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Conor McCoid

December 15th, 2025

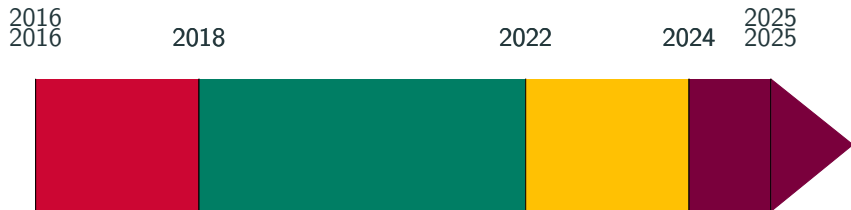
Joint work with Felix Kwok at the Université Laval  
and Blaise Bourdin at McMaster University



# Hobbies

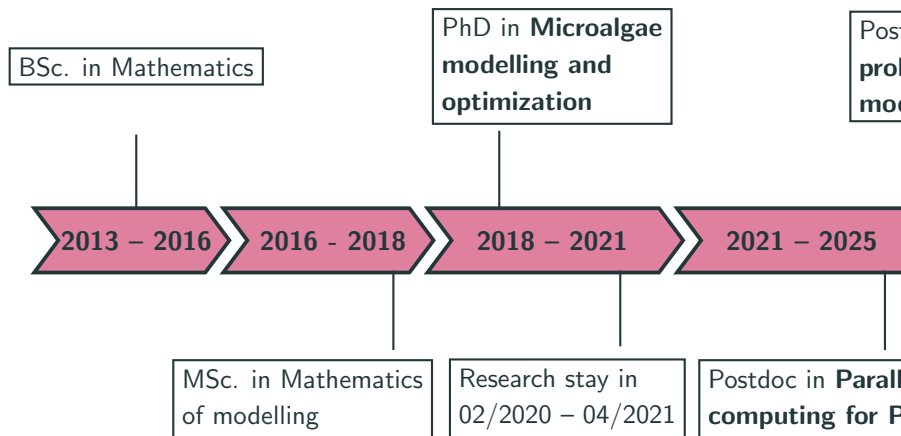


# Career timeline



ed Math: PhD in Applied Mathematics CRM-uLaval P PDF, McMaster University

# Career timeline



# CV highlights

- 6 journal articles, 3 proceedings
- 15 invited presentations at conferences
- Co-organized 2 minisymposia
- Founded careers workshop at McMaster
- Serving on EDII committee at McMaster as postdoc representative
- Prix Henri Fehr for best thesis in mathematics at the University of Geneva

1. Introduction to domain decomposition and Schwarz methods
2. Newton-Schwarz methods
  - 2.1 Phasefield fracture model and AltMin
  - 2.2 MSPIN and parallelogram minimization
3. Adaptive optimized Schwarz methods
  - 3.1 Symmetrized cells
  - 3.2 Asymmetric systems and FOM
  - 3.3 Symmetric systems and CG
4. Geometric intersections

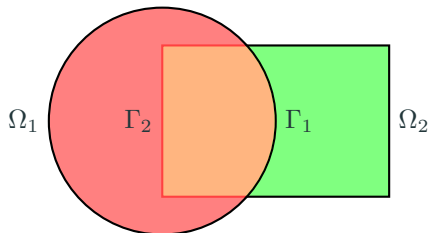
# **Introduction to domain decomposition and Schwarz methods**

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How do we solve the Laplace equation on complicated domains?

We split the domain into simpler subdomains.



