

STAT253/317 Lecture 12

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Chapter 6 Continuous-Time Markov Chains

6.2 Continuous-Time Markov Chains (CTMC)

Definitions. A stochastic process $\{X(t), t \geq 0\}$ with state space \mathcal{X} is called a *continuous-time Markov chain* if for any two states $i, j \in \mathcal{X}$,

$$\begin{aligned} & \underbrace{P(X(t+s) = j)}_{\text{future}} \underbrace{| X(s) = i}_{\text{present}}, \underbrace{X(u) = x(u), \text{ for } 0 \leq u < s}_{\text{past}} \\ &= \underbrace{P(X(t+s) = j)}_{\text{future}} \underbrace{| X(s) = i}_{\text{present}} \end{aligned}$$

If $P(X(t+s) = j | X(s) = i)$ does not depend on s for all $i, j \in \mathcal{X}$, then it is denoted as

$$P_{ij}(t) = P(X(t+s) = j | X(s) = i),$$

and we say the CTMC is *homogeneous* in time.

In STAT253/317, we focus on homogeneous CTMC only.

Exponential Waiting Time

Let $\{X(t), t \geq 0\}$ be a homogeneous continuous-time Markov chain. For $i \in \mathcal{X}$, let T_i denote the amount of time that $X(t)$ stays in state i before making a transition into a different state.

Claim: T_i has the *memoryless property*.

$$\begin{aligned} & \mathbb{P}(T_i \geq t + s | T_i \geq s) \\ &= \mathbb{P}(X(u) = i, \text{ for } s \leq u \leq s + t | X(u) = i, \text{ for } 0 \leq u \leq s) \\ &= \mathbb{P}(X(u) = i, \text{ for } s \leq u \leq s + t | X(s) = i) \quad (\text{Markov property}) \\ &= \mathbb{P}(X(u) = i, \text{ for } 0 \leq u \leq t | X(0) = i) \quad (\text{time homogeneity}) \\ &= \mathbb{P}(T_i \geq t) \quad \Rightarrow \quad \text{So } T_i \text{ is memoryless.} \end{aligned}$$

Recall that the exponential distribution is the only continuous distribution having the memoryless property.

Thus $T_i \sim \text{Exp}(\nu_i)$ for some rate ν_i .

An Alternative Definition of CTMC

A stochastic process $\{X(t), t \geq 0\}$ with state space \mathcal{X} is a *continuous-time Markov chain* if

- ▶ (exponential waiting time) when the chain reaches a state i , the time it stays at state $i \sim \text{Exp}(\nu_i)$, where ν_i is the transition rate at state i
- ▶ (embedded with a discrete time Markov chain) when the process leaves state i , it enters another state j with probability P_{ij} , such that

$$P_{ii} = 0, \quad \sum_{j \in \mathcal{X}} P_{ij} = 1 \quad \text{for all } i, j \in \mathcal{X}.$$

Remark: The amount of time T_i the process spends in state i , and the next state visited, must be independent. For if the next state visited were dependent on T_i , then information as to how long the process has already been in state i would be relevant to the prediction of the next state—and this contradicts the Markovian assumption.

6.3 Birth and Death Processes

Let $X(t)$ = the number of people in the system at time t .

Suppose that whenever there are n people in the system, then

- (i) new arrivals enter the system at an exponential rate λ_n , and
- (ii) people leave the system at an exponential rate μ_n .

Such an $\{X(t), t \geq 0\}$ is called a *birth and death process*.

$$\begin{array}{ccccccccccccccc} & \lambda_0 & & \lambda_1 & & \lambda_2 & & \cdots & & \lambda_{n-1} & & \lambda_n & & \cdots \\ 0 & \rightleftharpoons & 1 & \rightleftharpoons & 2 & \rightleftharpoons & 3 & \cdots & n-1 & \rightleftharpoons & n & \rightleftharpoons & n+1 & \cdots \\ & \mu_1 & & \mu_2 & & \mu_3 & & \cdots & & \mu_n & & \mu_{n+1} & & \cdots \end{array}$$

Suppose the process is at state $i > 0$ at time t . Then

B_i = waiting time until the next birth $\sim \text{Exp}(\lambda_i)$

D_i = waiting time until the next death $\sim \text{Exp}(\mu_i)$

Hence, the waiting time until the next transition out of state i is $\min(B_i, D_i) \sim \text{Exp}(\lambda_i + \mu_i)$, from which we can get

$$\nu_i = \lambda_i + \mu_i, \text{ for } i > 0$$

6.3 Birth and Death Processes (Cont'd)

Moreover, given the process is at state $i > 0$ at time t , the probability that the next transition is a birth rather than a death is

$$P_{i,i+1} = P(B_i < D_i) = \frac{\lambda_i}{\lambda_i + \mu_i},$$

which implies $P_{i,i-1} = P(D_i < B_i) = \frac{\mu_i}{\lambda_i + \mu_i}$, for $i > 0$.

