

## **Spectral methods: $\ell_2$ perturbation theory**



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University of Chicago, Autumn 2021

# Matrix perturbation theory (spectral analysis)

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Let  $M^\star$  be a “simple” matrix, and  $E$  be a perturbation matrix  
— “simple” means spectral structure of  $M^\star$  is understood

## Goal of matrix perturbation theory:

Understand how eigenspaces (resp. eigenvalues) / singular subspaces (resp. singular values) of  $M^\star + E$  change w.r.t. perturbation  $E$

# Outline

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- Preliminaries: basic matrix analysis
- Distance between two subspaces
- Eigenspace perturbation theory
- Perturbation bounds for singular subspaces
- Eigenvector perturbation bounds for probability transition matrices

## **Basic matrix analysis**

# Unitarily invariant norms

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## Definition 3.1

A matrix norm  $\|\cdot\|$  on  $\mathbb{R}^{m \times n}$  is said to be unitarily invariant if

$$\|\mathbf{A}\| = \|\mathbf{U}^\top \mathbf{A} \mathbf{V}\|$$

holds for any matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  and any two square orthonormal matrices  $\mathbf{U} \in \mathcal{O}^{m \times m}$  and  $\mathbf{V} \in \mathcal{O}^{n \times n}$ .

Examples:

- $\|\mathbf{A}\|$ : spectral norm (largest singular value of  $\mathbf{A}$ )
- $\|\mathbf{A}\|_F$ : Frobenius norm ( $\|\mathbf{A}\|_F = \sqrt{\text{tr}(\mathbf{A}^\top \mathbf{A})} = \sqrt{\sum_{i,j} A_{i,j}^2}$ )

# Properties of unitarily invariant norms

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## Lemma 3.2

For any unitarily invariant norm  $\|\cdot\|$ , one has

$$\begin{aligned}\|AB\| &\leq \|A\| \cdot \|B\|, & \|AB\| &\leq \|B\| \cdot \|A\|; \\ \|AB\| &\geq \|A\| \sigma_{\min}(B), & \text{if } B \text{ is square;} \\ \|AB\| &\geq \|B\| \sigma_{\min}(A), & \text{if } A \text{ is square.}\end{aligned}$$

Exercise: prove this lemma for special cases  $\|\cdot\|$  and  $\|\cdot\|_F$

# Eigenvalue perturbation bounds

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## Lemma 3.3 (Weyl's inequality for eigenvalues)

*Let  $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{n \times n}$  be two real symmetric matrices. For every  $1 \leq i \leq n$ , the  $i$ -th largest eigenvalues of  $\mathbf{A}$  and  $\mathbf{A} + \mathbf{E}$  obey*

$$|\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{A} + \mathbf{E})| \leq \|\mathbf{E}\|.$$

# Eigenvalue perturbation bounds

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eigenvalues of real symmetric matrices are stable against perturbations



# Eigenvalue perturbation bounds

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$$|\lambda_i(\mathbf{A}) - \lambda_i(\mathbf{A} + \mathbf{E})| \leq \|\mathbf{E}\|.$$

— proof left as exercise

eigenvalues of real symmetric matrices are stable against perturbations

# Singular value perturbation bounds

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## Lemma 3.4 (Weyl's inequality for singular values)

*Let  $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{m \times n}$  be two general matrices. Then for every  $1 \leq i \leq \min\{m, n\}$ , the  $i$ -th largest singular values of  $\mathbf{A}$  and  $\mathbf{A} + \mathbf{E}$  obey*

$$|\sigma_i(\mathbf{A} + \mathbf{E}) - \sigma_i(\mathbf{A})| \leq \|\mathbf{E}\|.$$

singular values are stable against perturbations

## Proof of Lemma 3.4

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We begin with introducing a useful “dilation” trick:

### Definition 3.5 (Symmetric dilation)

For  $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2}$ , define its symmetric dilation  $\mathcal{S}(\mathbf{A})$  to be

$$\mathcal{S}(\mathbf{A}) = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^\top & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{(n_1+n_2) \times (n_1+n_2)}.$$

Then one has the following eigendecomposition for  $\mathcal{S}(\mathbf{A})$ :

$$\mathcal{S}(\mathbf{A}) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{U} & \mathbf{U} \\ \mathbf{V} & -\mathbf{V} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{\Sigma} & \mathbf{0} \\ \mathbf{0} & -\mathbf{\Sigma} \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{U} & \mathbf{U} \\ \mathbf{V} & -\mathbf{V} \end{bmatrix}^\top.$$

Two observations: for  $1 \leq i \leq \min\{m, n\}$ ,  $\lambda_i(\mathcal{S}(\mathbf{A})) = \sigma_i(\mathbf{A})$ , and  $\|\mathcal{S}(\mathbf{A})\| = \|\mathbf{A}\|$ . Apply Lemma 3.3 to finish the proof.

