Stochastic Optimal Control of Conditional McKean-Vlasov Equations with Jump and Markovian Switching

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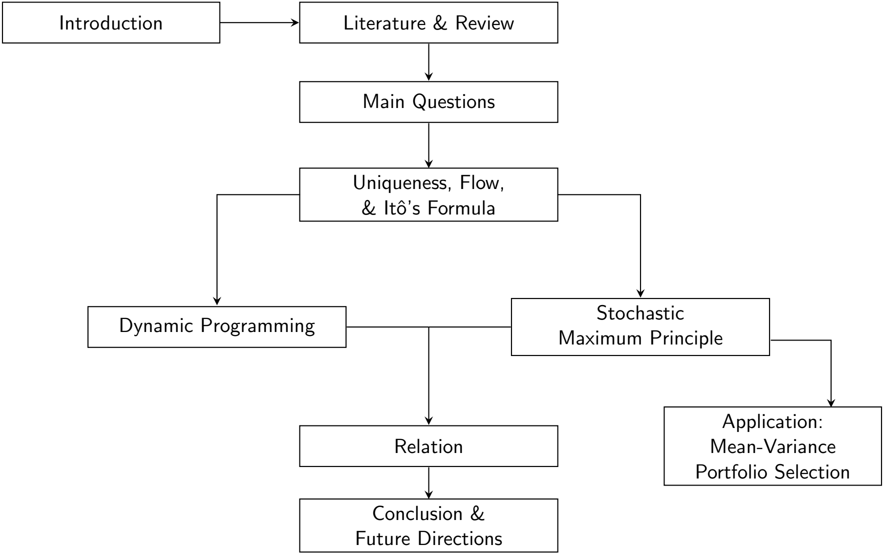
Talk Outline



McKean-Vlasov Equation

$$
\begin{aligned}
dX_s &= b(X_s, \mathbb{P}_{X_s})ds + \sigma(X_s, \mathbb{P}_{X_s})dW_s \\
X_0 &= \xi \in L^2(\mathcal{F}_t, \mathbb{R}^d)
\end{aligned}
$$

- $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d\times d}$
- $(W_s)_{s\geq 0}$ is a standard d-dimensional Brownian motion
- $\mathbb{P}_{X_s}$ denotes the law of X_s

This case is motivated by mean-field particle systems of the form

$$dX_t^{i,N} = b(X_t^{i,N}, \mu_{X_t^N})dt + \sigma(X_t^{i,N}, \mu_{X_t})dW_t^i, \quad \text{for all } i \in \{1, ..., N\}$$

- $W_t^1, ..., W_t^N$ are N independent Brownian motions;
- $\mu_x^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^i}$ is the empirical measure.

[1] L-P. Chaintron and A. Diez. Propagation of chaos: a review of models, methods and applications. I. Models and methods, I. Models and methods. 2022. ffhal-03585065v3

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- $\mu_x^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^i}$ is the empirical measure.

The empirical measure μ_{X^i} can be replaced by its expected limit, which is the law of X_t.[1]

[1] L-P. Chaintron and A. Diez. Propagation of chaos: a review of models, methods and applications. I. Models and methods, I. Models and methods. 2022. ffhal-03585065v3

Conditional McKean-Vlasov Equation with Jump and Markovian Switching

$$
\begin{aligned}
dX_s &= b(X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\
&\qquad + \int_{\mathbb{R}^q} \beta(X_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta), \\
X_0 &= \xi \in \mathcal{L}^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_0 = i_0,
\end{aligned}
$$

- $\tilde{N}([0,s] \times A) = N([0,s] \times A) - \int_0^s \int_A \nu(d\theta)dt$ is a martingale for any Borel subset A of $\mathbb{R}^q$ with $\nu(A) < \infty$,
- $(\alpha_s)_{s\geq 0}$ a Markov chain with finite state space $\mathcal{M} = \{1, 2, \dots, m_0\}$ and generator matrix $Q = (q_{i_0 j_0})_{i_0, j_0 \in \mathcal{M}}$ satisfying $q_{i_0 j_0 > 0}$ for $i_0 \neq j_0$ and $\sum_{j_0 \in \mathcal{M}} q_{i_0 j_0} = 0$.
- $\mathbb{P}^{\alpha}_{X_s} = \mathcal{L}(X_s | \mathcal{F}^{\alpha}_s)$ is the conditional probability measure of X_s given $\mathcal{F}^{\alpha}_s$.

This case is motivated by the mean-square limit as $N \to \infty$ of the stochastic controlled system of N weakly interacting agents

$$
\begin{aligned}
dX_s^{i,N} = {} & b(X_s^{i,N}, \mu_s^N, \alpha_s)ds + \sigma(X_s^{i,N}, \mu_s^N, \alpha_s)dW_s^i \\
& + \int_{\mathbb{R}^q} \beta(X_{s-}^{i,N}, \mu_s^N, \alpha_{s-}, \theta)N^i(ds, d\theta), \quad s \geq t, \qquad (2.1) \\
X_{t-} = {} & \xi^i \in L^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_{s-} = i_0,
\end{aligned}
$$

- $\{W, W_s^i : i = 1, 2, \ldots\}$ is a sequence of independent standard d-dimensional Brownian motions;
- $\{N, N_s^i : i = 1, 2, \ldots\}$ is a sequence of independent identically distributed Poisson random measure with the same finite intensity measure ν;
- $\{\xi, \xi_s^i : i = 1, 2, \ldots\}$ is a sequence of independent identically distributed random variables;
- $\mu_s^N = \mu_s^{N;\, t,\xi,i_0,u} = \dfrac{1}{N} \sum_{j=1}^N \delta_{X_s^{j,N}}$.

The existence and uniqueness for both McKean-Vlasov equations are known,

- The classical case for example by Carmona[2]

[2]Carmona R. *Lectures on BSDEs, Stochastic Control, and Stochastic Differential Games with Financial Applications*, SIAM, 2016

[3]Shao J., Tian T., and Wang S. Conditional McKean-Vlasov SDEs with jumps and Markovian regime-switching: Wellposedness, propagation of chaos, averaging principle. preprint (2022).

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State Equation

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Cost Functional

$$J(s, y, u(\cdot)) = \mathbb{E}\Big[\int_s^T f(t, x_t, u_t)dt + g(x_T)\Big]$$

DPP

In Dynamic Programming, we have three main pieces:

- The Dynamic Programming Principle;
- The Hamilton-Jacobi-Bellman Equation;
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HJB

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,x) + \inf_{u \in \mathcal{U}} H(t,x,u,\frac{\partial V}{\partial x},\frac{\partial^2 V}{\partial x^2}) = 0, \quad (t,x) \in [p,T] \times \mathbb{R}^d, \\ &V(T,x) = g(x), \quad x \in \mathbb{R}^d \end{aligned} \tag{3.2}$$

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Verification Theorem

$$\frac{\partial V}{\partial t}(t, x^*(t)) = H(t, x^*(t), u^*(t), \frac{\partial V}{\partial x^*}, \frac{\partial^2 V}{\partial x^{*2}}) = \min_{u^* \in \bar{\mathcal{U}}} H(t, x^*(t), u, \frac{\partial V}{\partial x^*}, \frac{\partial^2 V}{\partial x^{*2}})$$

An important facet of this depends on what is known as the *flow property*

$$\left(X_r^{s,X_s^{t,x,\xi},X_s^{t,\xi}}, X_r^{s,X_s^{t,\xi}}\right) = \left(X_r^{t,x,\xi}, X_r^{t,\xi}\right), \qquad r \in [s, T].$$

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and that of *law invariance* for X or the value function $V(t,\mu)$,i.e., $X^{t,x,\xi}$ does not depend on ξ itself but only on its law $\mathbb{P}_\xi$. This is usually given by an assumption of the form

$$\begin{aligned}
&|b(x,\mu) - b(x',\mu')| + |\sigma(x,\mu) - \sigma(x',\mu')| \\
&\leq C\Big(|x - x'| + W_2(\mu,\mu')\Big).
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To derive the HJB equation or prove the verification theorem, we need to use Itô's formula, so naturally the question arises for how this would work for an equation which depends on $\mathcal{P}_2(\mathbb{R}^d)$...

Lions Derivative

- $\mathcal{P}_p(A)$: For any $p \in \mathbb{N}, \mathcal{P}_p(A)$ is the set of all probability measures of p-th order on $(A, ||\cdot||)$,

  defined $\mathcal{P}_p(A) = \{\mu \in \mathcal{P}(A) : ||\mu||_p := (\int_A |x|^p \mu(dx))^{\frac{1}{p}} < \infty\}$.

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- **p-th order Wasserstein Metric:** For any μ, μ' in $\mathcal{P}_p(A)$, the p-th order Wasserstein distance on $\mathcal{P}_p(A)$ is defined as

  $W_p(\mu, \mu') = \inf_\pi (\int_{A\times A} |y - y'|^p \pi(dy, dy'))^{\frac{1}{p}}$
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- $\mathcal{L}^2(\mathcal{F}, A)$: The space of all A-valued square integrable random variables on our probability triple $(\Omega, \mathcal{F}, \mathcal{P})$. i.e., $\mathbb{E}|X|^2 < \infty$.

  For any $\tau \in L(\mathcal{F}, A)$, we set $||\tau||_{L^2} = \mathbb{E}[|\tau|^2]^{\frac{1}{2}} = \int_{\mathbb{R}^d} |x|^2 \mathbb{P}_\xi(dx)$.

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- Note that $\mathcal{L}^2(\mathcal{F}, A)$ is indeed a Hilbert Space under the inner product: $\langle X, Y\rangle_{\mathcal{L}^2(\Omega)} = \mathbb{E}[XY]$

- We endow $\mathcal{P}_2(\mathbb{R}^d)$ with the the 2-Wasserstein metric $W_2(\cdot,\cdot)$, which **is a metric space**, although not a normed space.
- We also assume the our $\mathcal{F}$ is *atomless*, that is there is no set $A \subset \mathcal{F}$ such that for a measure μ on the measurable space $(\Omega, \mathcal{F})$, $\mu(A) > 0$ and any measurable subset $B \subset A$ with $\mu(B) < \mu(A)$, the set B is zero measure.
- When $\mathcal{F}$ is atomless, we can consider $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\tau : \tau \in L^2(\mathcal{F}, \mathbb{R}^d)\}$

Definition (Frechet Derivative)

A functional f is Frechet differentiable at ξ_0 if there exists a bounded linear continuous mapping $Df(\xi_0) : L^2(\mathcal{F}, \mathbb{R}^d) \mapsto \mathbb{R}$ such that

$$f(\xi_0 + \zeta) - f(\zeta) = Df(\xi_0)(\zeta) + o(||\zeta||_{\mathcal{L}^2})$$

as $||\zeta||_{L^2} \mapsto 0$, where $\zeta \in L^2(\mathcal{F}, \mathbb{R}^d)$.

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Now consider the functional $f : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ and define its *lift* $\tilde{f}$, which is a function $\tilde{f} : L^2(\mathcal{F}, \mathbb{R}^d) \mapsto \mathbb{R}$ such that $\tilde{f}(\xi) = f(\mathbb{P}_\xi)$ for any $\xi \in L^2(\mathcal{F}, \mathbb{R}^d)$.

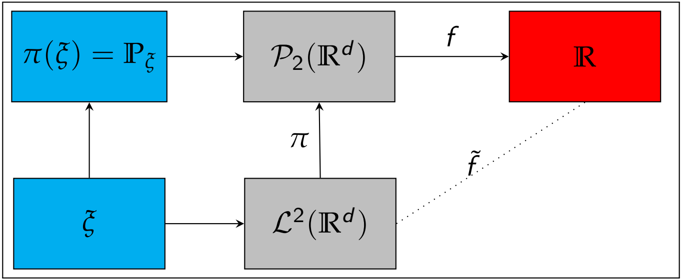

Figure: Schematic for the lift $\tilde{f}$ of f.

Lions Derivative

When $\tilde{\xi}_0$ is fixed, $D\tilde{f}(\tilde{\xi}_0)$ is a bounded linear functional, thus we can invoke the Riesz Representation Theorem.

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Theorem (Riesz Representation Theorem)

Every bounded linear functional f on a Hilbert Space H can be represented in terms of the inner product, namely, $f(x) = \langle x, z \rangle$ where z depends on f, is uniquely determined by f and has norm $||z|| = ||f||$.

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Every bounded linear functional f on a Hilbert Space H can be represented in terms of the inner product, namely, $f(x) = \langle x, z\rangle$ where z depends on f, is uniquely determined by f and has norm $||z|| = ||f||$.

