots, n$

T = time until the next new immigrant comes in $\sim \text{Exp}(\theta)$

So the time until the population size increase from n to $n + 1$ is

$$\min(B_1, \dots, B_n, T) \sim \text{Exp}(\lambda + \dots + \lambda + \theta) = \text{Exp}(n\lambda + \theta)$$

So the rate until the next birth is $\lambda_n = n\lambda + \theta$.

Similarly, one can show that the death rate is $\mu_n = n\mu$.

Example: $M/M/s$ Queueing Model

- s servers
- Poisson arrival of customers, rate $= \lambda$
- Exponential service time, rate $= \mu$

\Rightarrow a birth and death process with constant birth rate $\lambda_n = \lambda$, and death (departure) rate $\mu_n = \min(n, s)\mu$.

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Reason: Suppose, there are n customer in the system at time t . At most $\min(n, s)$ of them are being served. Let S_i be remaining service time of the i th server $\sim \text{Exp}(\mu)$. Then, the waiting time until the next departure is

$$\min(S_1, \dots, S_{\min(s, n)}) \sim \text{Exp}(\min(s, n)\mu).$$

6.4 The Transition Probability Function $P_{ij}(t)$

Recall the transition probability function $P_{ij}(t)$ of a CTMC $\{X(t), t \geq 0\}$ is

$$P_{ij}(t) = P(X(t+s) = j | X(s) = i)$$

Example. (Poisson Processes with rate λ)

$$\begin{aligned} P_{ij}(t) &= P(N(t+s) = j | N(s) = i) \\ &= P(N(t+s) - N(s) = j - i) = \begin{cases} e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases} \end{aligned}$$

Properties of Transition Probability Functions

- $P_{ij}(t) \geq 0$ for all $i, j \in \mathcal{X}$ and $t \geq 0$
- (Row sums are 1) $\sum_j P_{ij}(t) = 1$ for all $i \in \mathcal{X}$ and $t \geq 0$

Lemma 6.3 Chapman-Kolmogorov Equation

For all $i, j \in \mathcal{X}$ and $t \geq 0$,

$$P_{ij}(t+s) = \sum_{k \in \mathcal{X}} P_{ik}(t)P_{kj}(s)$$

Proof.

$$\begin{aligned} &P_{ij}(t+s) \\ &= P(X(t+s) = j | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} P(X(t+s) = j, X(t) = k | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} P(X(t+s) = j | X(t) = k, X(0) = i) P(X(t) = k | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} P(X(t+s) = j | X(t) = k) P(X(t) = k | X(0) = i) \text{ (Markov Property)} \\ &= \sum_{k \in \mathcal{X}} P_{kj}(s) P_{ik}(t) \end{aligned}$$

The matrix notation

Let $\mathbf{P}(t) = [P_{ij}(t)]$ be the transition matrix at time t .
We have $\mathbf{P}(0) = \mathbf{I}$. And C-K equations read

$$\mathbf{P}(t + s) = \mathbf{P}(t)\mathbf{P}(s)$$

One way to specify a CTMC is through $\{\mathbf{P}(t)\}_{t \geq 0}$. But this requires an infinite number of matrices. Can we simplify it?

Key: use derivatives $\mathbf{P}'(t)$

Transition rate matrix / infinitesimal generator \mathbf{Q}

Assume that

$$\mathbf{P}'(0) = \lim_{h \rightarrow 0} \frac{\mathbf{P}(h) - \mathbf{P}(0)}{h} \quad \text{exists.}$$

In other words, for each i, j ,

$$P'_{ij}(0) = \lim_{h \rightarrow 0} \frac{P_{ij}(h) - P_{ij}(0)}{h} \quad \text{exists.}$$

We will denote such limit as $\mathbf{Q} = [q_{ij}]$, the transition rate matrix.
What does this tell us about $\mathbf{P}'(t)$ for $t > 0$?

Kolmogorov's equations

By definition, one has

$$\begin{aligned}\mathbf{P}'(t) &= \lim_{h \rightarrow 0} \frac{\mathbf{P}(t+h) - \mathbf{P}(t)}{h} = \lim_{h \rightarrow 0} \frac{\mathbf{P}(t)\mathbf{P}(h) - \mathbf{P}(t)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\mathbf{P}(t)(\mathbf{P}(h) - \mathbf{I})}{h} = \mathbf{P}(t)\mathbf{Q}.\end{aligned}$$

This is the so-called Kolmogorov's forward equations.

Similarly you can prove backward equations

$$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t).$$

These imply $\mathbf{P}(t) = \exp(t\mathbf{Q})$.

Transition rate matrix

How to compute Q ?

