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Conor McCoid

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Joint work with Felix Kwok at the Université Laval
and Blaise Bourdin at McMaster University

CV highlights

- 6 journal articles, 3 proceedings
- 15 invited presentations at conferences
- Co-organized 2 minisymposia
- Founded careers workshop at McMaster
- Serving on EDII committee at McMaster as postdoc reppresentative
- Prix Henri Fehr for best thesis in mathematics at the University of Geneva

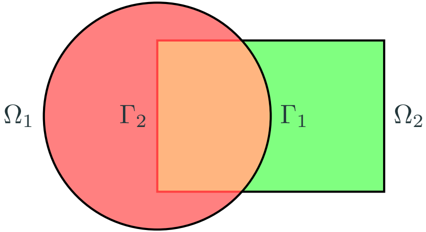
Outline

Introduction to domain decomposition and Schwarz methods

H.A. Schwarz, 1869

How do we solve the Laplace equation on complicated domains?

We split the domain into simpler subdomains.

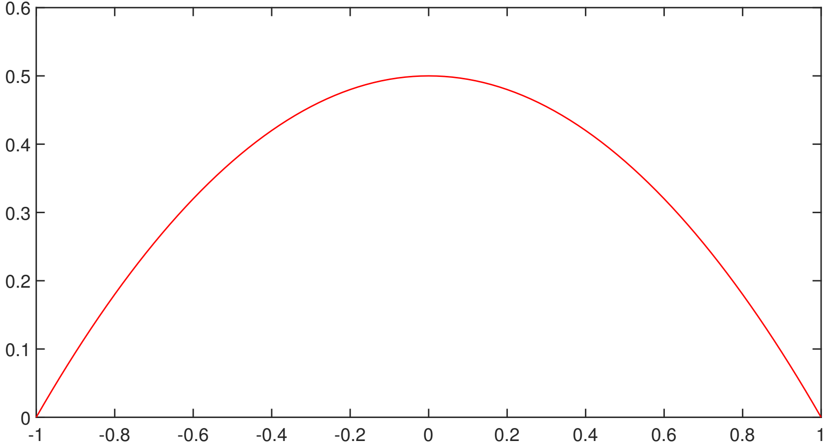


Alternating Schwarz method:

$$\begin{cases} \Delta u_1^{n+1} = 0 & \text{in } \Omega_1, \\ u_1^{n+1} = u_2^n & \text{on } \Gamma_1, \end{cases} \qquad \begin{cases} \Delta u_2^{n+1} = 0 & \text{in } \Omega_2, \\ u_2^{n+1} = u_1^{n+1} & \text{on } \Gamma_2. \end{cases}$$

Simple example

$$u''(x) = -1, \quad x \in [-1, 1], \quad u(-1) = u(1) = 0$$
$$\Omega_1 = [-1, 0.18], \quad \Omega_2 = [-0.22, 1]$$



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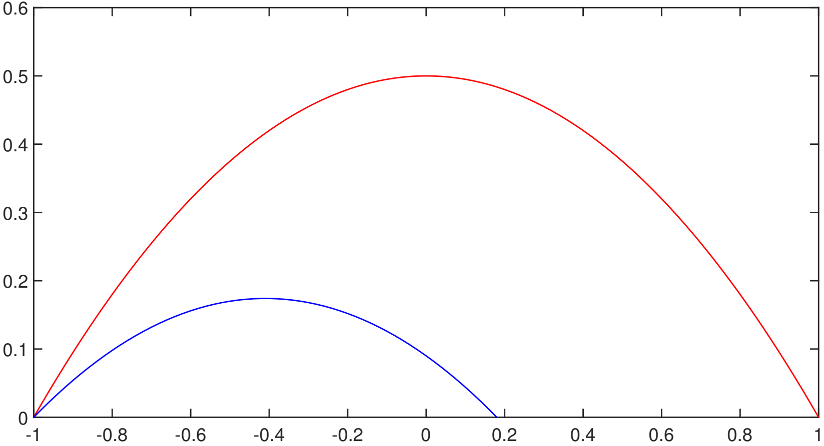

