proofs of matroid exercises and problems in András Frank's book *Connections in Combinatorial Optimization*

Contents

1 ch5 - Elements of matroid theory

1.1 independent set

Proof (Exercise 5.1.1): Assume I1 and I2 hold, prove I3' is equivalent to I3.

1. I3 $\rightarrow$ I3'. Suppose I3' is not true. There must be some independent set $U \supset K$ by I2. Take X in I3 to be the union of K and N, then there are no element in X can be added to K while remaining independence. Thus the set U does not exist, a contradiction.

2. I3' $\rightarrow$ I3. Take any two subset A, B of X and $A, B \in \mathcal{F}$. Suppose $|A| < |B|$, apply I3' then A and B will be of the same size.

$\square$

Proof (Exercise 5.1.2): similar to the previous one. Assume I1 and I2 hold,

1. I3 $\rightarrow$ I3". I3" is a weaker version of I3'. so I3 $\rightarrow$ I3' $\rightarrow$ I3".

2. I3" $\rightarrow$ I3. Take any two independent subset A, B of X. Suppose $|A| < |B|$. One can always find a independent subset D of B s.t. $A \cap B \subseteq D$ and $|D| = |A| + 1$ as I2 holds. Applying I3" adds one element to A. Since this works for any A, B, finally every independent subset of X will be of the same size.

$\square$

Proof (Problem 5.1.3): I3"" is a weaker version of I3", with additional constraint $|K \setminus N| = 1$.

1. I3 $\rightarrow$ I3" $\rightarrow$ I3"".

2. I3"" $\rightarrow$ I3. similar to previous proofs.

$\square$

Proof (every affine matroid can be represented as a linear matroid, and vice versa): Consider a vector space over any field,

1. affine $\rightarrow$ linear. For every element $\mathbf{x} = (x_1, ..., x_n)$ in S, add one dimension, $\mathbf{x}' = (x_1, ..., x_n, 1)$. Verification is simple.

2. linear $\rightarrow$ affine. Suppose the ground set of the linear matroid is $X = \{\mathbf{x_1}, ..., \mathbf{x_n}\}$. The ground set of the affine matroid is $Y = \{\mathbf{x_1} + e, \ldots, \mathbf{x_n} + e, e\}$ where $e \notin X$. Note that $\forall X' = \{\mathbf{x_i}, ..., \mathbf{x_j}\} \subseteq X$, X' is linearly independent, if and only if the corresponding subset $Y' = \{\mathbf{x_i} + e, ..., \mathbf{x_j} + e, e\} \subseteq Y$ is affinely independent.

$\square$

Proof (Exercise 5.1.4 circuit matroid is linear over any field): We prove that a set of edges contains a cycle if and only if the corresponding columns in Q are linearly dependent over any field. If the set of edges contains a cycle, let C be the set of edges in the cycle. One can easily see that, regardless of the edge orentation of edges, the set of columns in C either add up to zero or they can be divided into 2 parts and their sums are the same vector. Both case leads to linear dependence over any field. On the other side, if the set edges form a forest, we claim that for any subset of edges there will be at least one row of the corresponding columns contains only one non-zero value. This is because any subgraph of a forest contains at least one degree one vertex. Thus the corresponding row will contain only one +1 or -1. Thus the columns are linear independent.

$\square$

Note that non-isomorphic graphs may have isomorphic circuit matroids. A deep theorem of Whitney states that the circuit matroids of two non-isomorphic 3-connected graphs are not isomorphic.

1.2 circuits

Proof (Theorem 5.2.1 Weak circuit axiom): Suppose there exists two distinct circuits C_1 and C_2 violating the statement. Then $C_1 \cup C_2 - e$ is independent. We know that $C_1 \cap C_2$ is independent. Consider the maximal independent set I of $C_1 \cup C_2$ containing $C_1 \cap C_2$. Note that I can not have more than $|C_1 \cup C_2| - 2$ elements since independent set can not contain a circuit. Thus $C_1 \cup C_2 - e$ is a larger independent set than the maximal independent set of $C_1 \cup C_2$, a contradiction. $\square$

Theorem 1.1 (Theorem 5.2.3 Strong circuit axiom) *Let C_1 and C_2 be two distinct circuits, $e \in C_1 \cap C_2$, $e_1 \in C_1 - C_2$. Then there is a circuit C for which $e_1 \in C \subseteq C_1 \cup C_2 - e$*

Proof: (This is not easy... Finding the counter example with minimum union size is important here. The following is from the book.)