Thus, there exists a unique $h \in L^2(\mathcal{F}, \mathbb{R}^d)\}$ such that $D\tilde{f}(\xi_0)(\zeta) = \mathbb{E}[h \cdot \zeta]$ where $\zeta \in L^2(\mathcal{F}, \mathbb{R}^d)$.

It has been shown that $h = h_0(\xi_0)$, almost surely in probability where h_0 is a Borel function, $h_0 : \mathbb{R}^d \mapsto \mathbb{R}^d$ which only depends on the law of ξ. Thus the Frechet differentiable $\tilde{f}$ can be rewritten as

$$f(\mathbb{P}_{\xi_0+\zeta}) - f(\mathbb{P}_{\xi_0}) = \mathbb{E}[h_0(\xi_0) \cdot \zeta] + o(||\zeta||_{L^2}) \ .$$

Finally, we can define the Lion Derivative as follows:

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$f : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is Lions differentiable at $\mu_0 = \mathbb{P}_{\xi_0}$ if its *lift function* $\tilde{f}$ is Frechet differentiable at ξ_0.

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$f : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R}$ is Lions differentiable at $\mu_0 = \mathbb{P}_{\xi_0}$ if its *lift function* $\tilde{f}$ is Frechet differentiable at ξ_0.
In this case, we denote the function $\partial_\mu f(\mu_0, y) := h_0(y), y \in \mathbb{R}^d$ as the *Lions derivative* of f at μ_0.

For the general case of McKean-Vlasov equations,

[4] R. Buckdahn, J. Li, S. Peng, and C. Rainer. Mean-field stochastic differential equations and associated PDEs arXiv:1407.1215 (2014).

[5] T. Hao and J. Li. Mean-field SDEs with jumps and nonlocal integral-PDEs, Nonlinear Differential Equations and Applications. (2016)

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Despite these results, there is still a lack of an Itô's formula for general semimartingales.

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For semimartingales, we have from Guo et. al.[6],

(Guo et. al.) Corollary 3.6: Itô's Formula for Jump Process

For $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$,

$$\Phi(\mathbb{P}^\alpha_{X_s}) = \Phi(\mathbb{P}_\xi) + \int_t^s \hat{\mathbb{E}}^\alpha \Big[\partial_\mu \Phi(\mathbb{P}^\alpha_{X_r}, \hat{X}_r) \cdot \hat{b}_r + \frac{1}{2} \mathrm{Tr}\big(\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Phi(\mathbb{P}^\alpha_{X_r}, \hat{X}_r)\big) + \int_{\mathbb{R}^q} \Big(\frac{\delta \Phi}{\delta \mu}(\mathbb{P}^\alpha_{X_r}, \hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Phi}{\delta \mu}(\mathbb{P}^\alpha_{X_r}, \hat{X}_{r-}) \Big) d\nu(\theta) \Big] dr. \tag{3.3}$$

[6] X. Guo, H. Pham, and X. Wei. Itô's formula for flows of measures on semimartingales. arXiv:2010.05288 (2022).

The stochastic maximum principle requires an adjoint equation

Stochastic Maximum Principle

The stochastic maximum principle requires an adjoint equation

$$dp_t = -\Big[\nabla_x H(t, X_t, u_t, p_t, P_t)dt\Big] + P_t dW_t,$$
$$dp_T = \nabla g(X_T)$$

From which for a control u^* and adjoint solution processes (p_t^*, P_t^*), if

$$\mathcal{H}(t, x_t^*, u_t^*, p_t^*, P_t^*) = \inf_{u \in \bar{U}} \mathcal{H}(t, x_t^*, u_t, p_t^*, P_t^*)$$

then u^* is an optimal control.

Again, the literature is rife with results pertaining to the stochastic maximum principle for related SDEs & McKean-Vlasov Equations, such as

[7] M. Hafayed, S. Meherrem, and S. Eren, and D.Z. Guçoglu. On optimal singular control problem for general Mckean-Vlasov differential equations: Necessary and sufficient optimality conditions. Optimal Control Applied Methods (2017).

[8] N.C. Framstad, B. Øksendal, and A. Sulem. Sufficient Stochastic Maximum Principle for the Optimal Control of Jump Diffusions and Applications to Finance. Journal of Optimization Theory and Applications: Vol. 121, No. 1, pp. 77–98, (2004).

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Literature: Stochastic Maximum Principle

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- A sufficient maximum principle for the general McKean-Vlasov equation.[7]
- A sufficient maximum principle for a jump-diffusion model.[8]
- A sufficient maximum principle for a Markovian switching SDE.[9]

[7] M. Hafayed, S. Meherrem, and S. Eren, and D.Z. Guçoglu. On optimal singular control problem for general Mckean-Vlasov differential equations: Necessary and sufficient optimality conditions. Optimal Control Applied Methods (2017).

[8] N.C. Framstad, B. Øksendal, and A. Sulem. Sufficient Stochastic Maximum Principle for the Optimal Control of Jump Diffusions and Applications to Finance. Journal of Optimization Theory and Applications: Vol. 121, No. 1, pp. 77–98, (2004).

[9] C. Donnelly. Sufficient Stochastic Maximum Principle in a Regime-Switching Diffusion Model. Appl Math Optim 64:155–169, (2011).

Again, the literature is rife with results pertaining to the stochastic maximum principle for related SDEs & McKean-Vlasov Equations, such as

- A sufficient maximum principle for the general McKean-Vlasov equation.[7]
- A sufficient maximum principle for a jump-diffusion model.[8]
- A sufficient maximum principle for a Markovian switching SDE.[9]

Additionally, [8] & [9] detail the relation between the stochastic maximum principle and the dynamic programming principle through the expression of the adjoint processes p_t, P_t as derivatives of the value function $V(x, t)$.

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Controlled CMVE with Jump and Switching

$$\begin{aligned} dX_s &= b(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\ &\qquad + \int_{\mathbb{R}^q} \beta(X_{s-}, u_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta), \quad s \geq t, \qquad (3.4) \\ X_0 &= \xi \in L^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_0 = i_0, \end{aligned}$$

New Formulations

Controlled CMVE with Jump and Switching

$$\begin{aligned} dX_s &= b(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\ &\quad + \int_{\mathbb{R}^q} \beta(X_{s-}, u_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta), \quad s \geq t, \qquad (3.4) \\ X_0 &= \xi \in L^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_0 = i_0, \end{aligned}$$

- Can we build off the literature provided?

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Controlled CMVE with Jump and Switching

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(i) Wellposedness of (3.3);

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- Can we build off the literature provided?

The literature discussed gives us

(i) Wellposedness of (3.3);

(ii) Relevant Itô's formula;

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Controlled CMVE with Jump and Switching

$$\begin{aligned} dX_s &= b(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\ &\qquad + \int_{\mathbb{R}^q} \beta(X_{s-}, u_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta), \quad s \geq t, \\ X_0 &= \xi \in L^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_0 = i_0, \end{aligned} \tag{3.4}$$

- Can we build off the literature provided?

The literature discussed gives us

(i) Wellposedness of (3.3);

(ii) Relevant Itô's formula;

(iii) Relevant results for a Verification Theorem and Stochastic Maximum Principle

Main Problems

Question: Can We Derive

1. Uniqueness of solutions? The flow property?
2. A relevant version of Itô's formula?
3. A verification theorem for the DPP?
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$$\left(X_r^{s,X_s^{t,x,\xi,i_0},X_s^{t,\xi,i_0},\alpha_s}, X_r^{s,X_s^{t,\xi,i_0},\alpha_s}\right) = \left(X_r^{t,x,\xi,i_0}, X_r^{t,\xi,i_0}\right), \qquad r \in [s, T],$$

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$$(p_t, P_t) \overset{?}{=} (\partial_\mu V, \partial_x V)$$

Theorem (Uniqueness of Solutions)

Under Assumption $(\mathbf{A}_0)$, *the equations* (4.1) *and* (4.2) *admit unique solutions* $X^{t,\xi,i_0} = (X_s^{t,\xi,i_0})_{s\in[t,T]}$ *and* $X^{t,x,\xi,i_0} = (X_s^{t,x,\xi,i_0})_{s\in[t,T]}$ *in* $\mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}^d)$. *The solution* $X^{t,x,\xi}$ *is independent of* $\mathcal{F}_t$.

$$X_s^{t,\xi,i_0} = \xi + \int_t^s b(X_r^{t,\xi,i_0}, \mathbb{P}^\alpha_{X_r^{t,\xi,i_0}}, \alpha_r)dr + \int_t^s \sigma(X_r^{t,\xi,i_0}, \mathbb{P}^\alpha_{X_r^{t,\xi,i_0}}, \alpha_r)dW_r + \int_t^s \int_{\mathbb{R}^q} \beta(X_{r-}^{t,\xi,i_0}, \mathbb{P}^\alpha_{X_{r-}^{t,\xi,i_0}}, \alpha_{r-}, \theta) N(dr, d\theta), \qquad s \in [t,T], \tag{4.1}$$

$$X_s^{t,x,\xi,i_0} = x + \int_t^s b(X_r^{t,x,\xi,i_0}, \mathbb{P}^\alpha_{X_r^{t,\xi,i_0}}, \alpha_r)dr + \int_t^s \sigma(X_r^{t,x,\xi,i_0}, \mathbb{P}^\alpha_{X_r^{t,\xi,i_0}}, \alpha_r)dW_r + \int_t^s \int_{\mathbb{R}^q} \beta(X_{r-}^{t,x,\xi,i_0}, \mathbb{P}^\alpha_{X_{r-}^{t,\xi,i_0}}, \alpha_{r-}, \theta) N(dr, d\theta), \qquad s \in [t,T]. \tag{4.2}$$

BDG & Gronwall Inequality

Lemma (Burkholder-Davis-Gundy Inequality)

There exist some constants $c_p \in (0, \infty), p \geq 1$*, such that, for any local martingale* U *with* $U_0 = 0$*,*

$$c_p^{-1}\mathbb{E}[U]_t^{\frac{p}{2}} \leq \mathbb{E}U^{*p} \leq c_p\mathbb{E}[U]_t^{\frac{p}{2}}, \quad p \geq 1$$

where $[U]$ *denotes the quadratic variation of the process and* $U_t^* = \sup_{s \leq t} |M_s|$*.*

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Lemma (Gronwall Inequality)

Let $a(t)$ be a nonnegative function such that

$$a(t) \leq C + A\int_0^t a(s)ds \quad \text{for } 0 \leq t \leq T$$

for some constants C, A where $A \geq 0$. Then,

$$a(t) \leq Ce^{At} \quad \text{for } 0 \leq t \leq T.$$

Existence & Uniqueness: Proof [1/2]

- Introduce $\hat{X}^i = X^{i+1} - X^i$

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Existence & Uniqueness: Proof [1/2]

- Introduce $\hat{X}^i = X^{i+1} - X^i$

$$\begin{aligned}\hat{X}^i_s = &\int_t^s \left[\sigma(X^i_r, \mathbb{P}^{\alpha}_{X^i_r}, \alpha_r) - \sigma(X^{i-1}_r, \mathbb{P}^{\alpha}_{X^{i-1}_r}, \alpha_r)\right] dW_r \\ &+ \int_t^s \int_{\mathbb{R}^q} \left[\beta(X^i_{r-}, \mathbb{P}^{\alpha}_{X^i_{r-}}, \alpha_{r-}, \theta) - \beta(X^{i-1}_{r-}, \mathbb{P}^{\alpha}_{X^{i-1}_{r-}}, \alpha_{r-}, \theta)\right] N(dr, d\theta), \quad s \in [t, T].\end{aligned}$$

- Apply BDG and Lipschitz

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- Introduce $\hat{X}^i = X^{i+1} - X^i$

$$\hat{X}^i_s = \int_t^s \left[\sigma(X^i_r, \mathbb{P}^\alpha_{X^i_r}, \alpha_r) - \sigma(X^{i-1}_r, \mathbb{P}^\alpha_{X^{i-1}_r}, \alpha_r)\right] dW_r \\ + \int_t^s \int_{\mathbb{R}^q} \left[\beta(X^i_{r-}, \mathbb{P}^\alpha_{X^i_{r-}}, \alpha_{r-}, \theta) - \beta(X^{i-1}_{r-}, \mathbb{P}^\alpha_{X^{i-1}_{r-}}, \alpha_{r-}, \theta)\right] N(dr, d\theta), \quad s \in [t, T].$$

