Combinadics adjacency matrix

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Combinadics

The combinatorial number system, or combinadics, is a way to enumerate all m-combinations. Each combination of m natural numbers is given a ranking from 0 to $\binom{n}{m}-1$.

The ranking $N(J)$ of an m–combination $J=\{c_1,\dots,c_m\}$, arranged in lexicographical order, of natural numbers $\{0,\dots,n-1\}$ is equal to

$$N(J)=\binom{c_m}{m}+\dots+\binom{c_1}{1}.$$

Unranking

Generating J from $N(J)$, called unranking, can be achieved through a greedy algorithm using the same coefficients. Let c be the largest number in $\{0,\dots,n-1\}$ such that $N(J)\geq\binom{c}{m}$. Then $c\in J$. Subtract $\binom{c}{m}$ from $N(J)$ and repeat, now considering the numbers $\{0,\dots,c-1\}$ and the $(m-1)$–combination $J\setminus\{c\}$.

Important lemma

Lemma 1. *For m-combination J*

$$N(J^{\complement})+N(J)=\binom{n}{m}-1$$

Anti-transpose

The anti-transpose of a matrix A, denoted A^τ, is its reflection over its northeast-to-southwest diagonal. That is, if $A\in\mathbb{R}^{m\times m}$ then $(A^\tau)_{i,j}=(A)_{m-j+1,m-i+1}$, where $i,j=\{1,\dots,m\}$.

Adjacency matrix

Definition 1 (Adjacency)**.** *Two combinations J and K with the same cardinality are said to be adjacent if they differ by one element. That is,*

$$|K\cap J|=|K|-1=|J|-1.$$

Let A^n_m be the adjacency matrix for m-combinations taken over n elements. The following lemma gives a recurrence relation to construct A^n_m.

Lemma 2. *For $n>m>1$*

$$A^n_m=\begin{bmatrix}A^{n-1}_m & \tilde{A}^n_m\\ (\tilde{A}^n_m)^\top & A^{n-1}_{m-1}\end{bmatrix},$$

$$\tilde{A}^n_m=\begin{bmatrix}\tilde{A}^{n-1}_m & 0_{\binom{n-2}{m-2}\times\binom{n-2}{m}}\\ I_{\binom{n-2}{m-1}} & \tilde{A}^{n-1}_{m-1}\end{bmatrix},$$

with starting conditions

$$A^n_1=\begin{bmatrix}0 & 1 & \dots & 1\\ 1 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 1\\ 1 & \dots & 1 & 0\end{bmatrix},\quad \tilde{A}^4_2=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}.$$

Proof

	J_i	J_j
K_i	A^{n-1}_m	$\tilde{A}^n_m$
K_j	$(\tilde{A}^n_m)^\tau$	A^{n-1}_{m-1}

The combinations J_i and K_i are m-combinations over $n-1$ elements, so the upper left block is A^{n-1}_m:

$$\left(A^{n-1}_m\right)_{N(J_i)+1,N(K_i)+1}=\left(A^n_m\right)_{N(J_i)+1,N(K_i)+1}.$$

Both J_j and K_j have the element $n-1$, so

$$N(J_j)=\binom{n-1}{m}+N(J_j\setminus\{n-1\}),$$
$$N(K_j)=\binom{n-1}{m}+N(K_j\setminus\{n-1\}).$$

This makes the bottom right block A^{n-1}_{m-1}.

Subdivide the combinations J_i and K_j into sets that do or do not contain $n-1$ or $n-2$.

	$n-1\notin$	$n-1\in$
$n-2\notin$	$\tilde{K}_i$	$\tilde{J}_i$
$n-2\in$	$\tilde{K}_j$	$\tilde{J}_j$

$\tilde{K}_i$ and $\tilde{J}_j$ differ by at least two elements, so the relevant block is a zero matrix. $\tilde{J}_i$ and $\tilde{K}_j$ are adjacent if and only if $\tilde{J}_i\setminus\{n-1\}=\tilde{K}_j\setminus\{n-2\}$. This makes the relevant block an identity matrix. $\tilde{J}_j$ and $\tilde{K}_j$ are adjacent if and only if $\tilde{J}_j\setminus\{n-1\}$ and $\tilde{K}_j\setminus\{n-2\}$ are adjacent. The relevant block is then a copy of $\tilde{A}^{n-1}_{m-1}$.

For the final block, consider a new set $J_k=\tilde{J}_i\setminus\{n-1\}\cup\{n-2\}$. Then $\tilde{J}_i$ is adjacent to $\tilde{K}_i$ if and only if J_k is adjacent to $\tilde{K}_i$. Thus,

$$\left(A^n_m\right)_{N(\tilde{J}_i)+1,N(\tilde{K}_i)+1}=\left(A^{n-1}_m\right)_{N(J_k)+1,N(\tilde{K}_i)+1}.$$

The block is then a copy of $\tilde{A}^{n-1}_m$.

Since adjacency is bidirectional, A^n_m is symmetric, and the bottom left block is the transpose of the upper right block.

The following corollary means only half of all A^n_m need to be found.

Corollary 1.

$$(A^n_m)^\tau=A^n_{n-m}.$$

Union of adjacent combinations

Define a matrix B^n_m that stores the values of $N(J\cup K)$ for when J and K are adjacent:

$$\left(B^n_m\right)_{N(J)+1,N(K)+1}=\begin{cases}N(J\cup K)+1 & J,K\text{ adjacent}\\ 0 & \text{otherwise}\end{cases}$$

Lemma 3.

$$B^n_1=\begin{bmatrix}0 & 1 & 2 & \dots & \binom{n}{2}+1\\ & 0 & 3 & \dots & \binom{n}{2}+2\\ & & \ddots & \dots & \vdots\\ & & & 0 & \binom{n}{2}+n\\ & & & & 0\end{bmatrix},\quad B^n_{n-1}=A^n_1.$$

Lemma 4.

$$B^n_m=\begin{bmatrix}B^{n-1}_m & \tilde{B}^n_m\\ (\tilde{B}^n_m)^\top & B^{n-1}_{m-1}\end{bmatrix}+\binom{n-1}{m+1}\begin{bmatrix}0 & \tilde{A}^n_m\\ 0 & A^{n-1}_{m-1}\end{bmatrix},$$

$$\tilde{B}^n_m=\begin{bmatrix}\tilde{B}^{n-1}_m & 0\\ \mathrm{diag}(K) & \tilde{B}^{n-1}_{m-1}\end{bmatrix}-\binom{n-2}{m+1}\begin{bmatrix}\tilde{A}^{n-1}_m & 0\\ 0 & 0\end{bmatrix},$$

where $\mathrm{diag}(K)$ *is the diagonal matrix with entries equal to the row index.*

Intersections

Corollary 2. *Let J and K be two adjacent m–combinations, then*

$$(B^n_m)^\tau_{N(J)+1,N(K)+1}=\binom{n}{m}-N(J\cap K).$$

$$\begin{aligned}(B^n_m)^\tau_{N(J)+1,N(K)+1} &=(B^n_m)_{\binom{n}{m}-N(K),\binom{n}{m}-N(J)}\\ &=(B^n_m)_{N(K^{\complement})+1,N(J^{\complement})+1}\\ &=N(K^{\complement}\cup J^{\complement})+1\\ &=N((J\cap K)^{\complement})+1\\ &=\binom{n}{m}-N(J\cap K).\end{aligned}$$

References

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