Suppose the statement is no true. Find two distinct circuits C_1 and C_2 violating the statement with minimum $|C_1 \cup C_2|$. Weak circuit axiom shows that $C_1 \cup C_2 - e$ is not independent. Then there does not exists a circuit $C \subseteq C_1 \cup C_2 - e$ containing e_1. Then there exists $C_3 \subseteq C_1 \cup C_2 - e$ which does not contain e_1. Now we consider C_3 and C_2. C_3 and C_2 follow the statement since $C_1 \cup C_2$ is the counter example with minimum size and $|C_3 \cup C_2| < |C_1 \cup C_2|$. Thus there exists $C_4 \subseteq C_3 \cup C_2 - f$ containing $e \in C_2$ for some $f \in C_3 \cap C_2$. Now we consider C_4 and C_1. $C_4 \cup C_1 \subseteq C_1 \cup C_2$ since C_4 is a proper subset of $C_1 \cup C_2$. Hence for C_1 and C_4 we can apply strong circuit axiom and there should be a circuit $C \subseteq C_1 \cup C_4 - e \subseteq C_1 \cup C_2 - 2$ containing e_1, a contradiction. $\square$

There is a even stronger property of circuits.

Lemma 1.2 (Theorem 3 in [1]) *Let $C_1, \ldots, C_n$ be distinct circuits with $C_i \nsubseteq \bigcup_{k<i} C_k, i \in [n]$, If $D \subseteq E$ with $|D| = r < n$, then there exists $n - r$ circuits $C'_1, \ldots, C'_{n-r}$ such that $C'_i \subseteq \bigcup_k C_k \setminus D$ and $C'_i \nsubseteq \bigcup_{j \neq i} C'_j$*

For $n = 2$ and $|D| = 1$ this is weak circuit axiom.

Proof: By induction on n and r.

Case $n, r = 0$. We need to prove that if $C_i \not\subseteq \bigcup_{j<i} C_j$, we can find n circuits $C'_1, \ldots, C'_n$ s.t. $C'_i \not\subseteq \bigcup_{j \neq i} C'_j$. That is each C'_i contains an unique element given that C_i has an unique element in the prefix $C_1, \ldots, C_{i-1}$. We can construct C'_i inductively. Let u_i be the unique element in C'_i and set u_1 to be any element in C_1. Let $C'_1 = C_1$. For all $j \in (1, n]$, if $u_1 \notin C_j$ let $C'_j = C_j$, otherwise let C'_j be the new circuit in $C_j \cup C_1 - u_1$ by the weak circuit property. Note that no circuit in $\{C_2, \ldots, C_n\}$ contains u_1. Thus in each iteration we can fix one C'_i and one unique element in C'_i.

Case $n, r > 0$. WLOG we can assume that $C_i \not\subseteq \bigcup_{j \neq i} C_j$ for $i \in [n]$ by Case $n, r = 0$ and any element in D is contained in some C_i. There are 2 cases,

- $\exists e \in D$ s.t. $e \notin \bigcup C_i$. Then it is safe to delete any C_i and e and reduce to $n-1, r-1$ case.
- Otherwise suppose $\exists e \in D$ s.t. $e \in C_n$. Then we can apply the strong circuit property for $C_1, \ldots, C_{n-1}$ and C_n to get $C'_1, \ldots, C'_{n-1}$ such that $e \notin C'_i$ but C'_i contains the unique element u_i in C_i. Thus we reduce the problem to $n-1, r-1$ case by deleting C_n and e.

Thus every $r > 0$ case can be reduce to $n - r, 0$.

□

Proof (Theorem 5.2.4 circuit → independent set): Let $\mathcal{C}$ be the set of circuits. $\mathcal{I} = \{I \subseteq E | \nexists C \in \mathcal{C}, C \subseteq I\}$. We need to prove $\mathcal{I}$ follows I1, I2 and I3'(independent set exchange property). I1 and I2 holds trivially. Suppose I3' does not hold on $\mathcal{I}$. We can find $I_1, I_2 \in \mathcal{I}$ such that $|I_1| > |I_2|$ and $\forall e \in I_1 \setminus I_2$, $I_2 + e$ contains a circuit. There are two cases,

1. $I_2 \subseteq I_1$. This is trivial.
2. $I_2 \not\subseteq I_1$, Then $|I_1 \setminus I_2| \geq 2$. Take two elements $e, f \in I_1 \setminus I_2$, $I_2 + e$ and $I_2 + f$ contains two unique circuits C_e and C_f.

