Advances in Schwarz methods

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December 15th, 2025

Joint work with Felix Kwok at the Université Laval
and Blaise Bourdin at McMaster University

Hobbies

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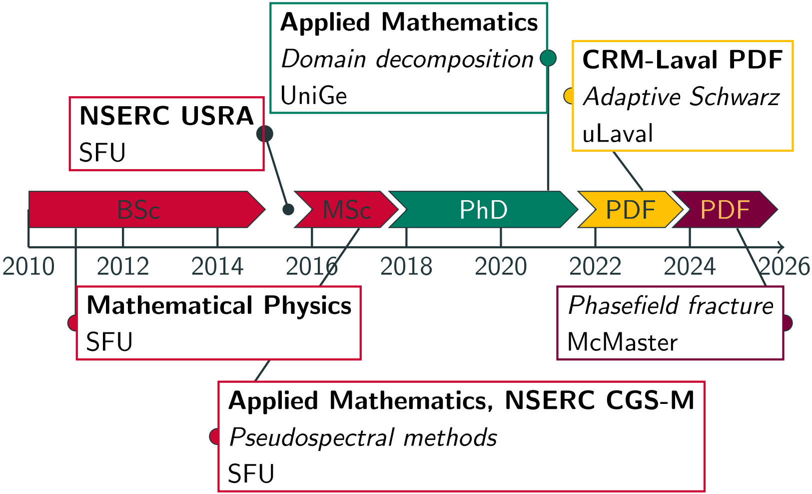
Career timeline



CV highlights

- 6 journal articles in Q1 journals, 3 conference proceedings papers
- 15 invited presentations at international conferences
- Co-organized 2 minisymposia
- 8 seminar presentations
- Instructor for 5 courses
- Founded careers workshop at McMaster
- Serving on EDII committee at McMaster as postdoc representative
- Prix Henri Fehr for best thesis in mathematics at the University of Geneva

Outline

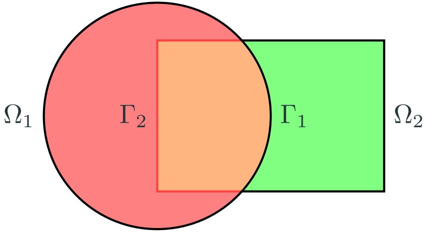
1. Introduction to domain decomposition and Schwarz methods
2. Newton-Schwarz methods
    2.1 Phasefield fracture model and AltMin
    2.2 MSPIN and parallelogram minimization
3. Adaptive optimized Schwarz methods
    3.1 Symmetrized cells
    3.2 Equivalence to Krylov subspace methods

Introduction to domain decomposition and Schwarz methods

H.A. Schwarz, 1869

How do we solve the Laplace equation ($\Delta u = 0$) on complicated domains?

We split the domain into simpler subdomains.

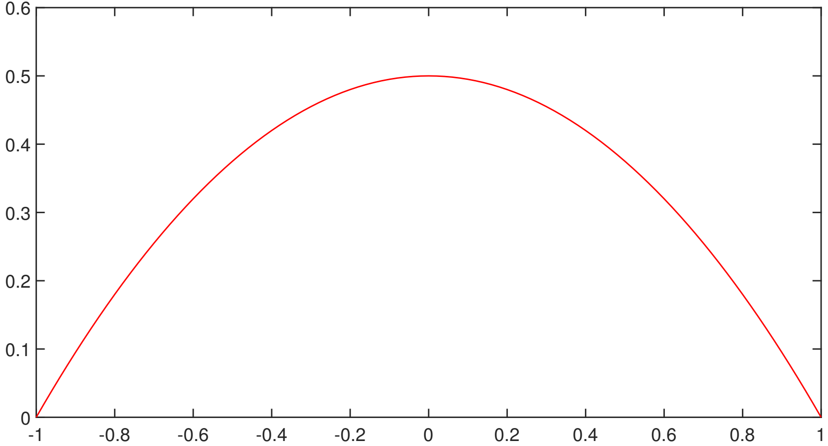


Alternating Schwarz method:

$$\begin{cases} \Delta u_1^{n+1} = 0 & \text{in } \Omega_1, \\ u_1^{n+1} = u_2^n & \text{on } \Gamma_1, \end{cases} \quad \begin{cases} \Delta u_2^{n+1} = 0 & \text{in } \Omega_2, \\ u_2^{n+1} = u_1^{n+1} & \text{on } \Gamma_2. \end{cases}$$

Simple example

$$u''(x) = -1, \quad x \in [-1, 1], \quad u(-1) = u(1) = 0$$
$$\Omega_1 = [-1, 0.18], \quad \Omega_2 = [-0.22, 1]$$



