Problems

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1. Let $X_1, \cdots, X_n$ be a sequence of i.i.d. random variables, of which cdf is $F_X(\cdot)$.

   (a) What is CDF of $X_{\max} = \max_i X_i$?
   Solution.

   The event $X_{\max} \leq x$ occurs iff $X_i \leq x$ for all i.

$$\begin{aligned} F_{X_{\max}}(x) =& \mathbb{P}[X_{\max} \leq x] \\ =& \mathbb{P}[X_1 \leq x, \cdots, X_n \leq x] \\ =& \prod_i F_X(x) = (F_X(x))^n . \end{aligned}$$

   (b) What is CDF of $X_{\min} = \min_i X_i$?
   Solution.
   The event $X_{\min} > x$ occurs iff $X_i > x$ for all i.

$$\begin{aligned} F_{X_{\min}}(x) &= \mathbb{P}[X_{\min} \leq x] \\ &= 1 - \mathbb{P}[X_{\min} > x] \\ &= 1 - \mathbb{P}[X_1 > x, \cdots, X_n > x] \\ &= 1 - (1 - F_X(x))^n . \end{aligned}$$

2. Show how the Chebyshev inequality can be derived from Markov inequality.

   Solution.

   Markov inequality is that if X is a non-negative random variable, then for all $r > 0$,

$$\mathbb{E}[X \geq r] \leq \frac{\mathbb{E}[X]}{r}.$$

   The Chebyshev inequality can be derived as

$$\mathbb{E}\left[|X - \mathbb{E}[X]| \geq r\right] = \mathbb{E}\left[(X - \mathbb{E}[X])^2 \geq r^2\right] \leq \frac{\mathrm{Var}(X)}{r^2}.$$

3. If X is a continuous random variable having CDF F_X, show that the random variable $Y = F_X(X)$ is uniformly distributed in $(0, 1)$.

   Solution.

   Define $F_X^{-1}(x)$ as

   $$F_X^{-1}(x) = \inf \{t \ : \ F(t) \geq x\}.$$

   CDF of Y is

   $$\begin{aligned} \mathbb{P}[Y \leq x] &= \mathbb{P}[F_X(X) \leq x] = \mathbb{P}\left[X \leq F_X^{-1}(x)\right] \\ &= F_X\left(F_X^{-1}(x)\right) = x, \quad \forall\, x \in (0, 1), \end{aligned}$$

   which is the CDF of uniform random variable in $(0, 1)$.

4. Suppose you can generate a random variable U uniformly distributed in $(0, 1)$. How would you use it to simulate a continuous random variable X having a arbitrary distribution function $F(\cdot)$?

   Solution.

   Generate $X = F^{-1}(U)$, where $F^{-1}(x)$ is defined as

   $$F^{-1}(x) = \inf \{t \ : \ F(t) \geq x\}.$$

   Then, X has CDF F, since

   $$\mathbb{P}[X \leq x] = \mathbb{P}\left[F^{-1}(U) \leq x\right] = \mathbb{P}[U \leq F(x)] = F(x).$$

5. Suppose you have access to two independent random variables U_1 and U_2, both uniformly distributed in $[0, 1]$. How would you use them to simulate two continuous random variables X_1 and X_2 with a given joint distribution $F(\cdot, \cdot)$?

   Solution.

   Compute the marginal CDF f as

   $$f(x) = \lim_{x_2 \to \infty} F(x, x_2).$$

   Define $g_t(x)$ as

   $$g_t(x) = \frac{\frac{d}{dt} F(t, x)}{f'(t)}.$$

   Simulate X_1 and X_2 as

   $$X_1 = f^{-1}(U_1), \quad X_2 = g_{X_1}^{-1}(U_2).$$

Then, X_1 and X_2 have CDF F, because

$$\begin{aligned}
\mathbb{P}[X_1 \leq x_1,\ X_2 \leq x_2] &= \mathbb{P}\left[U_1 \leq f(x_1),\ U_2 \leq \frac{\frac{d}{dt}F(X_1, x_2)}{f'(X_1)}\right] \\
&= \int_0^{f(x_1)} \mathbb{P}\left[U_2 \leq \frac{\frac{d}{dt}F(f^{-1}(u), x_2)}{f'(f^{-1}(u))}\right] du \\
&= \int_0^{f(x_1)} \frac{\frac{d}{dt}F(f^{-1}(u), x_2)}{f'(f^{-1}(u))} du \\
&= \int_{-\infty}^{x_1} \frac{d}{dt}F(t, x_2) dt = F(x_1, x_2).
\end{aligned}$$

6. Prove the weak law of large number using Chebyshev's inequality.

   Solution.

   Let $X_1, \cdots$, be a sequence of i.i.d. random variables, each having $\mathbb{E}[X_i] = \mu$. Then,

$$\mathbb{E}\left[\frac{X_1 + \cdots + X_n}{n}\right] = \mu$$

   and

$$\mathrm{Var}\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{\sigma^2}{n}.$$

   Chebyshev's inequality implies that

$$\mathbb{P}\left[\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right] \leq \frac{\sigma^2}{n\epsilon^2}.$$

   Thus,

$$\lim_{n\to\infty} \mathbb{P}\left[\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right] = 0.$$

7. Let $W \sim \mathcal{N}(0, 1)$ and $Q(x) = \mathbb{P}[W > x]$.

   (a) Show that

$$Q(x) < \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right).$$

   Solution.

$$
\begin{aligned}
Q(x) &= \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \\
&\le \int_x^\infty \frac{1}{\sqrt{2\pi}} \frac{t}{x} \exp\left(-\frac{t^2}{2}\right) dt \\
&= \frac{1}{\sqrt{2\pi}x} \int_x^\infty \exp\left(-\frac{t^2}{2}\right) d\left(\frac{1}{2}t^2\right) \\
&= \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right).
\end{aligned}
$$