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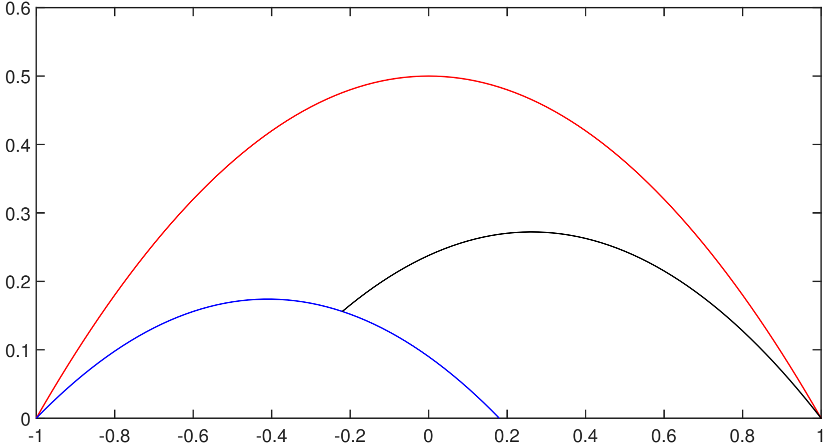

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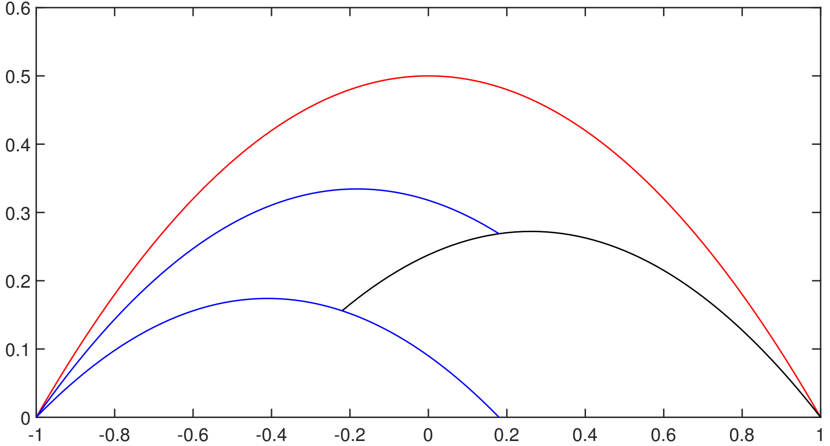

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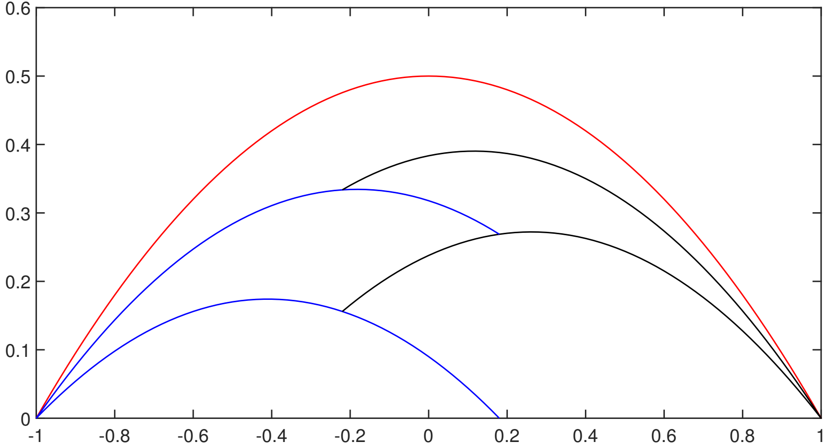

Field-split Schwarz methods

Instead of splitting up the physical domain, we can split up the problem into fields.

Suppose a problem can be written as

$$\begin{bmatrix} F(u,v) \\ G(u,v) \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

where F depends more strongly on u and G depends more strongly on v.

