STAT253/317 Lecture 8 Generating Functions

For a non-negative-integer-valued random variable T, the generating function of T is the expected value of s^T as a function of s

$$G(s) = \mathrm{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k),$$

in which s^T is defined as 0 if $T = \infty$.
Since $0 \leq \mathrm{P}(T = k) \leq 1$, the generating function is always defined for $-1 \leq s \leq 1$

Examples of Generating Functions

- If T has a geometric distribution: $P(T=k)=p(1-p)^k$, $k=0,1,2,\ldots$, the generating function of T is

$$G(s)=\sum_{k=0}^{\infty} s^k \mathrm{P}(T=k)=\sum_{k=0}^{\infty} s^k p(1-p)^k=\frac{p}{1-(1-p)s}$$

- If T has a Binomial distribution $P(T=k)=\binom{n}{k}p^k(1-p)^{n-k}$, $k=0,1,2,\ldots,n$, the generating function of T is

$$G(s)=\sum_{k=0}^{\infty} s^k \mathrm{P}(T=k)=\sum_{k=0}^{\infty} s^k \binom{n}{k} p^k(1-p)^{n-k}$$
$$=(ps+(1-p))^n$$

Properties of Generating Function

$$G(s) = \mathrm{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k)$$

- $G(s)$ is a power series converging absolutely for all $-1 \le s \le 1$.
  since $0 \le \mathrm{P}(T = k) \le 1$ and $\sum_k \mathrm{P}(T = k) \le 1$.
- $G(1) = \mathrm{P}(T < \infty) \begin{cases} = 1 & \text{if } T \text{ is finite w/ prob. } 1 \\ < 1 & \text{otherwise} \end{cases}$
- $P(T = k) = \dfrac{G^{(k)}(0)}{k!}$

  Knowing $G(s) \Leftrightarrow$ Knowing $\mathrm{P}(T = k)$ for all $k = 0, 1, 2, \ldots$

More Properties of Generating Functions

$$G(s) = \mathsf{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k)$$

- $\mathsf{E}[T] = \lim_{s \to 1^-} G'(s)$ if it exists because

$$G'(s) = \frac{d}{ds}\mathsf{E}[s^T] = \mathsf{E}[Ts^{T-1}] = \sum_{k=1}^{\infty} s^{k-1} k \mathrm{P}(T = k).$$

- $\mathsf{E}[T(T-1)] = \lim_{s \to 1^-} G''(s)$ if it exists because

$$G''(s) = \mathsf{E}[T(T-1)s^{T-2}] = \sum\nolimits_{k=2}^{\infty} s^{k-2} k(k-1) \mathrm{P}(T = k)$$

- If T and U are **independent** non-negative-integer-valued random variables, with generating function $G_T(s)$ and $G_U(s)$ respectively, then the generating function of $T + U$ is

$$G_{T+U}(s) = \mathsf{E}[s^{T+U}] = \mathsf{E}[s^T]\mathsf{E}[s^U] = G_T(s)G_U(s)$$

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space $= \{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p,\ P_{i,i-1} = 1 - p = q,$ for $i = 1, 2, 3 \ldots$
- Only one class, irreducible
- For $i < j$, define

$$\begin{aligned} N_{ij} &= \min\{m > 0 : X_m = j | X_0 = i\} \\ &= \text{time to reach state } j \text{ starting in state } i \end{aligned}$$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$
  By the Markov property, N_{01}, $N_{12}, \ldots, N_{n-1,n}$ are indep.
- Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i + 1 \\ 1 + N^*_{i-1,i} + N^*_{i,i+1} & \text{if } X_1 = i - 1 \end{cases} \tag{1}$$

where $N^*_{i-1,i} \sim N_{i-1,i}$, $N^*_{i,i+1} \sim N_{i,i+1}$, and $N^*_{i-1,i}$, $N^*_{i,i+1}$ are indep.

