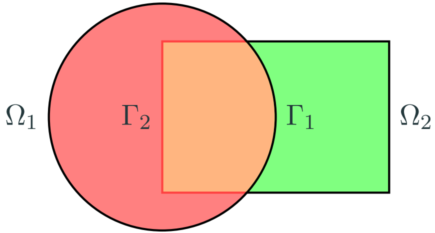
H.A. Schwarz, 1869

How do we solve the Laplace equation on complicated domains?

We split the domain into simpler subdomains.



Alternating Schwarz method:

$$\begin{cases} \Delta u_1^{n+1} = 0 & \text{in } \Omega_1, \\ u_1^{n+1} = u_2^n & \text{on } \Gamma_1, \end{cases} \qquad \begin{cases} \Delta u_2^{n+1} = 0 & \text{in } \Omega_2, \\ u_2^{n+1} = u_1^{n+1} & \text{on } \Gamma_2. \end{cases}$$

Algebraic decomposition

The subproblem on Ω_1 is

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} + T_{2\to 1} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1^{n+1} \\ \boldsymbol{u}_{1\Gamma}^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_\Gamma \end{bmatrix} + \begin{bmatrix} \\ -A_{\Gamma 2}\boldsymbol{u}_2^n + T_{2\to 1}\boldsymbol{u}_{2\Gamma}^n \end{bmatrix}$$

and the subproblem on Ω_2 is

$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} + T_{1\to 2} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_2^{n+1} \\ \boldsymbol{u}_{2\Gamma}^{n+1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_2 \\ \boldsymbol{f}_\Gamma \end{bmatrix} + \begin{bmatrix} \\ -A_{\Gamma 1}\boldsymbol{u}_1^n + T_{1\to 2}\boldsymbol{u}_{1\Gamma}^n \end{bmatrix}.$$

$\boldsymbol{u}_\Gamma$ appears in both subproblems since the interface is shared between the two subdomains. These copies are distinct and must be recombined in some way at the end, for example:

$$\boldsymbol{u}_\Gamma = \frac{\boldsymbol{u}_{1\Gamma} + \boldsymbol{u}_{2\Gamma}}{2}.$$

What is a symmetrized cell?

For each subdomain, take a copy of it and stitch it together along their shared interface. This pair is now perfectly symmetric, and one subproblem describes both copies.

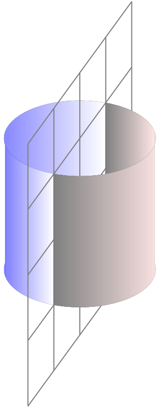


Figure 1: A symmetrized square domain with interfaces on two opposing edges