**Distance between two subspaces**

# Setup and notation

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- Two  $r$ -dimensional subspaces  $\mathcal{U}^*$  and  $\mathcal{U}$  in  $\mathbb{R}^n$
- Two orthonormal matrices  $\mathbf{U}^*$  and  $\mathbf{U}$  in  $\mathbb{R}^{n \times r}$
- Orthogonal complements:  $[\mathbf{U}^*, \mathbf{U}_\perp^*]$ , and  $[\mathbf{U}, \mathbf{U}_\perp]$

# Question: how to measure distance?

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- $\|U - U^*\|_F$  and  $\|U - U^*\|$  are not appropriate, since they fall short of accounting for global orthonormal transformation  
 $\forall$  orthonormal  $R \in \mathbb{R}^{r \times r}$ ,  $U$  and  $UR$  represent same subspace

# Three valid choices of distance

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- Distance modulo *optimal rotation*
- Distance using *projection matrices*
- Geometric construction via *principal/canonical angles*

# Distance modulo optimal rotation

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Given global rotation ambiguity, it is natural to adjust for rotation before computing distance:

$$\text{dist}_{\|\cdot\|}(\mathbf{U}, \mathbf{U}^*) := \min_{\mathbf{R} \in \mathcal{O}^{r \times r}} \|\mathbf{U}\mathbf{R} - \mathbf{U}^*\|$$



# Distance using projection matrices

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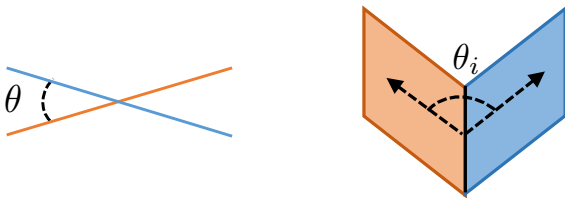
Key observation: projection matrix  $UU^\top$  associated with subspace  $\mathcal{U}$  is unique

$$\text{dist}_{p, \|\cdot\|} (U, U^\star) := \|\| UU^\top - U^\star U^{\star\top} \|\|$$

# Principal angles between two eigen-spaces

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In addition to “distance”, one might also be interested in “angles”



We can quantify the similarity between two lines (represented resp. by unit vectors  $\mathbf{u}$  and  $\mathbf{u}^*$ ) by an angle between them

$$\theta = \arccos\langle \mathbf{u}, \mathbf{u}^* \rangle$$

# Principal angles between two eigen-spaces

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More generally, for  $r$ -dimensional subspaces, one needs  $r$  angles

Specifically, given  $\|U^\top U^\star\| \leq 1$ , we write the singular value decomposition (SVD) of  $U^\top U^\star \in \mathbb{R}^{r \times r}$  as

$$U^\top U^\star = X \underbrace{\begin{bmatrix} \cos \theta_1 & & \\ & \ddots & \\ & & \cos \theta_r \end{bmatrix}}_{=:\cos \Theta} Y^\top =: X \cos \Theta Y^\top$$

where  $\{\theta_1, \dots, \theta_r\}$  are called the **principal angles** between  $U$  and  $U^\star$

# Distance using principal angles

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With principal angles in place, we can define  $\sin \Theta$  distance between subspaces as

$$\text{dist}_{\sin, \|\cdot\|} (U, U^*) := \|\sin \Theta\|$$

where

$$\Theta := \begin{bmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_r \end{bmatrix}, \quad \sin \Theta := \begin{bmatrix} \sin \theta_1 & & \\ & \ddots & \\ & & \sin \theta_r \end{bmatrix}$$

# Link between projections and principal angles

## Lemma 3.6

*The following identities are true:*

$$\begin{aligned}\|UU^\top - U^*U^{*\top}\| &= \|\sin \Theta\| = \|U_\perp^\top U^*\| = \|U^\top U_\perp^*\|; \\ \frac{1}{\sqrt{2}}\|UU^\top - U^*U^{*\top}\|_F &= \|\sin \Theta\|_F = \|U_\perp^\top U^*\|_F = \|U^\top U_\perp^*\|_F.\end{aligned}$$