Generating Function of $N_{i,i+1}$

Let $G_i(s)$ be the generating function of $N_{i,i+1}$. From (1), and by the independence of $N^*_{i-1,i}$ and $N^*_{i,i+1}$, we get that

$$G_i(s) = ps + q\mathrm{E}[s^{1+N^*_{i-1,i}+N^*_{i,i+1}}] = ps + qsG_{i-1}(s)G_i(s)$$

So

$$G_i(s) = \frac{ps}{1 - qsG_{i-1}(s)} \tag{2}$$

Since N_{01} is always 1, we have $G_0(s) = s$. Using the iterative relation (2), we can find

$$G_1(s) = \frac{ps}{1 - qsG_0(s)} = \frac{ps}{1 - qs^2} = ps\sum_{k=0}^{\infty}(qs^2)^k = \sum_{k=0}^{\infty} pq^k s^{2k+1}$$

So $\mathrm{P}(N_{12} = n) = \begin{cases} pq^k & \text{if } n = 2k+1 \text{ for } k = 0, 1, 2 \ldots \\ 0 & \text{if } n \text{ is even} \end{cases}$

Similarly,

$$
\begin{aligned}
G_2(s) &= \frac{ps}{1 - qsG_1(s)} = \frac{ps(1-qs^2)}{1-q(1+p)s^2} \\
&= \frac{ps}{1-q(1+p)s^2} - \frac{pqs^3}{1-q(1+p)s^2} \\
&= ps \sum_{k=0}^{\infty} (q(1+p)s^2)^k - pqs^3 \sum_{k=0}^{\infty} (q(1+p)s^2)^k \\
&= \sum_{k=0}^{\infty} pq^k(1+p)^k s^{2k+1} - \sum_{k=0}^{\infty} pq^{k+1}(1+p)^k s^{2k+3} \\
&= ps + \sum_{k=1}^{\infty} pq^k[(1+p)^k - (1+p)^{k-1}]s^{2k+1} \\
&= ps + \sum_{k=1}^{\infty} p^2q^k(1+p)^{k-1}s^{2k+1}
\end{aligned}
$$

So

$$
\mathrm{P}(N_{23} = n) = \begin{cases} p & \text{if } n = 1 \\ p^2q^k(1+p)^{k-1} & \text{if } n = 2k+1 \text{ for } k = 1, 2, \ldots \\ 0 & \text{if } n \text{ is even} \end{cases}
$$

Mean of $N_{i,i+1}$

Recall that $G_i'(1) = E(N_{i,i+1})$. Let $m_i = E(N_{i,i+1}) = G_i'(1)$.

$$G_i'(s) = \frac{p(1 - qsG_{i-1}(s)) + ps(qG_{i-1}(s) + qsG_{i-1}'(s))}{(1 - qsG_{i-1}(s))^2}$$
$$= \frac{p + pqs^2 G_{i-1}'(s)}{(1 - qsG_{i-1}(s))^2}$$

Since $N_{i,i+1} < \infty$, $G_i(1) = 1$ for all $i = 0, 1, \ldots, n-1$. We have

$$m_i = G_i'(1) = \frac{p + pqG_{i-1}'(1)}{(1-q)^2} = \frac{1 + qG_{i-1}'(1)}{p} = \frac{1}{p} + \frac{q}{p} m_{i-1}$$

We get the same iterative equation as in Lecture 7.

4.7 Branching Processes Revisit

Recall a Branching Process is a population of individuals in which

- all individuals have the same lifespan, and
- each individual will produce a random number of offsprings at the end of its life

Let $X_n =$ size of the nth generation, $n = 0, 1, 2, \ldots$. Let $Z_{n,i} = \#$ of offsprings produced by the ith individuals in the nth generation. Then

$$X_{n+1} = \sum_{i=1}^{X_n} Z_{n,i} \tag{3}$$

Suppose $Z_{n,i}$'s are i.i.d with probability mass function

$$P(Z_{n,i} = j) = P_j, \; j \geq 0.$$

We suppose the non-trivial case that $P_j < 1$ for all $j \geq 0$.
$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \ldots\}$.