Field-split multiplicative Schwarz:

$$F(u^{(n+1)}, v^{(n)}) = f, \quad G(u^{(n+1)}, v^{(n+1)}) = g.$$

Adaptive optimized Schwarz methods

More complicated DD example

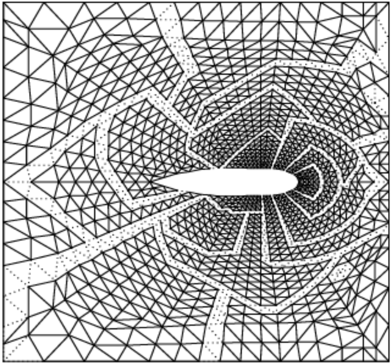

Figure 1: Also from Saad's "Iterative Methods"

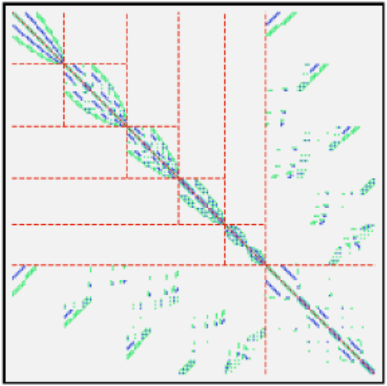

Figure 2: Possible matrix for example

A general form of the matrix

Let us consider matrices that can take the form

$$\begin{bmatrix} A_{11} & & & & A_{1\Gamma} \\ & A_{22} & & & A_{2\Gamma} \\ & & \ddots & & \vdots \\ & & & A_{nn} & A_{n\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & \dots & A_{\Gamma n} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_n \\ \boldsymbol{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_n \\ \boldsymbol{f}_\Gamma \end{bmatrix}, \tag{1}$$

where A_{ii} are square. This system represents n subdomains connected through a global interface represented by Γ.

Each subdomain now has its own subproblem:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + S_i \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i \\ \tilde{\boldsymbol{f}}_i \end{bmatrix}, \tag{2}$$

where $\tilde{\boldsymbol{f}}_i$ is some modification of $\boldsymbol{f}_\Gamma$, and S_i is some global transmission matrix.

How to choose S_i and $\tilde{\boldsymbol{f}}_i$

There are perfect choices of S_i and $\tilde{\boldsymbol{f}}_i$ such that each subproblem gives the exact solution to the global problem on its respective subdomain. However, these perfect choices are expensive to compute.

Instead, the standard procedure is to make *a priori* choices that give convergent iterative methods. These appear as:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + S_i \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_i^{(k+1)} \\ \boldsymbol{u}_{\Gamma i}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i \\ \boldsymbol{f}_\Gamma \end{bmatrix} + \sum_{j \neq i} \begin{bmatrix} & \\ -A_{\Gamma j} & T_j \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_j^{(k)} \\ \boldsymbol{u}_{\Gamma j}^{(k)} \end{bmatrix}, \quad (3)$$

where

$$S_i = \sum_{j \neq i} T_j. \quad (4)$$

Choices for T_j

The local transmission matrices T_j can represent boundary conditions between the subdomains. Some common options:

- Dirichlet, setting the interface variables on connected subdomains to be the same
- Neumann, setting the derivatives to be the same
- Optimized, setting Robin boundary conditions to be the same, using a Robin parameter that optimizes convergence rates

The strategy we'll employ here is to adapt the transmission conditions at each iteration so that they're closer to the perfect choices.

Symmetrized cells

For each subdomain, take a copy of it and stitch it together along their shared interface. This pair is now perfectly symmetric, and one subproblem describes both copies.