Lemma 6.2a

For any $i, j \in \mathcal{X}$, we have

$$q_{ii} = \lim_{h \rightarrow 0} \frac{P_{ii}(h) - 1}{h} = -\nu_i$$

Proof. Let T_i be the amount of time the process stays in state i before moving to other states.

$$\begin{aligned} P_{ii}(h) &= \mathbb{P}(X(h) = i | X(0) = i) \\ &= \mathbb{P}(X(h) = i, \text{no transition in } (0, h] | X(0) = i) \\ &\quad + \mathbb{P}(X(h) = i, 2 \text{ or more transition in } (0, h] | X(0) = i) \\ &= \mathbb{P}(T_i > h) + o(h) \\ &= e^{-\nu_i h} + o(h) \\ &= 1 - \nu_i h + o(h) \end{aligned}$$

Lemma 6.2b

For any $i \neq j \in \mathcal{X}$, we have

$$q_{ij} = \lim_{h \rightarrow 0} \frac{P_{ij}(h)}{h} = \nu_i P_{ij}$$

Proof.

$$\begin{aligned} P_{ij}(h) &= P(X(h) = j | X(0) = i) \\ &= P(X(h) = j, 1 \text{ transition in } (0, h] | X(0) = i) \\ &\quad + P(X(h) = j, 2 \text{ or more transition in } (0, h] | X(0) = i) \\ &= P(T_i < h) P_{ij} + o(h) \\ &= (1 - e^{-\nu_i h}) P_{ij} + o(h) \\ &= \nu_i P_{ij} h + o(h) \end{aligned}$$

For finite state space case $\mathcal{X} = \{1, 2, \dots, m\}$, define the matrices

$$\mathbf{P}(t) = \begin{bmatrix} P_{11}(t) & \cdots & P_{1m}(t) \\ \vdots & & \vdots \\ P_{m1}(t) & \cdots & P_{mm}(t) \end{bmatrix}, \quad \mathbf{P}'(t) = \begin{bmatrix} P'_{11}(t) & \cdots & P'_{1m}(t) \\ \vdots & & \vdots \\ P'_{m1}(t) & \cdots & P'_{mm}(t) \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & & \vdots \\ q_{m1} & \cdots & q_{mm} \end{bmatrix} = \begin{bmatrix} -\nu_1 & \nu_1 P_{12} & \cdots & \nu_1 P_{1m} \\ \nu_2 P_{21} & -\nu_2 & \cdots & \nu_2 P_{2m} \\ \vdots & \vdots & & \vdots \\ \nu_m P_{m1} & \nu_m P_{m2} & \cdots & -\nu_m \end{bmatrix}$$

In matrix notation,

Forward equation: $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

Backward equation: $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$

Limiting Probabilities

6.5. Limiting Probabilities

Definition. Just like discrete-time Markov chains, if the probability that a continuous-time Markov chain will be in state j at time t , $P_{ij}(t)$, converges to a limiting value P_j independent of the initial state i , for all $i \in \mathcal{X}$

$$P_j = \lim_{t \rightarrow \infty} P_{ij}(t) > 0$$

then we say P_j is the *limiting probability* of state j . If P_j exists for all $j \in \mathcal{X}$, we say $\{P_j\}_{j \in \mathcal{X}}$ is the *limiting distribution* of the process.

Remark. If $\lim_{t \rightarrow \infty} P_{ij}(t)$ exists, we must have

$$\lim_{t \rightarrow \infty} P'_{ij}(t) = 0.$$

Balanced equations

Recall the forward equations $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

If you set $t \rightarrow \infty$, you have

$$0 = p^\top \mathbf{Q},$$

where $p = (P_1, P_2, \dots)^\top$

This is the same as saying that

$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} \quad \text{for all } j \in \mathcal{X}$$

Interpretation of the Balanced Equations

$$\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$$

$\nu_j P_j$ = rate at which the process **leaves** state j

$\sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$ = rate at which the process **enters** state j

Balanced equations means that the rates at which the process enters and leaves state j are equal.

The limiting distribution $\{P_j\}_{j \in \mathcal{X}}$ can be obtained by solving the balanced equations along with the equation $\sum_{j \in \mathcal{X}} P_j = 1$.

Remarks. Just like discrete-time Markov chains, a sufficient condition for the existence of a limiting distribution is that the chain is irreducible and positive recurrent.

Examples

- **Poisson processes:** $\mu_n = 0$, $\lambda_n = \lambda$ for all $n \geq 0$

$$\nu_i = \lambda, \quad P_{i,i+1} = 1, \quad q_{i,i+1} = \nu_i P_{i,i+1} = \lambda$$

Balanced equations:

$$\nu_j P_j = P_{j-1} q_{j-1,j} \quad \Rightarrow \quad \lambda P_j = \lambda P_{j-1} \quad \Rightarrow \quad P_j = P_{j-1}$$

No limiting distribution exists. The chain is not irreducible. All states are transient.

