Phase-based algorithm for modeling time-rescaling distortion of rhythmic data

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Introduction

- Empirical evidence from rhythmic biomechanical data sets:
  - Many such data sets can be brought to excellent correspondence by rescaling time.
  - One stride of an animal gait is very similar to any other stride of the self-same animal
  - Only difference is time-varying rate.
- Our algorithm brings such data sets to a consistent time scale.

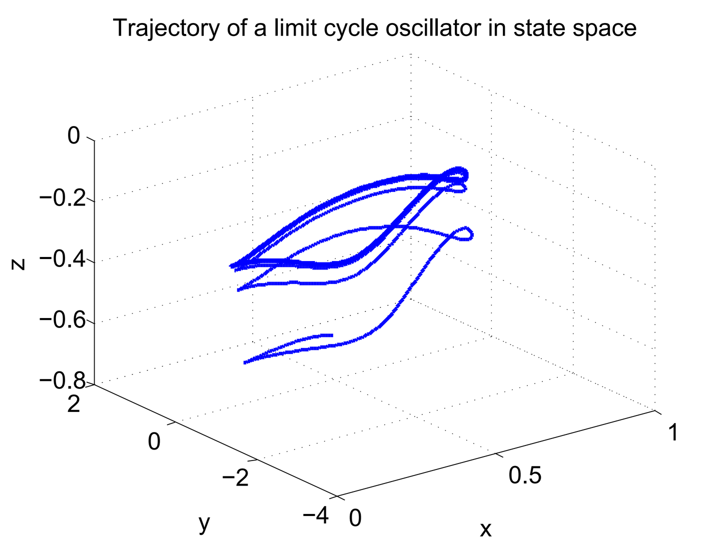

Figure 1: An example of a rhythmic system is shown above

Background

- Assumptions:
  - The biomechanical systems in question are well-modeled by mathematical *oscillators*
    - ∗ *Oscillator*: an attracting *limit cycle* of a differentiable dynamical system together with the dynamics in the stability basin of the limit cycle [2].
  - These oscillators are *hyperbolically stable*.
    - ∗ Implication: stability basin is partitioned into disjoint sets indexed by their *asymptotic phase*.
      - · *Asymptotic phase*: two points have the same asymptotic phase if their images under the flow asympotically coalesce in time [1].
    - ∗ Equivalent implication: existence of such a partition is equivalent to the existence of a useful closed differential one-form $d\phi$
      - · Line integral of $d\phi$ along any path yields the asymptotic phase difference of the endpoints [3].

Theorem

1. Given: a solution $x \in \mathcal{C}^1([a,b];\mathcal{B})$ of the differential equation describing an oscillator and a time-rescaling $s : [a,b] \to \mathbb{R}$ which is a continuous function of bounded variation OR a diffusion process
2. $\forall t \in [a,b]$, let $\tau(t) := \min\{\alpha \in [a,b] | \int_{x \circ s([a,b])} d\phi = t\}$ and let $\hat{x}(t) := x \circ s(\tau(t))$.
3. Then $\exists C \in \mathbb{R} : \forall t \in [a,b] : \hat{x}(t) = x(t+C)$, and
4. If $s(a) = a$, then $C = 0$

Algorithm

Given a data set $\{x(t_i)\}_{i=1}^N$ with samples at periodic times $I := \{t_i\}_{i=1}^N$, and *a priori* knowledge of the temporal one form $d\phi$:

τ =zeros$(1,N)$
$i = 1$
for $j = 1$ to N
  while $\tau(j) < t_j$
    $\tau(j) = \int_{t_1}^{t_i} \langle d\phi, dx(t)\rangle$
    $i = i + 1$
  end
  index = find(min(abs($\tau(j) - I$)))
  $\hat{x}(j) = x$(index)
end

