Bài 1 : Tìm đạo hàm của các hàm số sau :

a. $y = \frac{x-1}{5x-2}$

b. $y = \frac{2x+3}{7-3x}$

c. $y = \frac{x^2+2x+3}{3-4x}$

d. $y = \frac{x^2+7x+3}{x^2-3x}$

Lời giải:

a. $y' = \left(\frac{x-1}{5x-2}\right)' = \frac{(x-1)'(5x-2)-(x-1)(5x-2)'}{(5x-2)^2} = \frac{3}{(5x-2)^2}$

b. $y' = \left(\frac{2x+3}{7-3x}\right)' = \frac{(2x+3)'(7-3x)-(2x+3)(7-3x)'}{(7-3x)^2} = \frac{23}{(7-3x)^2}$

c. $y' = \left(\frac{x^2+2x+3}{3-4x}\right)' = \frac{(x^2+2x+3)'(3-4x)-(x^2+2x+3)(3-4x)'}{(3-4x)^2}$

$$= \frac{-4x^2+6x+18}{(3-4x)^2} = \frac{-2(2x^2-3x-9)}{(3-4x)^2}$$

d. $y' = \left(\frac{x^2+7x+3}{x^2-3x}\right) = \frac{(x^2+7x+3)'(x^2-3x)-(x^2+7x+3)(x^2-3x)'}{(x^2-3x)^2}$

$$= \frac{(2x+7)(x^2-3x)-(x^2+7x+3)(2x-3)}{(x^2-3x)^2} = \frac{-10x^2-6x+9}{x^2(x-3)^2}$$

Bài 2 : Giải các bất phương trình sau :

a. $y' < 0$ với $y = \frac{x^2+x+2}{x-1}$

b. $y' \geq 0$ với $y = \frac{x^2+3}{x+1}$

c. $y' > 0$ với $y = \frac{2x-1}{x^2+x+4}$

Lời giải:

a. Ta có : $y' = \frac{(2x+1)(x-1)-(x^2+x+2)}{(x-1)^2} = \frac{x^2-2x-3}{(x-1)^2}$

$y' < 0 \Leftrightarrow \begin{cases} x \neq 1 \\ x^2-2x-3<0 \end{cases} \Leftrightarrow x \in (-1;1) \cup (1;3)$

b. Ta có : $y' = \left(\frac{x^2+3}{x+1}\right)' = \frac{x^2+2x-3}{(x+1)^2}$

$y' \geq 0 \Leftrightarrow \begin{cases} x \neq -1 \\ x^2+2x-3 \geq 0 \end{cases} \Leftrightarrow x \in (-\infty;-3] \cup [1;+\infty)$

c. Ta có : $y' = \left(\frac{2x-1}{x^2+x+4}\right)' = \frac{-2x^2+2x+9}{(x^2+x+4)^2}$

$$y' > 0 \Leftrightarrow \frac{-2x^2+2x+9}{(x^2+x+4)^2} > 0 \Leftrightarrow 2x^2-2x-9<0$$

$$\Leftrightarrow x \in \left(\frac{1-\sqrt{19}}{2};\frac{1+\sqrt{19}}{2}\right)$$

Bài 3 : Tìm đạo hàm của các hàm số sau :

a. $y = 5\sin x - 3\cos x$

b. $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

c. $y = x.\cot x$

d. $y = \frac{\sin x}{x} + \frac{x}{\sin x}$

e. $y = \sqrt{1+2\tan x}$

f. $y = \sin\sqrt{1+x^2}$

Lời giải:

a. y’ = 5(sinx)’ – 3(cosx)’ = 5cosx + 3sinx

b. $$y' = \frac{(\sin x + \cos x)'(\sin x - \cos x) - (\sin x - \cos x)'.(\sin x + \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2}$$

c. y’ = (x.cotx)’ = x’.cotx + (cotx)’.x

$$= \cot x - \frac{1}{\sin^2 x}.x = \cot x - \frac{x}{\sin^2 x}$$

d. $$y' = \left(\frac{\sin x}{x}\right)' + \left(\frac{x}{\sin x}\right)'$$

Ta có : $$\left(\frac{\sin x}{x}\right)' = \frac{\cos x.x - \sin x}{x^2}$$

$$\left(\frac{x}{\sin x}\right)' = \frac{\sin x - x.\cos x}{(\sin x)^2}$$

$$y' = \frac{\cos x.x - \sin x}{x^2} + \frac{\sin x - x.\cos x}{\sin^2 x} = (x.\cos x - \sin x)\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$$

e. $$y' = (\sqrt{1 + 2\tan x})' = \frac{(1 + 2\tan x)'}{2\sqrt{1 + 2\tan x}} = \frac{1}{\cos^2 x\sqrt{1 + 2\tan x}}$$

f. $$y' = \left(\sin\sqrt{1 + x^2}\right) = (\sqrt{1 + x^2})'\cos\sqrt{1 + x^2} \quad (\sin U' = U'.\cos U)$$