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} & \\ A_{\Gamma i} & A_{\Gamma\Gamma} & A_{\Gamma i} \\ & A_{i\Gamma} & A_{ii} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}}_i \\ \hat{\boldsymbol{u}}_\Gamma \\ \hat{\boldsymbol{u}}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i \\ \hat{\boldsymbol{f}}_i \\ \boldsymbol{f}_i \end{bmatrix}. \tag{5}$$

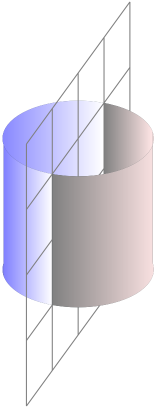

Figure 3: A symmetrized square domain with interfaces on two opposing edges

Adaptive optimized Schwarz on symmetrized cells

We solve the following iterations to correct the initial solution:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + T_i^{(k+1)} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_i^{(k+1)} \\ \boldsymbol{d}_{\Gamma i}^{(k+1)} \end{bmatrix} = \begin{bmatrix} & \\ -A_{\Gamma i} & T_i^{(k)} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_i^{(k)} \\ \boldsymbol{d}_{\Gamma i}^{(k)} \end{bmatrix}, \tag{6}$$

where $\boldsymbol{d}_i^{(k+1)} = \boldsymbol{u}_i^{(k+1)} - \boldsymbol{u}_i^{(k)}$.

We can use techniques from static condensation to reduce the form of this system to acting only on the global interface:

$$\left(\hat{A}_i + E_i^{(k+1)}\right) \boldsymbol{d}_{\Gamma i}^{(k+1)} = E_i^{(k)} \boldsymbol{d}_{\Gamma i}^{(k)}, \tag{7}$$

where

$$\hat{A}_i = A_{\Gamma\Gamma} - 2A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma}, \quad E_i^{(k)} = T_i^{(k)} + A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma}.$$

Updates to the transmission condition

We have one input, $\boldsymbol{d}_{\Gamma i}^{(k)}$, and one output, $E_i^{(k)}\boldsymbol{d}_{\Gamma i}^{(k)}$. This is enough to give a rank one approximation of $E_i^{(k)}$, which we can use to update the transmission conditions:

$$T_i^{(k+1)} := T_i^{(k)} - E_i^{(k)} \frac{\boldsymbol{d}_{\Gamma i}^{(k)} \left(\boldsymbol{d}_{\Gamma i}^{(k)}\right)^\top}{\left\|\boldsymbol{d}_{\Gamma i}^{(k)}\right\|^2}.$$

For subsequent iterations, we have to orthogonalize the vectors $\boldsymbol{d}_{\Gamma i}^{(k)}$:

Inputs: $\boldsymbol{d}_{\Gamma i}^{(k)}$, $E_i^{(k)}\boldsymbol{d}_{\Gamma i}^{(k)}$, all previous $\boldsymbol{d}_{\Gamma i}^{(j)}$ and $E_i^{(j)}\boldsymbol{d}_{\Gamma i}^{(j)}$
Set $\boldsymbol{w}_k := \boldsymbol{d}_{\Gamma i}^{(k)}$ and $\boldsymbol{v}_k := E_i^{(k)}\boldsymbol{d}_{\Gamma i}^{(k)}$
for $j = 1 : k-1$ **do**
    $h \leftarrow \langle \boldsymbol{d}_{\Gamma i}^{(j)}, \boldsymbol{w}_k \rangle$, $\boldsymbol{w}_k \leftarrow \boldsymbol{w}_k - h\boldsymbol{d}_{\Gamma i}^{(j)}$
    $\boldsymbol{v}_k \leftarrow \boldsymbol{v}_k - hE_i^{(j)}\boldsymbol{d}_{\Gamma i}^{(j)}$
end for
Output: $E_i^{(k+1)} := E_i^{(k)} - \boldsymbol{v}_k \boldsymbol{w}_k^\top$

Equivalence to a Krylov subspace method

From static condensation (7) we have that the vectors $\boldsymbol{d}_{\Gamma i}^{(k)}$ form a Krylov subspace:

$$\boldsymbol{d}_{\Gamma i}^{(k)} \in \mathcal{K}_k\left(\left(\hat{A}_i + E_i^{(1)}\right)^{-1} E_i^{(1)}, \boldsymbol{d}_{\Gamma i}^{(1)}\right) = \mathcal{K}_k.$$

