GSoC24 CODING PROJECT PROPOSAL
Develop a new rounding method for convex polytopes

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Abstract

This proposal aims to develop a new rounding method for convex polytopes in the GeomScale project.

Rounding a convex polytope is crucial to improve the mixing rate of several MCMC methods that sample uniformly from the polytope. A standard method to round a convex polytope is to bring it to John position. This can be achieved by computing the maximum volume ellipsoid (MVE) of the polytope (or the John ellipsoid) and apply to it the transformation that maps the ellipsoid to the unit ball. The final polytope would be in John position.

An alternative method is to apply the fast transformation of the MVE problem in [2] to compute the minimum volume enclosing ellipsoid (MVEE) of a set of points.

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1 Project Information

Project title:	Develop a new rounding method for convex polytopes
Project short title:	rounding convex polytope
url:	https://github.com/GeomScale/gsoc24/wiki/Develop-a-new-rounding-method-for-convex-polytopes

1.1 Motivation

The goal of this project is to provide a new rounding method for convex polytopes in `VolEsti` package. Rounding a polytope is important for Markov chain Monte Carlo methods that sample uniformly from the polytope. A key procedure of the traditional methods is computing the maximum volume inscribed ellipsoid(MVIE) of the polytope. Currently there are 3 rounding method for convex polytopes, svd rounding, MVIE rounding and MVEE(minimum volume enclosing ellipsoid) rounding in `VolEsti` package. The MVIE rounding method is an implementation of the algorithm described in [3] and seems to be not efficient enough. The MVEE rounding method randomly samples points on the polytope and uses the Khachiyan's Algorithm [1] to compute the MVEE. This method also leads to low efficiency. It would greatly optimize the polytope rounding procedure of `VolEsti` if we can implement a new efficient method for computing MVIE.

2 Biographical Information

Contact Information

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I am currently a 4th year undergraduate student at University of Electronic Science and Technology of China majoring in computer science. I will become a PhD student studying combinatorial optimization at the same school in September. I have research experiences in computational geometry which is important preliminary knowledge of this project. I implemented the k-level algorithm[1] for finding the curve consisting of the points that lie on one of the lines and have exactly k lines above them.

This project requires reading papers on math programming and geometry and implementing algorithms in `C++`, which is exactly what I am insterested in and good at. I am familiar with `C++` and math softwares such as `sagemath`. I enjoy reading theoretical papers and implementing algorithms. For example I implemented pebble game algorithm[2] for deciding graph sparsity in `sagemath` and lots of data structures in `C++`[3].

I have a lot of free time during the summer since I will complete undergraduate studies before July and start my PhD program in September. Thus I can trate GSoC as a full-time job and concentrate on it.

[1] code: https://github.com/congyu711/k-level, k-level alg: https://tmc.web.engr.illinois.edu/pub_kset.html Remarks on k-level algorithms in the plane

[2] https://gist.github.com/congyu711/7c783a54d82ecce1cedbdd227791ecd3

[3] for example, https://gist.github.com/congyu711/bbe11fbdee17f34b7c30ece4ad62f5b4

3 Mentors

Main mentor name:	Vissarion Fisikopoulos
Main mentor email:	vissarion.fisikopoulos@gmail.com
Co-mentor name:	Apostolos Chalkis
Co-mentor email:	tolis.chal@gmail.com

I have been in touch with mentors through emails since March 26th.

4 Coding Plan and Methods

⟪I havn't develop a coding plan yet.⟫ ⟸

5 Timeline

5.1 Schedule Conflicts

GSoC will become my full-time job from July 10th to August 30th. Maybe I need to attend a conference from August 3rd to August 9th(this will be settled on April 16th). Before July 10th I will trate GSoC as a part-time job and report my progress weekly.