- sanity check: if  $U = U^*$ , then everything is 0

## Proof of Lemma 3.6

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We prove the claim for spectral norm; the claim for Frobenius norm follows similar argument. Note that

$$\begin{aligned}\|U^\top U_\perp^\star\| &= \|U^\top \underbrace{U_\perp^\star U_\perp^{\star\top}}_{=I-U^\star U^{\star\top}} U\|^{\frac{1}{2}} \\&= \|U^\top U - U^\top U^\star U^{\star\top} U\|^{\frac{1}{2}} \\&= \|I - X \cos^2 \Theta X^\top\|^{\frac{1}{2}} \quad (\text{write } U^\top U^\star = X \cos \Theta Y^\top) \\&= \|I - \cos^2 \Theta\|^{\frac{1}{2}} \\&= \|\sin^2 \Theta\|^{\frac{1}{2}} \\&= \|\sin \Theta\|\end{aligned}$$

## Proof of Lemma 3.6 (cont.)

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Given that singular values are unitarily invariant, it suffices to look at the singular values of the following matrix

$$\begin{bmatrix} U^\top \\ U_\perp^\top \end{bmatrix} (UU^\top - U^*U^{*\top}) [U_\perp^*, U^*] = \begin{bmatrix} U^\top U_\perp^* & \mathbf{0} \\ \mathbf{0} & -U_\perp^\top U^* \end{bmatrix}$$

which further implies

$$\begin{aligned} \|UU^\top - U^*U^{*\top}\| &= \max \{ \|U^\top U_\perp^*\|, \|U_\perp^\top U^*\| \}; \\ \|UU^\top - U^*U^{*\top}\|_F &= \left( \|U^\top U_\perp^*\|_F^2 + \|U_\perp^\top U^*\|_F^2 \right)^{1/2} \end{aligned}$$

# Link between optimal rotations and projections

## Lemma 3.7

*The following identities are true:*

$$\begin{aligned}\|UU^\top - U^*U^{*\top}\| &\leq \min_{R \in \mathcal{O}^{r \times r}} \|UR - U^*\| \leq \sqrt{2}\|UU^\top - U^*U^{*\top}\|; \\ \frac{1}{\sqrt{2}}\|UU^\top - U^*U^{*\top}\|_F &\leq \min_{R \in \mathcal{O}^{r \times r}} \|UR - U^*\|_F \leq \|UU^\top - U^*U^{*\top}\|_F.\end{aligned}$$

— proof left as exercise



# Summary of distance metrics

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So far we have discussed

- 1)  $|||UU^\top - U^*U^{*\top}|||$
- 2)  $|||\sin \Theta|||$
- 3)  $|||U_\perp^\top U^*||| = |||U^\top U_\perp^*|||$
- 4)  $\min_{R \in \mathcal{O}^{r \times r}} |||UR - U^*|||$

# Summary of distance metrics

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So far we have discussed

- 1)  $||| \mathbf{U}\mathbf{U}^\top - \mathbf{U}^\star \mathbf{U}^{\star\top} |||$
- 2)  $||| \sin \mathbf{\Theta} |||$
- 3)  $||| \mathbf{U}_\perp^\top \mathbf{U}^\star ||| = ||| \mathbf{U}^\top \mathbf{U}_\perp^\star |||$
- 4)  $\min_{\mathbf{R} \in \mathcal{O}^{r \times r}} ||| \mathbf{U}\mathbf{R} - \mathbf{U}^\star |||$

Our choice of distance:

$$\begin{aligned} \text{dist}(\mathbf{U}, \mathbf{U}^\star) &:= \min_{\mathbf{R} \in \mathcal{O}^{r \times r}} \|\mathbf{U}\mathbf{R} - \mathbf{U}^\star\|; \\ \text{dist}_F(\mathbf{U}, \mathbf{U}^\star) &:= \min_{\mathbf{R} \in \mathcal{O}^{r \times r}} \|\mathbf{U}\mathbf{R} - \mathbf{U}^\star\|_F \end{aligned}$$

## **Eigenspace perturbation theory**

# Setup and notation

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Consider 2 symmetric matrices  $M^*$ ,  $M = M^* + E \in \mathbb{R}^{n \times n}$  with eigen-decompositions

$$M^* = \sum_{i=1}^n \lambda_i^* \mathbf{u}_i^* \mathbf{u}_i^{*\top} = \begin{bmatrix} U^* & U_{\perp}^* \end{bmatrix} \begin{bmatrix} \Lambda^* & \mathbf{0} \\ \mathbf{0} & \Lambda_{\perp}^* \end{bmatrix} \begin{bmatrix} U^{*\top} \\ U_{\perp}^{*\top} \end{bmatrix};$$
$$M = \sum_{i=1}^n \lambda_i \mathbf{u}_i \mathbf{u}_i^{\top} = \begin{bmatrix} U & U_{\perp} \end{bmatrix} \begin{bmatrix} \Lambda & \mathbf{0} \\ \mathbf{0} & \Lambda_{\perp} \end{bmatrix} \begin{bmatrix} U^{\top} \\ U_{\perp}^{\top} \end{bmatrix}$$