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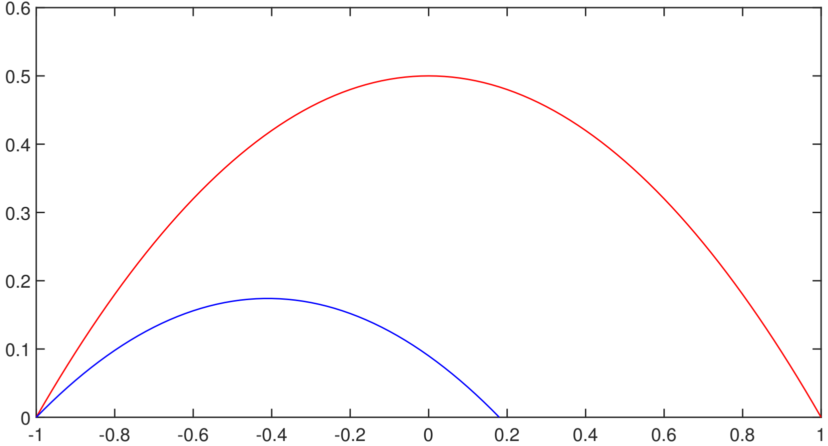

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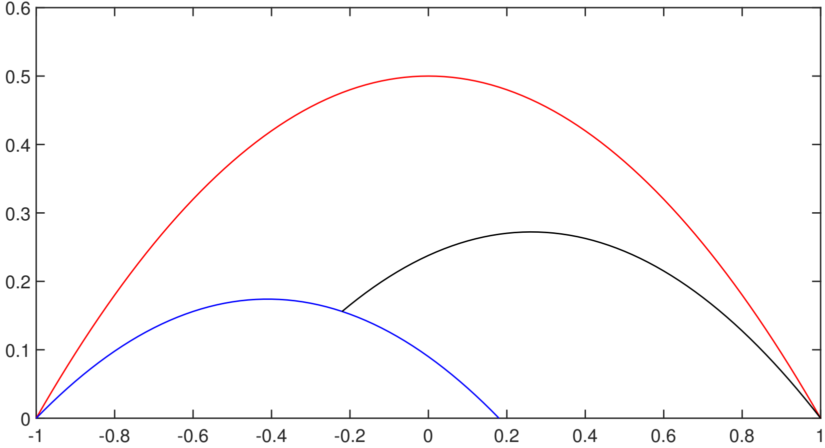

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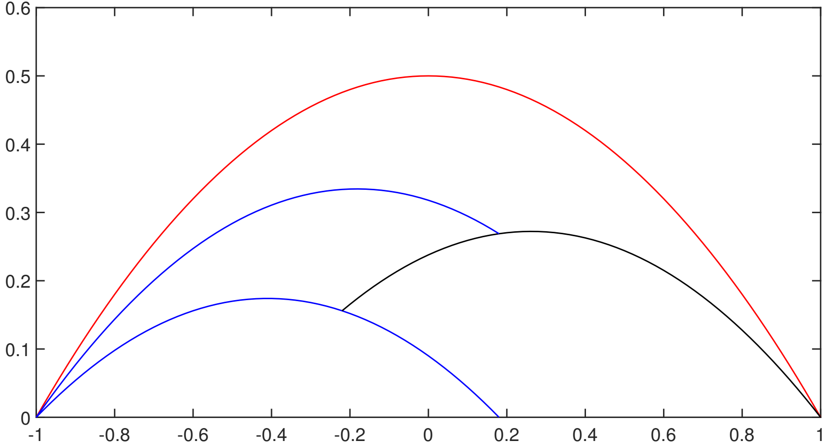

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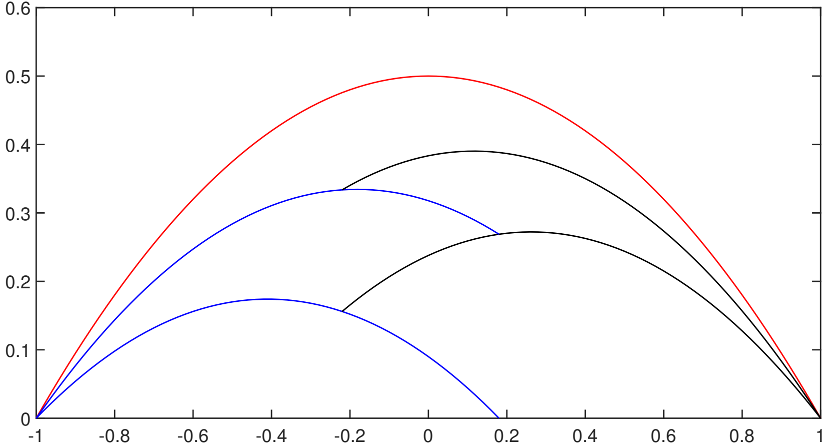


Field-split Schwarz methods

Instead of splitting up the physical domain, we can split up the problem into fields.

Suppose a problem can be written as

$$\begin{bmatrix} F(u,v) \\ G(u,v) \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

where F depends more strongly on u and G depends more strongly on v.

