

# STAT253/317 Lecture 5: 4.4 Limiting Distribution II

## Positive Recurrence and Null Recurrence

For a Markov chain, consider the return time to a recurrent state  $i$

$$T_i = \min\{n > 0 : X_n = i | X_0 = i\}$$

We say a state  $i$  is

- ▶ **positive recurrent** if  $i$  is recurrent and  $\mathbb{E}[T_i] < \infty$ .
- ▶ **null recurrent** if  $i$  is recurrent but  $\mathbb{E}[T_i] = \infty$ .

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We say a state is **ergodic** if it is aperiodic and positive recurrent.

# The Fundamental Limit Theorem of Markov Chains I

Consider an irreducible aperiodic Markov chain. Then

$$\lim_{n \rightarrow \infty} P_{ij}^{(n)} = \frac{1}{\mathbb{E}[T_j]}$$

Moreover, if a Markov chain is irreducible and every state is ergodic, then limiting distribution exists:

$$\pi_j = \lim_{n \rightarrow \infty} P_{ij}^{(n)} = \frac{1}{\mathbb{E}[T_j]}$$

and is uniquely determined by the set of equations

$$\pi_j \geq 0, \quad \sum_{j \in \mathfrak{X}} \pi_j = 1, \quad \pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$$

**Proof.** See Theorem 1.1, 1.2, 1.3 on p.81-86 in Karlin & Taylor (1975).

## Why $\pi_i = 1/\mathbb{E}(T_i)$ ?

Consider a Markov chain started from state  $j$ . Let  $S_k$  be the time till the  $k$ -th visit to state  $i$ . Then

$$S_k = T_{ji} + T_{ii}(1) + \dots + T_{ii}(k-1)$$

Here

- ▶  $T_{ji}$  = the first time the process visits state  $i$  from state  $j$ , and
- ▶  $T_{ii}(m)$  = the time between the  $m$ th and  $(m+1)$ st visit to state  $i$ .

Observe that  $T_{ii}(1), T_{ii}(2), \dots, T_{ii}(k-1)$  are i.i.d. and have the same distribution as  $T_i$ .

For  $k$  large, the Law of Large Numbers tells us

$$\frac{1}{k}[T_{ji} + T_{ii}(1) + T_{ii}(2) + \dots + T_{ii}(k-1)] \approx \mathbb{E}(T_i)$$

i.e., the chain visits state  $i$  about  $k$  times in  $k\mathbb{E}(T_i)$  steps.

In  $n$  steps, we expect about  $n\pi(i)$  visits to the state  $i$ . Hence setting  $n = k\mathbb{E}(T_i)$ , we get the relation

$$\pi_i = 1/\mathbb{E}(T_i).$$

## Proposition 4.5 Positive Recurrence is a Class Property

- From the Fundamental Limit Theorem of Markov Chains I

$$\pi_i = 1/\mathbb{E}[T_i]$$

and that a state  $i$  is positive recurrent if and only if  $\mathbb{E}[T_i] < \infty$   
it follows that a state  $i$  is positive recurrent if and only if  $\pi_i > 0$

- If a state  $j$  communicate with a positive recurrent state  $i$ , then state  $j$  is also positive recurrent.

*Proof.* Since  $i \leftrightarrow j$ , there exists  $n$  such that  $P_{ij}^{(n)} > 0$ . Along with the fact that  $i$  is positive recurrent,  $\pi_i > 0$ , we know  $\pi_j = \sum_k \pi_k P_{kj}^{(n)} \geq \pi_i P_{ij}^{(n)} > 0$ . So  $j$  is also positive recurrent.

## Corollary: Null Recurrence is a Class Property

If state  $i$  is null recurrent and  $i \leftrightarrow j$ , then state  $j$  is also null recurrent.

*Proof.* Since recurrence is a class property, state  $j$  can only be positive or null recurrent as it communicates with a null recurrent state  $i$ . Suppose state  $j$  is positive recurrent. As positive recurrence is a class property, state  $i$  must also be positive recurrent not null recurrent if it communicates with state  $j$ . So state  $j$  can only be null recurrent.

# Finite-State Markov Chains Have No Null Recurrent States

In an irreducible finite-state Markov chain all states are positive recurrent.

*Proof.*

Recall an irreducible Markov chain must be recurrent. (Why?)

Also recall that positive/null recurrence is a class property. Thus if one state is null recurrent, then all states are null recurrent.

However, since  $\sum_{j \in \mathfrak{X}} P_{ij}^{(n)} = 1$ . As there are only finite number of states, it is impossible that  $\lim_{n \rightarrow \infty} P_{ij}^{(n)} = 0$  for all  $j \in \mathfrak{X}$ . Thus no state can be null recurrent.

Remark. For a finite state Markov chain, a limiting distribution exists if it is irreducible and aperiodic

# The Fundamental Limit Theorem of Markov Chain II

(★★★★★)

If a Markov chain is **irreducible**, then the Markov chain is **positive recurrent** if and only if there exists a solution to the set of equations:

$$\pi_i \geq 0, \quad \sum_{i \in \mathcal{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathcal{X}} \pi_i P_{ij}$$

If a solution exists then

► it will be unique, and

$$\frac{1}{\mathbb{E}[T_j]} = \pi_j = \begin{cases} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{ij}^{(k)} & \text{if the chain is periodic} \\ \lim_{n \rightarrow \infty} P_{ij}^{(n)} & \text{if the chain is aperiodic} \end{cases}$$

**Remark.** When a Markov chain is periodic, though its limiting distribution  $\lim_{n \rightarrow \infty} P_{ij}^{(n)}$  doesn't exist, another limit

$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{ij}^{(k)}$  exists and is equal to the stationary distribution. The later limit can be interpret as the **long run proportion of time that the Markov chain is in state  $j$ .**

## Example 1: One-Dimensional Random Walk

In Lecture 4, we have shown that 1-dim symmetric random walk has no stationary distribution.

