STAT253/317 Winter 2014 Lecture 26

Yibi Huang

Mar 12, 2014

- Quadratic Variation
-

Total Variation (First-Order Variation)

For a function $f(t)$, we wish to compute the amount of up and down oscillation undergone by this function between 0 and T. Let $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a *partition* of $[0, T]$, which is a set of times

$$0 = t_0 < t_1 < t_2 < \ldots t_n = T.$$

The mesh size of the partition is defined as

$$\|\Pi\| = \max_{0 \leq j \leq n-1} |t_{j+1} - t_j|.$$

The **total variation** of a function $f(t)$ on the interval $[0, T]$ is defined as

$$TV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|.$$

Total Variation (First-Order Variation) (Cont'd)

Remark 1. If the function $f(x)$ is monotone on $[0, T]$, then the total of f on the on the interval $[0, T]$ is simply $|f(0) - f(T)|$.

Remark 2. If the function $f(x)$ is monotone increasing on $[0, c]$ and monotone decreasing on $[c, T]$, then the total of f on the on the interval $[0, T]$ is $|f(0) - f(c)| + |f(c) - f(T)|$.

$\boxed{\text{The total variation of Brownian motion in } [0, T] \text{ is } \infty \text{ for all } T > 0.}$

The proof will be given later

Quadratic Variation (Second-Order Variation)

For a function $f(t)$ defined on the interval $[0, T]$, the **quadratic variation** of $f(t)$ in $[0, T]$ is defined as

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2.$$

For a smooth function f on $[0, T]$ with continuous derivative f', by mean-value theorem, there exists some t_j^* between t_j and t_{j+1} such that

$$|f(t_{j+1}) - f(t_j)| = |f'(t_j^*)|(t_{j+1} - t_j).$$

The quadratic variation is then

$$\begin{aligned} &\sum_{j=0}^{n-1} [f'(t_j)^*]^2 (t_{j+1} - t_j)^2 \\ \leq\ &\|\Pi\| \sum_{j=0}^{n-1} [f'(t_j)^*]^2 (t_{j+1} - t_j) \to \|\Pi\| \int_0^T |f'(t)|^2 dt. \end{aligned}$$

If $f'(t)$ is continuous, then $\int_0^T |f'(t)|^2 dt < \infty$. As the mesh size $\|\Pi\| \to 0$, the quadratic variation of f must be 0.

A Useful Result

Several proofs in this lecture use the following results.

Proposition. If $X_1, X_2, \ldots, X_n \ldots$ is a sequence of random variable with

$$\lim_{n\to\infty} \mathbb{E}[X_n] = c \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Var}(X_n) = 0$$

then $X_n \to c$ in probability.

Quadratic Variation of Standard Brownian Motion

The quadratic variation of standard Brownian motion on the interval $[0, T]$ is T. Here T is a fixed constant.

Proof. For any partition $\Pi = \{t_0, t_1, \ldots, t_n\}$, since $B(t_{j+1}) - B(t_j) \sim N(0, t_{j+1} - t_j)$ for $j = 0, 1 \ldots, n-1$, and is independent of each other, we have

$$\begin{aligned}
\mathbb{E}\Big[\sum_{j=0}^{n-1}[B(t_{j+1}) - B(t_j)]^2\Big] &= \sum_{j=0}^{n-1}(t_{j+1} - t_j) = T \\
\mathrm{Var}\Big(\sum_{j=0}^{n-1}[B(t_{j+1}) - B(t_j)]^2\Big) &= \sum_{j=0}^{n-1} 3(t_{j+1} - t_j)^2 \\
&\leq 3T\|\Pi\| \longrightarrow 0 \quad \text{as } \|\Pi\| \to 0
\end{aligned}$$

Thus

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1}[B(t_{j+1}) - B(t_j)]^2 = T.$$

Proof of that Brownian Motion Has Infinite Total Variation

Suppose to the contrary that Brownian motion has finite total variation,

$$\sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j)]^2$$
$$\leq \max_{0 \leq j \leq n-1} |B(t_{j+1}) - B(t_j)| \underbrace{\sum_{j=0}^{n-1} |B(t_{j+1}) - B(t_j)|}_{\to \text{ total variation}}$$

Since the Brownian motion path is continuous with probability 1 on $[0, T]$, it is necessarily uniformly continuous on $[0, T]$. Therefore as the mesh size $\|\Pi\| \to 0$,

$$\max_{0 \leq j \leq n-1} |B(t_{j+1}) - B(t_j)| \to 0 \text{ with prob. } 1.$$

from which we conclude that $\sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j)]^2 \to 0$ with probability 1. This is a contradiction to the result on the previous slide.

Review of Riemann–Stieltjes Integral

The Riemann–Stieltjes integral of a real-valued function f of a real variable with respect to a real function g is defined as the limit of the approximating sum

$$\int_a^b f(t)dg(t) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} f(t_j^*)[g(t_{j+1}) - g(t_j)]$$

where t_j^* is in the jth subinterval $[t_{j+1}, t_j]$ and $\|\Pi\|$ is the mesh size $\max_{0 \leq j \leq n-1} |t_{j+1} - t_j|$ of the partition

$$\Pi = \{a = t_0 < t_1 < \ldots < t_n = b\}.$$

$\int_0^T B(t)dB(t) = ?$

In Riemann–Stieltjes integral, the limit does not depend on the selection of t_j^* in the subinterval $[t_{j+1}, t_j]$.