# Setup and notation

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$$M = \left[ \underbrace{u_1 \cdots u_r}_U \underbrace{u_{r+1} \cdots u_n}_{U_\perp} \right]$$

$$\cdot \left[ \begin{array}{cc} \lambda_1 & \\ & \ddots \\ & \underbrace{\lambda_r}_{\Lambda} \\ & & \lambda_{r+1} \\ & & & \ddots \\ & & & & \underbrace{\lambda_n}_{\Lambda_\perp} \end{array} \right]$$

$$\left[ \begin{array}{c} u_1^\top \\ \vdots \\ u_r^\top \\ u_{r+1}^\top \\ \vdots \\ u_n^\top \end{array} \right] \left\{ \begin{array}{l} U^\top \\ U_\perp^\top \end{array} \right.$$

# Davis-Kahan's $\sin \Theta$ theorem: a simple case



Chandler Davis



William Kahan

## Theorem 3.8 (Davis-Kahan's $\sin \Theta$ theorem: simple version)

Suppose  $M^* \succeq 0$  and is rank- $r$ . If  $\|E\| < (1 - 1/\sqrt{2})\lambda_r(M^*)$ , then

$$\text{dist}(U, U^*) \leq \sqrt{2} \|\sin \Theta\| \leq \frac{2\|EU^*\|}{\lambda_r(M^*)} \leq \frac{2\|E\|}{\lambda_r(M^*)};$$

$$\text{dist}_F(U, U^*) \leq \sqrt{2} \|\sin \Theta\|_F \leq \frac{2\|EU^*\|_F}{\lambda_r(M^*)} \leq \frac{2\sqrt{r}\|E\|}{\lambda_r(M^*)}.$$

# Interpretations of Davis-Kahan's $\sin \Theta$ theorem

Suppose  $M^\star \succeq 0$  and is rank- $r$ . If  $\|E\| < (1 - 1/\sqrt{2})\lambda_r(M^\star)$ , then

$$\text{dist}(U, U^\star) \leq \sqrt{2} \|\sin \Theta\| \leq \frac{2\|EU^\star\|}{\lambda_r(M^\star)} \leq \frac{2\|E\|}{\lambda_r(M^\star)}.$$

Remarks:

- Eigen-gap  $\lambda_r(M^\star) = \lambda_r(M^\star) - \lambda_{r+1}(M^\star)$
- Perturbation size  $\|E\|$
- Signal-to-noise (SNR) ratio  $\frac{\lambda_r(M^\star)}{\|E\|}$
- $\|EU^\star\|$  is sometimes useful; we will see benefit later
- Necessity of  $\|E\| \lesssim \lambda_r(M^\star)$

# What happens when SNR is small?

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A toy example (with  $0 < \epsilon < 1$ )

$$\mathbf{M}^\star = \begin{bmatrix} 1 + \epsilon & 0 \\ 0 & 1 - \epsilon \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} -\epsilon & \epsilon \\ \epsilon & \epsilon \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} 1 & \epsilon \\ \epsilon & 1 \end{bmatrix}$$

Leading eigenvectors of  $\mathbf{M}^\star$  and  $\mathbf{M}$  are given respectively by

$$\mathbf{u}_1^\star = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Consequently, we have

$$\|\mathbf{u}_1 \mathbf{u}_1^\top - \mathbf{u}_1^\star \mathbf{u}_1^{\star\top}\| = \frac{1}{\sqrt{2}}, \quad \text{and} \quad \|\mathbf{u}_1 \mathbf{u}_1^\top - \mathbf{u}_1^\star \mathbf{u}_1^{\star\top}\|_F = 1$$

— *large regardless of size of  $\epsilon$  or size of the perturbation  $\|\mathbf{E}\|$*



## Proof of Theorem 3.8

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We intend to control  $U_{\perp}^{\top} U^{\star}$  by studying their interactions through  $E$ :

$$U_{\perp}^{\top} E U^{\star} = U_{\perp}^{\top} (M - M^{\star}) U^{\star} = \Lambda_{\perp} U_{\perp}^{\top} U^{\star} - U_{\perp}^{\top} U^{\star} \Lambda^{\star},$$

which together with triangle inequality implies

$$\begin{aligned} |||U_{\perp}^{\top} E U^{\star}||| &\geq |||U_{\perp}^{\top} U^{\star} \Lambda^{\star}||| - |||\Lambda_{\perp} U_{\perp}^{\top} U^{\star}||| \\ &\geq \sigma_{\min}(\Lambda^{\star}) |||U_{\perp}^{\top} U^{\star}||| - \|\Lambda_{\perp}\| \cdot |||U_{\perp}^{\top} U^{\star}||| \end{aligned} \quad (3.6)$$

