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On monotone iteration and Schwarz methods for nonlinear parabolic PDEs☆

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Abstract

The Schwarz Alternating Method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomains.

In this paper, proofs of convergence of some Schwarz Alternating Methods for nonlinear parabolic problems which are known to have solutions by the monotone method (also known as the method of subsolutions and supersolutions) are given. In particular, an additive Schwarz method for scalar as well some coupled nonlinear PDEs are shown to converge to the solution on finitely many subdomains. In the coupled system case, each subdomain PDE is linear, decoupled and can be solved concurrently with other subdomain PDEs. These results are applicable to several models in population biology. The convergence behavior is illustrated by two numerical examples.

MSC: 65M55

Keywords: Domain decomposition; Nonlinear parabolic PDE; Schwarz alternating method; Monotone methods; Subsolution; Supersolution

1. Introduction

The Schwarz Alternating Method was devised by H.A. Schwarz more than one hundred years ago to solve linear boundary value problems. It has garnered interest recently because of its potential as an efficient algorithm for parallel computers. See the fundamental work in [16,17]. The

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literature on this method for the linear boundary value problem is huge, see the recent reviews in [7,34,29,25]. The literature for nonlinear problems is rather sparse. Besides Lions' works, see also [2,36,5,30,23,35,8,32,33,18,19,20,21], and references therein. The effectiveness of Schwarz methods for nonlinear problems (especially those in fluid mechanics) has been demonstrated in many papers. See proceedings of the annual domain decomposition conferences beginning with [12].

While the above papers study nonlinear elliptic PDEs, this paper investigates the application of Schwarz methods to nonlinear parabolic PDEs, extending the analysis in [21] to time-dependent problems. It can also be considered as an extension of the paper of Gander [10] where he considered one-dimensional nonlinear heat equations. Specifically, we consider time-dependent PDEs which are amenable to analysis by the monotone method (also known as the method of subsolutions and supersolutions). The paper [15] was among the first to employ such method to solve boundary value problems. Subsequent works by these two authors as well as in [27,1] and many others have made this method into one of the important tools in nonlinear analysis. Parter [24] is an early work on the numerical solution of nonlinear PDEs by this method. See [22] for a very complete reference with many applications as well as a good bibliography. See also [14,6] and the references therein for additional related works. Our results on coupled systems can be applied to the three types of Lotka–Volterra models in population biology: competition, cooperation and predator–prey.

One approach to solving time-dependent problems is to use backward Euler or Crank–Nicholson to discretize the time derivative. At each time step, an elliptic PDE can be solved using classical Schwarz methods. See [16,3,4]. An alternative approach is by waveform relaxation. Here, the time-dependent problem is solved in each subdomain to the final time T. Boundary data is exchanged and the process is repeated. See [11,10]. There are several advantages of this method. One is that subdomain problems are truely independent except for the exchange of boundary data after time T. Thus no communication among different subdomains is necessary during integration and that the time step used in different subdomains can be different. Second, when T is small, it is known that the iteration converges superlinearly for a class of nonlinear heat equations in one dimension. (See [10]. In that paper, the subdomain problems are nonlinear where as in this paper, all subdomain problems are linear, offering a considerable practical advantage.) The disadvantage is of course that when T is large, then the algorithm is inefficient. In this case, it is necessary to perform several waveform relaxations over smaller time intervals. It is a nontrivial task to determine the optimal time interval (which is both problem and hardware dependent) one should take. On parallel computers, waveform relaxation turns out to be quite efficient. See [13,28]. See [31,26] for further results and references for domain decomposition methods for linear parabolic problems.

In Section 2, we prove convergence of two Schwarz methods for a class of scalar nonlinear parabolic PDEs. In Section 3, we treat the so-called quasi-monotone nonincreasing case of a coupled system of PDEs, giving a proof of convergence of an additive Schwarz method on finitely many subdomains. The other two cases (quasi-monotone nondecreasing and mixed quasi-monotone) will be discussed in Section 4. This is followed by some numerical results and a conclusion in the final section. In the remaining part of this introduction, we set some notations.