However, if $g(t)$ is not sufficiently smooth, the limit may depend on the selection of t_j^*. For example, if $f(t) = g(t) =$ the standard Brownian Motion $B(t)$, we will show that

$$\lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} B(t_j^*)[B(t_{j+1}) - B(t_j)]$$

$$= \begin{cases} \frac{1}{2}B(T)^2 - \frac{1}{2}T & \text{if } t_j^* = t_j & \text{(Ito integral)} \\ \frac{1}{2}B(T)^2 & \text{if } t_j^* = \frac{t_{j+1}+t_j}{2} & \text{(Stratonovich integral)} \\ \frac{1}{2}B(T)^2 + \frac{1}{2}T & \text{if } t_j^* = t_{j+1}, \end{cases} \quad (1)$$

Which definition should we choose?

Proof of Equation (1)

Observe that

$$\sum_{j=0}^{n-1} B(t_j^*)[B(t_{j+1}) - B(t_j)] = I + II$$

where

$$\begin{aligned}
I &= \sum_{j=0}^{n-1} \frac{1}{2}[B(t_{j+1}) + B(t_j)][B(t_{j+1}) - B(t_j)] \\
&= \sum_{j=0}^{n-1} \frac{1}{2}[B(t_{j+1})^2 - B(t_j)^2] \\
&= \frac{1}{2}[B(t_n)^2 - B(t_0)^2] = \frac{1}{2}B(T)^2 \\
II &= \sum_{j=0}^{n-1} \{B(t_j^*) - \frac{1}{2}[B(t_{j+1}) + B(t_j)]\}[B(t_{j+1}) - B(t_j)] \\
&= \frac{1}{2}\sum_{j=0}^{n-1} \{B(t_j^*) - B(t_j) - [B(t_{j+1}) - B(t_j^*)]\} \\
&\qquad\qquad \times [B(t_{j+1}) - B(t_j^*) + B(t_j^*) - B(t_j)] \\
&= \frac{1}{2}\sum_{j=0}^{n-1} \{[B(t_j^*) - B(t_j)]^2 - [B(t_{j+1}) - B(t_j^*)]^2\}
\end{aligned}$$

Proof of Equation (1) (Cont'd)

For $t_j^* = t_j$, observe that

$$
\begin{aligned}
II &= \frac{1}{2}\Big\{ \sum_{j=0}^{n-1} [B(t_j^*) - B(t_j)]^2 - \sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j^*)]^2 \Big\} \\
&= \frac{1}{2}\Big\{ \sum_{j=0}^{n-1} \underbrace{[B(t_j) - B(t_j)]^2}_{=0} - \underbrace{\sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j)]^2}_{\text{quadratic variation}} \Big\}
\end{aligned}
$$

$$
\to \frac{1}{2}(0 - T) = -\frac{T}{2} \quad \text{in probability as } \max_{0 \le j \le n-1} |t_{j+1} - t_j| \to 0
$$

Similarly, for $t_j^* = t_{j+1}$, observe that

$$
II = -\frac{1}{2}\Big\{ \underbrace{\sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j)]^2}_{\text{quadratic variation}} + \sum_{j=0}^{n-1} \underbrace{[B(t_{j+1}) - B(t_{j+1})]^2}_{=0} \Big\}
$$

$$
\to \frac{1}{2}(T - 0) = \frac{T}{2} \quad \text{in probability as } \max_{0 \le j \le n-1} |t_{j+1} - t_j| \to 0
$$

Proof of Equation (1) (Cont'd)

For $t_j^* = (t_{j+1} + t_j)/2$,

$$II = \frac{1}{2}\Big\{ \sum_{j=0}^{n-1} [B(\frac{t_{j+1}+t_j}{2}) - B(t_j)]^2 - \sum_{j=0}^{n-1} [B(t_{j+1}) - B(\frac{t_{j+1}+t_j}{2})]^2 \Big\}$$

So

$$\begin{aligned}
\mathbb{E}[II] &= \frac{1}{2} \sum\nolimits_{j=0}^{n-1} \frac{(t_{j+1}-t_j)}{2} - \frac{(t_{j+1}-t_j)}{2} = 0 \\
\mathrm{Var}(II) &= \frac{1}{4} \sum\nolimits_{j=0}^{n-1} 3(\frac{t_{j+1}-t_j}{2})^2 + 3(\frac{t_{j+1}-t_j}{2})^2 \\
&= \frac{3}{8} \sum\nolimits_{j=0}^{n-1} (t_{j+1}-t_j)^2 \leq \frac{3}{8} \|\Pi\| T \to 0 \text{ as } \|\Pi\| \to 0
\end{aligned}$$

The above shows $II \longrightarrow 0$ in probability as $\|\Pi\| \to 0$.