# 1 Fixed point iterations

We examine in detail two fixed point iterations:

$$g_1(z) = e^{-z}, \quad g_2(z) = -\log(z). \tag{1}$$

First and foremost, these functions have fixed points at the real root of

$$ze^z - 1 = 0 \tag{2}$$

and are inverses of each other.

The function $g_1(z)$ is $2\pi$–periodic in the imaginary direction. It is for this reason that it cannot converge to any of the complex roots of function 2. Likewise, this means we need only consider $-\pi \leq Im(z) < \pi$. The function $g_2(z)$ also does not converge to complex roots by choice of branch cut. This can be changed with the addition of $in2\pi$, where $n$ is the branch cut of interest.

We are concerned with where each function will converge. We can guarantee convergence in a region $D$ by the fixed point theorem.

**Theorem 1** (Fixed point iteration theorem)**.** *If a function $g(z)$ satisfies*

**(i)** $g(z) \in D$

**(ii)** $|g'(z)| < 1$

*for all $z \in D$ then it has a unique fixed point in $D$ and the iteration $z_{n+1} = g(z_n)$ converges to this fixed point.*

Consider $g_1(z)$: condition (ii) of theorem 1 is satisfied when $Re(z) > 0$; for condition (i) note that $g_1(z)$ rotates off the real axis by angle $-Im(z)$. For $g_1(z)$ to satisfy $Re(g_1(z)) > 0$ it is necessary that $|Im(z)| < \pi/2$. Our region $D_0$ (the region for which $g_1(z)$ satisfies theorem 1) is therefore:

$$D_0 = \{z \in \mathbb{C} | Re(z) > 0, \ -\pi/2 < Im(z) < \pi/2\}. \tag{3}$$

Given that $g_2(z)$ is the inverse of $g_1(z)$ and $D_0$ exists, there is no such region for $g_2(z)$.

Figure 1 gives a representation of the region $D_0$ (purple) and its images and pre-images. For ease of notation we define the sets $D_k$ as:

$$D_{k+1} = g_1(D_k), \quad D_{k-1} = g_2(D_k).$$

Since $D_1 \subset D_0$ by definition of $D_0$ and $g_2(g_1(z)) = z$ there exists a hierarchy of sets: $D_{k+1} \subset D_k \subset D_{k-1}$ for all $k \in \mathbb{Z}$. Each set $D_{k-1}$ is the pre-image of $D_k$ under the $g_1(z)$ function. As such, $D_{-\infty}$ represents the basin of attraction of $g_1(z)$.

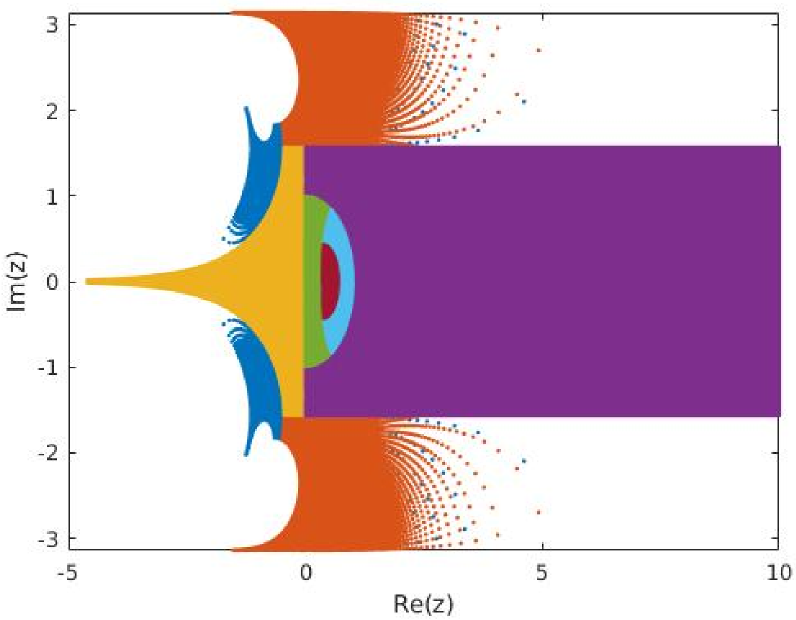


Figure 1: The region $D_0$ and its images under $g_1(z)$ and $g_2(z)$.

## 2 Preconditioned Newton

We now look at applying Newton's method to the functions $g_1(z)-z$ and $g_2(z)-z$. This will give the following fixed point iteration functions:

$$f_1(z) = \frac{zg_1'(z) - g_1(z)}{g_1'(z) - 1} = \frac{1+z}{1+e^z}, \quad f_2(z) = \frac{z(1-\log(z))}{1+z}. \tag{4}$$

Note that $f_1(z) = f_2(e^{-z})$ and $f_2(z) = f_1(-\log(z))$.

The function $f_1(z)$ has singularities at all branches of $\log(-1)$. Unlike $g_1(z)$, it is not periodic in the imaginary direction. The function $f_2(z)$ has an erroneous fixed point at $z = 0$, a singularity at $z = -1$ and a root at $z = e$. These points will be problematic and must be excluded from the basins of attraction.