Field-split multiplicative Schwarz:

$$F(u^{(n+1)}, v^{(n)}) = f, \quad G(u^{(n+1)}, v^{(n+1)}) = g.$$

Phasefield fracture and Newton-Schwarz

Brittle fracture

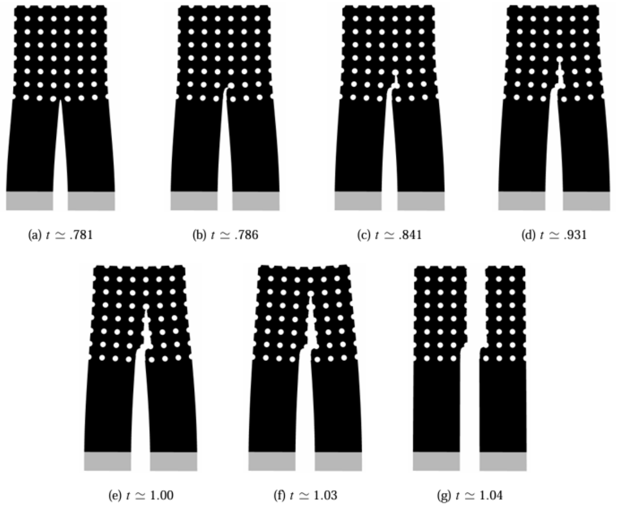


Figure 1: Crack propagation in brittle material, image taken from paper by Blaise Bourdin

Phasefield fracture model

$$E_\ell(u,\alpha) = \int \frac{1}{2}(1-\alpha)^2 Ae(u)\cdot e(u)dx + \frac{G_c}{4c_w}\int \frac{w(\alpha)}{\ell} + \ell\left|\nabla\alpha\right|^2 dx$$

- u is a vector field which defines the displacement
- α is a scalar field that is 0 away from the crack and 1 on the crack
- $e(u)$ is the strain tensor, A the stiffness tensor
- ℓ determines the accuracy of the approximation of the Hausdorff measure of the crack
- $w(0) = 0$, $w(1) = 1$, $w'(x) \geq 0$, $w \in C^1(0,1)$, $c_w = \int_0^1 \sqrt{w(s)}ds$
- G_c is the critical energy release rate
- The model is quasi-static: energy is minimized for a given time step/external work, then the model steps forward in time

Alternate minimization

Cracks propagate to minimize $E_\ell(u,\alpha)$. To model this, we then need to minimize over both u and α:

$$\begin{bmatrix} F(u,v) \\ G(u,v) \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial u} E_\ell(u,\alpha) \\ \frac{\partial}{\partial \alpha} E_\ell(u,\alpha) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The energy E_ℓ is not convex in both variables, but it is convex if either u or α is kept constant. This leads to alternate minimization.

Algorithm 1 Alternate Minimization (AltMin)

1: Make initial guess α_0 and set $n = 0$
2: **while** $\|\alpha_{n+1} - \alpha_n\|$ is greater than tolerance **do**
3: Find $u_{n+1} = \operatorname{argmin}_u E_\ell(u, \alpha_n)$
4: Find $\alpha_{n+1} = \operatorname{argmin}_\alpha E_\ell(u_{n+1}, \alpha)$
5: **end while**

(animation)

Applying Newton (Kopanicakova, Kothari & Krause, 2023)

AltMin can be thought of as a fixed point iteration over α:

$$\alpha_{n+1} = \mathsf{AltMin}(\alpha_n),$$

which stops when α_{n+1} is sufficiently close to α_n.

To accelerate the otherwise linear convergence, we can apply Newton's method to AltMin(α) - α after the minimization steps:

$$\begin{bmatrix} J_{uu} & \\ J_{\alpha u} & J_{\alpha\alpha} \end{bmatrix}^{-1} \begin{bmatrix} J_{uu} & J_{u\alpha} \\ J_{\alpha u} & J_{\alpha\alpha} \end{bmatrix} \begin{bmatrix} u_* - u_n \\ \alpha_* - \alpha_n \end{bmatrix} = \begin{bmatrix} u_{n+1} - u_n \\ \alpha_{n+1} - \alpha_n \end{bmatrix},$$

where J_{ij} is the second derivative of $E_\ell(u, \alpha)$ with respect to i then j.

In practice, we solve this system inexactly and usually with some kind of backtracking to globalize Newton's method. The resulting method is called multiplicative Schwarz preconditioning inexact Newton's method (MSPIN).

Globalization techniques

Newton's method needs globalization techniques to converge when the starting guess is far from a minimum. The default choice is cubic backtracking:

Algorithm 2 Cubic backtracking

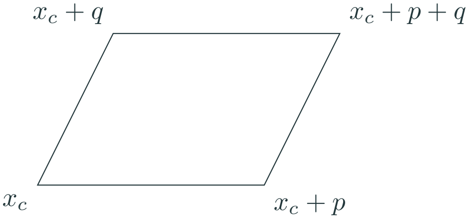
```
Inputs: current iterate $x_c$, search direction $p$, objective function $F(x)$
Set $\lambda_0 = 1$
Set $\lambda_1$ such that $x_p = x_c + \lambda_1 p$ minimizes the quadratic polynomial interpolating $F(x_c)$, $F(x_c + p)$ and $\nabla F(x_c)^\top p$
while $F(x_c + \lambda_1 p) \geq F(x_c)$ and $\|p\| <$ some tolerance do
    Set $\lambda_2$ such that $x_c + \lambda_2 p$ minimizes the cubic polynomial interpolating $F(x_c)$, $F(x_c + \lambda_0 p)$, $F(x_c + \lambda_1 p)$ and $\nabla F(x_c)^\top p$
    $\lambda_0 \leftarrow \lambda_1$, $\lambda_1 \leftarrow \lambda_2$
end while
```

Parallelogram minimization

MSPIN is well-suited for a new kind of globalization technique because it requires finding two solutions for the same problem:

- the result from multiplicative Schwarz/AltMin, $x_c + q$, and;
- the result from the Newton's method step, $x_c + p$.