In view of Weyl's inequality, one has  $\|\Lambda_{\perp}\| \leq \|E\|$ . In addition, we have  $\sigma_{\min}(\Lambda^{\star}) = \lambda_r(M^{\star})$ . These combined with relation (3.6) give

$$|||U_{\perp}^{\top} U^{\star}||| \leq \frac{|||U_{\perp}^{\top} E U^{\star}|||}{\lambda_r(M^{\star}) - \|E\|} \leq \frac{\sqrt{2} \|U_{\perp}\| \cdot |||E U^{\star}|||}{\lambda_r(M^{\star})} = \frac{\sqrt{2} |||E U^{\star}|||}{\lambda_r(M^{\star})}$$

This together with Lemmas 3.6-3.7 completes the proof

# Davis-Kahan's $\sin \Theta$ theorem: general case

—  $\text{eigenvalues}(A)$ : set of eigenvalues of  $A$

## Theorem 3.9 (Davis-Kahan's $\sin \Theta$ theorem: general version)

Assume that

$$\text{eigenvalues}(\Lambda^*) \subseteq (-\infty, \alpha - \Delta] \cup [\beta + \Delta, \infty); \quad (3.7a)$$

$$\text{eigenvalues}(\Lambda_{\perp}) \subseteq [\alpha, \beta]. \quad (3.7b)$$

for some quantities  $\alpha, \beta \in \mathbb{R}$  and eigengap  $\Delta > 0$ . Then one has

$$\text{dist}(U, U^*) \leq \sqrt{2} \|\sin \Theta\| \leq \frac{\sqrt{2} \|EU^*\|}{\Delta} \leq \frac{\sqrt{2} \|E\|}{\Delta};$$

$$\text{dist}_F(U, U^*) \leq \sqrt{2} \|\sin \Theta\|_F \leq \frac{\sqrt{2} \|EU^*\|_F}{\Delta} \leq \frac{\sqrt{2r} \|E\|}{\Delta}.$$

— conclusion remains valid if Assumption (3.7) is reversed

# **Perturbation theory for singular subspaces**

# Singular value decomposition

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Let  $M^\star$  and  $M = M^\star + E$  be two matrices in  $\mathbb{R}^{n_1 \times n_2}$  (WLOG, we assume  $n_1 \leq n_2$ ), whose SVDs are given respectively by

$$M^\star = \sum_{i=1}^{n_1} \sigma_i^\star \mathbf{u}_i^\star \mathbf{v}_i^{\star\top} = \begin{bmatrix} U^\star & U_\perp^\star \end{bmatrix} \begin{bmatrix} \Sigma^\star & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma_\perp^\star & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}^{\star\top} \\ \mathbf{V}_\perp^{\star\top} \end{bmatrix}$$
$$M = \sum_{i=1}^{n_1} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top = \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} \Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Sigma_\perp & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V}^\top \\ \mathbf{V}_\perp^\top \end{bmatrix}$$

- $\sigma_1 \geq \dots \geq \sigma_{n_1}$  (resp.  $\sigma_1^\star \geq \dots \geq \sigma_{n_1}^\star$ ) stand for the singular values of  $M$  (resp.  $M^\star$ ) arranged in descending order
- $U, U^\star \in \mathbb{R}^{n_1 \times r}$  have orthonormal columns

## Wedin's $\sin \Theta$ theorem

Davis-Kahan's theorem generalizes to singular subspace perturbation:

### Theorem 3.10 (Wedin's $\sin \Theta$ theorem)

If  $\|E\| < \sigma_r^* - \sigma_{r+1}^*$ , then one has

$$\max \{ \text{dist}(U, U^*), \text{dist}(V, V^*) \} \leq \frac{\sqrt{2} \max \{ \|E^\top U^*\|, \|EV^*\| \}}{\sigma_r^* - \sigma_{r+1}^* - \|E\|};$$
$$\max \{ \text{dist}_F(U, U^*), \text{dist}_F(V, V^*) \} \leq \frac{\sqrt{2} \max \{ \|E^\top U^*\|_F, \|EV^*\|_F \}}{\sigma_r^* - \sigma_{r+1}^* - \|E\|}$$