Experimental Results

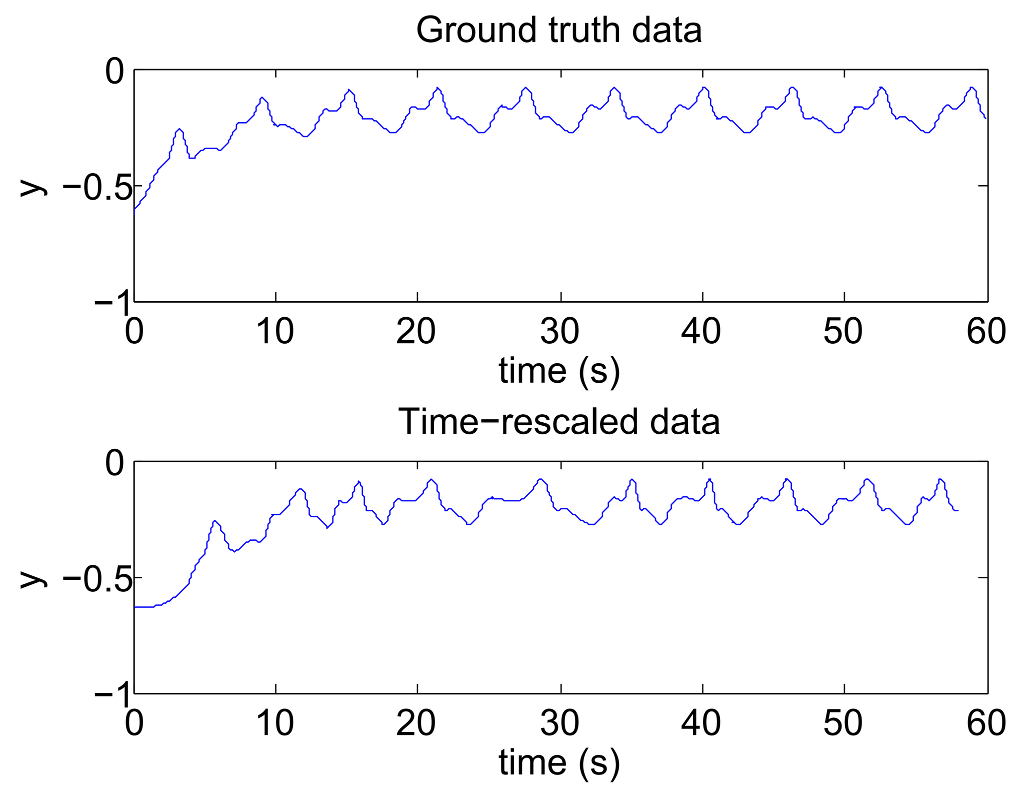

Figure 2: Ground truth data before and after time-rescaling is shown above

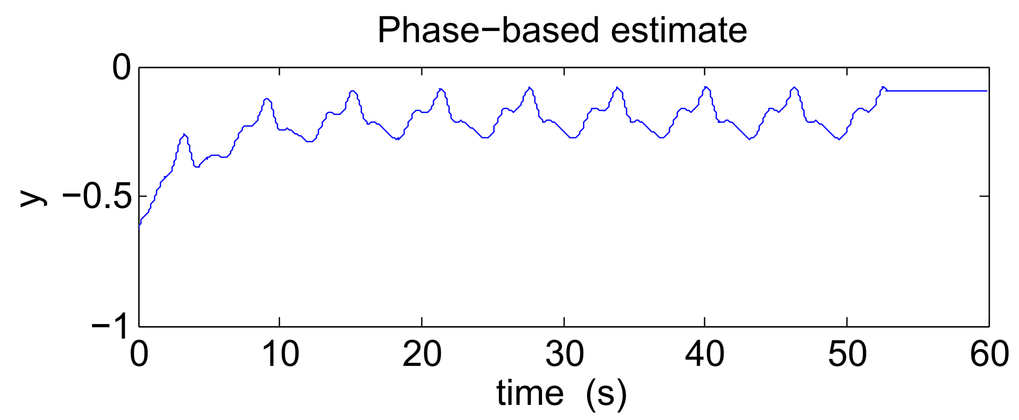

Figure 3: Results of our algorithm applied to the same data set are shown above.

Conclusions

- Data sets which can be modeled by "reasonable" oscillators often experience frequency drift.
- Our theoretical result suggests these distortions can be removed if $d\phi$ is known, and our algorithm exploits this.
- Algorithm was tested on data from simulated limit cycle oscillators with good results.
- Future work:
  - Test algorithm on data sets from animal locomotion experiments.
  - Investigate other applications of the one form $d\phi$.

Acknowledgements

I would like to thank Simon Wilshin, John Guckenheimer, and Clayton Scott, for their research on estimating the temporal one form (paper in preparation) paved the way for these results.

References

[1] J. Guckenheimer. Isochrons and phaseless sets. *Journal of Mathematical Biology*, 1:259–273, 1975.
[2] J. Guckenheimer. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields.* Springer-Verlag, New York, New York, 1 edition, 1983.
[3] Simon Wilshin, Clayton Scott, John Guckenheimer, and Shai Revzen. Estimating phase from observed trajectories using the temporal 1-form IN PREP.

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