By solving the system (6) the vector $\boldsymbol{d}_{\Gamma i}^{(k+1)}$ is equal to

$$\boldsymbol{d}_{\Gamma i}^{(k+1)} = \left(\hat{A}_i + E_i^{(k)}\right)^{-1} E_i^{(k)} \boldsymbol{x},$$

where $\boldsymbol{x} \in \mathcal{K}_k$ such that

$$\left(I - \left(\hat{A}_i + E_i^{(1)}\right)^{-1} E_i^{(1)}\right)\left(\boldsymbol{u}_{\Gamma i}^{(k-1)} + \boldsymbol{x}\right) \\ - \left(\hat{A}_i + E_i^{(1)}\right)^{-1}\left(\boldsymbol{f}_\Gamma - 2A_{\Gamma i}A_i^{-1}\boldsymbol{f}_i\right) \perp \mathcal{K}_k.$$

This is a Galerkin condition on the pre-image of the next difference. This makes the AOSM equivalent to the full orthogonalization method (FOM), a precursor to the generalized minimal residual method (GMRES).

Convergence of the AOSM

Because of the equivalence to FOM, we can use the research there to tell us how AOSM will converge. Unfortunately, that research tells us we can't know anything *a priori*:

Greenbaum, Ptak and Strakos (YYYY)

Recent work has brought us closer to understanding the convergence properties of these methods. One project I propose is to build on this work in the specific context of Schwarz-preconditioned systems.

Project proposal: AOSM/GMRES convergence

Project idea: Investigate the convergence of GMRES and FOM by applying recent analysis methods to Schwarz-preconditioned systems. Also study known worst-case convergence curves and if preconditioning techniques can improve rates.

Possible collaborations: Currently proposed as a postdoctoral project with Chen Greif (UBC) and Manfred Trummer (SFU), could also involve

Mentoring opportunities: Studying GMRES convergence curves makes for a great PhD project. The problem is well known in the community and any progress on it would make for an impressive thesis.

Numerical results of AOSM

Simple comparison with other methods

$$\begin{cases} \Delta u(x,y) = f(x,y), & (x,y) \in \Omega = [-1,1] \times [-1,1], \\ u(x,y) = g(x,y), & (x,y) \in \partial\Omega. \end{cases}$$

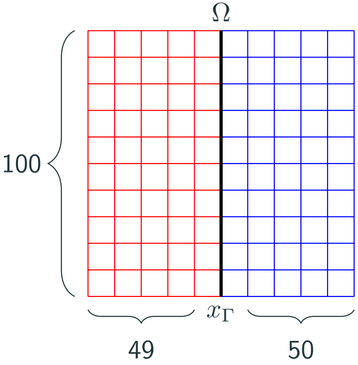

Figure 4: 100×100 evenly spaced grid split into two subdomains along the 50th value of x

Simple comparison with other methods

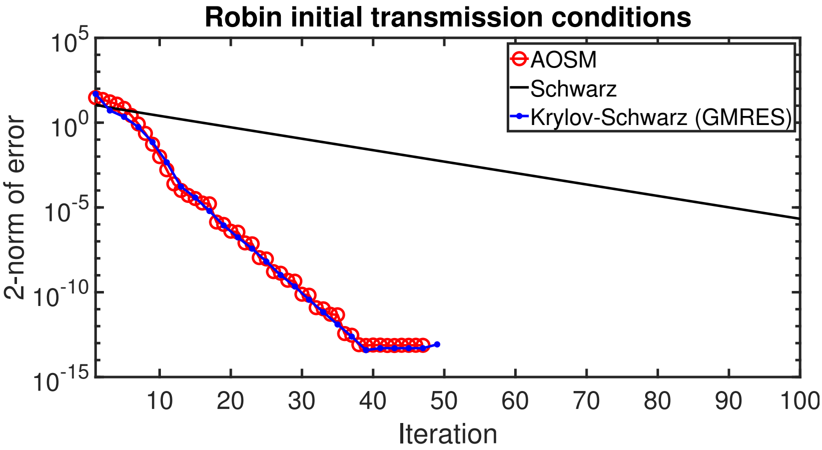

Figure 5: Comparison between Schwarz, AOSM and Krylov-Schwarz on simple elliptic PDE