Generating Functions of the Branching Processes

Let $g(s) = \mathrm{E}[s^{Z_{n,i}}] = \sum_{k=0}^{\infty} P_k s^k$ be the generating function of $Z_{n,i}$, and $G_n(s)$ be the generating function of X_n, $n = 0, 1, 2, \ldots$. Then $\{G_n(s)\}$ satisfies the following two iterative equations.

$$\text{(i) } G_{n+1}(s) = G_n(g(s)) \quad \text{for } n = 0, 1, 2, \ldots$$

$$\text{(ii) } G_{n+1}(s) = g(G_n(s)) \quad \text{if } X_0 = 1, \text{ for } n = 0, 1, 2, \ldots$$

Proof of (i).

$$\begin{aligned}
\mathrm{E}[s^{X_{n+1}}|X_n] = \mathrm{E}\left[s^{\sum_{i=1}^{X_n} Z_{n,i}}\right] &= \mathrm{E}\Big[\prod_{i=1}^{X_n} s^{Z_{n,i}}\Big] \\
&= \prod_{i=1}^{X_n} \mathrm{E}[s^{Z_{n,i}}] \qquad \text{by indep. of } Z_{n,i}\text{'s} \\
&= \prod_{i=1}^{X_n} g(s) \qquad \text{as } g(s) = \mathrm{E}[s^{Z_{n,i}}] \\
&= g(s)^{X_n}
\end{aligned}$$

From which, we have

$$G_{n+1}(s) = \mathrm{E}[s^{X_{n+1}}] = \mathrm{E}[\mathrm{E}[s^{X_{n+1}}|X_n]] = \mathrm{E}[g(s)^{X_n}] = G_n(g(s))$$

since $G_n(s) = \mathrm{E}[s^{X_n}]$.

Proof of (ii) $G_{n+1}(s) = g(G_n(s))$ if $X_0 = 1$

Suppose there are k individuals in the first generation ($X_1 = k$). Let Y_i be the number offspring of the ith individual in the first generation in the $(n+1)$st generation. Obviously,

$$X_{n+1} = Y_1 + \ldots + Y_k.$$

Observe $Y_1, \ldots, Y_k$'s are indep and each has the same distn. as X_n since they are all the size of the nth generation of a single ancestor. Thus, by ndep. of Y_i's

$$\mathrm{E}[s^{X_{n+1}}|X_1 = k] = \mathrm{E}\left[s^{Y_1+\ldots+Y_k}\right] = \mathrm{E}\left[\prod_{i=1}^{k} s^{Y_i}\right] = \prod_{i=1}^{k} \mathrm{E}[s^{Y_i}]$$

Since Y_i's have the same dist'n as X_n and $G_n(s) = \mathrm{E}[s^{X_n}]$, we have

$$\mathrm{E}[s^{X_{n+1}}|X_1 = k] = \prod_{i=1}^{k} G_n(s) = (G_n(s))^k$$

Since $X_0 = 1$, $X_1 = Z_{1,1}$, and hence $\mathrm{P}(X_1 = k) = P_k$.

$$G_{n+1}(s) = \mathrm{E}[s^{X_{n+1}}] = \sum_{k=0}^{\infty} \mathrm{E}[s^{X_{n+1}}|X_1 = k]P_k = \sum_{k=0}^{\infty} (G_n(s))^k P_k = g(G_n(s)),$$

where the last equality comes from that $g(s) = \sum_{k=0}^{\infty} P_k s^k$.

Example

Suppose $X_0 = 1$, and $(P_0, P_1, P_2) = (1/4, 1/2, 1/4)$. Find the distribution of X_2.

Sol.

$$g(s) = \frac{1}{4}s^0 + \frac{1}{2}s^1 + \frac{1}{4}s^2 = (1+s)^2/4.$$

Since $X_0 = 1$, $G_0(s) = \mathrm{E}[s^{X_0}] = \mathrm{E}[s^1] = s$. From (i) we have

$$\begin{aligned} G_1(s) &= G_0(g(s)) = g(s) = (1+s)^2/4 \\ G_2(s) &= G_1(g(s)) = \frac{1}{4}(1 + \frac{1}{4}(1+s)^2)^2 = \frac{1}{64}(5 + 2s + s^2)^2 \\ &= \frac{1}{64}(25 + 20s + 14s^2 + 4s^3 + s^4) = \sum_{k=0}^{\infty} \mathrm{P}(X_2 = k)s^k \end{aligned}$$