— can be simplified if  $\|E\| < (1 - 1/\sqrt{2})(\sigma_r^* - \sigma_{r+1}^*)$

## Proof of Theorem 3.10

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Similar to proof of Davis-Kahan theorem, we concentrate on  $U_{\perp}^{\top} U^{\star}$

$$\begin{aligned}
 U_{\perp}^{\top} U^{\star} &= U_{\perp}^{\top} (U^{\star} \Sigma^{\star} V^{\star \top}) V^{\star} \Sigma^{\star -1} \\
 &= U_{\perp}^{\top} \left( M - E - U_{\perp}^{\star} \Sigma_{\perp}^{\star} V_{\perp}^{\star \top} \right) V^{\star} \Sigma^{\star -1} \\
 &= U_{\perp}^{\top} \left( U \Sigma V^{\top} + U_{\perp} \Sigma_{\perp} V_{\perp}^{\top} - E - U_{\perp}^{\star} \Sigma_{\perp}^{\star} V_{\perp}^{\star \top} \right) V^{\star} \Sigma^{\star -1} \\
 &= \Sigma_{\perp} V_{\perp}^{\top} V^{\star} \Sigma^{\star -1} - U_{\perp}^{\top} E V^{\star} \Sigma^{\star -1}.
 \end{aligned} \tag{3.9}$$

Applying triangle inequality and Lemma 3.2 to identity (3.9) yields

$$\begin{aligned}
 ||| U_{\perp}^{\top} U^{\star} ||| &\leq \| \Sigma_{\perp} \| \cdot ||| V_{\perp}^{\top} V^{\star} ||| \cdot \| \Sigma^{\star -1} \| + \| U_{\perp}^{\top} \| \cdot ||| E V^{\star} ||| \cdot \| \Sigma^{\star -1} \| \\
 &= \sigma_{r+1} \cdot ||| V_{\perp}^{\top} V^{\star} ||| \cdot \frac{1}{\sigma_r^{\star}} + ||| E V^{\star} ||| \cdot \frac{1}{\sigma_r^{\star}} \\
 &\leq \frac{\sigma_{r+1}^{\star} + \| E \|}{\sigma_r^{\star}} ||| V_{\perp}^{\top} V^{\star} ||| + \frac{||| E V^{\star} |||}{\sigma_r^{\star}}
 \end{aligned} \tag{3.10}$$

## Proof of Theorem 3.10 (cont.)

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Repeating the same argument yields

$$\|V_{\perp}^{\top} V^{\star}\| \leq \frac{\|E^{\top} U^{\star}\|}{\sigma_r^{\star}} + \frac{\sigma_{r+1}^{\star} + \|E\|}{\sigma_r^{\star}} \|U_{\perp}^{\top} U^{\star}\| \quad (3.11)$$

To finish up, combine inequalities (3.10) and (3.11) to obtain

$$\begin{aligned} \max \{ \|U_{\perp}^{\top} U^{\star}\|, \|V_{\perp}^{\top} V^{\star}\| \} &\leq \frac{\max \{ \|E^{\top} U^{\star}\|, \|E V^{\star}\| \}}{\sigma_r^{\star}} \\ &+ \frac{\sigma_{r+1}^{\star} + \|E\|}{\sigma_r^{\star}} \max \{ \|U_{\perp}^{\top} U^{\star}\|, \|V_{\perp}^{\top} V^{\star}\| \}. \end{aligned}$$

When  $\|E\| < \sigma_r^{\star} - \sigma_{r+1}^{\star}$ , we can rearrange terms to obtain desired results

# Extensions of Wedin's theorem

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- Single rotation matrix: Wedin shows us existence of two unitary matrices  $\mathbf{R}_U, \mathbf{R}_V$  such that

$$\max \{ \|\mathbf{U}\mathbf{R}_U - \mathbf{U}^*\|_F, \|\mathbf{V}\mathbf{R}_V - \mathbf{V}^*\|_F \} \quad \text{is small}$$

- Can actually take same unitary matrix (exercise; hint “dilation”)
- Separate bounds for left and right singular vectors:
  - Can treat  $\mathbf{U}$  and  $\mathbf{V}$  differently and obtain sharper bounds
  - Useful when  $n_1$  and  $n_2$  are drastically different



# **Eigenvector perturbation for probability transition matrices**

# Eigen-decomposition for asymmetric matrices

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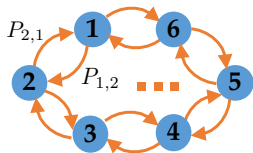
Eigen-decomposition for asymmetric matrices is trickier:

1. both eigenvalues and eigenvectors might be complex-valued
2. eigenvectors might not be orthogonal to each other

This lecture focuses on a special case: **probability transition matrices**

# Probability transition matrices

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Consider a Markov chain  $\{X_t\}_{t \geq 0}$

- $n$  states
- transition probability  $\mathbb{P}\{X_{t+1} = j \mid X_t = i\} = P_{i,j}$
- transition matrix  $\mathbf{P} = [P_{i,j}]_{1 \leq i,j \leq n}$