Precursor to multiple subdomains: red-black decompositions

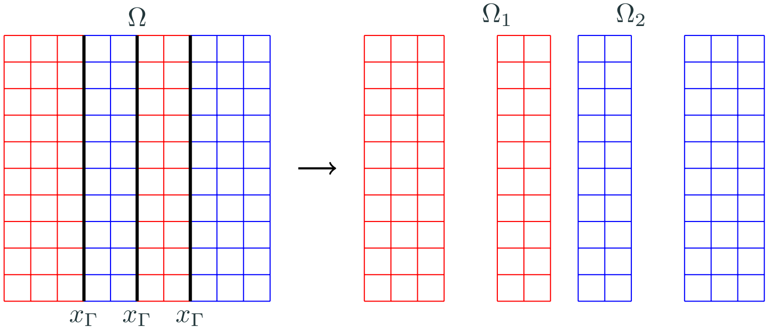

Figure 6: Splitting the 100×100 grid into four strips, then pairing the strips into two algebraic subdomains

Comparison: stripwise

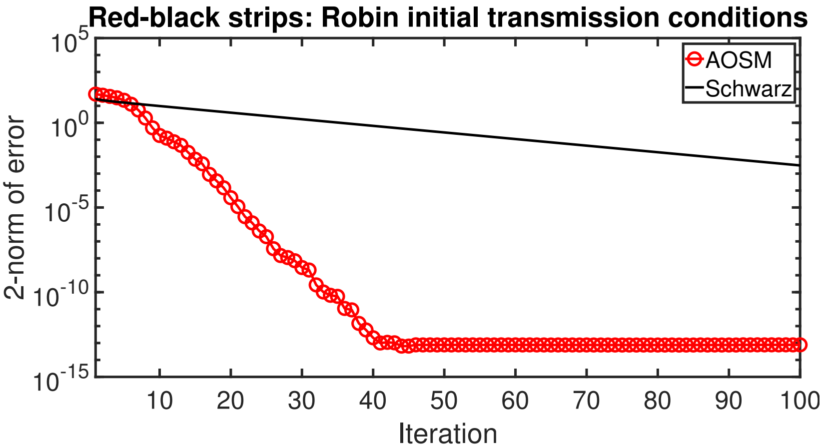

Figure 7: Convergence for AOSM and Schwarz on the stripwise decomposition

Adapted transmission conditions: stripwise

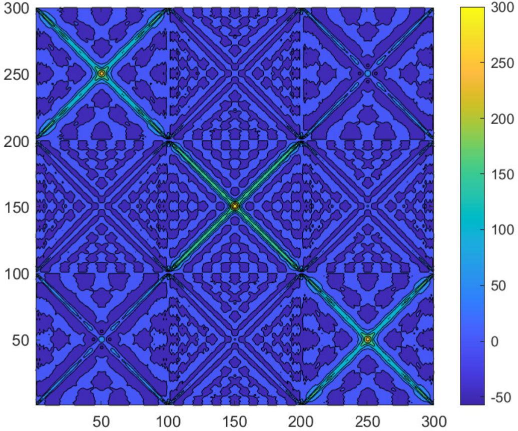

Figure 8: Matrix $T_{1\to 2}$ from AOSM after convergence of stripwise example

Heterogeneous elliptic PDE

$$\begin{cases} -\nabla\left(\alpha(x,y)\cdot\nabla u(x,y)\right) = f(x,y), & (x,y)\in\Omega=[0,1]\times[0,1], \\ u(x,y) = g(x,y), & (x,y)\in\partial\Omega. \end{cases}$$