- Apply BDG and Lipschitz

$$\mathbb{E}\Big[\sup_{s\in[t,u]} |\hat{X}^i_s|^2\Big] \le C\mathbb{E}\Big[\int_t^u \left(|\hat{X}^{i-1}_r|^2 + W_2(\mathbb{P}^\alpha_{X^i_r}, \mathbb{P}^\alpha_{X^{i-1}_r})^2\right) dr \\ + \int_t^u \int_{\mathbb{R}^q} (1 \wedge |\theta|^2)\left(|\hat{X}^{i-1}_r|^2 + W_2(\mathbb{P}^\alpha_{X^i_r}, \mathbb{P}^\alpha_{X^{i-1}_r})^2\right) \nu(d\theta) dr\Big] \le C \int_t^u \mathbb{E}|\hat{X}^{i-1}_r|^2 dr$$

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- Introduce $\hat{X}^i = X^{i+1} - X^i$

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$$+ \int_t^s \int_{\mathbb{R}^q} \Big[\beta(X^i_{r-}, \mathbb{P}^\alpha_{X^i_{r-}}, \alpha_{r-}, \theta) - \beta(X^{i-1}_{r-}, \mathbb{P}^\alpha_{X^{i-1}_{r-}}, \alpha_{r-}, \theta)\Big] N(dr, d\theta), \quad s \in [t, T].$$

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- Apply recursively

$$\mathbb{E}|\hat{X}^i_r|^2 \le C \int_t^r \mathbb{E}|\hat{X}^{i-1}_{r_1}|^2 dr_1 \le C^2 \int_t^r \int_t^{r_1} \mathbb{E}|\hat{X}^{i-2}_{r_2}|^2 dr_2 dr_1$$
$$\le \ldots \le C^i \int_t^r \int_t^{r_1} \cdots \int_t^{r_{i-1}} \mathbb{E}|\hat{X}^0_{r_{i-1}}|^2 dr_{i-1} \cdots dr_2 dr_1 \le K \frac{[C(r-t)]^i}{i!},$$

Thus,

$$\sum_{i=0}^{\infty}\left(\mathbb{E}\left[\sup_{s\in[t,u]}|\hat{X}_s^i|^2\right]\right)^{1/2}<\infty \implies \mathbb{E}\left[W_2(\mathbb{P}^{\alpha}_{X_r^i},\mathbb{P}^{\alpha}_{X_r})^2\right]\leq \|X^i-X\|^2_{\mathcal{S}^2_{\mathbb{F}}(t,T;\mathbb{R}^d)}\to 0$$

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and taking $i\to\infty$ we get

$$X_s=\xi+\int_t^s\sigma(X_r,\mathbb{P}^{\alpha}_{X_r},\alpha_r)dW_r+\int_t^s\int_{\mathbb{R}^q}\beta(X_{r-},\mathbb{P}^{\alpha}_{X_r},\alpha_{r-},\theta)N(dr,d\theta),\quad s\in[t,T].$$

Thus,

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By the same process as before, if we consider two solutions X_1 and X_2, we have the following inequality

$$\mathbb{E}\left[|X_u^1-X_u^2|^2\right]\leq C\int_t^u\mathbb{E}\left[|X_r^1-X_r^2|^2\right]dr.$$

From which Gronwall's inequality yields $X_1=X_2$.

From the uniqueness of (4.1) and (4.2) we have

$$X_s^{t,\xi,i_0} = X_s^{t,x,\xi,i_0}\Big|_{x=\xi} = X_s^{t,\xi,\xi,i_0}, \qquad s \in [t, T].$$

and the flow property

$$\left(X_r^{s,X_s^{t,x,\xi,i_0},X_s^{t,\xi,i_0},\alpha_s}, X_r^{s,X_s^{t,\xi,i_0},\alpha_s}\right) = (X_r^{t,x,\xi,i_0}, X_r^{t,\xi,i_0}), \qquad r \in [s, T].$$

Itô's Formula: Theorem

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Theorem (Itô's Formula)

For any function $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ *such that for each* $i_0 \in \mathcal{M}$, $\Psi(\cdot,\cdot,\cdot,i_0) \in \mathcal{C}^{1,2,(1,1)}([0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ *we have*

$$
\begin{aligned}
&\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) \\
&= \int_s^t \left\{ \partial_t \Psi_r + \partial_x \Psi_r \cdot b_r + \frac{1}{2}\left[\partial_{xx}\Psi_r \sigma_r \sigma_r^T\right] + Q\Psi_r(\alpha_r) \right\} dr \\
&\quad + \int_s^t \hat{\mathbb{E}}^{\alpha} \left\{ \partial_\mu \Psi_r(\hat{X}_r) \cdot \hat{b}_r + \frac{1}{2} Tr\left[\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Psi_r\right] \right. \\
&\quad \left. + \int_{\mathbb{R}^q} \left(\frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-}) \right) d\nu(\theta) \right\} dr \\
&\quad + \int_s^t \partial_x \Psi_r \sigma_r dW_r + \int_s^t \int_{\mathbb{R}^q} \left[\Psi_{r-}(X_{r-} + \beta_{r-}) - \Psi_{r-} \right] N(dr, d\theta) \\
&\quad + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \left(\Psi_s(j_0) - \Psi_s(i_0) \right) dM_{i_0 j_0}(s).
\end{aligned} \tag{4.3}
$$

Itô's Formula: Proof [1/2]

- $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$

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- Fix a positive integer n and let $\pi^m_{t,s} = (t = t^m_0 < t^m_1 < \ldots < t^m_{m+1} = s)$ be a partition of $[s,t]$. Consider $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,(1,1)}([0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$

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- $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$

$$\Phi(\mathbb{P}^\alpha_{X_s}) = \Phi(\xi) + \int_t^s \hat{\mathbb{E}}^\alpha \Big[\partial_\mu \Phi(\mathbb{P}^\alpha_{X_r}, \hat{X}_r) \cdot \hat{b}_r + \frac{1}{2} \mathrm{Tr}\big(\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Phi(\mathbb{P}^\alpha_{X_r}, \hat{X}_r)\big) + \int_{\mathbb{R}^q} \Big(\frac{\delta \Phi}{\delta \mu}(\mathbb{P}^\alpha_{X_r}, \hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Phi}{\delta \mu}(\mathbb{P}^\alpha_{X_r}, \hat{X}_{r-}) \Big) d\nu(\theta) \Big] dr. \tag{4.4}$$

- Fix a positive integer n and let $\pi^m_{t,s} = (t = t^m_0 < t^m_1 < \ldots < t^m_{m+1} = s)$ be a partition of $[s,t]$. Consider $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,(1,1)}([0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$

- $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$

$$\Phi(\mathbb{P}^{\alpha}_{X_s}) = \Phi(\xi) + \int_t^s \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\big(\hat{\sigma}_r\hat{\sigma}_r^T \partial_x\partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\big)$$
$$+ \int_{\mathbb{R}^q}\Big(\frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-})\Big)d\nu(\theta)\Big]dr. \tag{4.4}$$

- Fix a positive integer n and let $\pi^m_{t,s} = (t = t^m_0 < t^m_1 < \ldots < t^m_{m+1} = s)$ be a partition of $[s,t]$. Consider $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,(1,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}) = \sum_{j=0}^{m}\Big[\Psi(t^m_{j+1}, X_{t^m_{j+1}}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}})\Big]$$
$$+ \sum_{j=0}^{m}\Big[\Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_j}})\Big].$$

- $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$

$$\Phi(\mathbb{P}^{\alpha}_{X_s}) = \Phi(\xi) + \int_t^s \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\big(\hat{\sigma}_r\hat{\sigma}_r^T \partial_x\partial_\mu\Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\big) + \int_{\mathbb{R}^q}\Big(\frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-})\Big)d\nu(\theta)\Big]dr. \tag{4.4}$$

- Fix a positive integer n and let $\pi^m_{t,s} = (t = t^m_0 < t^m_1 < \ldots < t^m_{m+1} = s)$ be a partition of $[s,t]$. Consider $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,(1,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}) = \sum_{j=0}^{m}\Big[\Psi(t^m_{j+1}, X_{t^m_{j+1}}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}})\Big] + \sum_{j=0}^{m}\Big[\Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_j}})\Big].$$

Itô's Formula: Proof [1/2]

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- $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$

$$\Phi(\mathbb{P}^{\alpha}_{X_s}) = \Phi(\xi) + \int_t^s \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\big(\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\big) + \int_{\mathbb{R}^q}\Big(\frac{\delta \Phi}{\delta \mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Phi}{\delta \mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-})\Big) d\nu(\theta)\Big] dr. \tag{4.4}$$

- Fix a positive integer n and let $\pi^m_{t,s} = (t = t^m_0 < t^m_1 < \ldots < t^m_{m+1} = s)$ be a partition of $[s,t]$. Consider $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,(1,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}) = \sum_{j=0}^{m}\Big[\Psi(t^m_{j+1}, X_{t^m_{j+1}}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}})\Big] + \sum_{j=0}^{m}\Big[\Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_j}})\Big].$$

- Invoke standard Itô formula for jump-diffusions

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- $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$

$$\Phi(\mathbb{P}^{\alpha}_{X_s}) = \Phi(\xi) + \int_t^s \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\big(\hat{\sigma}_r\hat{\sigma}_r^T \partial_x\partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\big) + \int_{\mathbb{R}^q}\Big(\frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-})\Big)d\nu(\theta)\Big]dr. \tag{4.4}$$

- Fix a positive integer n and let $\pi^m_{t,s} = (t = t_0^m < t_1^m < \ldots < t_{m+1}^m = s)$ be a partition of $[s,t]$. Consider $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,(1,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}) = \sum_{j=0}^{m}\Big[\Psi(t_{j+1}^m, X_{t_{j+1}^m}, \mathbb{P}^{\alpha}_{X_{t_{j+1}^m}}) - \Psi(t_j^m, X_{t_j^m}, \mathbb{P}^{\alpha}_{X_{t_{j+1}^m}})\Big] + \sum_{j=0}^{m}\Big[\Psi(t_j^m, X_{t_j^m}, \mathbb{P}^{\alpha}_{X_{t_{j+1}^m}}) - \Psi(t_j^m, X_{t_j^m}, \mathbb{P}^{\alpha}_{X_{t_j^m}})\Big].$$

- Invoke standard Itô formula for jump-diffusions

- $\Phi \in \mathcal{C}^{1,1}(\mathcal{P}_2(\mathbb{R}^d))$

$$\Phi(\mathbb{P}^{\alpha}_{X_s}) = \Phi(\xi) + \int_t^s \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\big(\hat{\sigma}_r\hat{\sigma}_r^T \partial_x\partial_\mu\Phi(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_r)\big)$$
$$+ \int_{\mathbb{R}^q}\Big(\frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta\Phi}{\delta\mu}(\mathbb{P}^{\alpha}_{X_r}, \hat{X}_{r-})\Big)d\nu(\theta)\Big]dr. \quad (4.4)$$

- Fix a positive integer n and let $\pi^m_{t,s} = (t = t^m_0 < t^m_1 < \ldots < t^m_{m+1} = s)$ be a partition of $[s,t]$. Consider $\Psi(\cdot,\cdot,\cdot) \in \mathcal{C}^{1,2,(1,1)}([0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d))$

$$\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}) = \sum_{j=0}^{m}\Big[\Psi(t^m_{j+1}, X_{t^m_{j+1}}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}})\Big]$$
$$+ \sum_{j=0}^{m}\Big[\Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_{j+1}}}) - \Psi(t^m_j, X_{t^m_j}, \mathbb{P}^{\alpha}_{X_{t^m_j}})\Big].$$

- Invoke standard Itô formula for jump-diffusions
- Use (4.4)

- Consider $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ such that for each $i_0 \in \mathcal{M}$, $\Psi(\cdot, \cdot, \cdot, i_0) \in \mathcal{C}^{1,2,(1,1)}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$

Itô's Formula: Proof [2/2]

- Consider $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ such that for each $i_0 \in \mathcal{M}$, $\Psi(\cdot, \cdot, \cdot, i_0) \in \mathcal{C}^{1,2,(1,1)}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$
- Let $t = \tau_0 < \tau_1 < \ldots < \tau_k < s$ be the jump times of the Markov chain. Denote $\tau_{k+1} := s$

- Consider $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ such that for each $i_0 \in \mathcal{M}$, $\Psi(\cdot, \cdot, \cdot, i_0) \in \mathcal{C}^{1,2,(1,1)}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$
- Let $t = \tau_0 < \tau_1 < \ldots < \tau_k < s$ be the jump times of the Markov chain. Denote $\tau_{k+1} := s$

$$\begin{aligned}\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)\text{-}\Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) &= \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}})\text{-}\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}\text{-}})\right] \\ &+ \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}-}) - \Psi(\tau_i, X_{\tau_i}, \mathbb{P}^{\alpha}_{X_{\tau_i}}, \alpha_{\tau_i})\right].\end{aligned}$$