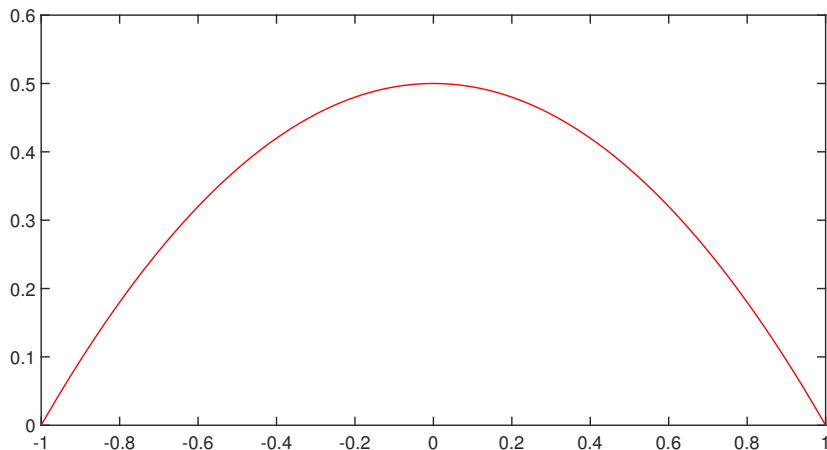
Alternating Schwarz method:

$$\begin{cases} \Delta u_1^{n+1} = 0 & \text{in } \Omega_1, \\ u_1^{n+1} = u_2^n & \text{on } \Gamma_1, \end{cases} \quad \begin{cases} \Delta u_2^{n+1} = 0 & \text{in } \Omega_2, \\ u_2^{n+1} = u_1^{n+1} & \text{on } \Gamma_2. \end{cases}$$

## Simple example

$$u''(x) = -1, \quad x \in [-1, 1], \quad u(-1) = u(1) = 0$$

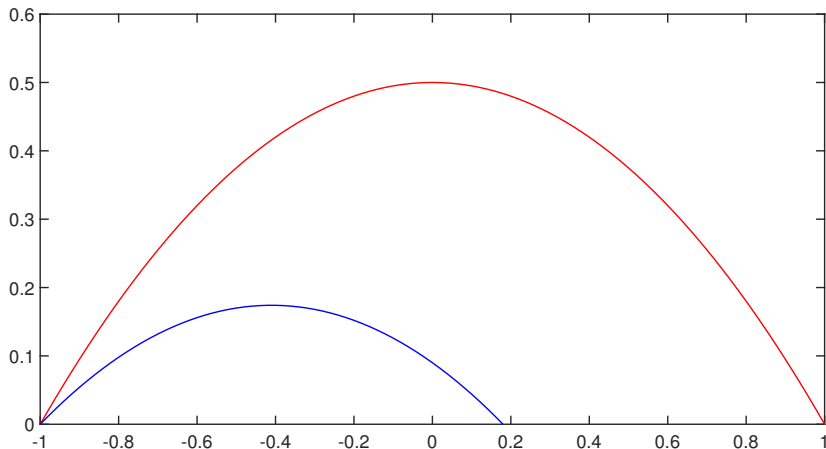
$$\Omega_1 = [-1, 0.18], \quad \Omega_2 = [-0.22, 1]$$