$$= \frac{x.\cos\sqrt{1 + x^2}}{\sqrt{1 + x^2}}$$

Bài 4 : Tìm đạo hàm của các hàm số sau:

a. $y=(9-2x)(2x^3-9x^2+1)$

b. $y=\left(6\sqrt{x}-\dfrac{1}{x^2}\right)(7x-3)$

c. $y=(x-2)\sqrt{x^2+1}$

d. $y=\tan^2 x-\cot^2 x$

e. $y=\cos\dfrac{x}{1+x}$

Lời giải:

a. $y'=[(9-2x)(2x^3-9x^2+1)]'$

$=(9-2x)'(2x^3-9x^2+1)+(9-2x)(2x^3-9x^2+1)'$

$=-2(2x^3-9x^2+1)+(9-2x)(6x^2-18x)$

$=-16x^3+108x^2-162x-2$

b. Đặt $U=6\sqrt{x}-\dfrac{1}{x^2}\Rightarrow U'=\dfrac{3}{\sqrt{x}}+\dfrac{2}{x^3}$

$V=7x-3\Rightarrow V'=7$

$y'=(U.V)'=U'V+UV'$

$$=\left(\frac{3}{\sqrt{x}}+\frac{2}{x^3}\right)(7x-3)+7\left(6\sqrt{x}-\frac{1}{x^2}\right)=63\sqrt{x}-\frac{9}{\sqrt{x}}+\frac{7}{x^2}-\frac{6}{x^3}$$

c. $y'=(x-2)'\sqrt{x^2+1}+(x-2)\left(\sqrt{x^2+1}\right)'$

$$=\sqrt{x^2+1}+\frac{x(x-2)}{\sqrt{x^2+1}}=\frac{2x^2-2x+1}{\sqrt{x^2+1}}$$

d. $y'=(\tan^2 x)'-(\cot^2 x)'=\dfrac{2\tan x}{\cos^2 x}+\dfrac{2x}{\sin^2 x^2}$

e. $y'=\left(\dfrac{x}{1+x}\right)\sin\dfrac{x}{1+x}=-\dfrac{1}{(1+x)^2}\sin\dfrac{x}{1+x}$

Bài 5 : Tính ...

Tính $\dfrac{f'(1)}{\varphi'(1)}$,

biết rằng $f(x) = x^2$ và $\varphi(x) = 4x + \sin\dfrac{\pi x}{2}$

Lời giải:

Ta có: $f'(x) = 2x \quad \Rightarrow f'(1) = 2$

$\varphi'(x) = 4 + \dfrac{\pi}{2}\cos\dfrac{\pi x}{2}$

$\Rightarrow \varphi'(1) = 4 + \dfrac{\pi}{2}\cos\dfrac{\pi}{2}$

Vậy $\dfrac{f'(1)}{\varphi'(1)} = \dfrac{2}{4} = \dfrac{1}{2}$

Bài 6 : Chứng minh rằng các hàm số sau có đạo hàm không phụ thuộc x:

a. $y = \sin^6 x + \cos^6 x + 3\sin^2 x.\cos^2 x$;

b. $y = \cos^2\left(\dfrac{\pi}{3} - x\right) + \cos^2\left(\dfrac{\pi}{3} + x\right) + \cos^2\left(\dfrac{2\pi}{3} - x\right) + \cos^2\left(\dfrac{2\pi}{3} + x\right) - 2\sin^2 x$

Lời giải:

a. $y = \sin^6 x + \cos^6 x - 3\sin^2 x.\cos^2 x$

Ta có: $\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$= (\sin^2 x + \cos^2 x)[(\sin^2 x)^2 - \sin^2 x.\cos^2 x + (\cos^2 x)^2]$

$= (\sin^2 x)^2 + (\cos^2 x)^2 - \sin^2 x.\cos^2 x$

$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x.\cos^2 x - \sin^2 x\cos^2 x$

$= - 1 - 3\sin^2 x\cos^2 x$

$\Rightarrow y = 1 - 3\sin^2 x\cos^2 x + 3\sin^2 x\cos^2 x = 1$

Suy ra y' = 0 (đpcm)

$$b.y' = 2\cos\left(\frac{\pi}{3} - x\right)\sin\left(\frac{\pi}{3} - x\right) - 2\cos\left(\frac{\pi}{3} + x\right)\sin\left(\frac{\pi}{3} + x\right)$$

$$+ 2\cos\left(\frac{2\pi}{3} - x\right)\sin\left(\frac{2\pi}{3} - x\right)$$

$$- 2\cos\left(\frac{2\pi}{3} + x\right)\sin\left(\frac{2\pi}{3} + x\right) - 4\sin x\cos x$$

$$= \sin\left(\frac{2\pi}{3} - 2x\right) - \sin\left(\frac{2\pi}{3} + 2x\right) +$$