This gives us two directions to look in, meaning we should seek our minimum in a parallelogram.



Parallelogram minimization

Given an objective function $F(x)$, minimize the polynomial $P(i,j) = ai^2 + bij + cj^2 + di + ej + f$, where

$$
\begin{aligned}
f =& F(x_c), & e =& \nabla F(x_c)^\top p, & d =& \nabla F(x_c)^\top q, \\
c =& F(x_c + p) - e - f, & a =& F(x_c + q) - c - f,
\end{aligned}
$$

$$b = F(x_c + p + q) - F(x_c + p) - F(x_c + q) + f,$$

in the region where $0 < i, j < 1$.

$$
\begin{bmatrix} i_{\min} \\ j_{\min} \end{bmatrix} \in \left\{ \begin{bmatrix} \frac{-2cd+be}{4ac-b^2} \\ \frac{-2ae+bd}{4ac-b^2} \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{e}{2c} \end{bmatrix}, \begin{bmatrix} -\frac{d}{2a} \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{e+b}{2c} \end{bmatrix}, \begin{bmatrix} -\frac{d+b}{2a} \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}
$$

Adaptive optimized Schwarz methods

More complicated DD example

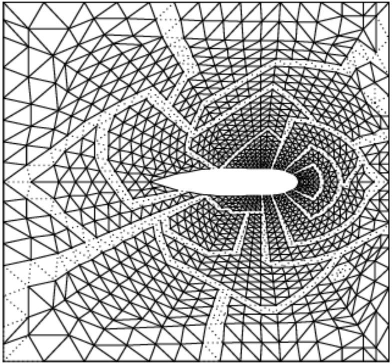


Figure 2: Airfoil from Saad's "Iterative Methods"

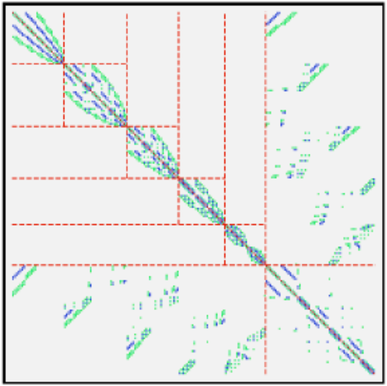


Figure 3: Possible matrix for example

A general form of the matrix

Let us consider matrices that can take the form

$$\begin{bmatrix} A_{11} & & & & A_{1\Gamma} \\ & A_{22} & & & A_{2\Gamma} \\ & & \ddots & & \vdots \\ & & & A_{nn} & A_{n\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & \dots & A_{\Gamma n} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \\ \vdots \\ \boldsymbol{u}_n \\ \boldsymbol{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \vdots \\ \boldsymbol{f}_n \\ \boldsymbol{f}_\Gamma \end{bmatrix}, \tag{1}$$

where A_{ii} are square. This system represents n subdomains connected through a global interface represented by Γ.

Each subdomain now has its own subproblem:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + S_i \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_i \\ \boldsymbol{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i \\ \tilde{\boldsymbol{f}}_i \end{bmatrix}, \tag{2}$$

where $\tilde{\boldsymbol{f}}_i$ is some modification of $\boldsymbol{f}_\Gamma$, and S_i is some global transmission matrix.

How to choose S_i and $\tilde{\boldsymbol{f}}_i$

There are perfect choices of S_i and $\tilde{\boldsymbol{f}}_i$ such that each subproblem gives the exact solution to the global problem on its respective subdomain. However, these perfect choices are expensive to compute.

Instead, the standard procedure is to make *a priori* choices that give convergent iterative methods. These appear as:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + S_i \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_i^{(k+1)} \\ \boldsymbol{u}_{\Gamma i}^{(k+1)} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i \\ \boldsymbol{f}_\Gamma \end{bmatrix} + \sum_{j \neq i} \begin{bmatrix} & \\ -A_{\Gamma j} & T_j \end{bmatrix} \begin{bmatrix} \boldsymbol{u}_j^{(k)} \\ \boldsymbol{u}_{\Gamma j}^{(k)} \end{bmatrix}, \quad (3)$$

where

$$S_i = \sum_{j \neq i} T_j. \quad (4)$$

Choices for T_j

The local transmission matrices T_j can represent boundary conditions between the subdomains. Some common options:

- Dirichlet, setting the interface variables on connected subdomains to be the same
- Neumann, setting the derivatives to be the same
- Optimized, setting Robin boundary conditions to be the same, using a Robin parameter that optimizes convergence rates

The strategy we'll employ here is to adapt the transmission conditions at each iteration so that they're closer to the perfect choices.