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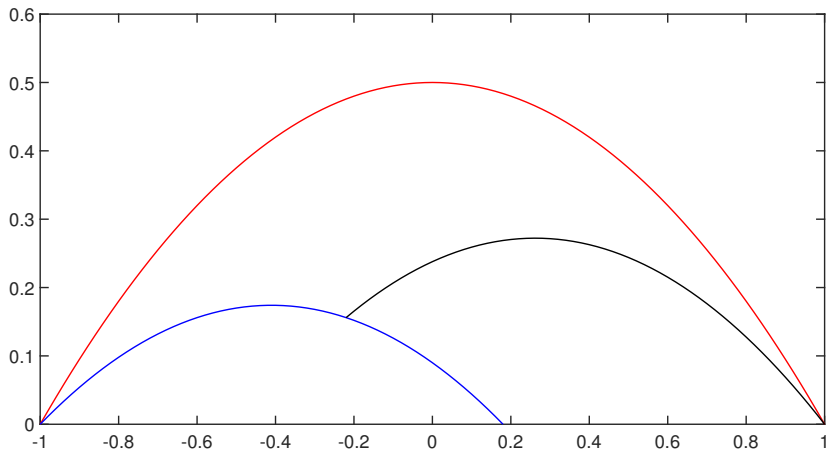
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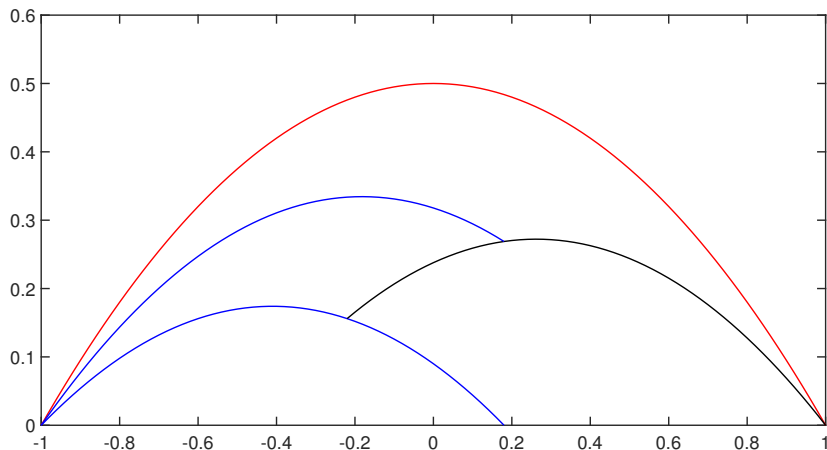
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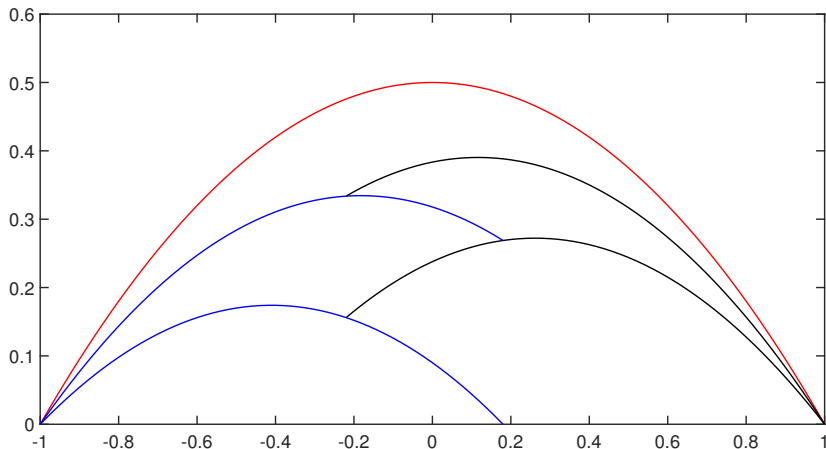
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# Field-split Schwarz methods

Instead of splitting up the physical domain, we can split up the problem into fields.

Suppose a problem can be written as

$$\begin{bmatrix} F(u, v) \\ G(u, v) \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

where  $F$  depends more strongly on  $u$  and  $G$  depends more strongly on  $v$ .

Field-split multiplicative Schwarz:

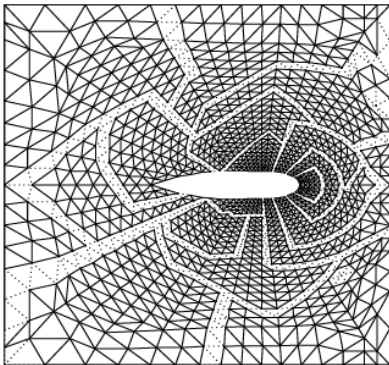
$$F(u^{(n+1)}, v^{(n)}) = f, \quad G(u^{(n+1)}, v^{(n+1)}) = g.$$

