Optimal Experiment Design

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1 Signal Equation

The signal equation is

$$S_{ij}\left(M_0, T_1, T_2\right) = M_0 f_i(T_1, T_2) \frac{1 - E_j(T_1)}{1 - E_j(T_1) f_T(T_1, T_2)}, \tag{1}$$

where M_0 is the initial longitudinal magnetization and T_1 and T_2 are relaxation parameters. The EPG function is

$$f_i(T_1, T_2) = \mathrm{EPG}\left(\theta_1^i, T_1, T_2, T_s\right), \tag{2}$$

where θ_1^i indicates the first i refocusing pulses and T_s is the echo spacing. The recovery function is

$$E_j(T_1) = e^{-\frac{(\mathrm{TR}_j - (T+1)\times T_s)}{T_1}}, \tag{3}$$

where TR_j is the j^{th} repetition time.

The Fisher Information Matrix (FIM) consists of the following six terms:

$$\mathbf{I} = \begin{bmatrix} \frac{2}{\sigma^2}\sum_{ij}\left(\frac{\partial S_{ij}}{\partial T_1}\frac{\partial S_{ij}}{\partial T_1}\right) & \frac{2}{\sigma^2}\sum_{ij}\left(\frac{\partial S_{ij}}{\partial T_1}\frac{\partial S_{ij}}{\partial T_2}\right) & \frac{2}{\sigma^2}\sum_{ij}\left(\frac{\partial S_{ij}}{\partial T_1}\frac{\partial S_{ij}}{\partial M_0}\right) \\ \cdot & \frac{2}{\sigma^2}\sum_{ij}\left(\frac{\partial S_{ij}}{\partial T_2}\frac{\partial S_{ij}}{\partial T_2}\right) & \frac{2}{\sigma^2}\sum_{ij}\left(\frac{\partial S_{ij}}{\partial T_2}\frac{\partial S_{ij}}{\partial M_0}\right) \\ \cdot & \cdot & \frac{2}{\sigma^2}\sum_{ij}\left(\frac{\partial S_{ij}}{\partial M_0}\frac{\partial S_{ij}}{\partial M_0}\right) \end{bmatrix}. \tag{4}$$

For ease of notation, a function $f(x, y, z)$ will be referred to as $f(x)$ when it is understood that the other parameters are constants in the expression. Also, the partial derivative of a function $f(x, y, z)$ w.r.t. parameter x will be denoted by $f'(x)$. We have

$$S'_{ij}(M_0) = f_i \frac{1 - E_j}{1 - E_j f_T}, \tag{5}$$

$$\begin{aligned} S'_{ij}(T_1) = M_0\left(1 - E_j(T_1)\right) \frac{\left(f'_i(T_1)\left(1 - E_j(T_1)\right) - f_i(T_1)E'_j(T_1)\right)}{\left(1 - f_T E_j\right)^2} \times \\ \frac{\left(1 - f_T(T_1)E_j(T_1)\right) + f_i(T_1)\left(f'_T(T_1)E_j(T_1) + f_T(T_1)E'_j(T_1)\right)}{\left(1 - f_T E_j\right)^2}, \end{aligned} \tag{6}$$

$$S'_{ij}(T_2) = M_0\left(1 - E_j\right) \frac{\left(f'_i(T_2)\left(1 - f_T(T_2)E_j\right) + f_i(T_2) f'_T(T_2) E_j\right)}{\left(1 - f_T E_j\right)^2}. \tag{7}$$

The inverse of the FIM is a lower bound on the covariance matrix of an unbiased estimator of the parameters $\boldsymbol{\theta} = (T_1, T_2, M_r)$. Thus it reasons to minimize some metric of sub-matrix of $\mathbf{I}^{-1}$. Group the FIM into

$$\mathbf{I} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{I}_{12} \\ \mathbf{I}_{21} & \mathbf{I}_{22}, \end{bmatrix} \tag{8}$$

where $\mathbf{I}_{11}$ represents the 2×2 sub-matrix. Then the Schur complement of $\mathbf{I}$ w.r.t. $\mathbf{I}_{22}$ is

2 EPG Function

The EPG function is a recursive function of the signal magnetization and refocusing flip angles. For T refocusing pulses, the EPG function $\mathbf{f} : \mathbb{R}_+^2 \to \mathbb{R}_+^T$ is

$$\mathbf{f}(T_1, T_2, \theta_1^T) = \begin{bmatrix} f_1 & \cdots & f_T \end{bmatrix}^T. \tag{9}$$

Let the magnetization state after the i^{th} refocusing pulse be $\mathbf{s}_i \in \mathbb{C}^{3(T+1)}$. The first entry of $\mathbf{s}_i$ represents the coherent transverse magnetization, M_{xy}, and the third entry represents the coherent longitudinal magnetization, M_z. The observed signal after the i^{th} pulse is then

$$f_i(T_1, T_2, \theta_1^i) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{s}_i(T_1, T_2, \theta_1^i) \tag{10}$$

The initial magnetization state (after the 90 degree excitation pulse) is given by

$$\mathbf{s}_0 = \operatorname{vec}\left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \right). \tag{11}$$

One progression of the EPG states is given by

$$\mathbf{s}_i = \mathbf{E}\left(\mathbf{G}\mathbf{R}_i\mathbf{G}\mathbf{E}\mathbf{s}_{i-1} + \mathbf{E}_0\right) + \mathbf{E}_0, \tag{12}$$

where $\mathbf{E}$, $\mathbf{E}_0$, $\mathbf{G}$, and $\mathbf{R}_i$ are defined in the conventional way for EPG. Thus,

$$\begin{aligned} \nabla\mathbf{s}_i(T_1) &= \nabla\mathbf{E}(T_1)\left(\mathbf{G}\mathbf{R}_i\mathbf{G}\mathbf{E}(T_1)\mathbf{s}_{i-1}(T_1) + \mathbf{E}_0(T_1)\right) \\ &\quad + \mathbf{E}(T_1)\left(\mathbf{G}\mathbf{R}_i\mathbf{G}\left(\nabla\mathbf{E}(T_1)\mathbf{s}_{i-1}(T_1) + \mathbf{E}(T_1)\nabla\mathbf{s}_{i-1}(T_1)\right) + \nabla\mathbf{E}_0(T_1)\right) + \nabla\mathbf{E}_0(T_1), \end{aligned} \tag{13}$$

$$\begin{aligned} \nabla\mathbf{s}_i(T_2) &= \nabla\mathbf{E}(T_2)\mathbf{G}\mathbf{R}_i\mathbf{G}\left(\mathbf{E}(T_2)\mathbf{s}_{i-1}(T_2) + \mathbf{E}_0\right) \\ &\quad + \mathbf{E}(T_2)\left(\mathbf{G}\mathbf{R}_i\mathbf{G}\left(\nabla\mathbf{E}(T_2)\mathbf{s}_{i-1}(T_2) + \mathbf{E}(T_2)\nabla\mathbf{s}_{i-1}(T_2)\right)\right). \end{aligned} \tag{14}$$