   A few words on matroid proof techniques. Initially i want to show that any C_e always contains a unique element in $I_2 \setminus I_1$, other circuits can not contain it. Thus I can show the contradiction that $I_2 \setminus I_1$ is not smaller than $I_1 \setminus I_2$. However this is not true. Just consider uniform matroids. Every C_e contains $I_2 \setminus I_1$. So it is always useful to consider if your techniques work on special matroids. At least try uniform and circuit matroid first.

   There is a proof in [5]. If weaker results are usable, just prove the weaker one. Instead of proving I3' we prove that for two independent set I and J such that $|I \setminus J| = 1$ and $|J \setminus I| = 2$, $\exists e \in J \setminus I$, $I + e$ is independnet. Suppose we find I and J contains no circuits and volating this weak exchange property. Suppose $\{y\} = I \setminus J$. If $y + J$ contains no circuit, then $I \cup J$ contains no circuits. I and J are not violating weak exchange property. Consider then case $J + y$ contains a circuit C. $C \cap J \setminus I \neq \emptyset$ since otherwise I will contain a circuit. Let x be an element in $C \cap J \setminus I$. Note that C is the unique circuit in $J \cup I$, so $J \cup I - x$ should be independent. Then for the other element $z \in J \setminus I$, $I + z$ contains no circuit.

□

Similar methods can be used to prove edges in (k,ℓ)-sparse subgraphs form the independent set of matroid.

Proof ((k,ℓ)-sparsity matroid): We prove the set of edges of (k,ℓ)-sparse subgraphs in $G = (V,E)$ for all $k \geq 1, \ell \leq 2k-1$ satisfy axioms of independent sets of matroid. Again $\emptyset \in \mathcal{I}$ and the hereditary property hold trivially. For simplicity we will mix the symbols for graphs and their edge sets. We need to prove that for two (k,ℓ)-sparsesubgraph I and J on G such that $|I \setminus| = 1$ and $|J \setminus I| = 2$, $\exists e \in J \setminus I$, $I + e$ is (k,ℓ)-sparse. Suppose This weak exchange property does not hold. Let $\{y = (u,v)\} = I \setminus J$. Then $J + y$ can not be (k,ℓ)-sparsesince if it is then $I \cup J$ will be independent in the sparsity matroid on G. Since $J + y$ is not (k,ℓ)-sparse, we can find tight subgraphs $G_1 = (V_1,E_1), G_2 = (V_2,E_2), \ldots$ of J containing two endpoints of y. Let $V' = \cup V_i$. By theorem 5(1) in [4], V' induces a tight subgraph H in J. Note that $E[H] \nsubseteq I$. Now we need to show $I - x + z$ is (k,ℓ)-sparsefor $x \in E[H] \cap J, z \in J \setminus I - x$. H is the minimal subgraph of J s.t. adding y breaks the sparsity since any other subgraph would have been one of G_i and V' is contained in it. Thus $I \cup J - x$ is (k,ℓ)-sparse. I,J is not a counterexample. □

Proof (Exercise 5.2.1): Note that minimal cuts are actually cocircuits of circuit matroids. Bonds in undirected graphs are non-empty minimal edge cuts. C1 and C2 holds trivially since bonds are non-empty and minimal. Suppose C3 doesn't hold. Find two bonds B_1 and B_2 violating C3 in graph $G = (V,E)$. Then our assumption imples that for some $e \in B_1 \cap B_2$, $G - B_1 \cup B_2 + e$ has the same number of components as G. Let $\#\mathrm{C}(G)$ be the number of components in G. For any $e \in B_1 \cap B_2$, $\#\mathrm{C}(G - B_2 + e) = \#\mathrm{C}(G - B_1 + e) = \#\mathrm{C}(G)$ since B_1 and B_2 are minimal cuts. Suppose B_1 divides V into V_1 and V_2, B_2 divides V into V_3 and V_4. Note that V_i may not be connected. Assume WLOG e connects V_1 and V_4. We claim that V_2 and V_3 are disconnected in $G - B_1 - B_2 + e$. One can see that in $G - B_1 - B_2 + e$ the only edge between V_is is e. Thus V_2 and V_3 are disconnected. However in G V_2 and V_3 are connected since both B_1 and B_1 are bonds. Thus $\#\mathrm{C}(G - B_1 - B_2 + e) > \#\mathrm{C}(G)$. $B_1 \cup B_2 - e$ contains a non-empty cut. □