# **Adaptive optimized Schwarz methods**

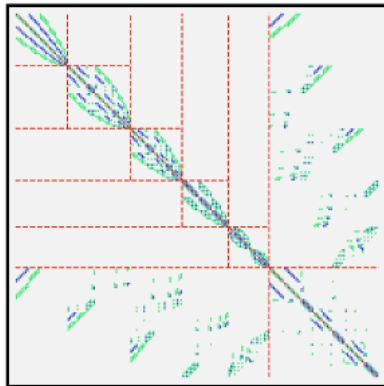
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## More complicated DD example



**Figure 1:** Also from Saad's "Iterative Methods"



**Figure 2:** Possible matrix for example

# A general form of the matrix

Let us consider matrices that can take the form

$$\begin{bmatrix} A_{11} & & & & A_{1\Gamma} \\ & A_{22} & & & A_{2\Gamma} \\ & & \ddots & & \vdots \\ & & & A_{nn} & A_{n\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & \dots & A_{\Gamma n} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \\ \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_n \\ \mathbf{f}_\Gamma \end{bmatrix}, \quad (1)$$

where  $A_{ii}$  are square. This system represents  $n$  subdomains connected through a global interface represented by  $\Gamma$ .

Each subdomain now has its own subproblem:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + S_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i \\ \tilde{\mathbf{f}}_i \end{bmatrix}, \quad (2)$$

where  $\tilde{\mathbf{f}}_i$  is some modification of  $\mathbf{f}_\Gamma$ , and  $S_i$  is some global transmission matrix.

## How to choose $S_i$ and $\tilde{f}_i$

There are perfect choices of  $S_i$  and  $\tilde{f}_i$  such that each subproblem gives the exact solution to the global problem on its respective subdomain. However, these perfect choices are expensive to compute.

Instead, the standard procedure is to make *a priori* choices that give convergent iterative methods. These appear as:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + S_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_i^{(k+1)} \\ \mathbf{u}_{\Gamma i}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{f}_{\Gamma} \end{bmatrix} + \sum_{j \neq i} \begin{bmatrix} -A_{\Gamma j} & T_j \end{bmatrix} \begin{bmatrix} \mathbf{u}_j^{(k)} \\ \mathbf{u}_{\Gamma j}^{(k)} \end{bmatrix}, \quad (3)$$

where

$$S_i = \sum_{j \neq i} T_j. \quad (4)$$

## Choices for $T_j$

The local transmission matrices  $T_j$  can represent boundary conditions between the subdomains. Some common options:

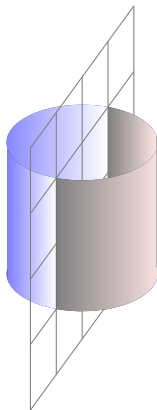
- Dirichlet, setting the interface variables on connected subdomains to be the same
- Neumann, setting the derivatives to be the same
- Optimized, setting Robin boundary conditions to be the same, using a Robin parameter that optimizes convergence rates

The strategy we'll employ here is to adapt the transmission conditions at each iteration so that they're closer to the perfect choices.

# Symmetrized cells

For each subdomain, take a copy of it and stitch it together along their shared interface. This pair is now perfectly symmetric, and one subproblem describes both copies.

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} & \\ A_{\Gamma i} & A_{\Gamma\Gamma} & A_{\Gamma i} \\ & A_{i\Gamma} & A_{ii} \end{bmatrix} \begin{bmatrix} \hat{u}_i \\ \hat{u}_\Gamma \\ \hat{u}_i \end{bmatrix} = \begin{bmatrix} f_i \\ \hat{f}_i \\ f_i \end{bmatrix}. \quad (5)$$



**Figure 3:** A symmetrized square domain with interfaces on two opposing edges

# Adaptive optimized Schwarz on symmetrized cells

We solve the following iterations to correct the initial solution:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + T_i^{(k+1)} \end{bmatrix} \begin{bmatrix} \mathbf{d}_i^{(k+1)} \\ \mathbf{d}_{\Gamma i}^{(k+1)} \end{bmatrix} = \begin{bmatrix} & \\ -A_{\Gamma i} & T_i^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{d}_i^{(k)} \\ \mathbf{d}_{\Gamma i}^{(k)} \end{bmatrix}, \quad (6)$$

where  $\mathbf{d}_i^{(k+1)} = \mathbf{u}_i^{(k+1)} - \mathbf{u}_i^{(k)}$ .