Symmetrized cells

For each subdomain, take a copy of it and stitch it together along their shared interface. This pair is now perfectly symmetric, and one subproblem describes both copies.

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} & \\ A_{\Gamma i} & A_{\Gamma\Gamma} & A_{\Gamma i} \\ & A_{i\Gamma} & A_{ii} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{u}}_i \\ \hat{\boldsymbol{u}}_\Gamma \\ \hat{\boldsymbol{u}}_i \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_i \\ \hat{\boldsymbol{f}}_i \\ \boldsymbol{f}_i \end{bmatrix}. \tag{5}$$

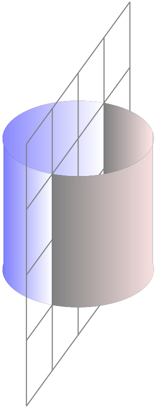


Figure 4: A symmetrized square domain with interfaces on two opposing edges

Adaptive optimized Schwarz on symmetrized cells

We solve the following iterations to correct the initial solution:

$$\begin{bmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma} + T_i^{(k+1)} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_i^{(k+1)} \\ \boldsymbol{d}_{\Gamma i}^{(k+1)} \end{bmatrix} = \begin{bmatrix} & \\ -A_{\Gamma i} & T_i^{(k)} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_i^{(k)} \\ \boldsymbol{d}_{\Gamma i}^{(k)} \end{bmatrix}, \qquad (6)$$

where $\boldsymbol{d}_i^{(k+1)} = \boldsymbol{u}_i^{(k+1)} - \boldsymbol{u}_i^{(k)}$.

We can use techniques from static condensation to reduce the form of this system to acting only on the global interface:

$$\left(\hat{A}_i + E_i^{(k+1)}\right) \boldsymbol{d}_{\Gamma i}^{(k+1)} = E_i^{(k)} \boldsymbol{d}_{\Gamma i}^{(k)}, \qquad (7)$$

where

$$\hat{A}_i = A_{\Gamma\Gamma} - 2A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma}, \quad E_i^{(k)} = T_i^{(k)} + A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma}.$$

$E_i^{(k)}\boldsymbol{d}_{\Gamma i}^{(k)}$ is then the non-zero part of the right hand side of (6).

Updates to the transmission condition

We have one input, $\boldsymbol{d}_{\Gamma i}^{(k)}$, and one output, $E_i^{(k)} \boldsymbol{d}_{\Gamma i}^{(k)}$. This is enough to give a rank one approximation of $E_i^{(k)}$, which we can use to update the transmission conditions:

$$T_i^{(k+1)} := T_i^{(k)} - E_i^{(k)} \frac{\boldsymbol{d}_{\Gamma i}^{(k)} \left(\boldsymbol{d}_{\Gamma i}^{(k)} \right)^\top}{\left\| \boldsymbol{d}_{\Gamma i}^{(k)} \right\|^2}.$$

For subsequent iterations, we have to orthogonalize the vectors $\boldsymbol{d}_{\Gamma i}^{(k)}$, i.e. with modified Gram-Schmidt.

If the matrix is symmetric, then the target transmission conditions are as well. We want to choose updates that preserve that symmetry:

$$T_i^{(k+1)} := T_i^{(k)} - \frac{E_i^{(k)} \boldsymbol{d}_{\Gamma i}^{(k)} \left(E_i^{(k)} \boldsymbol{d}_{\Gamma i}^{(k)} \right)^\top}{\left(\boldsymbol{d}_{\Gamma i}^{(k)} \right)^\top E_i^{(k)} \boldsymbol{d}_{\Gamma i}^{(k)}}.$$

The orthogonalization is now with respect to an A-norm.

AOSM is equivalent to Krylov subspace methods

Equivalence theorem of general AOSM, CM. & Kwok (2025)

The AOSM for general matrices is equivalent to the full orthogonalization method (FOM).

FOM is a precursor to the generalized minimal residual method (GMRES), considered by some the default iterative method for solving asymmetric linear systems.

By adding in a small least squares problem to the AOSM to correct the vectors $\boldsymbol{d}_{\Gamma i}^{(k)}$ and $\boldsymbol{d}_{i}^{(k)}$, we can get equivalence to GMRES. This would minimize the residual within the Krylov subspace at each iteration.

Equivalence theorem of symmetric AOSM, CM. (preprint)

The AOSM for symmetric matrices is equivalent to the conjugate gradient method (CG).

Convergence of the AOSM

Convergence of AOSM for symmetric matrices

The symmetric AOSM converges linearly and if the eigenvalues of $I - \left(\hat{A}_i + E_i^{(1)}\right)^{-1} E_i^{(1)}$ are clustered, then the AOSM converges superlinearly.

Unfortunately, for general matrices research tells us we can't know anything about the convergence of FOM or GMRES *a priori*:

Greenbaum, Ptak and Strakos (1996)

Given a non-increasing positive sequence $\{f_i\}$ of real numbers and an arbitrary set of eigenvalues, there exists a matrix A with these eigenvalues and a vector $\boldsymbol{b}$ such that the sequence of residuals of GMRES applied to the linear system $A\boldsymbol{x} = \boldsymbol{b}$ has norms equal to $\{f_i\}$.