As only birth is possible at state 0, we know $\nu_0 = \lambda_0$ and $P_{01} = 1$.

To sum up, a birth and death process is a CTMC with state space $\mathcal{X} = \{0, 1, 2, \dots\}$ such that

$$\begin{aligned}\nu_i &= \lambda_i + \mu_i, i > 0, & \nu_0 &= \lambda_0, \\ P_{i,i+1} &= \frac{\lambda_i}{\lambda_i + \mu_i}, & P_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i}, i > 0 \\ P_{01} &= 1, & P_{i,j} &= 0 \quad \text{if } |i - j| > 1\end{aligned}$$

The parameters $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ are called, respectively, the arrival (or birth) and departure (or death) rates.

Examples of Birth and Death Processes

- ▶ Poisson Processes: $\mu_n = 0$, $\lambda_n = \lambda$ for all $n \geq 0$
- ▶ Pure Birth Process:

$$\mu_n = 0 \quad \Rightarrow \quad \nu_i = \lambda_i, \quad P_{i,i+1} = 1, \quad P_{i,i-1} = 0$$

- ▶ Yule Processes (Pure Birth Process with Linear Growth rate):
If there are n people and each independently gives birth at an exponential rate λ , then the total rate at which births occur is $n\lambda$.

$$\mu_n = 0, \quad \lambda_n = n\lambda$$

Reason: Let

B_i = time until the i th individual give birth $\sim \text{Exp}(\lambda)$, $i = 1, \dots, n$

So the time until the next (first) birth when there are n individuals in the population is

$$\min(B_1, B_2, \dots, B_n) \sim \text{Exp}(\lambda + \lambda + \dots + \lambda) = \text{Exp}(n\lambda)$$

So the rate until the next birth is $\lambda_n = n\lambda$.

Example: Linear Growth Model with Immigration

- ▶ each individual independently gives birth at an exponential rate λ
- ▶ each individual independently die at an exponential rate μ
- ▶ new immigrants come in at an exponential rate θ

Such a process is a birth-death process with birth and death rates

$$\mu_n = n\mu, \quad \lambda_n = n\lambda + \theta$$

Reason: Let

B_i = time until the i th individual give birth $\sim \text{Exp}(\lambda)$, $i = 1, \dots, n$

T = time until the next new immigrant comes in $\sim \text{Exp}(\theta)$

So the time until the population size increase from n to $n + 1$ is

$$\min(B_1, \dots, B_n, T) \sim \text{Exp}(\lambda + \dots + \lambda + \theta) = \text{Exp}(n\lambda + \theta)$$

So the rate until the next birth is $\lambda_n = n\lambda + \theta$.

Similarly, one can show that the death rate is $\mu_n = n\mu$.

Example: $M/M/s$ Queueing Model

- ▶ s servers
- ▶ Poisson arrival of customers, rate $= \lambda$
- ▶ Exponential service time, rate $= \mu$

\Rightarrow a birth and death process with constant birth rate $\lambda_n = \lambda$, and death (departure) rate $\mu_n = \min(n, s)\mu$.

Reason: Suppose, there are n customer in the system at time t . At most $\min(n, s)$ of them are being served. Let S_i be remaining service time of the i th server $\sim \text{Exp}(\mu)$. Then, the waiting time until the next departure is

$$\min(S_1, \dots, S_{\min(s, n)}) \sim \text{Exp}(\min(s, n)\mu).$$

6.4 The Transition Probability Function $P_{ij}(t)$

Recall the transition probability function $P_{ij}(t)$ of a CTMC $\{X(t), t \geq 0\}$ is

$$P_{ij}(t) = \mathbb{P}(X(t+s) = j | X(s) = i)$$

Example. (Poisson Processes with rate λ)

$$\begin{aligned} P_{ij}(t) &= \mathbb{P}(N(t+s) = j | N(s) = i) \\ &= \mathbb{P}(N(t+s) - N(s) = j - i) = \begin{cases} e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!} & \text{if } j \geq i \\ 0 & \text{if } j < i \end{cases} \end{aligned}$$