We can use techniques from static condensation to reduce the form of this system to acting only on the global interface:

$$\left( \hat{A}_i + E_i^{(k+1)} \right) \mathbf{d}_{\Gamma i}^{(k+1)} = E_i^{(k)} \mathbf{d}_{\Gamma i}^{(k)}, \quad (7)$$

where

$$\hat{A}_i = A_{\Gamma\Gamma} - 2A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma}, \quad E_i^{(k)} = T_i^{(k)} + A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma}.$$

## Updates to the transmission condition

We have one input,  $\mathbf{d}_{\Gamma i}^{(k)}$ , and one output,  $E_i^{(k)} \mathbf{d}_{\Gamma i}^{(k)}$ . This is enough to give a rank one approximation of  $E_i^{(k)}$ , which we can use to update the transmission conditions:

$$T_i^{(k+1)} := T_i^{(k)} - E_i^{(k)} \frac{\mathbf{d}_{\Gamma i}^{(k)} \left( \mathbf{d}_{\Gamma i}^{(k)} \right)^\top}{\left\| \mathbf{d}_{\Gamma i}^{(k)} \right\|^2}.$$

For subsequent iterations, we have to orthogonalize the vectors  $\mathbf{d}_{\Gamma i}^{(k)}$ :

- 1: Inputs:  $\mathbf{d}_{\Gamma i}^{(k)}$ ,  $E_i^{(k)} \mathbf{d}_{\Gamma i}^{(k)}$ , all previous  $\mathbf{d}_{\Gamma i}^{(j)}$  and  $E_i^{(j)} \mathbf{d}_{\Gamma i}^{(j)}$
- 2: Set  $\mathbf{w}_k := \mathbf{d}_{\Gamma i}^{(k)}$  and  $\mathbf{v}_k := E_i^{(k)} \mathbf{d}_{\Gamma i}^{(k)}$
- 3: **for**  $j = 1 : k - 1$  **do**
- 4:      $h \leftarrow \langle \mathbf{d}_{\Gamma i}^{(j)}, \mathbf{w}_k \rangle$ ,  $\mathbf{w}_k \leftarrow \mathbf{w}_k - h \mathbf{d}_{\Gamma i}^{(j)}$
- 5:      $\mathbf{v}_k \leftarrow \mathbf{v}_k - h E_i^{(j)} \mathbf{d}_{\Gamma i}^{(j)}$
- 6: **end for**
- 7: Output:  $E_i^{(k+1)} := E_i^{(k)} - \mathbf{v}_k \mathbf{w}_k^\top$

## Equivalence to a Krylov subspace method

From static condensation (7) we have that the vectors  $\mathbf{d}_{\Gamma i}^{(k)}$  form a Krylov subspace:

$$\mathbf{d}_{\Gamma i}^{(k)} \in \mathcal{K}_k \left( \left( \hat{A}_i + E_i^{(1)} \right)^{-1} E_i^{(1)}, \mathbf{d}_{\Gamma i}^{(1)} \right) = \mathcal{K}_k.$$