Recent work has brought us closer to understanding the convergence properties of these methods. One project I propose is to build on this work in the specific context of Schwarz-preconditioned systems.

Project proposal: AOSM/GMRES convergence

Project idea: Investigate the convergence of GMRES and FOM by applying recent analysis methods to Schwarz-preconditioned systems. Also study known worst-case convergence curves and if preconditioning techniques can improve rates.

Possible collaborations: Currently proposed as a postdoctoral project with Chen Greif (UBC) and Manfred Trummer (SFU), could also involve

Mentoring opportunities: Studying GMRES convergence curves makes for a great PhD project. The problem is well known in the community and any progress on it would make for an impressive thesis.

Numerical results of AOSM

Simple comparison with other methods

$$\begin{cases} \Delta u(x,y) = f(x,y), & (x,y) \in \Omega = [-1,1] \times [-1,1], \\ u(x,y) = g(x,y), & (x,y) \in \partial\Omega. \end{cases}$$

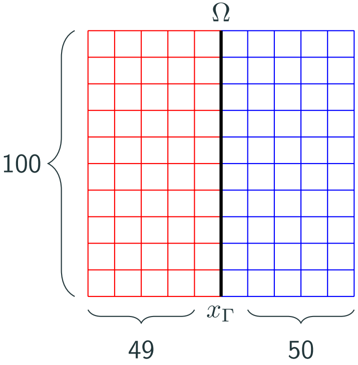


Figure 5: 100×100 evenly spaced grid split into two subdomains along the 50th value of x

Simple comparison with other methods

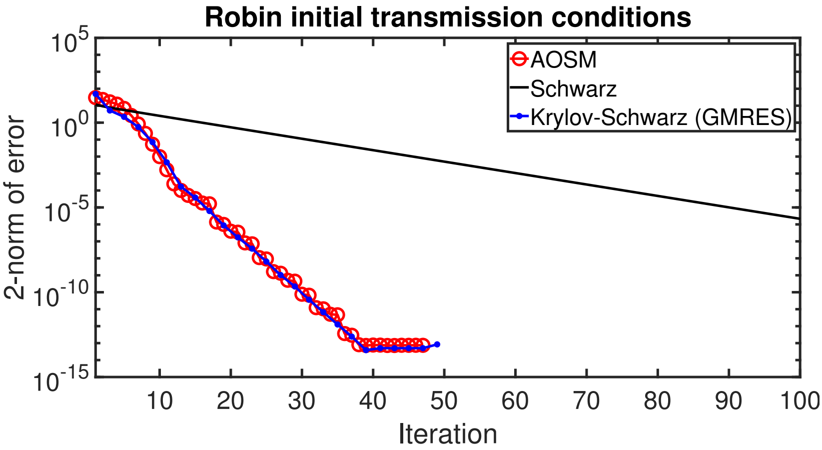


Figure 6: Comparison between Schwarz, AOSM and Krylov-Schwarz on simple elliptic PDE

Precursor to multiple subdomains: red-black decompositions

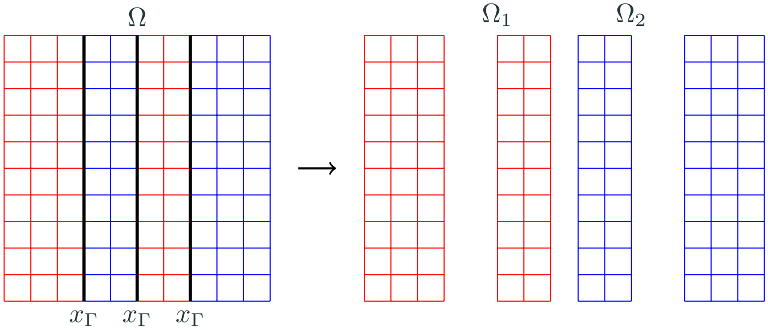


Figure 7: Splitting the 100×100 grid into four strips, then pairing the strips into two algebraic subdomains

Comparison: stripwise

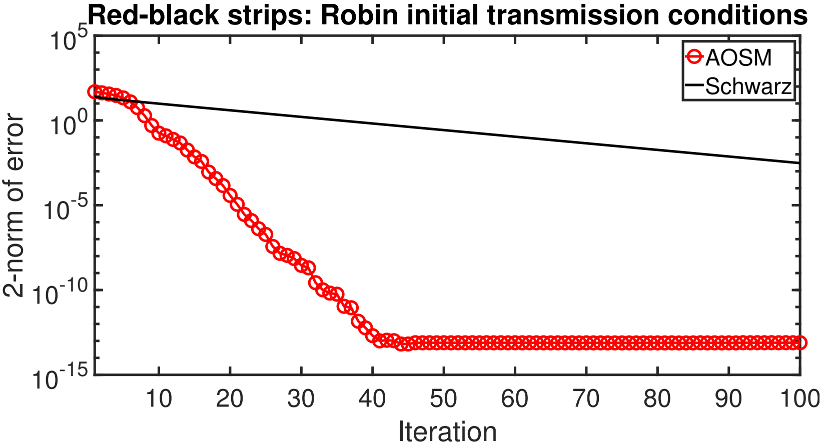


Figure 8: Convergence for AOSM and Schwarz on the stripwise decomposition

Heterogeneous elliptic PDE

$$\begin{cases} -\nabla\left(\alpha(x,y)\cdot\nabla u(x,y)\right) = f(x,y), & (x,y)\in\Omega=[0,1]\times[0,1], \\ u(x,y) = g(x,y), & (x,y)\in\partial\Omega. \end{cases}$$

