proofs of matroid exercises and problems in András Frank's book *Connections in Combinatorial Optimization*

1 ch5 - Elements of matroid theory

Proof (Exercise 5.1.1): Assume I1 and I2 hold, prove I3' is equivalent to I3.

1. I3 $\rightarrow$ I3'. Suppose I3' is not true. There must be some independent set $U \supset K$ by I2. Take X in I3 to be the union of K and N, then there are no element in X can be added to K while remaining independence. Thus the set U does not exist, a contradiction.

2. I3' $\rightarrow$ I3. Take any two subset A, B of X and $A, B \in \mathcal{F}$. Suppose $|A| < |B|$, apply I3' then A and B will be of the same size.

$\square$

Proof (Exercise 5.1.2): similar to the previous one. Assume I1 and I2 hold,

1. I3 $\rightarrow$ I3". I3" is a weaker version of I3'. so I3 $\rightarrow$ I3' $\rightarrow$ I3".

2. I3" $\rightarrow$ I3. Take any two independent subset A, B of X. Suppose $|A| < |B|$. One can always find a independent subset D of B s.t. $A \cap B \subseteq D$ and $|D| = |A| + 1$ as I2 holds. Applying I3" adds one element to A. Since this works for any A, B, finally every independent subset of X will be of the same size.

$\square$

Proof (Problem 5.1.3): I3"" is a weaker version of I3", with additional constraint $|K \setminus N| = 1$.

1. I3 $\rightarrow$ I3" $\rightarrow$ I3"".

2. I3"" $\rightarrow$ I3. similar to previous proofs.

$\square$

Proof (every affine matroid can be represented as a linear matroid, and vice versa): Consider a vector space over any field,

1. affine $\rightarrow$ linear. For every element $\mathbf{x} = (x_1, \ldots, x_n)$ in S, add one dimension, $\mathbf{x}' = (x_1, \ldots, x_n, 1)$. Verification is simple.

2. linear $\rightarrow$ affine. For every element $\mathbf{x} = (x_1, \ldots, x_n)$ in S, remove one dimension, **⟨⟨can't do it...⟩⟩** $\Longleftarrow$ ✠

$\square$

Proof (Exercise 5.1.4 circuit matroid is linear over any field): We prove that a set of edges contains a cycle if and only if the corresponding columns in Q are linearly dependent over any field. If the set of edges contains a cycle, let C be the set of edges in the cycle. One can easily see that, regardless of the edge orientation of edges, the set of columns in C either add up to zero or they can be divided into 2 parts and their sums are the same vector. Both case leads to linear dependence over any field. On the other side, if the set edges form a forest, we claim that for any subset of edges there will be at least one row of the corresponding columns contains only one non-zero value. This is because any subgraph of a forest contains at least one degree one vertex. Thus the corresponding row will contain only one +1 or -1. Thus the columns are linear independent.

□

Note that non-isomorphic graphs may have isomorphic circuit matroids. A deep theorem of Whitney states that the circuit matroids of two non-isomorphic 3-connected graphs are not isomorphic.

Proof (Theorem 5.2.1 Weak circuit axiom): Suppose there exists two distinct circuits C_1 and C_2 violating the statement. Then $C_1 \cup C_2 - e$ is independent. We know that $C_1 \cap C_2$ is independent. Consider the maximal independent set I of $C_1 \cup C_2$ containing $C_1 \cap C_2$. Note that I can not have more than $|C_1 \cup C_2| - 2$ elements since independent set can not contain a circuit. Thus $C_1 \cup C_2 - e$ is a larger independent set than the maximal independent set of $C_1 \cup C_2$, a contradiction. □

Theorem 1.1 (Theorem 5.2.3 Strong circuit axiom) *Let C_1 and C_2 be two distinct circuits, $e \in C_1 \cap C_2$, $e_1 \in C_1 - C_2$. Then there is a circuit C for which $e_1 \in C \subseteq C_1 \cup C_2 - e$*

Proof: (I can't prove this. Finding the counter example with minimum union size is important here. The following is from the book.)