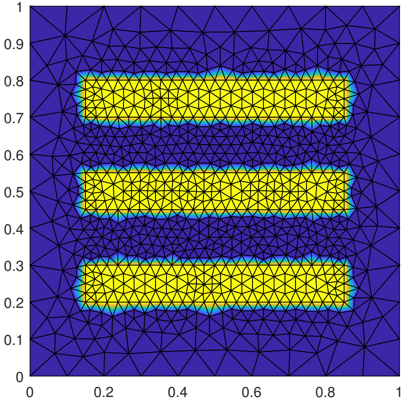

Figure 9: $\alpha(x,y)=1$ except along three thin channels where $\alpha(x,y)=1000$

Unstructured grid

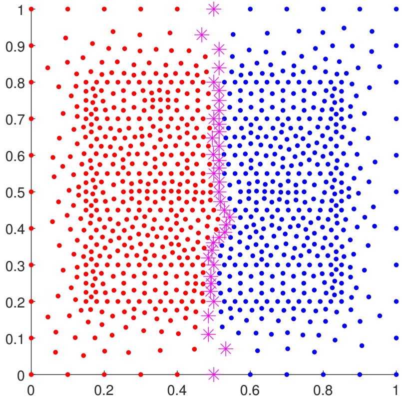

Figure 10: Splitting an unstructured grid into two subdomains

First round of solves

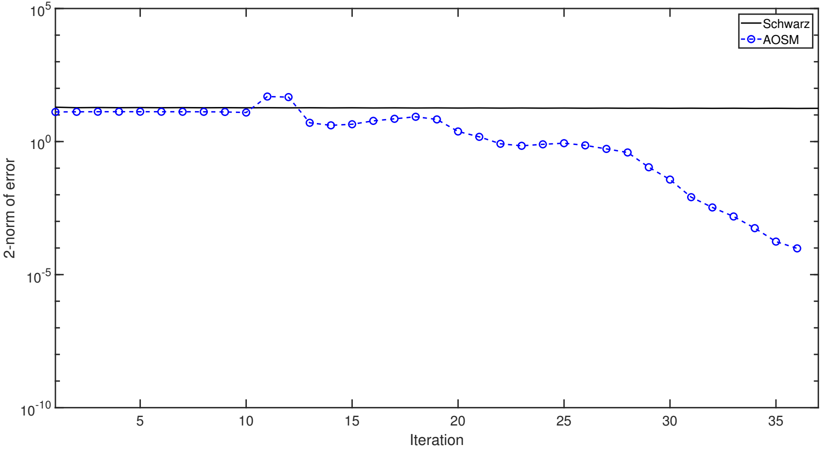

Figure 11: (Lack of) convergence for Schwarz and AOSM for heterogeneous elliptic PDE

Second round of solves

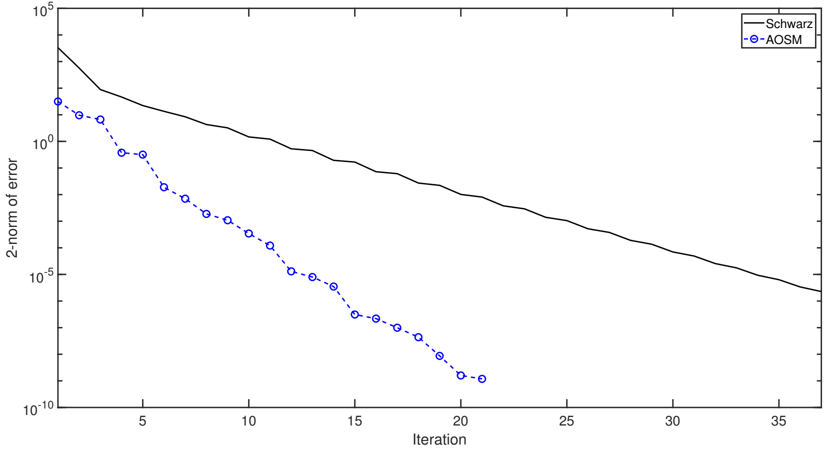

Figure 12: Convergence for Schwarz and AOSM using adapted transmission conditions