Examples

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$$\nu_i = \lambda, \quad P_{i,i+1} = 1, \quad q_{i,i+1} = \nu_i P_{i,i+1} = \lambda$$

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$$\nu_j P_j = P_{j-1} q_{j-1,j} \quad \Rightarrow \quad \lambda P_j = \lambda P_{j-1} \quad \Rightarrow \quad P_j = P_{j-1}$$

No limiting distribution exists. The chain is not irreducible. All states are transient.

- **Pure birth processes with $\lambda_n > 0$ for all n**

No limiting distribution exists. All states are transient.

Birth and Death Processes

For a birth and death process,

$$\begin{aligned}\nu_0 &= \lambda_0, \\ \nu_i &= \lambda_i + \mu_i, \quad i > 0 \\ P_{01} &= 1, \\ P_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i}, \quad i > 0 \\ P_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i}, \quad i > 0 \\ P_{i,j} &= 0 \quad \text{if } |i - j| > 1\end{aligned} \quad \Rightarrow \quad \begin{aligned}q_{i,i+1} &= \nu_i P_{i,i+1} = \lambda_i, \quad i \geq 0 \\ q_{i,i-1} &= \nu_i P_{i,i-1} = \mu_i, \quad i \geq 1\end{aligned}$$

Balanced Equations for Birth and Death Processes

The balanced equations $\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}$ for a birth and death process are

$$\begin{aligned}\lambda_0 P_0 &= \mu_1 P_1 \\ (\mu_1 + \lambda_1) P_1 &= \lambda_0 P_0 + \mu_2 P_2, \\ (\mu_2 + \lambda_2) P_2 &= \lambda_1 P_1 + \mu_3 P_3, \\ &\vdots \\ (\mu_{n-1} + \lambda_{n-1}) P_{n-1} &= \lambda_{n-2} P_{n-2} + \mu_n P_n \\ (\mu_n + \lambda_n) P_n &= \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}\end{aligned}$$

Adding up all the equations above and eliminating the common terms on both sides, we get

$$\lambda_n P_n = \mu_{n+1} P_{n+1} \quad n \geq 0,$$

We hence just need to solve $\lambda_n P_n = \mu_{n+1} P_{n+1}$ for the limiting distribution.

6.6. Time Reversibility

Definition. A continuous-time Markov chain with state space \mathcal{X} is *time reversible* if

$$P_i q_{ij} = P_j q_{ji}, \quad \text{for all } i, j \in \mathcal{X} \quad (\text{detailed balanced equation})$$

If a distribution $\{P_j\}$ on \mathcal{X} satisfies the detailed balanced equation, then it is a stationary distribution for the process.

Example. We have just shown that for Birth and Death processes, the balanced equations would lead to the detailed balanced equations, which are

$$\lambda_n P_n = \mu_{n+1} P_{n+1}, \quad n \geq 0$$

Limiting Dist'n for Birth and Death Processes

Solving $\lambda_n P_n = \mu_{n+1} P_{n+1}$, $n \geq 0$ for the limiting distribution, we get

$$P_n = \frac{\lambda_{n-1}}{\mu_n} P_{n-1} = \frac{\lambda_{n-1}}{\mu_n} \frac{\lambda_{n-2}}{\mu_{n-1}} P_{n-2} = \dots = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} P_0$$

To meet the requirement $\sum_{n=0}^{\infty} P_n = 1$, we need

$$\sum_{n=0}^{\infty} P_n = P_0 + P_0 \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_n \mu_{n-1} \cdots \mu_1} = 1$$

In other words, to have a limiting distribution, it is necessary that

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$$

Limiting Dist'n for Birth and Death Processes (Cont'd)

If $\sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$ is finite, the limiting distribution is

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}}$$

and

$$P_k = \frac{\lambda_0 \lambda_1 \cdots \lambda_{k-1} / (\mu_1 \mu_2 \cdots \mu_k)}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}}, \quad k \geq 1$$

Lemma: (Ratio Test) If $a_n \geq 0$ for all n , then

$$\sum_{n=1}^{\infty} a_n \begin{cases} < \infty & \text{if } \lim_{n \rightarrow \infty} a_n/a_{n-1} < 1 \\ = \infty & \text{if } \lim_{n \rightarrow \infty} a_n/a_{n-1} > 1 \end{cases}$$

For $a_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}$, $a_n/a_{n-1} = \lambda_{n-1}/\mu_n$. By the ratio test, if

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\mu_n} < 1,$$

then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty$, the limiting distribution exists.