The coefficient of s^k in the polynomial of $G_2(s)$ is the chance that $X_2 = k$.

k	0	1	2	3	4
$\mathrm{P}(X_2 = k)$	$\frac{25}{64}$	$\frac{20}{64}$	$\frac{14}{64}$	$\frac{4}{64}$	$\frac{1}{64}$

and $\mathrm{P}(X_2 = k) = 0$ for $k \geq 5$.

Extinction Probability of a Branching Process

$$
\begin{aligned}
\text{Let } \pi_0 &= \lim_{n\to\infty} \mathrm{P}(X_n = 0 | X_0 = 1) \\
&= \mathrm{P}(\text{the population will eventually die out} | X_0 = 1)
\end{aligned}
$$

As $G_n(s) = \mathrm{E}[s^{X_n}] = \sum_{k=0}^{\infty} P(X_n = k)s^k$, plugging in $s = 0$, we get

$$G_n(0) = P(X_n = 0) = P(\text{extinct by the } n\text{th generation}).$$

Recall that if $X_0 = 1$, $G_1(s) = g(s)$, and $G_{n+1}(s) = g(G_n(s))$. We can compute $G_n(0)$ iteratively as follows

$$
\begin{aligned}
G_1(0) &= g(0) \\
G_{n+1}(0) &= g(G_n(0)), \quad n = 1, 2, 3, \ldots
\end{aligned}
$$

Finally, we can get the extinction probability by taking the limit

$$\pi_0 = \lim_{n\to\infty} G_n(0).$$

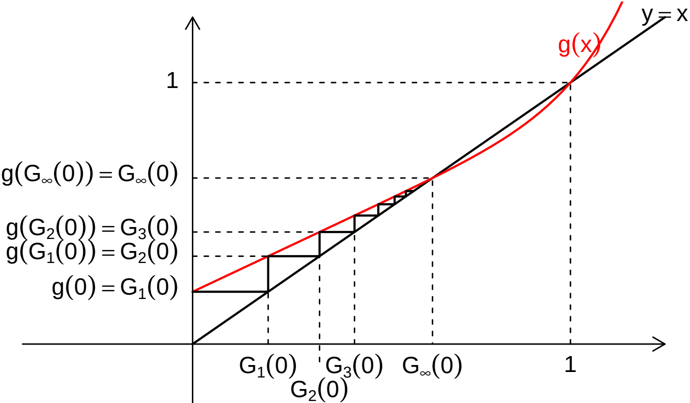
Extinction Probability of a Branching Process

If $X_0 = 1$, the extinction probability π_0 is a **smallest root** of the equation

$$g(s) = s \tag{4}$$

in the range $0 \leq s \leq 1$, where $g(s) = \sum_{k=0}^{\infty} P_k s^k$ is the generating function of $Z_{n,i}$.

Proof.

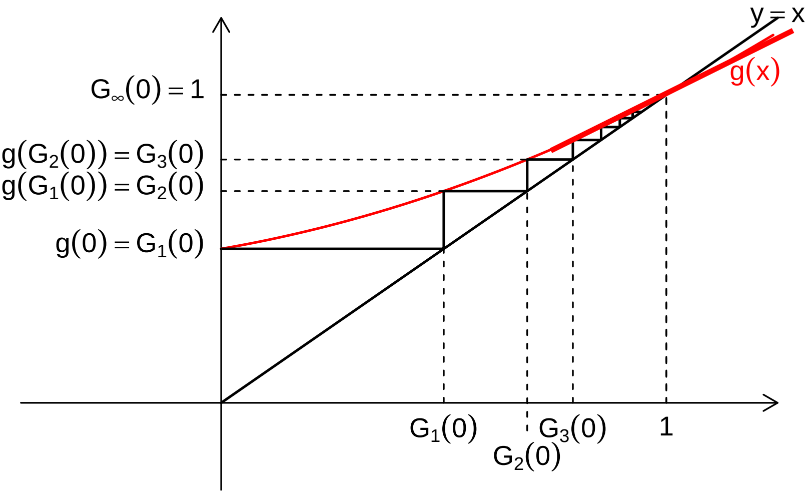


A Branching Process Will Become Extinct If $\mu \leq 1$

Let $\mu = \mathrm{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. If $\mu \leq 1$, the extinction probability π_0 is 1 unless $P_1 = 1$.

mu = g'(1) = slope of the curve g(x) at x = 1

Proof.