(b) Show that

$$
Q(x) > \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) \exp\left(-\frac{x^2}{2}\right) \quad \forall\ x > 1.
$$

Solution.

$$
\begin{aligned}
&\left(1 + \frac{1}{x^2}\right) Q(x) \\
&= \left(1 + \frac{1}{x^2}\right) \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt \\
&\ge \int_x^\infty \frac{1}{\sqrt{2\pi}} \left(1 + \frac{1}{t^2}\right) \exp\left(-\frac{t^2}{2}\right) dt \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_x^\infty \frac{1}{t} t \exp\left(-\frac{t^2}{2}\right) dt + \int_x^\infty \exp\left(-\frac{t^2}{2}\right) dt\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\frac{1}{t} \exp\left(-\frac{t^2}{2}\right)\Bigg|_x^\infty - \int_x^\infty \frac{1}{t^2} \exp\left(-\frac{t^2}{2}\right) dt + \int_x^\infty \frac{1}{t^2} \exp\left(-\frac{t^2}{2}\right) dt\right] \\
&= \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right).
\end{aligned}
$$

When $x > 1$,

$$
Q(x) \ge \frac{1}{\sqrt{2\pi}x} \frac{x^2}{x^2+1} \exp\left(-\frac{x^2}{2}\right) \ge \frac{1}{\sqrt{2\pi}x} \frac{x^2-1}{x^2} \exp\left(-\frac{x^2}{2}\right).
$$

8. Prove Cauchy-Schwarz inequality.

$$
\mathbb{E}[XY]^2 \le \mathbb{E}[X^2]\mathbb{E}[Y^2].
$$

Solution. For any $a, b \in \mathbb{R}$,

$$
\mathbb{E}\left[(aX + bY)^2\right] = a^2\mathbb{E}[X^2] + b^2\mathbb{E}[Y^2] + 2ab\mathbb{E}[XY] \ge 0, \tag{1}
$$
$$
\mathbb{E}\left[(aX - bY)^2\right] = a^2\mathbb{E}[X^2] + b^2\mathbb{E}[Y^2] - 2ab\mathbb{E}[XY] \ge 0. \tag{2}
$$

Let

$$a = \sqrt{\mathbb{E}[Y^2]},\ b = \sqrt{\mathbb{E}[X^2]}.$$

Then, equations (1) and (2) become

$$2ab\mathbb{E}[XY] \geq -2a^2b^2,$$
$$2ab\mathbb{E}[XY] \leq 2a^2b^2.$$

This results in

$$\mathbb{E}[XY] \geq -\sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]},$$
$$\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]}\sqrt{\mathbb{E}[Y^2]}.$$

9. The amount of weight, W, that a bridge can withstand without damage, is a Gaussian random variable with mean μ_W and variance σ_W^2. Suppose the weight of cars $X_1, X_2, \cdots, X_n$ are i.i.d. random variables with mean μ_X and variance σ_X^2. How many cars would have to be on the bridge for the probability of damage to exceed 0.1 ?

   Solution.

$$P_n \triangleq \mathbb{P}[X_1 + \cdots + X_n - W \geq 0].$$

Since X_i and W are Gaussian, $X_i - \frac{W}{n}$ is also a Gaussian.

$$\mathbb{E}\left[X_i - \frac{W}{n}\right] = \mu_X - \frac{\mu_W}{n}.$$

$$\mathrm{Var}\left(X_i - \frac{W}{n}\right) = n\sigma_X^2 + \sigma_W^2.$$

Using Central Limit theorem,

$$\begin{aligned} P_n &= \mathbb{P}\left[\sum\left(X_i - \frac{W}{n}\right) \geq 0\right] \\ &= \mathbb{P}\left[\frac{\sum_i \left(X_i - \frac{W}{n}\right) - n\left(\mu_X - \frac{\mu_W}{n}\right)}{\sqrt{n\sigma_X^2 + \sigma_W^2}} \geq \frac{-\left(n\mu_X - \mu_W\right)}{\sqrt{n\sigma_X^2 + \sigma_W^2}}\right]. \end{aligned}$$

Since $\mathbb{P}[Z \geq 1.28] \approx .1$, the number of cars should satisfy

$$\frac{\mu_W - n\mu_X}{\sqrt{n\sigma_X^2 + \sigma_W^2}} \leq 1.28.$$

10. Show that if X and Y are independent and

$$\begin{aligned}
&X \sim \mathcal{N}(\mu_X, \sigma_X^2),\\
&Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2),\\
&Z = X + Y,
\end{aligned}$$

then

$$Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2).$$

Solution.

For independent random variables X and Y, the distribution of the sum equals the convolution of PDF's of X and Y. Thus, PDF of Z is

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} f_Y(z-x) f_X(x) dx\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_Y^2}} \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(z-x-\mu_Y)^2}{2\sigma_Y^2} - \frac{(x-\mu_X)^2}{2\sigma_X^2}\right) dx\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sqrt{\sigma_X^2+\sigma_Y^2}} \exp\left(-\frac{(z-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}\right)\\
&\qquad \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}} \exp\left(-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_X^2+\sigma_Y^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sqrt{\sigma_X^2+\sigma_Y^2}}\right)^2}\right) dx\\
&= \frac{1}{\sqrt{2\pi(\sigma_X^2+\sigma_Y^2)}} \exp\left(-\frac{(z-(\mu_X+\mu_Y))^2}{2(\sigma_X^2+\sigma_Y^2)}\right).
\end{aligned}$$