Properties of Transition Probability Functions

- ▶ $P_{ij}(t) \geq 0$ for all $i, j \in \mathcal{X}$ and $t \geq 0$
- ▶ (Row sums are 1) $\sum_j P_{ij}(t) = 1$ for all $i \in \mathcal{X}$ and $t \geq 0$

Lemma 6.3 Chapman-Kolmogorov Equation

For all $i, j \in \mathcal{X}$ and $t \geq 0$,

$$P_{ij}(t + s) = \sum_{k \in \mathcal{X}} P_{ik}(t)P_{kj}(s)$$

Proof.

$$\begin{aligned} &P_{ij}(t + s) \\ &= \mathbb{P}(X(t + s) = j | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} \mathbb{P}(X(t + s) = j, X(t) = k | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} \mathbb{P}(X(t + s) = j | X(t) = k, X(0) = i) \mathbb{P}(X(t) = k | X(0) = i) \\ &= \sum_{k \in \mathcal{X}} \mathbb{P}(X(t + s) = j | X(t) = k) \mathbb{P}(X(t) = k | X(0) = i) \text{ (Markov Property)} \\ &= \sum_{k \in \mathcal{X}} P_{kj}(s)P_{ik}(t) \end{aligned}$$

The matrix notation

$$P(t + s) = P(t)P(s)$$

Transition rate matrix

Assume that

$$P'(0) = \lim_{h \rightarrow 0} \frac{P(h) - P(0)}{h} \quad \text{exists.}$$

We will denote such limit as Q , the transition rate matrix.

Kolmogorov's equations

How do one calculate $P'(t)$?

Forward equations:

$$\begin{aligned}P'(t) &= \lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h} = \lim_{h \rightarrow 0} \frac{P(t)P(h) - P(t)}{h} \\&= P(t)Q.\end{aligned}$$

Similarly you can prove backward equations

$$P'(t) = QP(t).$$

Transition rate matrix

How to compute Q ?

Lemma 6.2a

For any $i, j \in \mathcal{X}$, we have

$$-q_{ii} = \lim_{h \rightarrow 0} \frac{1 - P_{ii}(h)}{h} = \nu_i$$

Proof. Let T_i be the amount of time the process stays in state i before moving to other states.

$$\begin{aligned} P_{ii}(h) &= \mathbb{P}(X(h) = i | X(0) = i) \\ &= \mathbb{P}(X(h) = i, \text{ no transition in } (0, h] | X(0) = i) \\ &\quad + \mathbb{P}(X(h) = i, 2 \text{ or more transition in } (0, h] | X(0) = i) \\ &= \mathbb{P}(T_i > h) + o(h) \\ &= e^{-\nu_i h} + o(h) \\ &= 1 - \nu_i h + o(h) \end{aligned}$$

Lemma 6.2b

For any $i \neq j \in \mathcal{X}$, we have

$$q_{ij} = \lim_{h \rightarrow 0} \frac{P_{ij}(h)}{h} = \nu_i P_{ij}$$

Proof.

$$\begin{aligned} P_{ij}(h) &= \mathbb{P}(X(h) = j | X(0) = i) \\ &= \mathbb{P}(X(h) = j, 1 \text{ transition in } (0, h] | X(0) = i) \\ &\quad + \mathbb{P}(X(h) = j, 2 \text{ or more transition in } (0, h] | X(0) = i) \\ &= \mathbb{P}(T_i < h) P_{ij} + o(h) \\ &= (1 - e^{-\nu_i h}) P_{ij} + o(h) \\ &= \nu_i P_{ij} h + o(h) \end{aligned}$$

For finite state space case $\mathcal{X} = \{1, 2, \dots, m\}$, define the matrices

$$\mathbf{P}(t) = \begin{bmatrix} P_{11}(t) & \cdots & P_{1m}(t) \\ \vdots & & \vdots \\ P_{m1}(t) & \cdots & P_{mm}(t) \end{bmatrix}, \quad \mathbf{P}'(t) = \begin{bmatrix} P'_{11}(t) & \cdots & P'_{1m}(t) \\ \vdots & & \vdots \\ P'_{m1}(t) & \cdots & P'_{mm}(t) \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & & \vdots \\ q_{m1} & \cdots & q_{mm} \end{bmatrix} = \begin{bmatrix} -\nu_1 & \nu_1 P_{12} & \cdots & \nu_1 P_{1m} \\ \nu_2 P_{21} & -\nu_2 & \cdots & \nu_2 P_{2m} \\ \vdots & \vdots & & \vdots \\ \nu_m P_{m1} & \nu_m P_{m2} & \cdots & -\nu_m \end{bmatrix}$$

In matrix notation,

Forward equation: $\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$

Backward equation: $\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$