Itô's Formula: Proof [2/2]

- Consider $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ such that for each $i_0 \in \mathcal{M}$, $\Psi(\cdot, \cdot, \cdot, i_0) \in \mathcal{C}^{1,2,(1,1)}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$
- Let $t = \tau_0 < \tau_1 < \ldots < \tau_k < s$ be the jump times of the Markov chain. Denote $\tau_{k+1} := s$

$$\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)\text{-}\Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) = \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}})\text{-}\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}\text{-}})\right]$$
$$+ \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}-}) - \Psi(\tau_i, X_{\tau_i}, \mathbb{P}^{\alpha}_{X_{\tau_i}}, \alpha_{\tau_i})\right].$$

- Consider $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ such that for each $i_0 \in \mathcal{M}$, $\Psi(\cdot,\cdot,\cdot,i_0) \in \mathcal{C}^{1,2,(1,1)}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$
- Let $t = \tau_0 < \tau_1 < \ldots < \tau_k < s$ be the jump times of the Markov chain. Denote $\tau_{k+1} := s$

$$\Psi(s, X_s, \mathbb{P}^\alpha_{X_s}, \alpha_s) - \Psi(t, X_t, \mathbb{P}^\alpha_{X_t}, \alpha_t) = \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^\alpha_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}}) - \Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^\alpha_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}-})\right]$$
$$+ \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^\alpha_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}-}) - \Psi(\tau_i, X_{\tau_i}, \mathbb{P}^\alpha_{X_{\tau_i}}, \alpha_{\tau_i})\right].$$

- $\int_t^s Q\Psi(r, X_r, \mathbb{P}^\alpha_{X_r})(\alpha_r)dr + \sum_{i_0,j_0 \in \mathcal{M}} \int_t^s \left(\Psi(r, X_r, \mathbb{P}^\alpha_{X_r}, j_0) - \Psi(r, X_r, \mathbb{P}^\alpha_{X_r}, i_0)\right) dM_{i_0 j_0}(r)$

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- Consider $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ such that for each $i_0 \in \mathcal{M}$, $\Psi(\cdot, \cdot, \cdot, i_0) \in \mathcal{C}^{1,2,(1,1)}([0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$
- Let $t = \tau_0 < \tau_1 < \ldots < \tau_k < s$ be the jump times of the Markov chain. Denote $\tau_{k+1} := s$

$$\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s) \text{-} \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) = \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}}) \text{-} \Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}\text{-}})\right] + \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^{\alpha}_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}-}) - \Psi(\tau_i, X_{\tau_i}, \mathbb{P}^{\alpha}_{X_{\tau_i}}, \alpha_{\tau_i})\right].$$

- $\int_t^s Q\Psi(r, X_r, \mathbb{P}^{\alpha}_{X_r})(\alpha_r)dr + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \left(\Psi(r, X_r, \mathbb{P}^{\alpha}_{X_r}, j_0) - \Psi(r, X_r, \mathbb{P}^{\alpha}_{X_r}, i_0)\right) dM_{i_0 j_0}(r)$

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- Consider $\Psi : [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to \mathbb{R}$ such that for each $i_0 \in \mathcal{M}$, $\Psi(\cdot,\cdot,\cdot,i_0) \in \mathcal{C}^{1,2,(1,1)}([0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$
- Let $t = \tau_0 < \tau_1 < \ldots < \tau_k < s$ be the jump times of the Markov chain. Denote $\tau_{k+1} := s$

$$\Psi(s, X_s, \mathbb{P}^\alpha_{X_s}, \alpha_s) - \Psi(t, X_t, \mathbb{P}^\alpha_{X_t}, \alpha_t) = \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^\alpha_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}}) - \Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^\alpha_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}-})\right]$$
$$+ \sum_{i=0}^{k} \left[\Psi(\tau_{i+1}, X_{\tau_{i+1}}, \mathbb{P}^\alpha_{X_{\tau_{i+1}}}, \alpha_{\tau_{i+1}-}) - \Psi(\tau_i, X_{\tau_i}, \mathbb{P}^\alpha_{X_{\tau_i}}, \alpha_{\tau_i})\right].$$

- $\int_t^s Q\Psi(r, X_r, \mathbb{P}^\alpha_{X_r})(\alpha_r)dr + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \left(\Psi(r, X_r, \mathbb{P}^\alpha_{X_r}, j_0) - \Psi(r, X_r, \mathbb{P}^\alpha_{X_r}, i_0)\right) dM_{i_0 j_0}(r)$
- $\alpha_{\tau_i} = \alpha_{\tau_{i+1}-}$ for $0 \le i \le k \implies$ Use previous formula.

Altogether,

$$
\begin{aligned}
&\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) \\
&= \int_s^t \left\{ \partial_t \Psi_r + \partial_x \Psi_r \cdot b_r + \frac{1}{2}\left[\partial_{xx}\Psi_r \sigma_r \sigma_r^T\right] + Q\Psi_r(\alpha_r) \right\} dr \\
&\quad + \int_s^t \hat{\mathbb{E}}^{\alpha} \Bigg\{ \partial_\mu \Psi_r(\hat{X}_r) \cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Psi_r\right] \\
&\quad + \int_{\mathbb{R}^q} \left(\frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-}) \right) d\nu(\theta) \Bigg\} dr \\
&\quad + \int_s^t \partial_x \Psi_r \sigma_r dW_r + \int_s^t \int_{\mathbb{R}^q} \left[\Psi_{r-}(X_{r-} + \beta_{r-}) - \Psi_{r-} \right] N(dr, d\theta) \\
&\quad + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \Big(\Psi_s(j_0) - \Psi_s(i_0) \Big) dM_{i_0 j_0}(s).
\end{aligned}
$$

- Jump-diffusion terms
- Itô's formula for flows of measures
- Markov terms

Itô's Formula: Proof [3/3]

Altogether,

$$
\begin{aligned}
&\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) \\
&= \int_s^t \left\{ \partial_t \Psi_r + \partial_x \Psi_r \cdot b_r + \frac{1}{2}\left[\partial_{xx}\Psi_r \sigma_r \sigma_r^T\right] + Q\Psi_r(\alpha_r) \right\} dr \\
&\quad + \int_s^t \hat{\mathbb{E}}^{\alpha} \Bigg\{ \partial_\mu \Psi_r(\hat{X}_r) \cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Psi_r\right] \\
&\quad + \int_{\mathbb{R}^q} \left(\frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-}) \right) d\nu(\theta) \Bigg\} dr \\
&\quad + \int_s^t \partial_x \Psi_r \sigma_r dW_r + \int_s^t \int_{\mathbb{R}^q} \left[\Psi_{r-}(X_{r-} + \beta_{r-}) - \Psi_{r-}\right] N(dr, d\theta) \\
&\quad + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \Big(\Psi_s(j_0) - \Psi_s(i_0) \Big) dM_{i_0 j_0}(s).
\end{aligned}
$$

- Jump-diffusion terms
- Itô's formula for flows of measures
- Markov terms

Altogether,

$$
\begin{aligned}
&\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) \\
&= \int_s^t \left\{ \partial_t \Psi_r + \partial_x \Psi_r \cdot b_r + \frac{1}{2}\left[\partial_{xx}\Psi_r \sigma_r \sigma_r^T\right] + Q\Psi_r(\alpha_r) \right\} dr \\
&\quad + \int_s^t \hat{\mathbb{E}}^{\alpha} \left\{ \partial_\mu \Psi_r(\hat{X}_r) \cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Psi_r\right] \right. \\
&\quad \left. + \int_{\mathbb{R}^q} \left(\frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-}) \right) d\nu(\theta) \right\} dr \\
&\quad + \int_s^t \partial_x \Psi_r \sigma_r dW_r + \int_s^t \int_{\mathbb{R}^q} \left[\Psi_{r-}(X_{r-} + \beta_{r-}) - \Psi_{r-}\right] N(dr, d\theta) \\
&\quad + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \Big(\Psi_s(j_0) - \Psi_s(i_0) \Big) dM_{i_0 j_0}(s).
\end{aligned}
$$

- Jump-diffusion terms
- Itô's formula for flows of measures
- Markov terms

Altogether,

$$
\begin{aligned}
&\Psi(s, X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s) - \Psi(t, X_t, \mathbb{P}^{\alpha}_{X_t}, \alpha_t) \\
&= \int_s^t \left\{ \partial_t \Psi_r + \partial_x \Psi_r \cdot b_r + \frac{1}{2}\left[\partial_{xx}\Psi_r \sigma_r \sigma_r^T\right] + Q\Psi_r(\alpha_r) \right\} dr \\
&\quad + \int_s^t \hat{\mathbb{E}}^{\alpha} \left\{ \partial_\mu \Psi_r(\hat{X}_r) \cdot \hat{b}_r + \frac{1}{2}\mathrm{Tr}\left[\hat{\sigma}_r \hat{\sigma}_r^T \partial_x \partial_\mu \Psi_r\right] \right. \\
&\quad \left. + \int_{\mathbb{R}^q} \left(\frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-} + \hat{\beta}_{r-}(\theta)) - \frac{\delta \Psi}{\delta \mu}(\hat{X}_{r-}) \right) d\nu(\theta) \right\} dr \\
&\quad + \int_s^t \partial_x \Psi_r \sigma_r dW_r + \int_s^t \int_{\mathbb{R}^q} \left[\Psi_{r-}(X_{r-} + \beta_{r-}) - \Psi_{r-} \right] N(dr, d\theta) \\
&\quad + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \left(\Psi_s(j_0) - \Psi_s(i_0) \right) dM_{i_0 j_0}(s).
\end{aligned}
$$

- Jump-diffusion terms
- Itô's formula for flows of measures
- Markov terms

Recall:

- The State equation
- The Cost Functional
- The Hamiltonian

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$$
\begin{aligned}
dX_s &= \tilde{b}(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\
&\qquad + \int_{\mathbb{R}^q} \beta(X_{s-}, u_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta), \quad s \geq t, \\
X_{t-} &= \xi \in L^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_{s-} = i_0,
\end{aligned}
$$

Recall:

- The State equation
- The Cost Functional
- The Hamiltonian

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$$J(t, \xi, i_0, u) = \mathbb{E}\left[\int_t^T f(X_s^{t,\xi,i_0,u}, u_s, \mathbb{P}^{\alpha}_{X_s^{t,\xi,i_0,u}}, \alpha_s)ds + g(X_T^{t,\xi,i_0,u}, \mathbb{P}^{\alpha}_{X_T^{t,\xi,i_0,u}}, \alpha_T)\right].$$

Recall:

- The State equation
- The Cost Functional
- The Hamiltonian

$$
\begin{aligned}
dX_s &= \tilde{b}(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\
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$$
J(t, \xi, i_0, u) = \mathbb{E}\left[\int_t^T f(X_s^{t,\xi,i_0,u}, u_s, \mathbb{P}^{\alpha}_{X_s^{t,\xi,i_0,u}}, \alpha_s)ds + g(X_T^{t,\xi,i_0,u}, \mathbb{P}^{\alpha}_{X_T^{t,\xi,i_0,u}}, \alpha_T) \right].
$$

$$
\begin{aligned}
H(x, u, \mu, p, P, r, i_0) &= f(x, u, \mu, i_0) + b(x, u, \mu, i_0) \cdot p + \frac{1}{2}\mathrm{Tr}\left[\sigma(x, u, \mu, i_0)\sigma^T(x, u, \mu, i_0)P\right] \\
&\quad + \int_{\mathbb{R}^q} \left(r(x + \beta(x, u, \mu, i_0, \theta)) - r(x)\right)\nu(d\theta).
\end{aligned}
$$

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Theorem (DPP)

The following dynamic programming holds

$$V(t,\mu,i_0) = \inf_{u\in\mathcal{U}} \mathbb{E}\left\{ \left[\int_t^s f(X_r^{t,\xi,i_0,u}, u_r, \mathbb{P}^{\alpha}_{X_r^{t,\xi,i_0,u}}, \alpha_r)dr \right] + V(s, \mathbb{P}^{\alpha}_{X_s^{t,\xi,i_0,u}}, \alpha_s) \right\},$$

for any (s,t) such that $0 \le t < s \le T$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, and $\xi \in L^2(\mathcal{F}_t, \mathbb{R}^d)$ such that $\mathbb{P}_\xi = \mu$.

