A Summary of Brownian Motions

In the following, $\Phi(x)$ is the CDF of $N(0,1)$. That is $\Phi(x) = \mathrm{P}(X \le x)$, where $X \sim N(0,1)$. $\{B(t), t \ge 0\}$ is a Brownian motion process with drift μ and variance parameter σ^2.

1. Using the independent & stationary increment property to derive

   - the mean, variance, covariance of $B(t_1), B(t_2), \ldots, B(t_m)$
   - joint distribution of $B(t_1), B(t_2), \ldots, B(t_m)$
   - conditional mean, variance, and distribution of $B(s_1), \ldots, B(s_n)$ given $B(t_1), \ldots, B(t_m)$

   In particular, given $B(t) = x$ (p.10-11 in Lecture 22&23),

$$B(s) \sim \begin{cases} x + B(t-s) = x + N(\mu(t-s), \sigma^2(t-s)) & \text{if } s > t \\ N(\frac{s}{t}x, \sigma^2 \frac{s(t-s)}{t}) & \text{if } s < t \end{cases}$$

2. Maximum and Hitting times
   Let $T_x = \min\{t : B(t) = x\}$ be the first time t that $\{B(t)\}$ hits $x > 0$.

$$\begin{aligned} \mathrm{P}(\max_{0 \le s < t} B(s) > x) &= \mathrm{P}(T_x \le t) \\ &= \begin{cases} 2\mathrm{P}(B(t) \ge x) = 2 - 2\Phi\left(\frac{x}{\sigma\sqrt{t}}\right) & \text{if } \mu = 0 \\ e^{\frac{2\mu x}{\sigma^2}} P(B(t) < -x) + P(B(t) > x) = e^{\frac{2\mu x}{\sigma^2}} \Phi\left(\frac{-x-\mu t}{\sigma\sqrt{t}}\right) + 1 - \Phi\left(\frac{x - 2\mu t}{\sigma\sqrt{t}}\right) & \text{if } \mu \ne 0 \end{cases} \end{aligned}$$

3. Let $T_{-a,b} = \min(T_{-a}, T_b)$ be the first time t that $\{B(t)\}$ hit $-a$ or b, $a > 0$, $b > 0$,

$$T_{-a,b} = \min(T_{-a}, T_b) = \min\{t : B(t) = -a, \text{ or } B(t) = b\}.$$

   - drift $\mu = 0$ (Lecture 22&23)

$$\begin{aligned} &\mathrm{P}(\text{down } a \text{ before up } b) = \mathrm{P}(B(T) = -a) = \frac{b}{a+b}, \\ &\mathrm{P}(\text{up } b \text{ before down } a) = \mathrm{P}(B(T) = b) = \frac{a}{a+b}, \\ &\mathbb{E}[T_{-a,b}] = \frac{ab}{\sigma^2} \end{aligned}$$

   - drift $\mu \ne 0$ (Lecture 24)

$$\begin{aligned} \mathrm{P}(\text{up } b \text{ before down } a) = \mathrm{P}(B(T) = b) &= \frac{e^{2\mu a/\sigma^2} - 1}{e^{2\mu a/\sigma^2} - e^{-2\mu b/\sigma^2}} = \frac{1 - e^{-2\mu a/\sigma^2}}{1 - e^{-2\mu(a+b)/\sigma^2}} \\ \mathbb{E}[T_{-a,b}] &= \frac{b - (a+b)e^{-2\mu a/\sigma^2} + a e^{-2\mu(a+b)/\sigma^2}}{\mu(1 - e^{-2\mu(a+b)/\sigma^2})} \quad \text{(Exercise 10.23.)} \\ \mathbb{E}[T_{-a,a}] &= \frac{a(1 - e^{-2\mu a/\sigma^2})^2}{\mu(1 - e^{-4\mu a/\sigma^2})} \end{aligned}$$

4. $\max_{0 \le t \le \infty} B(t)$ & $\min_{0 \le t \le \infty} B(t)$

   - If $\mu < 0$

$$\begin{aligned} &\mathrm{P}(\max_{0 \le t \le \infty} B(t) < a) = 1 - \exp(-2|\mu| a/\sigma^2), \quad a \ge 0 \\ &\mathrm{P}(\min_{0 \le t \le \infty} B(t) = -\infty) = 1 \end{aligned}$$

- If $\mu > 0$

$$\mathrm{P}(\max_{0\le t\le\infty} B(t) = \infty) = 1$$

$$\mathrm{P}(\min_{0\le t\le\infty} B(t) > a) = 1 - \exp(-2\mu|a|/\sigma^2), \quad a \le 0$$

- If $\mu = 0$,

$$\mathrm{P}(\max_{0\le t\le\infty} B(t) = \infty) = \mathrm{P}(\min_{0\le t\le\infty} B(t) = -\infty) = 1$$

5. For $a > 0$, $x \le a$, by Reflection Principle,

$$\mathrm{P}\left(B(t) \le x, \max_{0\le s\le t} B(s) < a\right) = \mathrm{P}(B(t) \le x) - \mathrm{P}(B(t) \ge 2a - x) \quad \text{(Lecture 18-10)}$$