Proof (Problem 5.2.2): For $X \subseteq V$, let $E(X)$ be the set of edges induced by X. $E(X \cap Y) = E(X) \cap E(Y)$ and $E(X \cup Y) = E(X) \cup E(Y) \cup E(X,Y)$ where $E(X,Y)$ is the set of edges connecting $X \setminus Y$ and $Y \setminus X$. Then we have $r(E(X)) + r(E(Y)) \geq r(E(X) \cup E(Y)) + r(E(X) \cap E(Y)) \geq r(E(X \cup Y)) - r(E(X,Y)) + r(E(X \cap Y))$. Note that $|X| = r(E(X)) + c(X)$. We have

$$\begin{aligned}
c(X) + c(Y) &\leq |X| + |Y| - |X \cup Y| - |X \cap Y| + c(X \cup Y) + c(X \cap Y) + r(E(X,Y)) \\
&\leq C(X \cup Y) + C(X \cap Y) + d_G(X,Y)
\end{aligned}$$

□

Proof (Theorem 5.2.5): Let C_1 and C_2 be the counter example with minimal union. Suppose $x \in C_1, y \in C_2$ and there is no circuit $C \subseteq C_1 \cup C_2$ containing both x and y. By strong circuit axiom, there exists $C_1' \subseteq C_1 \cup C_2$ containing x but not $e \in C_1 \cap C_2$. Note that $C_1 \setminus C_1' = \emptyset$ since otherwise $C_1' \cup C_2 \subsetneq C_1 \cup C_2$ is not a counter example. Similarly we define C_2' to be the circuit containing y but not e. $C_2 \setminus C_2' = \emptyset$. Note that $C_1' \cup C_2'$ is a proper subset of $C_1 \cup C_2$ since $e \notin C_1' \cup C_2'$. Also $C_1' \cap C_2' \neq \emptyset$ since otherwise C_1' and C_2' will be subset of C_1 and C_2 and won't be circuits. Thus C_1' and C_2' should follow the theorem, C_1 and C_2 are not counter examples. □

1.3 separable,connected,direct sum...

Theorem 1.3 (theorem 5.2.7) *$\{S_1, S_2\}$ is a non-empty bipartition of the ground set S. $M = (S, \mathcal{F})$ is the direct sum of $M|S_1$ and $M|S_2$ if and only if $r(S_1) + r(S_2) = r(S)$, where r is the rank function of M.*

Proof: If M is indeed the direct sum of $M|S_1$ and $M|S_2$ then $r(S_1) + r(S_2) = r(S)$ by definition. If M is not a direct sum and $r(S_1) + r(S_2) = r(S)$. Then there must be a circuit intersecting both S_1 and S_2. Let this circuit be C. $r(C) = |C| - 1$ and $r(C \cap S_1) + r(C \cap S_2) = |C|$. $r(S_1) + r(C) \geq r(S_1 \cap C) + r(S_1 \cup C)$, $r(S_2) + r(C) \geq r(S_2 \cap C) + r(S_2 \cup C)$, add them up, $r(S_1) + r(S_2) + 2|C| - 2 \geq |C| + r(S_1 \cup C) + r(S_2 \cup C)$. Then we have $r(S_1) + r(S_2) \geq r(S_1 \cup C) + r(S_2 \cup C) - |C| + 2 \geq r(S) + |C| - 1 - |C| + 2 > r(S)$. □

connected ≡ non-separable ≡ $r(X) + r(S - X) > r(S)$, X is a proper non-empty subset of S.
interesting fact: circuit matroid is connected if and only if the graph is 2-connected.
⟪in the proof of theorem 5.2.9, why is CM connected if CB is connected? is it possible that CB is connected but CB' is not connected?⟫ ⟸

1.4 basis

Theorem 1.4 (basis ⟷ independent set) *$\mathcal{B}$ satisfies the basis axioms,*

1. *$\mathcal{B}$ is not empty*
2. *$B_1, B_2 \in \mathcal{B}$, $\forall e \in B_1 \setminus B_2$, $\exists f \in B_2 \setminus B_1$, s.t. $B_1 - e + f \in \mathcal{B}$*

$\mathcal{F} = \{I | \exists B \in \mathcal{B}, I \subseteq B\}$ satisfies the independent set axioms. The bases of matroid satisfy basis axioms.