Hamilton-Jacobi-Bellman Equation

$$\partial_t V(t,\mu,i_0) + \mathbb{E}\left[\inf_{u\in U} H(\xi, u, \mu, \partial_\mu V(t,\mu,i_0,\xi), \partial_x\partial_\mu V(t,\mu,i_0,\xi), \frac{\delta V}{\delta\mu}(t,\mu,i_0,\xi), i_0) \right]$$

$$+QV(t,\mu)(i_0) = 0,$$
$$V(T,\mu,i_0) = \langle g(\cdot,\mu,i_0),\mu\rangle.$$

Theorem (VT)

(a) *Suppose that V^* is a classical solution to the HJB equation, then for any $(t, \mu, i_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M}$, $V^*(t, \mu, i_0) \leq V(t, \mu, i_0)$, where V is the value function.*

(b) *If there exists a function $u^* : [0, T) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M} \to U$ which is continuous in (t, x, μ) for each $i_0 \in \mathcal{M}$ such that*

$$u^*(t, x, \mu, i_0) \in \arg\min_{u \in U} \mathcal{H}\left(x, u, \mu, \partial_\mu V_t^*, \partial_x \partial_\mu V_t^*, \frac{\delta V^*}{\delta \mu}(i_0)\right)$$

and the corresponding dynamics $(X_s)_{T \geq s \geq t}$ using the control $u^* \in \mathcal{U}$, $u_s^* = u^*(X_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)$ admits a unique solution denoted by X_s^*. Then $V^* = V$ and u^* is an optimal control.

Verification Theorem (b): Proof [1/2]

- Invoke our Itô's formula

$$
\begin{aligned}
&V^*(t,\mu,i_0)\\
&= V^*(T,\mathbb{P}^{\alpha}_{X^*_T},\alpha_T) - \int_t^T \Big\{ \partial_s V^*(s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s) + \mathbb{E}^{\alpha}\Big[\partial_\mu V^*(s,\mathbb{P}^{\alpha}_{X^*_s},X_s)\cdot b(X^*_s,u^*_s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s)\\
&\quad + \frac{1}{2}\mathrm{Tr}\Big(\sigma\sigma^T(X^*_s,u^*_s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s)\partial_x\partial_\mu V^*(s,\mathbb{P}^{\alpha}_{X^*_s},\alpha(s),X^*_s)\Big)\\
&\quad + \int_{\mathbb{R}^q}\Big(\frac{\delta V^*}{\delta\mu}(s,\mathbb{P}_{X^*_s},\alpha_{s-},X^*_{s-}+\beta(X^*_s,u^*_s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_{s-}\theta)) - \frac{\delta V^*}{\delta\mu}(s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s,X^*_s)\nu(d\theta)\Big)\Big]\\
&\quad + QV^*(s,\mathbb{P}^{\alpha}_{X^*_s})(\alpha_s)\Big\} ds + \sum_{j_0,k_0\in\mathcal{M}}\Big(V^*(s,\mathbb{P}^{\alpha}_{X^*_s},k_0) - V^*(s,\mathbb{P}^{\alpha}_{X^*_s},j_0)\Big) dM_{j_0k_0}(s)
\end{aligned}
$$

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$$
\begin{aligned}
&V^*(t,\mu,i_0)\\
&= V^*(T,\mathbb{P}^{\alpha}_{X^*_T},\alpha_T) - \int_t^T \Bigg\{\partial_s V^*(s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s) + \mathbb{E}^{\alpha}\Bigg[\partial_\mu V^*(s,\mathbb{P}^{\alpha}_{X^*_s},X_s)\cdot b(X^*_s,u^*_s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s)\\
&\quad + \frac{1}{2}\mathrm{Tr}\Big(\sigma\sigma^T(X^*_s,u^*_s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s)\partial_x\partial_\mu V^*(s,\mathbb{P}^{\alpha}_{X^*_s},\alpha(s),X^*_s)\Big)\\
&\quad + \int_{\mathbb{R}^q}\Big(\frac{\delta V^*}{\delta\mu}(s,\mathbb{P}_{X^*_s},\alpha_{s-},X^*_{s-}+\beta(X^*_s,u^*_s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_{s-}\theta)) - \frac{\delta V^*}{\delta\mu}(s,\mathbb{P}^{\alpha}_{X^*_s},\alpha_s,X^*_s)\nu(d\theta)\Big)\Bigg]\\
&\quad + QV^*(s,\mathbb{P}^{\alpha}_{X^*_s})(\alpha_s)\Bigg\}ds + \sum_{j_0,k_0\in\mathcal{M}}\Big(V^*(s,\mathbb{P}^{\alpha}_{X^*_s},k_0) - V^*(s,\mathbb{P}^{\alpha}_{X^*_s},j_0)\Big)dM_{j_0k_0}(s)
\end{aligned}
$$

- Matches our Hamiltonian, thus

- Invoke our Itô's formula

$$
\begin{aligned}
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&\quad + QV^*(s,\mathbb{P}^{\alpha}_{X^*_s})(\alpha_s)\Big\}ds + \sum_{j_0,k_0\in\mathcal{M}}\Big(V^*(s,\mathbb{P}^{\alpha}_{X^*_s},k_0) - V^*(s,\mathbb{P}^{\alpha}_{X^*_s},j_0)\Big)dM_{j_0k_0}(s)
\end{aligned}
$$

- Matches our Hamiltonian, thus

$$
\begin{aligned}
V^*(t,\mu,i_0) &= V^*(T,\mathbb{P}^{\alpha}_{X^*_T},\alpha_T) + \int_t^T \mathbb{E}^{\alpha}\Big[f(X^*_s,u^*_s,\mathbb{P}_{X^*_s},\alpha_s)\Big]ds - \int_t^T\Big\{\partial_s V^*(s,\mathbb{P}_{X^*_s},\alpha_s)\\
&\quad + \mathbb{E}^{\alpha}\Big[H\Big(X^*_s,u^*_s,\mathbb{P}_{X^*_s},\partial_\mu V^*(s,\mathbb{P}_{X^*_s},\alpha_s,X^*_s),\partial_x\partial_\mu V^*(s,\mathbb{P}_{X^*_s},\alpha_s,X^*_s),\frac{\delta V^*}{\delta\mu}(s,\mathbb{P}_{X^*_s},\alpha_s,X^*_s)\Big)\Big]\\
&\quad + QV^*(s,\mathbb{P}^{\alpha}_{X^*_s})(\alpha_s)\Big\}ds + \sum_{j_0,k_0\in\mathcal{M}}\Big(V^*(s,\mathbb{P}^{\alpha}_{X^*_s},k_0) - V^*(s,\mathbb{P}^{\alpha}_{X^*_s},j_0)\Big)dM_{j_0k_0}(s)
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By our assumption on u^*, we have

$$
\begin{aligned}
&V^*(t,\mu,i_0)\\
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$$\begin{aligned} & V^*(t, \mu, i_0) \\ & = V^*(T, \mathbb{P}^{\alpha}_{X^*_T}, \alpha_T) + \mathbb{E}^{\alpha}\Big[\int_t^T f(X^*_s, u^*_s, \mathbb{P}_{X^*_s}, \alpha_s)ds\Big] \\ & \quad + \sum_{j_0, k_0 \in \mathcal{M}} \Big(V^*(s, \mathbb{P}^{\alpha}_{X^*_s}, k_0) - V^*(s, \mathbb{P}^{\alpha}_{X^*_s}, j_0)\Big) dM_{j_0 k_0}(s). \end{aligned}$$

Taking the expectation yields

$$V^*(t, \mu, i_0) = \mathbb{E}\Big[g(X^*_T, \mathbb{P}^{\alpha}_{X^*_T}, \alpha_T) + \int_t^T f(X^*_s, u^*_s, \mathbb{P}_{X^*_s}, \alpha_s)ds\Big].$$

Theorem (a)

(a) $V^*(t, \mu, i_0) \le V(t, \mu, i_0)$

By our assumption on u^*, we have

$$\begin{aligned} & V^*(t, \mu, i_0) \\ & = V^*(T, \mathbb{P}^{\alpha}_{X^*_T}, \alpha_T) + \mathbb{E}^{\alpha}\Big[\int_t^T f(X^*_s, u^*_s, \mathbb{P}_{X^*_s}, \alpha_s) ds\Big] \\ & \quad + \sum_{j_0, k_0 \in \mathcal{M}} \Big(V^*(s, \mathbb{P}^{\alpha}_{X^*_s}, k_0) - V^*(s, \mathbb{P}^{\alpha}_{X^*_s}, j_0)\Big) dM_{j_0 k_0}(s). \end{aligned}$$

Taking the expectation yields

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$$\implies V^*(t, \mu, i_0) = V(t, \mu, i_0) \quad \square$$

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Theorem (Sufficient Condition for Optimality)

Assume that assumptions (H1)-(H4) hold. Let $\bar{u}$ be a control in $\mathcal{U}$ with corresponding state trajectory $X^{\bar{u}}(\cdot)$. Let $(\bar{p}(\cdot), \bar{P}(\cdot), \bar{r}(\cdot), \bar{\lambda}(\cdot))$ be the solution to the adjoint equation corresponding to $\bar{u}(\cdot) \in \mathcal{U}$. If

$$\mathcal{H}(X_t^{\bar{u}}, \bar{u}_t, \mathbb{P}_{X^{\bar{u}}}, \bar{p}_t, \bar{P}_t, \bar{r}_t) = \inf_{v \in \mathcal{U}} \mathcal{H}(X^{\bar{u}}, v, \mathbb{P}_{X^{\bar{u}}}, \bar{p}_t, \bar{P}_t, \bar{r}_t)$$

then $\bar{u}(\cdot)$ is an optimal control.

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$$\mathcal{H}(X_t^{\bar{u}}, \bar{u}_t, \mathbb{P}_{X^{\bar{u}}}, \bar{p}_t, \bar{P}_t, \bar{r}_t) = \inf_{v \in \mathcal{U}} \mathcal{H}(X^{\bar{u}}, v, \mathbb{P}_{X^{\bar{u}}}, \bar{p}_t, \bar{P}_t, \bar{r}_t)$$

then $\bar{u}(\cdot)$ is an optimal control.

Suppose that $u^* \in \mathcal{U}$ is an optimal control with the corresponding optimal state X^*. For each $v(\cdot) \in \mathcal{U}$ and $\gamma \in [0,1]$ denote the perturbed control

$$u_s^\gamma = u_s^* + \gamma(v_s - u_s^*), \qquad s \geq 0.$$

Lemma (5.1)

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t^* - X_t^\gamma|^2\right] \le C\gamma^2.$$

Stochastic Maximum Principle: Lemmas

Lemma (5.1)

$$\mathbb{E}\Big[\sup_{0\le t\le T}|X_t^* - X_t^\gamma|^2\Big] \le C\gamma^2.$$

Lemma (5.2)

Let $\phi(\cdot)$ be the solution of the following linear McKean-Vlasov equation

$$\begin{aligned} d\phi_t &= \Big[\bar{b}_x(t)\phi_t + \hat{\mathbb{E}}^\alpha\big(\hat{b}_\mu(t)\hat{\phi}_t\big) + \bar{b}_u(t)(v_t - u_t^*)\Big]dt \\ &\quad + \Big[\bar{\sigma}_x(t)\phi_t + \hat{\mathbb{E}}^\alpha\big(\hat{\sigma}_\mu(t)\hat{\phi}_t\big) + \bar{\sigma}_u(t)(v_t - u_t^*)\Big]dW_t \\ &\quad + \int_{\mathbb{R}^q}\Big[\bar{\beta}_x(t-,\theta)\phi_{t-} + \hat{\mathbb{E}}^\alpha\big(\hat{\beta}_\mu(t-,\theta)\hat{\phi}_{t-}\big) + \bar{\beta}_u(t-,\theta)(v_{t-} - u_{t-}^*)\Big]\tilde{N}(dt,d\theta), \\ \phi_0 &= 0. \end{aligned} \tag{4.5}$$

Then $\phi(\cdot)$ is the derivative of X^γ with respect to γ in the $\mathcal{L}_2$-sense. More precisely,

$$\lim_{t\to 0_+}\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\frac{X_t^\gamma - X_t^*}{\gamma} - \phi_t\Big|^2\Big] = 0.$$

Stochastic Maximum Principle: Lemmas

Lemma (5.3)

The Gateaux derivative of the cost functional $J(\cdot)$ is given by the following formula

$$\frac{d}{d\gamma} J(u^* + \gamma(v - u^*))\Big|_{\gamma=0} = \mathbb{E}\int_0^T \Big[\bar{f}_x(t)\phi(t) + \hat{\mathbb{E}}^\alpha\big(\hat{f}_\mu(t)\hat{\phi}(t)\big) + \bar{f}_u(t)(v_t - u_t^*)\Big]dt$$
$$+ \mathbb{E}\Big[\bar{g}_x(T)\phi(T) + \hat{\mathbb{E}}^\alpha\big(\hat{g}_\mu(T)\hat{\phi}(T)\big)\Big].$$