# Stationary distribution

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Recall  $\mathbf{P}$  is probability transition matrix

- $\boldsymbol{\pi} = [\pi_i]_{1 \leq i \leq n}$  is stationary distribution of  $\mathbf{P}$  if

$$\boldsymbol{\pi} \geq \mathbf{0}, \quad \mathbf{1}^\top \boldsymbol{\pi} = 1, \quad \text{and} \quad \boldsymbol{\pi}^\top \mathbf{P} = \boldsymbol{\pi}^\top$$

- $\boldsymbol{\pi}$  is in fact left eigenvector of  $\mathbf{P}$  with eigenvalue 1
- 1 is largest eigenvalue of  $\mathbf{P}$  in absolute sense:  $|\lambda_i(\mathbf{P})| \leq 1$  by Gershgorin circle theorem

# Reversible Markov chains

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- Markov chain  $\{X_t\}_{t \geq 0}$  with transition matrix  $P$  and stationary distribution  $\pi$  is said to be **reversible** if

$$\pi_i P_{i,j} = \pi_j P_{j,i} \quad \text{for all } i, j$$

— *detailed balance condition*

- Nice consequence: if  $P$  is reversible, all eigenvalues are real  
— *will see proof later*

# Setup

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- Probability transition matrix  $P^*$  of reversible Markov chain
- Perturbed transition matrix  $P = P^* + E$
- $\pi^*$ ,  $\pi$  are leading left eigenvectors of  $P^*$ ,  $P$ , respectively
- Question: how does  $E$  affect perturbation  $\pi - \pi^*$

# New norms

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Fix a strictly positive probability vector  $\boldsymbol{\pi} = [\pi_i]_{1 \leq i \leq n}$ , define

- Vector norm:  $\|\boldsymbol{x}\|_{\boldsymbol{\pi}} := \sqrt{\sum_i \pi_i x_i^2}$  with  $\boldsymbol{x} = [x_i]_{1 \leq i \leq n}$
- Matrix norm:  $\|\boldsymbol{A}\|_{\boldsymbol{\pi}} := \sup_{\|\boldsymbol{x}\|_{\boldsymbol{\pi}}=1} \|\boldsymbol{A}\boldsymbol{x}\|_{\boldsymbol{\pi}}$  with  $\boldsymbol{A} = [A_{i,j}]_{1 \leq i,j \leq n}$

# Eigenvector perturbation for transition matrices

## Theorem 3.11 (Chen, Fan, Ma, Wang '17)

Suppose that  $P^*$  represents a reversible Markov chain, whose stationary distribution vector  $\pi^*$  is strictly positive. Assume that

$$\|E\|_{\pi^*} < 1 - \max \{ \lambda_2(P^*), -\lambda_n(P^*) \}.$$

Then one has

$$\|\pi - \pi^*\|_{\pi^*} \leq \frac{\|\pi^{*\top} E\|_{\pi^*}}{1 - \max \{ \lambda_2(P^*), -\lambda_n(P^*) \} - \|E\|_{\pi^*}}.$$

- Similar to Davis-Kahan
- Eigengap:  $1 - \max \{ \lambda_2(P^*), -\lambda_n(P^*) \}$  since  $1 = \lambda_1(P)$
- Noise size:  $\|\pi^{*\top} E\|_{\pi^*}$



## Proof of Theorem 3.11

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By definitions of  $\pi^\star$  and  $\pi$ , we have

$$\pi^{\star\top} P^\star = \pi^{\star\top}, \quad \text{and} \quad \pi^\top P = \pi^\top,$$

which imply the following decomposition of  $\pi - \pi^\star$

$$\begin{aligned} \pi^\top - \pi^{\star\top} &= \pi^\top P - \pi^{\star\top} P^\star = (\pi - \pi^\star)^\top P + \pi^{\star\top} (P - P^\star) \\ &= (\pi - \pi^\star)^\top (P - P^\star) + (\pi - \pi^\star)^\top P^\star + \pi^{\star\top} (P - P^\star) \\ &= (\pi - \pi^\star)^\top (P - P^\star) + (\pi - \pi^\star)^\top (P^\star - \mathbf{1}\pi^{\star\top}) + \pi^{\star\top} (P - P^\star) \end{aligned}$$