6 Management of Coding Project

⟪I havn't develop a coding plan yet.⟫ ⟸

7 Tests

7.1 Compile and run volesti library. Run the rounding routines.

Compile with `test/CMakeLists.txt` and run `new_rounding_test`.

```
[doctest] doctest version is "2.4.9"
[doctest] run with "--help" for options
use MVEE

--- Testing rounding of H-skinny_cube5
Number type: d
Computed volume 3071.67
Expected volume = 3070.64
Relative error (expected) = 0.000336825
Relative error (exact) = 0.0401018
Computed volume 3129.59
Expected volume = 3188.25
Relative error (expected) = 0.0183995
Relative error (exact) = 0.0220038
Computed volume 3163.97
Expected volume = 3140.6
Relative error (expected) = 0.00744021
Relative error (exact) = 0.0112604

--- Testing rounding of H-skinny_cube10
Number type: d
```

```
Computed volume 102149
Expected volume = 122550
Relative error (expected) = 0.166472
Relative error (exact) = 0.00245223
Computed volume 93610.8
Expected volume = 108426
Relative error (expected) = 0.136638
Relative error (exact) = 0.0858316
Computed volume 97672.9
Expected volume = 105003
Relative error (expected) = 0.0698087
Relative error (exact) = 0.0461633
use MVIE

-- Testing rounding of H-skinny_cube5
Number type: d
Computed volume 3262.77
Expected volume = 3140.6
Relative error (expected) = 0.0388998
Relative error (exact) = 0.0196153

-- Testing rounding of H-skinny_cube5
Number type: d
Computed volume 3160.01
Expected volume = 3140.6
Relative error (expected) = 0.00617984
Relative error (exact) = 0.0124974
===============================================================================
[doctest] test cases: 3 | 3 passed | 0 failed | 0 skipped
[doctest] assertions: 8 | 8 passed | 0 failed |
[doctest] Status: SUCCESS!
```

7.2 Compare the existing roudning C++ implementation in volesti that solves the MVEE problem of a uniformly distributed sample inside the polytope with PolyRound.

7.3 Write a pseudocode of the transformation between MVE and MVEE problem.

For simplicity we assume the polytope always contains the origin as an interior point.

Problem 1 (maximum volume inscribed ellipsoid (MVIE)) *Given a full-dimensional polytope Q in $\mathbb{R}^n$ defined by linear inequalities,*

$$Q = \{x \in \mathbb{R}^n | c_i^T x \leq 1, i \in [m]\}$$

find a ellipsoid of maximum volume inscribed in Q.

Problem 2 (minimum volume enclosing ellipsoid (MVEE)) *Given a full-dimensional polytope Q in $\mathbb{R}^n$ defined by convex hull of m points in $\mathbb{R}^n$,*

$$Q = \text{conv. hull}\{x_1, \dots, x_m\}$$

find a ellipsoid of minimum volume circumscribed about Q.

Note that MVIE can be used to solve MVEE. The transformation is the following,

Algorithm 1: reduce MVEE to MVIE

Input : n dimensional polytope $Q = \text{conv.hull}\{v_1, \dots, v_m\}$
Output : the minimum volume enclosing ellipsoid of Q

```
/* MVEE → centered MVEE                                                  */
```
Add a new dimension, $Q \leftarrow \text{conv.hull}\{\pm(v_1, 1), \dots, \pm(v_m, 1)\}$;
```
/* centered MVEE → centered MVIE                                         */
```
$Q' \leftarrow \{x \in \mathbb{R}^{n+1} | e^T \cdot x \leq 1, \forall e \in Q\}$;
```
/* centered MVIE → MVIE                                                  */
```
$Q'' \leftarrow \{x \in \mathbb{R}^{n+1} | \pm e^T \cdot x \leq 1, \forall e \in Q\}$;
$E = \text{MVIE}(Q'')$; /* E is the solution to MVIE and centered MVIE */
```
/* solution to centered MVIE → solution to centered MVEE                 */
```
$E' \leftarrow$ the polar of E;
```
/* solution to centered MVEE → solution to MVEE                          */
```
E'' is the intersection of E' with the hyperplane $\{x \in \mathbb{R}^{n+1} | x_{n+1} = 1\}$;
return E''

However the inverse transformation is not known. In [2] there is an explaination. Centered MVIE can be solved through MVIE but the general MVIE can not be solve through centered MVIE. Besides this step, the inverse of every transformation in algorithm 1 exists. So currently it is not possible to solve MVIE by using some algorithm for MVEE.

References

[1] Leonid G. Khachiyan. Rounding of polytopes in the real number model of computation. *Mathematics of Operations Research*, 21(2):307–320, May 1996.

[2] Leonid G. Khachiyan and Michael J. Todd. On the complexity of approximating the maximal inscribed ellipsoid for a polytope. *Mathematical Programming*, 61(1–3):137–159, August 1993.

[3] Yin Zhang and Liyan Gao. On numerical solution of the maximum volume ellipsoid problem. *SIAM Journal on Optimization*, 14(1):53–76, January 2003.