By solving the system (6) the vector  $\mathbf{d}_{\Gamma i}^{(k+1)}$  is equal to

$$\mathbf{d}_{\Gamma i}^{(k+1)} = \left( \hat{A}_i + E_i^{(k)} \right)^{-1} E_i^{(k)} \mathbf{x},$$

where  $\mathbf{x} \in \mathcal{K}_k$  such that

$$\begin{aligned} & \left( I - \left( \hat{A}_i + E_i^{(1)} \right)^{-1} E_i^{(1)} \right) \left( \mathbf{u}_{\Gamma i}^{(k-1)} + \mathbf{x} \right) \\ & - \left( \hat{A}_i + E_i^{(1)} \right)^{-1} \left( \mathbf{f}_{\Gamma} - 2A_{\Gamma i}A_i^{-1}\mathbf{f}_i \right) \perp \mathcal{K}_k. \end{aligned}$$

This is a Galerkin condition on the pre-image of the next difference. This makes the AOSM equivalent to the full orthogonalization method (FOM), a precursor to the generalized minimal residual method (GMRES).



# Convergence of the AOSM

Because of the equivalence to FOM, we can use the research there to tell us how AOSM will converge. Unfortunately, that research tells us we can't know anything *a priori*:

**Greenbaum, Ptak and Strakos (YYYY)**

Recent work has brought us closer to understanding the convergence properties of these methods. One project I propose is to build on this work in the specific context of Schwarz-preconditioned systems.

## Project proposal: AOSM/GMRES convergence

**Project idea:** Investigate the convergence of GMRES and FOM by applying recent analysis methods to Schwarz-preconditioned systems. Also study known worst-case convergence curves and if preconditioning techniques can improve rates.

**Possible collaborations:** Currently proposed as a postdoctoral project with Chen Greif (UBC) and Manfred Trummer (SFU), could also involve

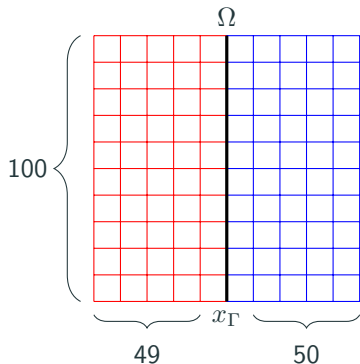
**Mentoring opportunities:** Studying GMRES convergence curves makes for a great PhD project. The problem is well known in the community and any progress on it would make for an impressive thesis.

## Numerical results of AOSM

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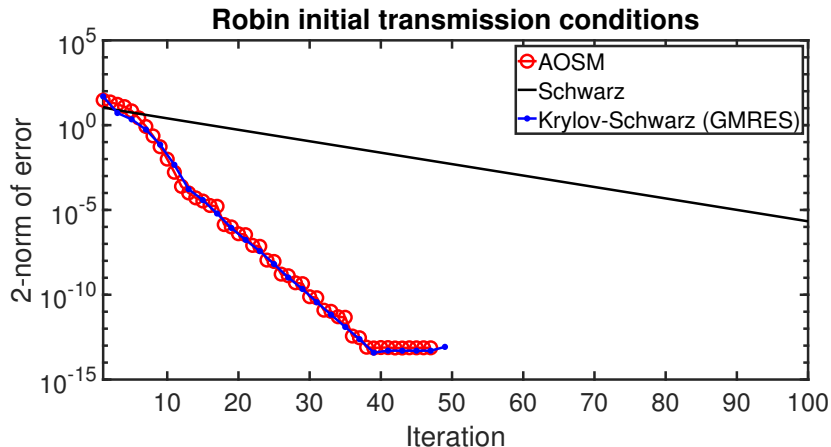
## Simple comparison with other methods

$$\begin{cases} \Delta u(x, y) = f(x, y), & (x, y) \in \Omega = [-1, 1] \times [-1, 1], \\ u(x, y) = g(x, y), & (x, y) \in \partial\Omega. \end{cases}$$