Proof: I1 and I2 holds trivially. We prove the weak independent set exchange property, i.e. $I_1, I_2 \in \mathcal{F}$, $|I_1 \setminus I_2| = 1$ and $|I_2 \setminus I_1| = 1$, $\exists e \in I_2 \setminus I_2$ such that $I_1 + e \in \mathcal{F}$. First we show that every base has the same size. Suppose $B_1, B_2 \in \mathcal{B}$ such that $|B_1| \leq |B_2|$ and $|B_1 \setminus B_2|$ is minimum. By basis axiom 2 we can find $B_2' = B_2 - x + y$, where $x \in B_2 \setminus B_1$, $y \in B_1 \setminus B_2$. Consider B_1 and B_2', $|B_1| \leq |B_2| = |B_2'|$ and $|B_1 \setminus B_2| \geq |B_1 \setminus B_2'|$. Thus every base has the same size.

Suppose I_1 is contained in a base B_1 and I_2 is contained in B_2. If $B_1 = B_2$ then $I_1 \cup I_2 \in \mathcal{F}$. Let $\{y\} = B_1 \setminus B_2$. Thus we assume $B_1 \neq B_2$ and $I_2 \setminus I_1 \nsubseteq B_1$. Now we modify B_1 by repeatedly removing elements in $B_1 \setminus B_2$(except y) from it and adding $B_2 \setminus B_1$ to it. Finally B_2 contains $y, I_1 \cap I_2$ and one element in $I_2 \setminus I_1$ at the same time. Thus I_1 and I_2 satisfy the weak independent set exchange property.

Bases of matroid(maximal independent set) satisfy basis axioms trivially by the independent set exchange axiom. □

The followings are some basis-exchange properties(see introduction of [2] for references).

Proof (symmetric basis exchange property): We need to prove that for two bases B_1, B_2 and $x \in B_2$, $\exists y \in B_1 \cap C_x$ such that $x \in C_y$. Instead of trying to find such a y, we try to find C_y such that $C_y - B_2 \subseteq C_x - B_2$ and $x \in C_y$. Note that if such C_y exists, $C_y - B_2$ should contain only one element y. So first we further require $|C_y - B_2|$ to be the minimum and try to prove $|C_y - B_2| = 1$.

Suppose $|C_y - B_2| \neq 1$, for any element z in $C_y - B_2$ there is a unique circuit $C_z \subseteq B_2 + z$. C_z does not contain x since $|C_z - B_2| < |C_y - B_2|$. Then by strong circuit exchange property we can find a new circuit $C_z' \subseteq C_z \cup C_x - z$ containing x. Thus $|C_y - B_2| = 1$ and $C_y \subseteq B_2 + y$ is the unique circuit of y. Since C_x and C_y are both unique, C_x and C_y are the same circuit. □

Theorem 1.5 (multiple symmetric exchange property) *$B_1, B_2 \in \mathcal{B}$, $X \subseteq B_1 - B_2$, then there exists $Y \subseteq B_2 - B_1$ such that both $B_1 - X + Y$ and $B_2 + X - Y$ in $\mathcal{B}$.*

Proof ([6]): The proof is short but not easy. Let M be the original matroid, consider the following matroids,

- $M_1 = (M/X)|B_2$(M with X contracted, restricted to B_2)
- $M_2 = (M/(B_1 - X))|B_2$
- $M_3 = M_2^*$.

We need to show that there exists $Y \subseteq B_2$ such that Y is a base of M_2 and $B_2 - Y$ is a base of M_1. In otherwords, M_1 and M_3 have a common base of size $|B_2 - X|$. Let n be the rank of M. We can easily compute the rank functions for $F \subseteq B_2$,

- $r_1(F) = r(F \cup X) - |X|$
- $r_2(F) = r(F \cup (B_1 - X)) - |B_1 - X| = r(F \cup (B_1 - X)) + |X| - n$
- $\begin{aligned}[t] r_3(F) &= |F| + r_2(B_2 - F) - r_2(B_2) \\ &= |F| + r((B_2 - F) \cup (B_1 - X)) - r(B_2 \cup (B_1 - X)) \\ &= |F| + r((B_2 - F) \cup (B_1 - X)) - n \end{aligned}$