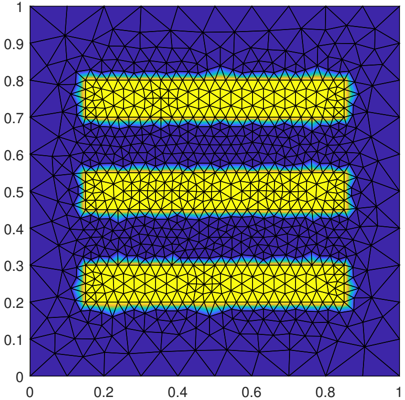


Figure 9: $\alpha(x,y)=1$ except along three thin channels where $\alpha(x,y)=1000$

Unstructured grid

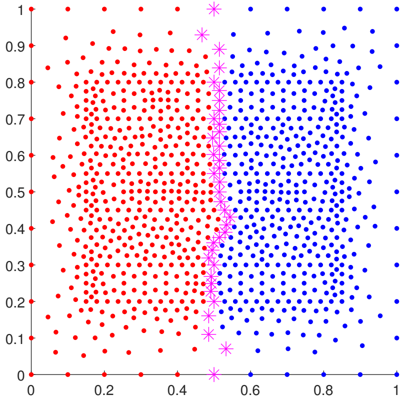


Figure 10: Splitting an unstructured grid into two subdomains

First round of solves

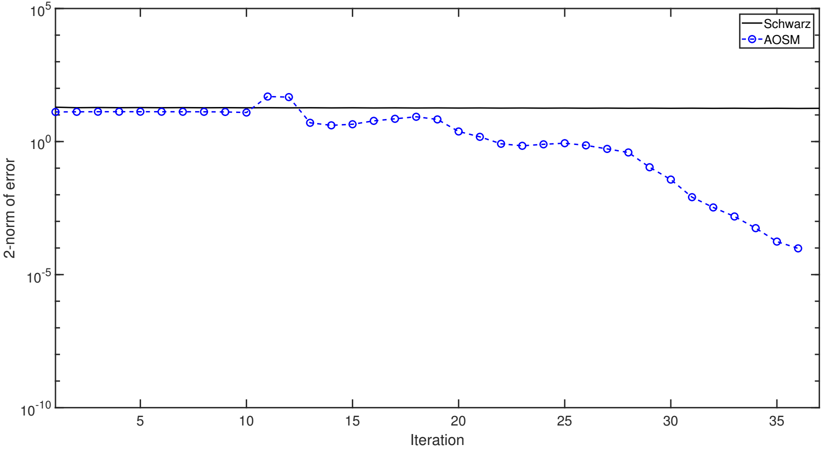


Figure 11: (Lack of) convergence for Schwarz and AOSM for heterogeneous elliptic PDE

Second round of solves

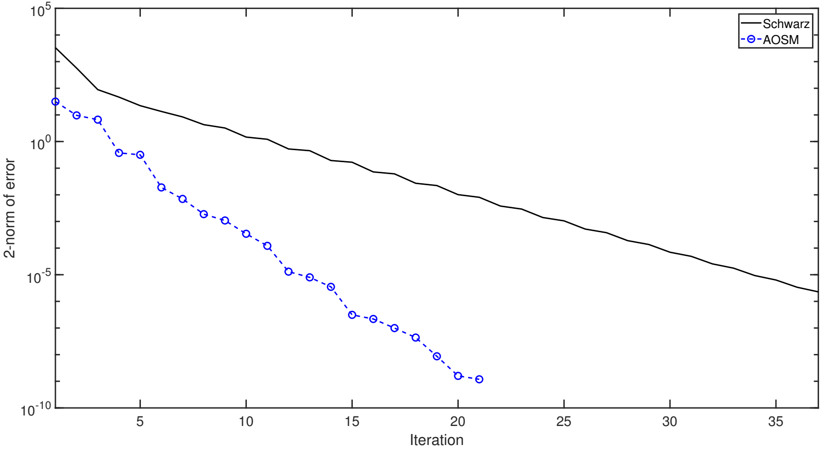


Figure 12: Convergence for Schwarz and AOSM using adapted transmission conditions

Thank you

Appendices

Project proposal: AOSM versions of Krylov-Schwarz

Project idea: Develop AOSM versions of existing Krylov subspace methods and compare against Krylov-Schwarz methods. Also investigate best practices for size and shape of global interface, passing information between subdomains, and restricting global interface to local interface.