Lemma (5.3)

The Gateaux derivative of the cost functional $J(\cdot)$ is given by the following formula

$$\frac{d}{d\gamma} J(u^* + \gamma(v - u^*))\Big|_{\gamma=0} = \mathbb{E} \int_0^T \Big[\bar{f}_x(t)\phi(t) + \hat{\mathbb{E}}^\alpha\big(\hat{f}_\mu(t)\hat{\phi}(t)\big) + \bar{f}_u(t)(v_t - u_t^*)\Big] dt$$
$$+ \mathbb{E}\Big[\bar{g}_x(T)\phi(T) + \hat{\mathbb{E}}^\alpha\big(\hat{g}_\mu(T)\hat{\phi}(T)\big)\Big].$$

Adjoint Equation

Let $\phi(\cdot)$ be the solution of the following linear McKean-Vlasov equation

$$dp_t = -\Big\{\bar{b}_x^\top(t)p_t + \hat{\mathbb{E}}^\alpha\big(\check{b}_\mu^\top(t)\hat{p}_t\big) + \bar{\sigma}_x^\top(t)P_t + \hat{\mathbb{E}}^\alpha\Big[\big(\check{\sigma}_\mu^\top(t)\hat{P}_t\big)\Big]$$
$$+ \int_{\mathbb{R}^q} \Big[\bar{\beta}_x^\top(t-,\theta)r(t-,\theta) + \hat{\mathbb{E}}^\alpha\big(\check{\beta}_\mu^\top(t-,\theta)\hat{r}(t-,\theta)\big)\Big]\nu(d\theta) + \bar{f}_x(t) + \hat{\mathbb{E}}^\alpha\big(\check{f}_\mu(t)\big)\Big\}dt$$
$$+ P_t dW_t + \lambda_t \bullet dM_t + \int_{\mathbb{R}^q} r(t-,\theta)\tilde{N}(dt, d\theta),$$
$$p_T = \bar{g}_x(T) + \hat{\mathbb{E}}^\alpha(\check{g}_\mu(T)).$$

Stochastic Maximum Principle: Lemmas

Lemma (5.5)

$$\mathbb{E}(\bar{p}_T\phi_T) = \mathbb{E}\bigg[\int_0^T \Big(\bar{p}_t\bar{b}_u(t)(v_t - u_t^*) - \bar{f}_x(t)\phi_t - \bar{\mathbb{E}}^{\alpha}[\hat{f}_\mu(t)]\phi_t + \bar{P}_t\bar{\sigma}_u(t)(v_t - u_t^*)\Big)dt + \int_{\mathbb{R}^q}\bar{\beta}_u(t-,\theta)(v_{t-} - u_{t-}^*)\nu(d\theta)dt\bigg].$$

Stochastic Maximum Principle: Lemmas

Lemma (5.5)

$$\mathbb{E}(\bar{p}_T\phi_T) = \mathbb{E}\Big[\int_0^T \Big(\bar{p}_t\bar{b}_u(t)(v_t - u_t^*) - \bar{f}_x(t)\phi_t - \bar{\mathbb{E}}^{\alpha}[\hat{\bar{f}}_{\mu}(t)]\phi_t + \bar{P}_t\bar{\sigma}_u(t)(v_t - u_t^*)\Big)dt + \int_{\mathbb{R}^q} \bar{\beta}_u(t-,\theta)(v_{t-} - u_{t-}^*)\nu(d\theta)dt\Big].$$

Lemma (5.6)

The Gateaux derivative of the cost functional can be expressed in terms of the Hamiltonian $\mathcal{H}$ in the following way

$$\begin{aligned}
&\frac{d}{d\gamma}J(u^* + \gamma(v - u^*))\Big|_{\gamma=0} \\
&= \mathbb{E}\Big(\int_0^T \bar{f}_u(t)(v_t - u_t^*) + \bar{p}_t\bar{b}_u(v_t - u_t^*) + \bar{\sigma}_u(v_t - u_t)\bar{P}(t) + \int_{\mathbb{R}^q} \bar{\beta}_u(t-,\theta)\bar{r}(t-,\theta)(v_t - u_t^*)dt\Big) \\
&= \mathbb{E}\Big(\int_0^T \frac{d}{du}\mathcal{H}(X_t^{u^*}, u_t^*, \mathbb{P}_{X_t^{u^*}}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)(v_t - u_t^*)\nu dt\Big).
\end{aligned}$$

Theorem (5.7 Necessary Condition for Optimality)

Under Assumption (**A**), *if* $u^*(\cdot)$ *is an optimal control with state trajectory* $X^* = X^{u^*}$, *then there exists a quadruple* $(\bar{p}(\cdot), \bar{P}(\cdot), \bar{r}(\cdot), \bar{\lambda}(\cdot)$ *of adapted processes that satisfies the BSDE (5.3) and (5.4) such that*

$$\frac{d}{du}\mathcal{H}(X_t^*, u_t^*, \mathbb{P}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_{t-})(v - u_t^*) \geq 0, \qquad dtd\mathbb{P} - a.s.$$

on $[0, T] \times \Omega$ *for any* $v \in U$

For any $u(\cdot), v(\cdot) \in \mathcal{U}$, we have using convexity assumptions & the Fubini theorem,

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$$
\begin{aligned}
J(u(\cdot)) - J(v(\cdot)) &= \mathbb{E}\Big[g(X_T^u, \mathbb{P}^{\alpha}_{X_T^u}, \alpha_T) - g(X_T^v, \mathbb{P}^{\alpha}_{X_T^v}, \alpha_T)\Big] \\
&\quad + \mathbb{E}\Big[\int_0^T f(X_t^u, \mathbb{P}^{\alpha}_{X_t^u}, u_t, \alpha_t) - f(X_t^v, \mathbb{P}^{\alpha}_{X_t^v}, v_t, \alpha_t)\Big] dt. \\
&\geq \mathbb{E}\Big\{\Big[g_x(X_T^v, \mathbb{P}^{\alpha}_{X_T^v}, \alpha_T) + \hat{\mathbb{E}}^{\alpha}\Big(g_\mu(\hat{X}_T^v, \mathbb{P}^{\alpha}_{X_T^v}, \alpha_T; X_T^v)\Big)\Big](X_T^u - X_T^v)\Big\} \\
&\quad + \mathbb{E}\Big[\int_0^T \Big(f(X_t^u, \mathbb{P}^{\alpha}_{X_t^u}, u_t, \alpha_t) - f(X_t^v, \mathbb{P}^{\alpha}_{X_t^v}, v_t, \alpha_t)\Big) dt\Big].
\end{aligned}
$$

For any $u(\cdot), v(\cdot) \in \mathcal{U}$, we have using convexity assumptions & the Fubini theorem,

$$\begin{aligned} J(u(\cdot)) - J(v(\cdot)) &= \mathbb{E}\Big[g(X_T^u, \mathbb{P}^\alpha_{X_T^u}, \alpha_T) - g(X_T^v, \mathbb{P}^\alpha_{X_T^v}, \alpha_T)\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T f(X_t^u, \mathbb{P}^\alpha_{X_t^u}, u_t, \alpha_t) - f(X_t^v, \mathbb{P}^\alpha_{X_t^v}, v_t, \alpha_t)\Big] dt. \\ &\geq \mathbb{E}\Big\{\Big[g_x(X_T^v, \mathbb{P}^\alpha_{X_T^v}, \alpha_T) + \hat{\mathbb{E}}^\alpha\Big(g_\mu(\hat{X}_T^v, \mathbb{P}^\alpha_{X_T^v}, \alpha_T; X_T^v)\Big)\Big](X_T^u - X_T^v)\Big\} \\ &\quad + \mathbb{E}\Big[\int_0^T \Big(f(X_t^u, \mathbb{P}^\alpha_{X_t^u}, u_t, \alpha_t) - f(X_t^v, \mathbb{P}^\alpha_{X_t^v}, v_t, \alpha_t)\Big) dt\Big]. \end{aligned}$$

Denote

$$p_T^v = g_x(X_T^v, \mathbb{P}^\alpha_{X_T^v}, \alpha_T) + \hat{\mathbb{E}}^\alpha\Big[g_\mu(\hat{X}_T^v, \mathbb{P}^\alpha_{X_T^v}, \alpha_T; X_T^v)\Big].$$

Terminal condition of adjoint equation $\implies p_T^{u^*} = \bar{p}_T = \bar{g}_x(T) + \hat{\mathbb{E}}^\alpha\big(\check{g}_\mu(T)\big).$

For $v = u^*$and by using integration by parts,

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$$
\begin{aligned}
&\mathbb{E}\Big[\bar{p}_T(X_T^u - X_T^*)\Big] \\
&= \mathbb{E}\int_0^T \Big[\mathcal{H}\big(X_t^u, u_t, \mathbb{P}^{\alpha}_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big) - \mathcal{H}\big(X_t^*, u_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big)\Big]dt \\
&\quad - \mathbb{E}\int_0^T \Big[f(X_t^u, \mathbb{P}^{\alpha}_{X_t^u}, u_t, \alpha_t) - f(X_t^*, \mathbb{P}^{\alpha}_{X_t^*}, u_t^*, \alpha_t)\Big]dt \\
&\quad - \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) \\
&\quad + \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\Big]\Big\}dt.
\end{aligned}
$$

For $v = u^*$and by using integration by parts,

$$
\begin{aligned}
&\mathbb{E}\Big[\bar{p}_T(X_T^u - X_T^*)\Big] \\
&= \mathbb{E}\int_0^T \Big[\mathcal{H}(X_t^u, u_t, \mathbb{P}^{\alpha}_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) - \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)\Big] dt \\
&\quad - \mathbb{E}\int_0^T \Big[f(X_t^u, \mathbb{P}^{\alpha}_{X_t^u}, u_t, \alpha_t) - f(X_t^*, \mathbb{P}^{\alpha}_{X_t^*}, u_t^*, \alpha_t)\Big] dt \\
&\quad - \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) \\
&\quad + \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\Big]\Big\} dt.
\end{aligned}
$$

Recall

$$
\begin{aligned}
J(u(\cdot)) - J(v(\cdot)) \geq \mathbb{E}\Big\{\Big[g_x(X_T^v, \mathbb{P}^{\alpha}_{X_T^v}, \alpha_T) + \hat{\mathbb{E}}^{\alpha}\Big(g_\mu(\hat{X}_T^v, \mathbb{P}^{\alpha}_{X_T^v}, \alpha_T; X_T^v)\Big)\Big](X_T^u - X_T^v)\Big\} \\
+ \mathbb{E}\Big[\int_0^T \Big(f(X_t^u, \mathbb{P}^{\alpha}_{X_t^u}, u_t, \alpha_t) - f(X_t^v, \mathbb{P}^{\alpha}_{X_t^v}, v_t, \alpha_t)\Big) dt\Big].
\end{aligned}
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For $v = u^*$ and by using integration by parts,

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\begin{aligned}
&\mathbb{E}\Big[\bar{p}_T(X_T^u - X_T^*)\Big] \\
&= \mathbb{E}\int_0^T \Big[\mathcal{H}(X_t^u, u_t, \mathbb{P}^\alpha_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) - \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)\Big] dt \\
&\quad - \mathbb{E}\int_0^T \Big[f(X_t^u, \mathbb{P}^\alpha_{X_t^u}, u_t, \alpha_t) - f(X_t^*, \mathbb{P}^\alpha_{X_t^*}, u_t^*, \alpha_t)\Big] dt \\
&\quad - \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) \\
&\quad + \hat{\mathbb{E}}^\alpha\Big[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^\alpha_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\Big]\Big\} dt.
\end{aligned}
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Recall

$$
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+ \mathbb{E}\Big[\int_0^T \Big(f(X_t^u, \mathbb{P}^\alpha_{X_t^u}, u_t, \alpha_t) - f(X_t^v, \mathbb{P}^\alpha_{X_t^v}, v_t, \alpha_t)\Big) dt\Big].
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$$

For $v = u^*$and by using integration by parts,

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&\mathbb{E}\Big[\bar{p}_T(X_T^u - X_T^*)\Big] \\
&= \mathbb{E}\int_0^T \Big[\mathcal{H}\big(X_t^u, u_t, \mathbb{P}^\alpha_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big) - \mathcal{H}\big(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big)\Big] dt \\
&\quad - \mathbb{E}\int_0^T \Big[f\big(X_t^u, \mathbb{P}^\alpha_{X_t^u}, u_t, \alpha_t\big) - f\big(X_t^*, \mathbb{P}^\alpha_{X_t^*}, u_t^*, \alpha_t\big)\Big] dt \\
&\quad - \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) \\
&\quad + \hat{\mathbb{E}}^\alpha\Big[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^\alpha_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\Big]\Big\} dt.
\end{aligned}$$