Suppose the statement is no true. Find two distinct circuits C_1 and C_2 violating the statement with minimum $|C_1 \cup C_2|$. Weak circuit axiom shows that $C_1 \cup C_2 - e$ is not independent. Then there does not exists a circuit $C \subseteq C_1 \cup C_2 - e$ containing e_1. Then there exists $C_3 \subseteq C_1 \cup C_2 - e$ which does not contain e_1. Now we consider C_3 and C_2. C_3 and C_2 follow the statement since $C_1 \cup C_2$ is the counter example with minimum size and $|C_3 \cup C_2| < |C_1 \cup C_2|$. Thus there exists $C_4 \subseteq C_3 \cup C_2 - f$ containing $e \in C_2$ for some $f \in C_3 \cap C_2$. Now we consider C_4 and C_1. $C_4 \cup C_1 \subsetneq C_1 \cup C_2$ since C_4 is a proper subset of $C_1 \cup C_2$. Hence for C_1 and C_4 we can apply strong circuit axiom and there should be a circuit $C \subseteq C_1 \cup C_4 - e \subseteq C_1 \cup C_2 - 2$ containing e_1, a contradiction. □

Proof (Theorem 5.2.4 circuit axioms define a matroid): Let $\mathcal{C}$ be the set of circuits. $\mathcal{I} = \{I \subseteq E | \nexists C \in \mathcal{C}, C \subseteq I\}$. We need to prove $\mathcal{I}$ follows I1, I2 and I3'(independent set exchange property). I1 and I2 holds trivially. Suppose I3' does not hold on $\mathcal{I}$. We can find $I_1, I_2 \in \mathcal{I}$ such that $|I_1| > |I_2|$ and $\forall e \in I_1 \setminus I_2$, $I_2 + e$ contains a circuit. There are two cases,

1. $I_2 \subseteq I_1$. This is trivial.
2. $I_2 \nsubseteq I_1$, Then $|I_1 \setminus I_2| \geq 2$. Take two elements $e, f \in I_1 \setminus I_2$, $I_2 + e$ and $I_2 + f$ contains two unique circuits C_e and C_f. Note that $C_e \cap C_f = \emptyset$ by Theorem 1.1. Hence all distinct

circuits formed by adding one element in $I_1 \setminus I_2$ to I_2 are disjoint. Also note that each circuit contains at least one element in $I_2 \setminus I_1$ since otherwise I_1 will contain a circuit. Thus $|I_2 \setminus I_1| \geq |I_1 \setminus I_2|$, contradicting to $|I_1| > |I_2|$.

□

Similar methods can be used to prove edges in (k,ℓ)-sparse subgraphs form the independent set of matroid.

Proof ((k,ℓ)-sparsity matroid): We prove the set of edges of (k,ℓ)-sparse subgraphs in $G = (V,E)$ for all $k \geq 1, \ell \leq 2k-1$ satisfy I1,I2 and I3'. Again I1 and I2 holds trivially. Suppose I3' does not hold. We can find two (k,ℓ)-sparsesubgraphs $N = (V_N, E_N)$ and $K = (V_K, E_K)$ violating I3' with $|E_N| > |E_K|$. For any edge $e \in E_N \setminus E_K$, $K + e$ is not (k,ℓ)-sparseand there is a maximal(in terms of edges) (k,ℓ)-tight subgraph G_e of K containing two endpoints of e. We claim that G_es are edge disjoint for different e. By Theorem 5(1) in [1], intersection and union(with respect to the vertex sets) of two (k,ℓ)-tight subgraphs with at least two common vertices induce (k,ℓ)-tight subgraphs. Hence if for different e and f, G_e and G_f have any common edge, G_e and G_f won't be maximal. Thus for all $e \in E_N \setminus E_K$, corresponding G_es are edge disjoint. We have $|E_K| \geq |E_N|$, a contradiction. □

Proof (Exercise 5.2.1): □

References

[1] Audrey Lee and Ileana Streinu. Pebble game algorithms and sparse graphs. *Discrete Mathematics*, 308(8):1425–1437, April 2008.