Possible collaborations:

Mentoring opportunities: Modifications to an established algorithm would make for a good PhD project, and a straightforward introduction to high-performance computing.

Equivalence to a Krylov subspace method

From static condensation (7) we have that the vectors $\boldsymbol{d}_{\Gamma i}^{(k)}$ form a Krylov subspace:

$$\boldsymbol{d}_{\Gamma i}^{(k)} \in \mathcal{K}_k \left(\left(\hat{A}_i + E_i^{(1)} \right)^{-1} E_i^{(1)}, \boldsymbol{d}_{\Gamma i}^{(1)} \right) = \mathcal{K}_k.$$

By solving the system (6) the vector $\boldsymbol{d}_{\Gamma i}^{(k+1)}$ is equal to

$$\boldsymbol{d}_{\Gamma i}^{(k+1)} = \left(\hat{A}_i + E_i^{(k)} \right)^{-1} E_i^{(k)} \boldsymbol{x},$$

where $\boldsymbol{x} \in \mathcal{K}_k$ such that

$$\left(I - \left(\hat{A}_i + E_i^{(1)} \right)^{-1} E_i^{(1)} \right) \left(\boldsymbol{u}_{\Gamma i}^{(k-1)} + \boldsymbol{x} \right) \\ - \left(\hat{A}_i + E_i^{(1)} \right)^{-1} \left(\boldsymbol{f}_\Gamma - 2 A_{\Gamma i} A_i^{-1} \boldsymbol{f}_i \right) \perp \mathcal{K}_k.$$

This is a Galerkin condition on the pre-image of the next difference.

Pseudocode for AOSM orthogonalization steps

Asymmetric matrices:

1: Inputs: $\boldsymbol{d}_{\Gamma i}^{(k)}$, $E_i^{(k)}\boldsymbol{d}_{\Gamma i}^{(k)}$, all previous $\boldsymbol{d}_{\Gamma i}^{(j)}$ and $E_i^{(j)}\boldsymbol{d}_{\Gamma i}^{(j)}$
2: Set $\boldsymbol{w}_k := \boldsymbol{d}_{\Gamma i}^{(k)}$ and $\boldsymbol{v}_k := E_i^{(1)}\boldsymbol{d}_{\Gamma i}^{(k)}$
3: **for** $j = 1 : k-1$ **do**
4: $\quad h \leftarrow \langle \boldsymbol{d}_{\Gamma i}^{(j)}, \boldsymbol{w}_k \rangle$, $\boldsymbol{w}_k \leftarrow \boldsymbol{w}_k - h\boldsymbol{d}_{\Gamma i}^{(j)}$
5: $\quad \boldsymbol{v}_k \leftarrow \boldsymbol{v}_k - h\boldsymbol{v}_j$
6: **end for**
7: Output: $E_i^{(k+1)} := E_i^{(k)} - \boldsymbol{v}_k \boldsymbol{w}_k^\top$

Symmetric matrices:

Inputs: $\boldsymbol{d}_{\Gamma i}^{(k)}$, $E_i^{(k)}\boldsymbol{d}_{\Gamma i}^{(k)}$, all previous $\boldsymbol{d}_{\Gamma i}^{(j)}$ and $E_i^{(j)}\boldsymbol{d}_{\Gamma i}^{(j)}$
Set $\boldsymbol{w}_k := \boldsymbol{d}_{\Gamma i}^{(k)}$ and $\boldsymbol{v}_k := E_i^{(1)}\boldsymbol{d}_{\Gamma i}^{(k)}$
for j=1:k-1 **do**
    $h \leftarrow \langle \boldsymbol{w}_k, \boldsymbol{v}_j \rangle$
    $\boldsymbol{w}_k \leftarrow \boldsymbol{w}_k - h\boldsymbol{w}_j / \boldsymbol{v}_j^\top \boldsymbol{w}_j$
    $\boldsymbol{v}_k \leftarrow \boldsymbol{v}_k - h\boldsymbol{v}_j / \boldsymbol{v}_j^\top \boldsymbol{w}_j$
end for
Output: $E_i^{(k+1)} := E_i^{(k)} - \frac{\boldsymbol{v}_k \boldsymbol{v}_k^\top}{\boldsymbol{v}_k^\top \boldsymbol{w}_k}$

Adapted transmission conditions: stripwise

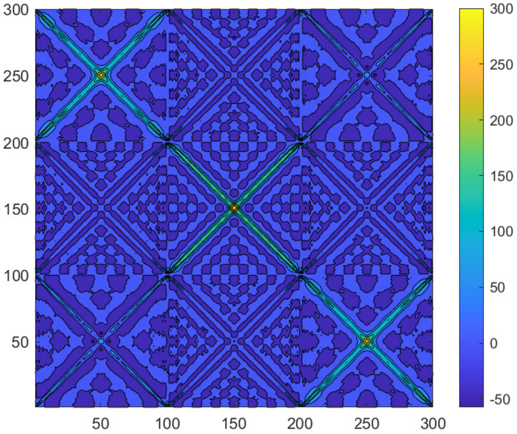


Figure 13: Matrix $T_{1\to 2}$ from AOSM after convergence of stripwise example