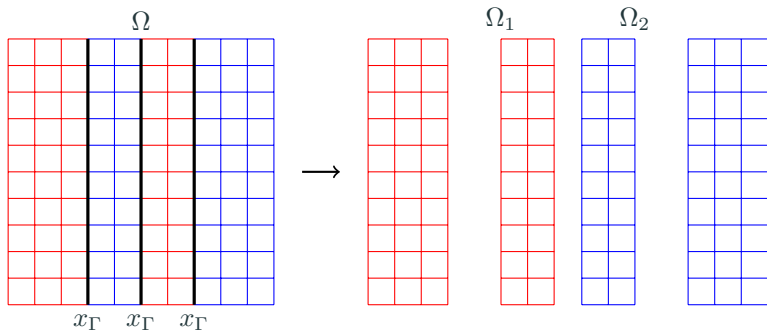
**Figure 4:**  $100 \times 100$  evenly spaced grid split into two subdomains along the 50th value of  $x$

## Simple comparison with other methods



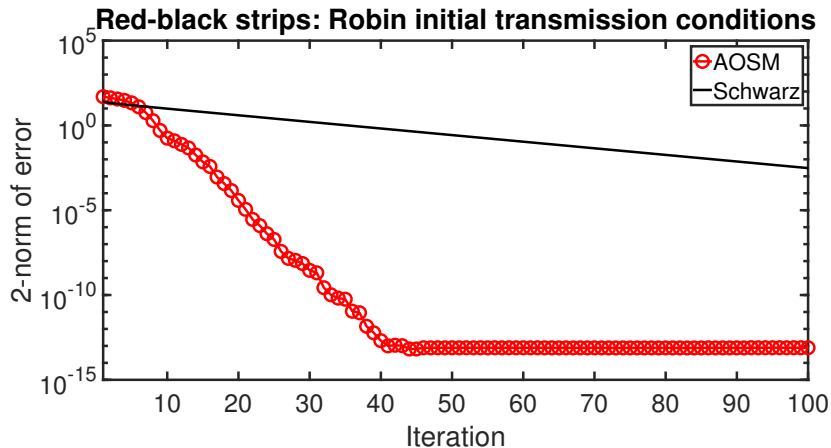
**Figure 5:** Comparison between Schwarz, AOSM and Krylov-Schwarz on simple elliptic PDE

## Precursor to multiple subdomains: red-black decompositions



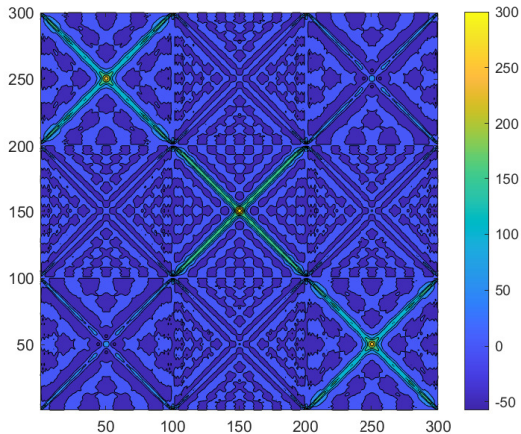
**Figure 6:** Splitting the  $100 \times 100$  grid into four strips, then pairing the strips into two algebraic subdomains