Example 6.4 Linear Growth Model with Immigration

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Using the Ratio Test,

$$\lim_{n \rightarrow \infty} \frac{\lambda_{n-1}}{\mu_n} = \lim_{n \rightarrow \infty} \frac{(n-1)\lambda + \theta}{n\mu} = \frac{\lambda}{\mu}$$

So the linear growth model has a limiting distribution if $\lambda < \mu$.

Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

- single-server service station. Service times are i.i.d. $\sim \text{Exp}(\mu)$
- Poisson arrival of customers with rate λ

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Q: What fraction of potential customers are lost?

Let $X(t)$ be the number of customers in the station at time t .

$\{X(t), t \geq 0\}$ is a birth-death process with the birth and death rates below

$$\mu_n = \begin{cases} 0 & \text{if } n = 0 \\ \mu & \text{if } n \geq 1 \end{cases} \quad \text{and} \quad \lambda_n = \begin{cases} \lambda & \text{if } 0 \leq n < N \\ 0 & \text{if } n \geq N \end{cases}$$

Example 5.5 (M/M/1 Queueing w/ Finite Capacity)

Solving $\lambda_n P_n = \mu_{n+1} P_{n+1}$ for the limiting distribution

$$P_1 = (\lambda/\mu)P_0$$

$$P_2 = (\lambda/\mu)P_1 = (\lambda/\mu)^2 P_0$$

$$\vdots$$

$$P_i = (\lambda/\mu)^i P_0, \quad i = 1, 2, \dots, N$$

Plugging $P_i = (\lambda/\mu)^i P_0$ into $\sum_{i=0}^N P_i = 1$, one can solve for P_0 and get

$$P_i = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^i$$

Answer: The fraction of customers lost is $P_N = \frac{1 - \lambda/\mu}{1 - (\lambda/\mu)^{N+1}} (\lambda/\mu)^N$

Duration Times for Birth and Death Processes

Let

T_i = time to move from state i to state $i + 1$, $i = 0, 1, \dots$

Suppose at some moment $X(t) = i$. Let

B_i = time until the next birth $\sim \text{Exp}(\lambda_i)$

D_i = time until the next death $\sim \text{Exp}(\mu_i)$

Then

$$\begin{aligned} T_i &= \begin{cases} B_i & \text{if the next step is } i \rightarrow i + 1, \text{ i.e., } B_i < D_i \\ D_i + T_{i-1} + T_i^* & \text{if the next step is } i \rightarrow i - 1, \text{ i.e., } D_i < B_i \end{cases} \\ &= \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i, \text{ occur w/ prob. } \frac{\lambda_i}{\lambda_i + \mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i, \text{ occur w/ prob. } \frac{\mu_i}{\lambda_i + \mu_i} \end{cases} \end{aligned}$$

Note

- T_i^* has the same distribution as T_i
- T_{i-1} and T_i^* are indep. of B_i and D_i because it's Markov

Duration Times for Birth and Death Processes

Taking expected value of

$$T_i = \min(B_i, D_i) + \begin{cases} 0 & \text{if } B_i < D_i, \text{ occur w/ prob. } \frac{\lambda_i}{\lambda_i + \mu_i} \\ T_{i-1} + T_i^* & \text{if } D_i < B_i, \text{ occur w/ prob. } \frac{\mu_i}{\lambda_i + \mu_i} \end{cases}$$

we get

$$\begin{aligned} \mathbb{E}[T_i] &= \mathbb{E}[\min(B_i, D_i)] + (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i]) \frac{\mu_i}{\lambda_i + \mu_i} \\ &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} (\mathbb{E}[T_{i-1}] + \mathbb{E}[T_i]) \end{aligned}$$

We obtain the recursive formula

$$\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$$

Duration Times for Birth and Death Processes (Cont'd)

Since $T_0 \sim \text{Exp}(\lambda_0)$, $\mathbb{E}[T_0] = 1/\lambda_0$.

Using the recursive formula $\lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}]$, we have

$$\mathbb{E}[T_0] = \frac{1}{\lambda_0}$$

$$\mathbb{E}[T_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} \mathbb{E}[T_0] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0}$$

$$\mathbb{E}[T_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} \left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0} \right) = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2 \lambda_1} + \frac{\mu_2 \mu_1}{\lambda_2 \lambda_1 \lambda_0}$$

\vdots

$$\begin{aligned} \mathbb{E}[T_i] &= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} \mathbb{E}[T_{i-1}] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i \lambda_{i-1}} + \cdots + \frac{\mu_i \mu_{i-1} \cdots \mu_2 \mu_1}{\lambda_i \lambda_{i-1} \cdots \lambda_2 \lambda_1 \lambda_0} \\ &= \frac{1}{\lambda_i} \left(1 + \sum_{k=1}^i \prod_{j=1}^k \frac{\mu_{i-j+1}}{\lambda_{i-j}} \right) \end{aligned}$$