We can perform the same analysis as before using theorem 1. Condition (ii) can be written in terms of the fixed point functions:

$$|f_1'(z)| = \left|\frac{g_1''(z)(g_1(z)-z)}{(g_1'(z)-1)^2}\right| < 1, \quad \left|\frac{g_2''(z)(g_2(z)-z)}{(g_2'(z)-1)^2}\right| < 1.$$

This ultimately requires

$$|f_1'(z)| = \left|\frac{1-ze^z}{(1+e^z)^2}\right| < 1, \quad |f_2'(z)| = \left|\frac{z+\log(z)}{(1+z)^2}\right| < 1.$$

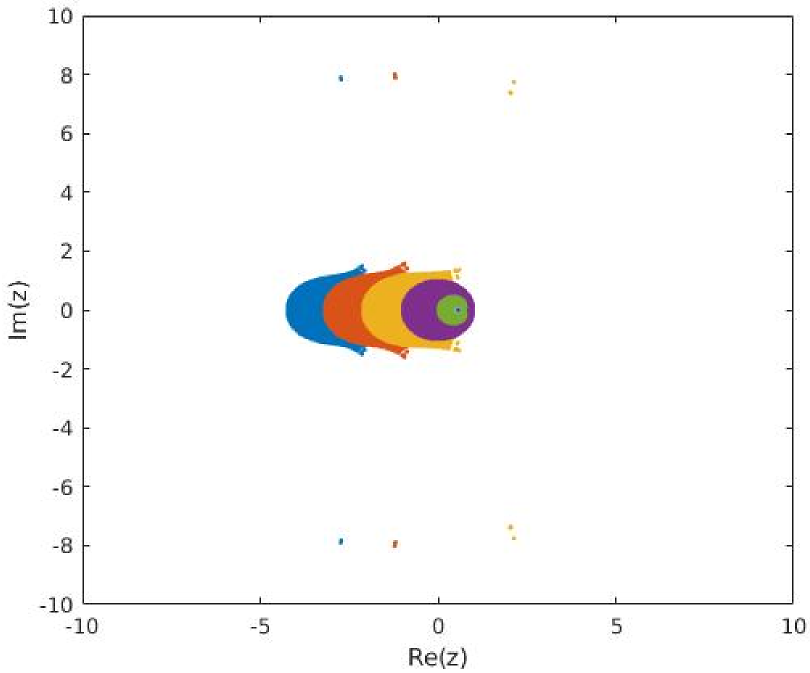


Figure 2: The region $D_0^1$, its images and pre-images under $f_1(z)$.

Condition (ii) holds for $f_2(z)$ except in an elliptical region containing $z = -1$. The region where the fixed point theorem is true for $f_2(z)$, hereafter called $D_0^2$, is then the complex plane without the pre-images of this ellipse. However, if $z \approx -1$ but not equal then $f_2(z)$ is not within this ellipse. More precisely, the image of the ellipse is outside the ellipse. Thus, $D_0^2$ is the entire complex plane except for the points -1, 0 and $e$ and their pre-images. This constitutes a countable set. (side note: the pre-image of 0 for $f_2(z)$ is $e$, so the definition of $D_0^2$ can be further simplified if desired)

Analysis for $f_1(z)$ is carried out numerically. Through experiments we can establish that the ball of radius 1 in the complex plane represents a region where theorem 1 is satisfied. Call this ball $D_0^1$. We can also show that the inverse of $f_1(z)$ is

$$f_1^{-1}(z) = z - 1 - W(-ze^{z-1})$$

where $W(z)$ is the Lambert W function (0 branch). Using this, we repeat figure 1.

Figure 2 shows the hierarchy of sets, with $D_0^1$ in purple. Its images are inset and converge rapidly to the root. Its pre-images extend onto the negative real line with some scattering. A more resolved $D_0^1$ (see figure 3) shows greater detail in this scattering, and reveals some fractal structure.

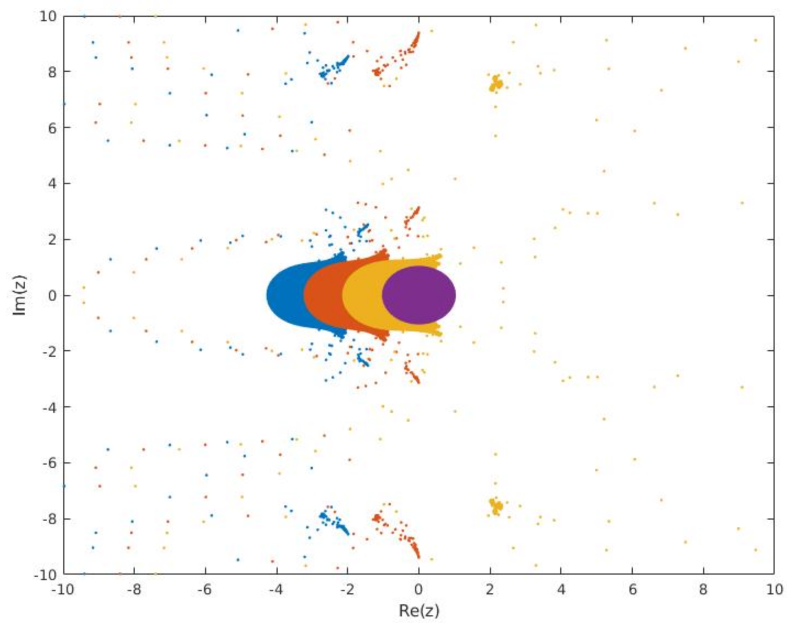


Figure 3: The region $D_0^1$ and its pre-images under $f_1(z)$.