## Comparison: stripwise



**Figure 7:** Convergence for AOSM and Schwarz on the stripwise decomposition

## Adapted transmission conditions: stripwise

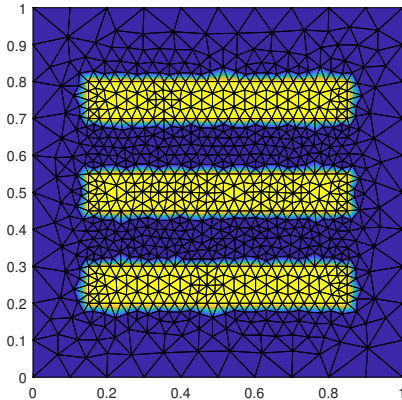


**Figure 8:** Matrix  $T_{1 \rightarrow 2}$  from AOSM after convergence of stripwise example



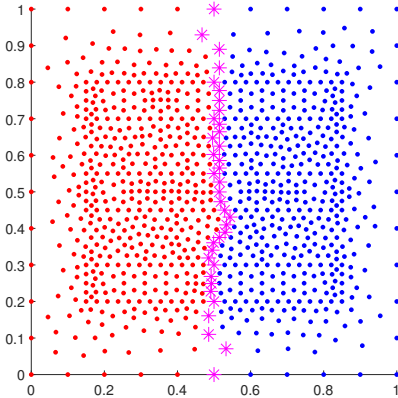
# Heterogeneous elliptic PDE

$$\begin{cases} -\nabla (\alpha(x, y) \cdot \nabla u(x, y)) = f(x, y), & (x, y) \in \Omega = [0, 1] \times [0, 1], \\ u(x, y) = g(x, y), & (x, y) \in \partial\Omega. \end{cases}$$



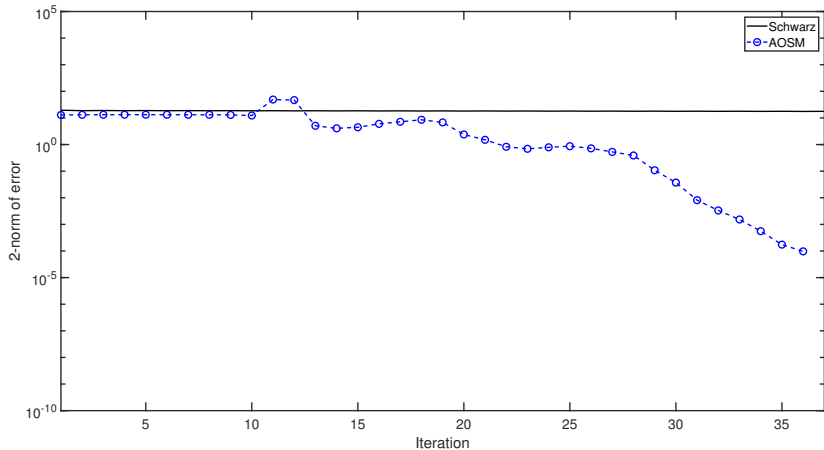
**Figure 9:**  $\alpha(x, y) = 1$  except along three thin channels where  $\alpha(x, y) = 1000$

# Unstructured grid



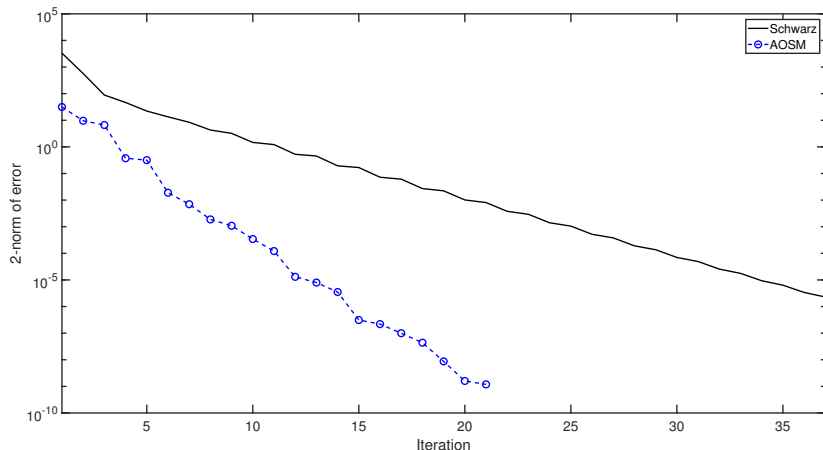
**Figure 10:** Splitting an unstructured grid into two subdomains

# First round of solves



**Figure 11:** (Lack of) convergence for Schwarz and AOSM for heterogeneous elliptic PDE

## Second round of solves



**Figure 12:** Convergence for Schwarz and AOSM using adapted transmission conditions