In last step, we use  $(\pi - \pi^\star)^\top \mathbf{1} = 1 - 1 = 0$

## Proof of Theorem 3.11 (cont.)

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Apply triangle inequality w.r.t. norm  $\|\cdot\|_{\pi^*}$  to obtain

$$\begin{aligned}\|\pi - \pi^*\|_{\pi^*} &\leq \|(\pi - \pi^*)^\top (P - P^*)\|_{\pi^*} + \|(\pi - \pi^*)^\top (P^* - \mathbf{1}\pi^{*\top})\|_{\pi^*} \\ &\quad + \|\pi^{*\top} (P - P^*)\|_{\pi^*} \\ &\leq \left( \|P - P^*\|_{\pi^*} + \|P^* - \mathbf{1}\pi^{*\top}\|_{\pi^*} \right) \|\pi - \pi^*\|_{\pi^*} \\ &\quad + \|\pi^{*\top} (P - P^*)\|_{\pi^*}\end{aligned}$$

Assuming  $\|P - P^*\|_{\pi^*} + \|P^* - \mathbf{1}\pi^{*\top}\|_{\pi^*} < 1$ , rearrangement gives

$$\|\pi - \pi^*\|_{\pi^*} \leq \frac{\|\pi^{*\top} (P - P^*)\|_{\pi^*}}{1 - \|P - P^*\|_{\pi^*} - \|P^* - \mathbf{1}\pi^{*\top}\|_{\pi^*}}$$

Proof will be complete if one can show

$$\|P^* - \mathbf{1}\pi^{*\top}\|_{\pi^*} = \max \{ \lambda_2(P^*), -\lambda_n(P^*) \} \quad (3.12)$$

## Proof of identity (3.12)

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Define diagonal matrix  $\mathbf{\Pi}^* = \text{diag}([\pi_1^*, \dots, \pi_n^*]) \in \mathbb{R}^{n \times n}$ . Observe

$$\begin{aligned} \|\mathbf{A}\|_{\pi^*} &= \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_{\pi^*}}{\|\mathbf{x}\|_{\pi^*}} = \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|(\mathbf{\Pi}^*)^{1/2} \mathbf{A} (\mathbf{\Pi}^*)^{-1/2} (\mathbf{\Pi}^*)^{1/2} \mathbf{x}\|_2}{\|(\mathbf{\Pi}^*)^{1/2} \mathbf{x}\|_2} \\ &= \sup_{\mathbf{v} \neq \mathbf{0}} \frac{\|(\mathbf{\Pi}^*)^{1/2} \mathbf{A} (\mathbf{\Pi}^*)^{-1/2} \mathbf{v}\|_2}{\|\mathbf{v}\|_2} = \|(\mathbf{\Pi}^*)^{1/2} \mathbf{A} (\mathbf{\Pi}^*)^{-1/2}\| \end{aligned}$$

As a consequence, one has

$$\begin{aligned} \|\mathbf{P}^* - \mathbf{1}\pi^{*\top}\|_{\pi^*} &= \|(\mathbf{\Pi}^*)^{1/2} (\mathbf{P}^* - \mathbf{1}\pi^{*\top}) (\mathbf{\Pi}^*)^{-1/2}\| \\ &= \|\mathbf{S}^* - \pi_{1/2}^* (\pi_{1/2}^*)^\top\| \end{aligned}$$

with  $\mathbf{S}^* = (\mathbf{\Pi}^*)^{1/2} \mathbf{P}^* (\mathbf{\Pi}^*)^{-1/2}$  and  $\pi_{1/2}^* = [(\pi_j^*)^{1/2}]_{1 \leq j \leq n}$

## Proof of identity (3.12) (cont.)

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Several properties of  $\mathbf{S}^*$ :

- Symmetric: all eigenvalues are real  
— *check detailed balance*
- Similar to  $\mathbf{P}^*$ :  $\mathbf{S}^*$  have same eigenvalues as  $\mathbf{P}^*$ , and

$$\mathbf{S}^* \boldsymbol{\pi}_{1/2}^* = \boldsymbol{\pi}_{1/2}^*$$

- Eigenvalues of  $\mathbf{S}^* - \boldsymbol{\pi}_{1/2}^* (\boldsymbol{\pi}_{1/2}^*)^\top$  are  $0, \lambda_2(\mathbf{S}^*), \dots, \lambda_n(\mathbf{S}^*)$

Combine all to see

$$\begin{aligned} \|\mathbf{S}^* - \boldsymbol{\pi}_{1/2}^* (\boldsymbol{\pi}_{1/2}^*)^\top\| &\stackrel{(i)}{=} \max \{ |\lambda_2(\mathbf{S}^*)|, |\lambda_n(\mathbf{S}^*)| \} \\ &= \max \{ \lambda_2(\mathbf{S}^*), -\lambda_n(\mathbf{S}^*) \} \stackrel{(ii)}{=} \max \{ \lambda_2(\mathbf{P}^*), -\lambda_n(\mathbf{P}^*) \}. \end{aligned}$$