Recall

$$\begin{aligned}
J(u(\cdot)) - J(v(\cdot)) \geq \mathbb{E}\Big\{\Big[g_x(X_T^v, \mathbb{P}^\alpha_{X_T^v}, \alpha_T) + \hat{\mathbb{E}}^\alpha\Big(g_\mu(\hat{X}_T^v, \mathbb{P}^\alpha_{X_T^v}, \alpha_T; X_T^v)\Big)\Big](X_T^u - X_T^v)\Big\} \\
+ \mathbb{E}\Big[\int_0^T \Big(f(X_t^u, \mathbb{P}^\alpha_{X_t^u}, u_t, \alpha_t) - f(X_t^v, \mathbb{P}^\alpha_{X_t^v}, v_t, \alpha_t)\Big) dt\Big].
\end{aligned}$$

We have

$$
\begin{aligned}
J(u(\cdot)) - J(u^*(\cdot)) \geq \mathbb{E}\int_0^T \Big[\mathcal{H}\big(X_t^u, u_t, \mathbb{P}^{\alpha}_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big) - \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)\Big] dt \\
- \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) \\
+ \hat{\mathbb{E}}^{\alpha}\Big[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\Big]\Big\} dt.
\end{aligned}
$$

We have

$$J(u(\cdot)) - J(u^*(\cdot)) \geq \mathbb{E}\int_0^T \Big[\mathcal{H}\big(X_t^u, u_t, \mathbb{P}^\alpha_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big) - \mathcal{H}\big(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big)\Big] dt$$
$$- \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)$$
$$+ \hat{\mathbb{E}}^\alpha\Big[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^\alpha_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\Big]\Big\} dt.$$

Recall: A function $\psi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is convex if for any (x, μ),

$$f(x', \mu') - f(x, \mu) \geq f_x(x, \mu) \cdot (x' - x) + \mathbb{E}\big[\partial_\mu f(x, \mu) \cdot (\xi' - \xi)\big],$$

We have

$$J(u(\cdot)) - J(u^*(\cdot)) \geq \mathbb{E}\int_0^T \left[\mathcal{H}(X_t^u, u_t, \mathbb{P}^\alpha_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t) - \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)\right] dt$$
$$- \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)$$
$$+ \hat{\mathbb{E}}^\alpha\left[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^\alpha_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\right]\Big\} dt.$$

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- $\mathcal{H}$ is convex with respect to (x, μ, u), thus for any $u(\cdot) \in \mathcal{U}$,

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Recall: A function $\psi : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is convex if for any (x, μ),

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Stochastic Maximum Principle: Proof [3/3]

$$\frac{d}{du}\mathcal{H}(X_t^*, u_t^*, \mathbb{P}_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_{t-})(v - u_t^*) \geq 0, \qquad dtd\mathbb{P} - a.s.$$

We have

$$J(u(\cdot)) - J(u^*(\cdot)) \geq \mathbb{E}\int_0^T \Big[\mathcal{H}\big(X_t^u, u_t, \mathbb{P}^\alpha_{X_t^u}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big) - \mathcal{H}\big(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big)\Big]dt$$
$$- \mathbb{E}\int_0^T (X_t^u - X_t^*)\Big\{\partial_x\mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t)$$
$$+ \hat{\mathbb{E}}^\alpha\Big[\partial_\mu\mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^\alpha_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t; X_t^*)\Big]\Big\}dt.$$

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$$J(u(\cdot)) - J(u^*(\cdot)) \geq \mathbb{E}\int_0^T \partial_u\mathcal{H}\big(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \bar{p}_t, \bar{P}_t, \bar{r}_t, \alpha_t\big)(u_t - u_t^*)dt \geq 0$$

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Theorem (Relation)

Assume that $V \in C^{1,(2,1)}([0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M}_t$. *If* $u^* \in \mathcal{U}$ *satisfies condition (1), then the solution of the adjoint equation is given by*

$$\bar{p}_t = \partial_\mu V(t, \mathbb{P}_{X_t^*}, \alpha_t; X_t^*), \tag{4.6}$$

$$\bar{P}_t = \partial_x \partial_\mu V(t, \mathbb{P}_{X_t^*}, \alpha_t; X_t^*)\sigma(X_t^*, u_t^*, \mathbb{P}_{X_t^*}, \alpha_t), \tag{4.7}$$

$$\bar{r}(t,\theta) = \partial_\mu V\left(t, \mathbb{P}^{\alpha}_{X_t^*}, \alpha_t; X_t^* + \beta(X_t^*, u_t^*, \mathbb{P}^{\alpha}_{X_t^*}, \alpha_t, \theta)\right) - \partial_\mu V(t, \mathbb{P}_{X_t^*}, \alpha_t; X_t^*) \tag{4.8}$$

$$\bar{\lambda}_{i_0 j_0}(t) = V(t, \mathbb{P}_{X_t^*}, j_0; X_t^*) - \partial_\mu V(t, \mathbb{P}_{X_t^*}, i_0; X_t^*) \tag{4.9}$$

Theorem (Relation)

Assume that $V \in C^{1,(2,1)}([0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{M}_t$. *If* $u^* \in \mathcal{U}$ *satisfies condition (1), then the solution of the adjoint equation is given by*

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$$\bar{r}(t,\theta) = \partial_\mu V\left(t, \mathbb{P}^\alpha_{X_t^*}, \alpha_t; X_t^* + \beta(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \alpha_t, \theta)\right) \text{-} \partial_\mu V(t, \mathbb{P}_{X_t^*}, \alpha_t; X_t^*) \tag{4.8}$$

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Condition (1)

$$u^*(t,x,\mu,i_0) \in \arg\min_{u \in U} \mathcal{H}\left(x, u, \mu, \partial_\mu V_t^*, \partial_x \partial_\mu V_t^*, \frac{\delta V^*}{\delta \mu}(i_0)\right)$$

Relation between DPP & SMP: Proof [1/4]

Denote

$$
\begin{aligned}
F(t,\mu,u,i_0) &= \partial_t V(t,\mu,i_0) + QV(t,\mu)(i_0), \\
&\quad + \mathbb{E}\left[H\left(x,u,\mu,\partial_\mu V(t,\mu,i_0,x),\partial_x\partial_\mu V(t,\mu,i_0,x),\frac{\delta V}{\delta\mu}(t,\mu,i_0,x),i_0\right)\right].
\end{aligned}
$$

Denote

$$
\begin{aligned}
F(t,\mu,u,i_0) &= \partial_t V(t,\mu,i_0) + QV(t,\mu)(i_0),\\
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\end{aligned}
$$

Next, it follows from the assumption that $F(t,\mu,u,i_0) \geq 0$ because

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Denote

$$F(t,\mu,u,i_0) = \partial_t V(t,\mu,i_0) + QV(t,\mu)(i_0),$$
$$+\,\mathbb{E}\Big[H\Big(x,u,\mu,\partial_\mu V(t,\mu,i_0,x),\partial_x\partial_\mu V(t,\mu,i_0,x),\frac{\delta V}{\delta\mu}(t,\mu,i_0,x),i_0\Big)\Big].$$

Next, it follows from the assumption that $F(t,\mu,u,i_0)\geq 0$ because

$$\min_{\mu,u} F(t,\mu,u,\alpha_t) = F\Big(t,\mathbb{P}^{\alpha}_{X_t^*},u^*(t,X_t^*,\mathbb{P}^{\alpha}_{X_t^*}),\alpha_t\Big) = 0.$$

- We can differentiate $F(t,\mu,\alpha)$ with respect to μ and then evaluate it at $\mathbb{P}^{\alpha}_{X_t^*}$ and $u_t^* = u^*(t,X_t^*,\mathbb{P}^{\alpha}_{X_t^*})$ to get

Denote

$$F(t,\mu,u,i_0) = \partial_t V(t,\mu,i_0) + QV(t,\mu)(i_0), \\ + \mathbb{E}\Big[H\Big(x,u,\mu,\partial_\mu V(t,\mu,i_0,x),\partial_x\partial_\mu V(t,\mu,i_0,x),\frac{\delta V}{\delta\mu}(t,\mu,i_0,x),i_0\Big)\Big].$$

Next, it follows from the assumption that $F(t,\mu,u,i_0)\geq 0$ because

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- We can differentiate $F(t,\mu,\alpha)$ with respect to μ and then evaluate it at $\mathbb{P}^{\alpha}_{X_t^*}$ and $u_t^* = u^*(t,X_t^*,\mathbb{P}^{\alpha}_{X_t^*})$ to get

$$\partial_\mu F(t,\mathbb{P}^{\alpha}_{X_t^*},u_t^*,\alpha) = 0.$$

A calculation of $\partial_\mu F(t, \mathbb{P}^{\alpha}_{X_t^*}, u_t^*, \alpha)$ yields

$$\begin{aligned}
&\partial_t \partial_\mu V(t, \mathbb{P}^{\alpha}_{X_t^*}, \alpha_t; X_t^*) \\
&= -\Big\{ \partial_x f_t^* + \hat{\mathbb{E}}^{\alpha}[\partial_\mu \hat{f}_t^*] + \Big(\partial_x b_t^* + \hat{\mathbb{E}}^{\alpha}[\partial_\mu \hat{b}_t^*]\Big)\partial_\mu V_t^* + b_t^* \Big(\partial_x \partial_\mu V_t^* + \hat{\mathbb{E}}^{\alpha}[\partial_\mu^2 \hat{V}_t^*]\Big) \\
&\quad + \frac{1}{2}\mathrm{Tr}\Big[\partial_x (\sigma_t^* (\sigma_t^*)^T)\partial_x \partial_\mu V_t^* + \hat{\mathbb{E}}^{\alpha}\big(\partial_\mu [\widehat{\sigma_t^* (\sigma_t^*)^T}]\partial_x \partial_\mu V_t^*\big)\Big] \\
&\quad + \frac{1}{2}\mathrm{Tr}\Big[\sigma_t^* (\sigma_t^*)^T \partial_x^2 \partial_\mu V_t^* + \hat{\mathbb{E}}^{\alpha}\big(\hat{\sigma}_t^* (\hat{\sigma}_t^*)^T \partial_x \partial_\mu^2 \hat{V}_t^*\big)\Big] \\
&\quad + \int_{\mathbb{R}^q} \Big[\frac{\delta(\partial_\mu V)}{\delta \mu}(t, \mathbb{P}^{\alpha}_{X_t^*}, \alpha_t; X_t + \beta^*(t,\theta)) - \frac{\delta(\partial_\mu V)}{\delta \mu}(t, \mathbb{P}^{\alpha}_{X_t^*}, \alpha_t; X_t^*)\Big] d\nu(\theta) \\
&\quad + Q \partial_\mu V^*(t, \mathbb{P}^{\alpha}_{X_t^*}; X_t^*)(\alpha_t)\Big\}.
\end{aligned}$$

Now apply our Itô's formula (3.6) to $\partial_\mu V(s, \mathbb{P}_{X_s^*}, \alpha_s; X_s^*)$ between t and T

$$\partial_\mu V(t, \mathbb{P}_{X_t^*}, X_t^*, \alpha(t)) - \partial_\mu V(T, \mathbb{P}_{X_T^*}, X_T^*, \alpha(T))$$

$$= -\Big(\int_t^T \partial_s \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) + \partial_x \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) b_r^* + Q \partial_\mu V(s, X_s^*, \mathbb{P}_{X_s^*}, \alpha(s))$$

$$+ \frac{1}{2} Tr(\sigma_s^* (\sigma_s^*)^T \partial_{xx} \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s + \hat{\mathbb{E}}^\alpha \Big[\partial_\mu^2 V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s, \hat{X}_s^*) \cdot \hat{b}_s^*$$

$$+ \frac{1}{2} Tr(\hat{\sigma}_s^* (\hat{\sigma}_s^*)^T) \partial_x \partial_\mu^2 V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s)$$

$$+ \int_{\mathbb{R}^q} \Big(\frac{\delta(\partial_\mu V)}{\delta \mu}(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \hat{X}_{s-} + \hat{\beta}_{s-}(\theta), \alpha_s) - \frac{\delta(\partial_\mu V)}{\delta \mu}(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, \hat{X}*_{s-}) \Big) d\nu(\theta) \Big] ds$$

$$+ \int_{\mathbb{R}^q} \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, X_{s-} + \beta_{s-}(\theta), \alpha_s) - \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, X*_{s-}) \Big) \tilde{N}(ds, d\theta)$$

$$+ \int_t^T \partial_x \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) \sigma_s^* dW_s + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \partial_\mu V(j_0) - \partial_\mu V(i_0) dM_{i_0 j_0}(s)$$