Then we want to show that M_1 and M_3 have a common base of size $|B_2 - X|$. Note that the size of maximum common independent set is $\min r_1(F) + r_3(B_2 - F)$,

$$\begin{aligned}
r_1(F) + r_3(B_2 - F) &= r(F \cup X) - |X| + |B_2 - F| + r(F \cup (B_1 - X)) - n \\
&\geq r(F \cup X \cup F \cup (B_1 - X)) + r((F \cup X) \cap (F \cup (B_1 - X))) + |B_2 - F| - |X| - n \\
&= n + r(F) + |B_2 - F| - |X| - n \\
&= n + |F| + n - |F| - |X| - n \\
&= |B_2 - X|
\end{aligned}$$

Thus max common base of M_1 and M_3 has size $|B_2 - X|$. □

Theorem 1.6 (bijective exchange property) *$B_1, B_2 \in \mathcal{B}$, then there exists a bijection $\sigma : B_1 \to B_2$ such that $\forall x \in B_1$, $B_1 - x + \sigma(x) \in \mathcal{B}$.*

Proof: $\sigma(e) = e$ for all $e \in B_1 \cap B_2$ so it is enough to find an injection from $B_1 - B_2$ to $B_2 - B_1$. There exists a unique circuit $C_e \subseteq B_2 + e$ for all $e \in B_1 - B_2$, so we need to show that the family of C_e satisfies Hall's condition. For n elements $\{e_1, \ldots, e_n\}$ in $B_1 - B_2$, $|\bigcup_{i \in [n]} C_{e_i}| \geq n$. Suppose $|\bigcup_{i \in [n]} C_{e_i}| < n$. This is a special case of Lemma 1.2.

□

base-orderable matroid A matroid is base-orderable if for any two bases $B_1, B_2 \in \mathcal{B}$, there exists a bijection $\sigma : B_1 \to B_2$ such that for every $e \in B_1 - B_2$, both $B_1 - e + \sigma(e)$ and $B_2 - \sigma(e) + e$ are bases.

The bijective exchange property holds for any matroid, however not every matroid is base-orderable.

Theorem 1.7 (partition basis-exchange property, Theorem 3.3 in [3]) *$B_1, B_2 \in \mathcal{B}$, for each partition $\{P_1, \ldots, P_m\}$ of B_1 there exists a partition of $B_2 = \{Q_1, \ldots, Q_m\}$ such that $B_1 - P_i + Q_i \in \mathcal{B}$ for $i \in [m]$.*

In [3] there are proofs of all exchange properties above. These proofs are all generalized from matrix determinant. It seems that extending algebra proofs for linear matroid is a good way to get proofs for general matroids. But I haven't read that paper. Maybe determinant proofs only works for exchange properties.

Proof (problem 5.3.1): By induction. x_k is the only element in B contained in C_{y_k}. Thus $B + y_k - x_k \in \mathcal{B}$. Consider the easier case $B + y_k - x_k$, $x_1, \ldots, x_{k-1}$ and $y_1, \ldots, y_{k-1}$. ☐

Proof (problem 5.3.2): We check all the new pairs.

- t and $x \in C(B', t)$. Note that $C(B', t) = C(B, s)$. By (*) $f(s) + 1 \geq f(x)$ for all $x \in C(B, t)$. Thus $f(t) = f(s) + 1 \geq f(x)$.
- $x \in S - B'$ and $s \in C(B', x)$. We need to prove that $f(s) \leq f(x) + 1$. Note that the intersection of $C(B', x)$ and $C(B, s)$ is not empty and $t \in C(B, s)$, there exists a circuit containing both x and t. Thus $f(s) + 1 = f(t) \leq f(x) + 1$.

☐

Proof (problem 5.3.3, circuit and cocircuit never intersect in exactly 1 element): Suppose cocircuit C^* and circuit C intersect in exactly 1 element e. Consider the matroid minor $M' = M \backslash (C^* - x)$($M$ with $C^* - x$ deleted). Thus x is a coloop in M' since C^* is the minimal set intersecting every base and $C^* - x$ is deleted. Note that C is still a dependent set in M' since nothing is deleted in C. We find a contradiction that circuit C contains a coloop x. ☐

1.5 Generalized partition matroid

Proof (problem 5.3.4): The only if part is easy. For the if part, one can see that $|F| \leq k$ and $|F \cap S_i| \leq g_i$ for every i. Greedily adding elements to F without violating $|F \cap S_i| \leq g_i$ makes a base. Thus F must be a subset of some base and F is independent. ☐

Proof (problem 5.3.5): what is cross-free...? ☐

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