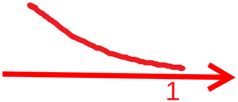


Formal Proof

Let $h(s) = g(s) - s$. Since $g(1) = 1$, $g'(1) = \mu$,

$$h(1) = g(1) - 1 = 0,$$

$$h'(s) = \left(\sum_{j=1}^{\infty} jP_j s^{j-1}\right) - 1 \leq \left(\sum_{j=1}^{\infty} jP_j\right) - 1 = \mu - 1 \quad \text{for } 0 \leq s < 1$$

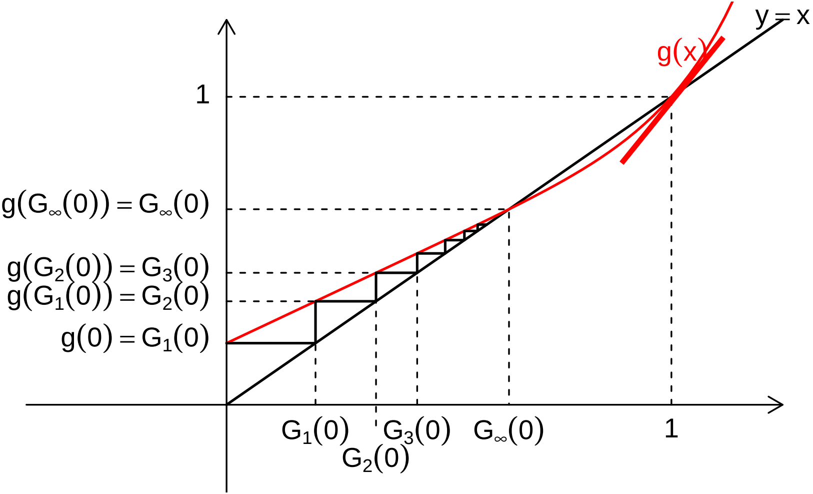


$$\begin{aligned}
\text{Thus } \mu \leq 1 &\Rightarrow h'(s) \leq 0 \text{ for } 0 \leq s < 1 \\
&\Rightarrow h(s) \text{ is non-increasing in } [0,1) \\
&\Rightarrow h(s) > h(1) = 0 \text{ for } 0 \leq s < 1 \\
&\Rightarrow g(s) > s \qquad \text{for } 0 \leq s < 1 \\
&\Rightarrow \text{ There is no root in } [0,1).
\end{aligned}$$

Extinction Probability When $\mu > 1$

If $\mu > 1$, there is a unique root of the equation $g(s) = s$ in the domain $[0, 1)$, and that is the extinction probability.

Proof.



Formal Proof

Let $h(s) = g(s) - s$. Observe that

$$\begin{aligned} h(0) &= g(0) = P_0 > 0 \\ h'(0) &= g'(0) - 1 = P_1 - 1 < 0 \\ \text{Then } \mu > 1 &\Rightarrow h'(1) = \mu - 1 > 0 \\ &\Rightarrow h(s) \text{ is increasing near } 1 \\ &\Rightarrow h(1-\delta) < h(1) = 0 \text{ for } \delta > 0 \text{ small enough} \end{aligned}$$

Since $h(s)$ is continuous in $[0,1)$, there must be a root to $h(s) = s$. The root is unique since

$$h''(s) = g''(s) = \sum_{j=2}^{\infty} j(j-1)P_j s^{j-2} \geq 0 \quad \text{for } 0 \leq s < 1$$

$h(s)$ is convex in [0,1).
g(s) is also convex.