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Now apply our Itô's formula (3.6) to $\partial_\mu V(s, \mathbb{P}_{X_s^*}, \alpha_s; X_s^*)$ between t and T

$$
\begin{aligned}
&\partial_\mu V(t, \mathbb{P}_{X_t^*}, X_t^*, \alpha(t)) - \partial_\mu V(T, \mathbb{P}_{X_T^*}, X_T^*, \alpha(T)) \\
= -\Big(&\int_t^T \partial_s \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) + \partial_x \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) b_r^* + Q \partial_\mu V(s, X_s^*, \mathbb{P}_{X_s^*}, \alpha(s)) \\
&+ \frac{1}{2} Tr(\sigma_s^* (\sigma_s^*)^T \partial_{xx} \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s + \hat{\mathbb{E}}^\alpha \Big[\partial_\mu^2 V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s, \hat{X}_s^*) \cdot \hat{b}_s^* \\
&+ \frac{1}{2} Tr(\hat{\sigma}_s^* (\hat{\sigma}_s^*)^T) \partial_x \partial_\mu^2 V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s) \\
&+ \int_{\mathbb{R}^q} \Big(\frac{\delta(\partial_\mu V)}{\delta \mu}(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \hat{X}_{s-} + \hat{\beta}_{s-}(\theta), \alpha_s) - \frac{\delta(\partial_\mu V)}{\delta \mu}(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, \hat{X}*_{s-}) \Big) d\nu(\theta) \Big] ds \\
&+ \int_{\mathbb{R}^q} \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, X_{s-} + \beta_{s-}(\theta), \alpha_s) - \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, X*_{s-}) \Big) \tilde{N}(ds, d\theta) \\
&+ \int_t^T \partial_x \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) \sigma_s^* dW_s + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \partial_\mu V(j_0) - \partial_\mu V(i_0) dM_{i_0 j_0}(s)
\end{aligned}
$$

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Now apply our Itô's formula (3.6) to $\partial_\mu V(s, \mathbb{P}_{X_s^*}, \alpha_s; X_s^*)$ between t and T

$$\begin{aligned}
&\partial_\mu V(t, \mathbb{P}_{X_t^*}, X_t^*, \alpha(t)) - \partial_\mu V(T, \mathbb{P}_{X_T^*}, X_T^*, \alpha(T)) \\
&= -\Big(\int_t^T \partial_s \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) + \partial_x \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) b_r^* + Q \partial_\mu V(s, X_s^*, \mathbb{P}_{X_s^*}, \alpha(s)) \\
&\quad + \frac{1}{2} Tr(\sigma_s^* (\sigma_s^*)^T \partial_{xx} \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s + \hat{\mathbb{E}}^\alpha \Big[\partial_\mu^2 V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s, \hat{X}_s^*) \cdot \hat{b}_s^* \\
&\quad + \frac{1}{2} Tr(\hat{\sigma}_s^* (\hat{\sigma}_s^*)^T) \partial_x \partial_\mu^2 V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha_s) \\
&\quad + \int_{\mathbb{R}^q} \Big(\frac{\delta(\partial_\mu V)}{\delta \mu}(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \hat{X}_{s-} + \hat{\beta}_{s-}(\theta), \alpha_s) - \frac{\delta(\partial_\mu V)}{\delta \mu}(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, \hat{X}*_{s-}) \Big) d\nu(\theta) \Big] ds \\
&\quad + \int_{\mathbb{R}^q} \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, X_{s-} + \beta_{s-}(\theta), \alpha_s) - \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, X*_{s-}) \Big) \tilde{N}(ds, d\theta) \\
&\quad + \int_t^T \partial_x \partial_\mu V(s, \mathbb{P}_{X_s^*}, X_s^*, \alpha(s)) \sigma_s^* dW_s + \sum_{i_0, j_0 \in \mathcal{M}} \int_t^s \partial_\mu V(j_0) - \partial_\mu V(i_0) dM_{i_0 j_0}(s)
\end{aligned}$$

Thus from substituting $\partial_t \partial_\mu V_t^*$, we have

$$
\begin{aligned}
& d\partial_\mu V(t, \mathbb{P}_{X_t^*}, X_t^*, \alpha(t)) \\
& = -\Big[\partial_x H(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, \partial_\mu V_t^*, \partial_x \partial_\mu V_t^* \sigma_t^*, \partial_\mu V_t^*(X_{t-} + \beta_{t-}(\theta)) - \partial_\mu V_t^*(X_{t-}^*)) \\
& \quad + \hat{\mathbb{E}}^\alpha[\partial_\mu H(\bar{X}_t^*, \bar{u}_t^*, \mathbb{P}^\alpha_{X_t^*}, \partial_\mu V_t^*, \partial_x \partial_\mu V_t^* \bar{\sigma}_t^*, \partial_\mu V_t^*(X_{t-} + \beta_{t-}(\theta)) - \partial_\mu V_t^*(X_{t-}^*))]\Big] dt \\
& \quad + \partial_x \partial_\mu V(t, \mathbb{P}_{X_t^*}, X_t^*, \alpha(t)) \sigma(X_t^*, u_t^*, \mathbb{P}_{X_t^*}, \alpha(t)) dW_t + \Big(\partial_\mu V(j_0) - \partial_\mu V(i_0)\Big) dM_{i_0 j_0}(s) \\
& \quad + \int_{\mathbb{R}^q} \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, X_{s-} + \beta_{s-}(\theta), \alpha_s) - \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, X*_{s-})\Big) \tilde{N}(ds, d\theta), \\
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& \quad + \partial_x \partial_\mu V(t, \mathbb{P}_{X_t^*}, X_t^*, \alpha(t)) \sigma(X_t^*, u_t^*, \mathbb{P}_{X_t^*}, \alpha(t)) dW_t + \Big(\partial_\mu V(j_0) - \partial_\mu V(i_0)\Big) dM_{i_0 j_0}(s) \\
& \quad + \int_{\mathbb{R}^q} \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, X_{s-} + \beta_{s-}(\theta), \alpha_s) - \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, X*_{s-})\Big) \tilde{N}(ds, d\theta), \\
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\end{aligned}$$

$$\begin{aligned}
dp_t &= -\Big\{\nabla_x \mathcal{H}(X_t^*, u_t^*, \mathbb{P}^\alpha_{X_t^*}, p_t, P_t, r_t, \alpha_t) + \hat{\mathbb{E}}^\alpha\Big[\partial_\mu \mathcal{H}(\hat{X}_t^*, \hat{u}_t^*, \mathbb{P}^\alpha_{X_t^*}, \hat{p}_t, \hat{P}_t, \hat{r}_t, \alpha_t, X_t^*)\Big]\Big\} dt \\
&\qquad + P_t dW_t + \lambda_t \bullet dM_t + \int_{\mathbb{R}^q} r(t-, \theta) \tilde{N}(dt, d\theta), \\
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& \quad + \partial_x \partial_\mu V(t, \mathbb{P}_{X_t^*}, X_t^*, \alpha(t)) \sigma(X_t^*, u_t^*, \mathbb{P}_{X_t^*}, \alpha(t)) dW_t + \Big(\partial_\mu V(j_0) - \partial_\mu V(i_0)\Big) dM_{i_0 j_0}(s) \\
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& \quad + \int_{\mathbb{R}^q} \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, X_{s-} + \beta_{s-}(\theta), \alpha_s) - \partial_\mu V(s, X_s^*, \mathbb{P}^\alpha_{X_s^*}, \alpha_r, X*_{s-})\Big) \tilde{N}(ds, d\theta), \\
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& \qquad + P_t dW_t + \lambda_t \bullet dM_t + \int_{\mathbb{R}^q} r(t-, \theta) \tilde{N}(dt, d\theta), \\
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\end{aligned}
$$

Mean-Variance Portfolio Selection

Consider a market which consists of the following two possible investments:

- A risk free security whose price is the determined by

$$dY_0(t) = \kappa(t) Y_0(t) dt, \quad Y_0(0) > 0$$

- A risky security where the price is determined by

$$dY_1(t) = Y_1(t-)\left[\tau(t)dt + \sigma(t)dW(t) + \int_{\mathbb{R}} \beta(t,\theta)\tilde{N}(dt,dz)\right], \quad Y_1(0) > 0$$

Wealth Dynamics

$$\begin{aligned} dX^u(t) &= \kappa(t)X^u(t)dt + u(t)\Big((\tau(t) - \kappa(t))dt + \sigma(t)dWt\Big) \\ &\quad + u(t-)\int_{\mathbb{R}} \beta(t,\theta)\tilde{N}(dt,d\theta), \\ X^u(0) &= X_0 \in \mathbb{R} \end{aligned}$$

Mean-Variance Portfolio Selection

Cost Functional

$$J(h(\cdot)) = \mathbb{E}\left[\frac{\delta}{2}X^u(T)^2 - X^u(T)\right] - \left(\frac{\delta}{2}\mu^X(T)\right)^2$$

Where $\mu^X(T) = \mathbb{E}[X^u(T)|\mathcal{F}_T^\alpha]$.

Hamiltonian

$$H(x,\mu,u,p,P,r,i_0) = \Big[\kappa(t)X(t) + (\tau(t) - \kappa(t)u(t)\Big]p(t)$$
$$+\sigma(t)u(t)P(t) + u(t)\int_{\mathbb{R}^q}\beta(t,\theta)r(t,\theta)d\nu(\theta)$$

Adjoint Equation

$$dp^*(t) = -\kappa(t)p^*(t)dt + P^*dWt + \int_{\mathbb{R}^q} r^*(t,\theta)\tilde{N}(dt,d\theta) + \lambda\cdot dM_t,$$
$$p(T) = \delta X(T) - \delta\mu^X(T) - 1$$

Where $\mu^X(T) = \mathbb{E}[X^u(T)|\mathcal{F}_T^\alpha]$.

Theorem

The solution $u^* \in \mathcal{U}$ of the mean-variance portfolio selection problem , when X obeys the given wealth dynamics is given in feedback form by

$$u^*(t) = \frac{(X^*(t) - \mu^{X^*}(t) + \frac{\xi_3}{\xi_1})(K)}{\sigma^2(t) + \int_{\mathbb{R}^q} \beta^2(t,\theta) d\nu(\theta)}$$

where

- $\xi_1(t) = -2\mathbb{E}\Big(exp\big[\int_t^T 2\kappa(s)ds\big]\big|\alpha(t)\Big)$
- $\xi_3(t) = -\mathbb{E}\Big(exp\big[\int_t^T 2\kappa(s)ds\big]\big|\alpha(t)\Big)$

Concluding Remarks

We have established:

1. Uniqueness, existence, and the flow property;
2. A relevant version of Itô's formula;
3. A verification theorem;
4. A sufficient stochastic maximum principle;
5. The relation between the SMP and DPP.

There are three main directions which come to mind directly:

- Singular Control;

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- Singular Control;
- Nonconvex Control Domain;

Future Directions

There are three main directions which come to mind directly:

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Singular Control

$$\begin{aligned} dX_s = {} & b(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\ & + \int_{\mathbb{R}^q} \beta(X_{s-}, u_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta) + h d\eta_s, \quad s \geq t, \\ X_0 = {} & \xi \in L^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_0 = i_0, \end{aligned}$$

There are three main directions which come to mind directly:

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Singular Control

$$
\begin{aligned}
dX_s &= b(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s \\
&\qquad + \int_{\mathbb{R}^q} \beta(X_{s-}, u_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta) + hd\eta_s, \quad s \geq t, \\
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\end{aligned}
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- Optimality conditions for the stochastic maximum principle.

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- Singular Control;
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Singular Control

$$dX_s = b(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)ds + \sigma(X_s, u_s, \mathbb{P}^{\alpha}_{X_s}, \alpha_s)dW_s$$
$$+ \int_{\mathbb{R}^q} \beta(X_{s-}, u_{s-}, \mathbb{P}^{\alpha}_{X_{s-}}, \alpha_{s-}, \theta)\tilde{N}(ds, d\theta) + hd\eta_s, \quad s \geq t,$$
$$X_0 = \xi \in L^2(\mathcal{F}_t, \mathbb{R}^d), \quad \alpha_0 = i_0,$$

- Optimality conditions for the stochastic maximum principle.
- Viscosity Solutions.

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