$$\sin\left(\frac{4\pi}{3} - 2x\right) - \sin\left(\frac{4\pi}{3} + 2x\right) - 2\sin 2x$$

$$= -2\cos\frac{2\pi}{3}\sin 2x - 2\cos\frac{4\pi}{3}\sin 2x - 2\sin 2x$$

$$= \sin 2x(1 + 1 - 2) = 0 \text{ (đpcm)}$$

Bài 7 : Giải phương trình f'(x) = 0, biết rằng:

$$a.f(x) = 3\cos x + 4\sin x + 5x$$

$$b.f(x) = 1 - \sin(\pi + x) + 2\cos\left(\frac{2\pi + x}{2}\right)$$

Lời giải:

a. $f(x) = 3\cos x + 4\sin x + 5x$

$f'(x) = -3\sin x + 4\cos x + 5$

$f'(x) = 0 \Leftrightarrow -3\sin x + 4\cos x + 5 = 0$

$\Leftrightarrow 3\sin x + 4\cos x = -5 \qquad (1)$

Chia 2 vế của (1) cho 5 ta được:

$$(1) \Leftrightarrow -\frac{3}{5}\sin x + \frac{4}{5}\cos x = -1$$

$$\Leftrightarrow \frac{3}{5}\sin x - \frac{4}{5}\cos x = 1 \qquad (2)$$

Đặt: $\cos\alpha = \dfrac{3}{5}$ và $\sin\alpha = \dfrac{4}{5}$, khi đó:

$\Leftrightarrow \cos\alpha.\sin x - \sin\alpha.\cos x = 1$

(2)

$$\Leftrightarrow \sin(x - \alpha) = 1 \quad \Leftrightarrow x - \alpha = \frac{\pi}{2} + k2\pi$$

$$\Leftrightarrow x = \alpha + \frac{\pi}{2} + k2\pi \ (k \in Z)$$

b. $f(x) = 1 - \sin(\pi + x)^2 + 2\cos\left(\dfrac{2\pi + x}{2}\right)$

$f'(x) = -\cos(\pi + x) - \sin\left(\dfrac{2\pi + x}{2}\right)$

$$f'(x) = 0 \qquad \Leftrightarrow -\cos(\pi + x) - \sin\left(\frac{2\pi + x}{2}\right) = 0$$

$$\Leftrightarrow \cos x - \sin\left(\pi + \frac{x}{2}\right) = 0$$

$$\Leftrightarrow \cos x + \sin\frac{x}{2} = 0 \qquad (1)$$

Đặt $U = \dfrac{x}{2} \Rightarrow x = 2U$

Khí đó: $(1) \Leftrightarrow \cos 2U + \sin U = 0$

$\Leftrightarrow 1 - 2\sin^2 U + \sin U = 0$

$$\Leftrightarrow \left[\begin{array}{l} \sin U = 1 \\ \sin U = -\dfrac{1}{2} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} U = \dfrac{\pi}{2} + k2\pi \\ \sin U = \sin\left(\dfrac{\pi}{6}\right) \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} U = \dfrac{\pi}{2} + k2\pi \\ U = -\dfrac{\pi}{6} + k2\pi \\ U = \pi + \dfrac{\pi}{6} - k2\pi \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \dfrac{x}{2} = \dfrac{\pi}{2} + k2\pi \\ \dfrac{x}{2} = -\dfrac{\pi}{6} + k2\pi \\ \dfrac{x}{2} = \pi + \dfrac{\pi}{6} - k2\pi \end{array}\right.$$

$$\Leftrightarrow \left[\begin{array}{l} x = \pi + k4\pi \\ x = -\dfrac{\pi}{3} + k4\pi \\ x = \dfrac{7\pi}{3} - k2\pi \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \pi + k4\pi \\ x = -\dfrac{\pi}{3} + k4\pi \end{array}\right. (k \in Z)$$

Bài 8 : Giải bất phương trình f'(x) > g'(x), biết rằng :

a. $f(x) = x^3 + x - \sqrt{2}$, $g(x) = 3x^2 + x + \sqrt{2}$

b. $f(x) = 2x^3 - x^2 + \sqrt{3}$, $g(x) = x^3 + \dfrac{x^2}{2} - \sqrt{3}$

Lời giải:

a. Ta có: $f'(x) = 3x^2 + 1$

$g'(x) = 6x + 1.$

$f'(x) > g'(x)$

$\Leftrightarrow\ 3x^2 + 1 > 6x + 1$

$\Leftrightarrow\ 3x^2 - 6x > 0 \quad \Rightarrow\ x \in (-\infty;0) \cup (2;+\infty)$

b. Ta có: $f'(x) = 6x^2 - 2x$

$g'(x) = 3x^2 + x.$

$f'(x) > g'(x)$

$\Leftrightarrow\ 6x^2 - 2x > 3x^2 + x \quad \Leftrightarrow\ 3x^2 - 3x > 0$

$\Leftrightarrow\ x \in (-\infty;0) \cup (1;+\infty).$