- Conclusion from 2nd limit theorem: 1-dim symmetric random walk is null recurrent, i.e.

$$\mathbb{E}[T_i] = \infty \quad \text{for all state } i$$

In fact, in Lecture 3 we have shown that

$$P_{ii}^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2} \left(\frac{1}{2}\right)^n \approx \sqrt{\frac{2}{\pi n}} & \text{if } n \text{ is even} \end{cases}$$

Thus  $\pi_i = \lim_{n \rightarrow \infty} P_{ii}^{(n)} = 0$ , and hence  $\mathbb{E}[T_i] = 1/\pi_i = \infty$ . (This relation holds even if the chain considered here is periodic.)



## Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$P_{i,i+1} = p \quad \text{for all } i = 0, 1, 2, \dots$$

$$P_{i,i-1} = 1 - p \quad \text{for all } i = 1, 2, \dots$$

$$p_{00} = 1 - p$$

Try to solve  $\pi_j = \sum_{i \in \mathbb{X}} \pi_i P_{ij}$

$$\pi_0 = \pi_0 P_{00} + \pi_1 P_{10} = (1 - p)(\pi_0 + \pi_1) \Rightarrow \pi_1 = \frac{p}{1-p} \pi_0$$

$$\pi_1 = \pi_0 P_{01} + \pi_2 P_{21} = p\pi_0 + (1 - p)\pi_2 \Rightarrow \pi_2 = \left(\frac{p}{1-p}\right)^2 \pi_0$$

$$\pi_2 = \pi_0 P_{12} + \pi_3 P_{32} = p\pi_1 + (1 - p)\pi_3 \Rightarrow \pi_3 = \left(\frac{p}{1-p}\right)^3 \pi_0$$

$\vdots$

$$\pi_j = p\pi_{j-1} + (1 - p)\pi_{j+1} \Rightarrow \pi_{j+1} = \left(\frac{p}{1-p}\right)^{j+1} \pi_0$$

## Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \left( \frac{p}{1-p} \right)^i = \begin{cases} \pi_0 \left( \frac{1-p}{1-2p} \right) & \text{if } p < 1/2 \\ \infty & \text{if } p \geq 1/2 \end{cases}$$

Conclusion: The process is positive recurrent iff  $p < 1/2$ , in which case

$$\pi_i = \frac{1-2p}{1-p} \left( \frac{p}{1-p} \right)^i, \quad i = 0, 1, 2, \dots$$

### Ex 3: Ehrenfest Diffusion Model with $N$ Balls

Recall that in Lecture 4, we show that Ehrenfest Diffusion Model is irreducible, has period = 2, and there exists a solution to the set of equations

$$\pi_i \geq 0, \quad \sum_{i \in \mathfrak{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$$

which is

$$\pi_i = \binom{N}{i} \left(\frac{1}{2}\right)^N \quad \text{for } i = 0, 1, 2, \dots, N$$

Though the limiting distribution  $\lim_{n \rightarrow \infty} P_{ij}^{(n)}$  does not exist, we can show that

$$\lim_{n \rightarrow \infty} P_{ij}^{(2n)} = 2 \binom{N}{j} \left(\frac{1}{2}\right)^N, \quad \lim_{n \rightarrow \infty} P_{ij}^{(2n+1)} = 0 \quad \text{if } i + j \text{ is even}$$

$$\lim_{n \rightarrow \infty} P_{ij}^{(2n)} = 0, \quad \lim_{n \rightarrow \infty} P_{ij}^{(2n+1)} = 2 \binom{N}{j} \left(\frac{1}{2}\right)^N \quad \text{if } i + j \text{ is odd}$$

From the above, one can verify that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n P_{ij}^{(k)} = \binom{N}{j} \left(\frac{1}{2}\right)^N = \pi_j.$$

## Exercise 4.50 on p.284

A Markov chain has transition probability matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{matrix}$$

Communicating classes:

$$\begin{array}{ccc} \{1, 2\} & \{3, 4\} & \{5, 6\} \\ \uparrow & \uparrow & \uparrow \\ \text{transient} & \text{recurrent} & \text{recurrent} \end{array}$$

Find  $\lim_{n \rightarrow \infty} P^{(n)}$ .

## Exercise 4.50 on p.284 (Cont'd)

Observe that  $\lim_{n \rightarrow \infty} P_{ij}^{(n)} = 0$  if  $j$  is transient, hence,

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{array}{c} \begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \left( \begin{array}{cccccc} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \end{array} \right) \end{array}$$

## Exercise 4.50 on p.284 (Cont'd)

Observe that  $\lim_{n \rightarrow \infty} P_{ij}^{(n)} = 0$  if  $j$  is NOT accessible from  $i$

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

The two classes  $\{3,4\}$  and  $\{5,6\}$  do not communicate and hence the transition probabilities in between are all 0.

## Exercise 4.50 on p.284 (Cont'd)

Recall we have shown that the limiting distribution of a two-state Markov chain with the transition matrix  $\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$  is  $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$ . As the Markov chain restricted to the class  $\{3,4\}$  is

also a Markov chain with the transition matrix  $\begin{matrix} & \begin{matrix} 3 & 4 \end{matrix} \\ \begin{matrix} 3 \\ 4 \end{matrix} & \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \end{matrix}$ .

Hence,

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{matrix}$$

## Exercise 4.50 on p.284 (Cont'd)

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{matrix}$$

For the same reason,

$$\lim_{n \rightarrow \infty} P^{(n)} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{matrix}$$