Let Ω be a bounded domain in $\mathbb{R}^N$ with a smooth boundary. Suppose Ω is composed of $m \geqslant 2$ subdomains, that is, $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$. The boundary of each subdomain is also assumed to be smooth. We are interested in the solution of a parabolic PDE from time 0 to T for some fixed positive T. Define $D = (0,T) \times \Omega$, $D_i = (0,T) \times \Omega_i$, $S = (0,T) \times \partial\Omega$ and $S_i = (0,T) \times \partial\Omega_i$. Let X denote the space of functions in $C(\bar{D})$ which are continuously differentiable in time and twice differentiable in Ω. We shall look for solutions of PDEs lying in this space. The functions that we are dealing with will in general be functions of time and space. We normally suppress the dependence on space in the notation. For instance, $u(t)$ means $u(t,x)$. Finally, for any normed space Z, recall that $H^1(0,T;Z)$ consists of all measurable functions $u:[0,T]\to Z$ such that

$$\int_0^T (\|u(t)\|_Z^2 + \|u_t(t)\|_Z^2)\,\mathrm{d}t < \infty.$$

2. Scalar equations

Consider the PDE

$$u_t - \Delta u = f(t,x,u) \text{ on } D, \quad u = h \text{ on } S, \quad u(0) = U \text{ on } \Omega. \tag{2.1}$$

A function $u \in X$ is a *subsolution* of the above PDE if

$$\underline{u}_t - \Delta \underline{u} - f(t,x,\underline{u}) \leqslant 0 \text{ on } D, \quad \underline{u} \leqslant h \text{ on } S, \quad \underline{u}(0) \leqslant U \text{ on } \Omega.$$

Similarly, a *supersolution* is one which satisfies the above with all inequalities reversed.

Let us now record the assumptions for the above PDE. Suppose that it has a subsolution $\underline{u}$ and a supersolution $\bar{u}$ which satisfy $\underline{u} \leqslant \bar{u}$ on $\bar{D}$. Define the sector of smooth functions

$$\mathscr{A} \equiv \{u \in X,\ \underline{u} \leqslant u \leqslant \bar{u} \text{ on } \bar{D}\}.$$

Assume f is a Holder continuous function defined on $\bar{\Omega}\times\mathscr{A}$ and h and U are sufficiently smooth in their domains of definition with $U = h(0)$ on $\partial\Omega$. In addition, suppose there exist some nonnegative Holder continuous functions $\underline{c}$ and $\bar{c}$ defined on $\bar{D}$ so that

$$-\underline{c}(u-v) \leqslant f(t,x,u) - f(t,x,v) \leqslant \bar{c}(u-v) \text{ on } D, \quad v \leqslant u \in \mathscr{A}. \tag{2.2}$$

With these assumptions, it is known (Section 3.2 in [22]) that the PDE has a unique solution in $\mathscr{A}$. Such PDEs have applications in population biology, chemical kinetics, etc.

A fundamental tool for monotone iteration schemes is the (weak) maximum principle. One useful form is the following.

Lemma 1. *Let $w \in X$ satisfy*

$$w_t - \Delta w + \underline{c}w \geqslant 0 \text{ on } D, \quad w \geqslant 0 \text{ on } S, \quad w(0) \geqslant 0 \text{ on } \Omega.$$

Then $w \geqslant 0$ on $\bar{D}$.

We shall need an extension of the above result to functions which are less smooth and on domains with a possibly nonsmooth boundary. We shall call the following the generalized maximum principle.

Lemma 2. *Suppose Y is an open set in $\mathbb{R}^N$ and $Y_T = (0,T)\times Y$. Let $w \in H^1(0,T;H^1(Y)\cap C(\bar{Y}))$ and satisfy*

$$\int_{Y_T} (\nabla w\cdot\nabla\phi + w_t\phi + \underline{c}w\phi) \geqslant 0, \quad \forall \text{ nonnegative } \phi \in H^1(0,T;H_0^1(Y)) \tag{2.3}$$

and $w(0) \geqslant 0$ on Y. Then $w \geqslant 0$ on $\bar{Y}_T$.

Proof. Let $w^+(t,x)=\max(w(t,x),0)$, $w^-(t,x)=\max(-w(t,x),0)$. Note that $w=w^+-w^-$ and that w^- is a nonnegative function in $H^1(0,T;H_0^1(Y))$. Taking $\phi=w^-$ in (2.3), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{Y_T}(w^-)^2 \leqslant -\int_{Y_T}(|\nabla w^-|^2+\underline{c}(w^-)^2).$$

Since $w^-(0)=0$, the above implies that $w^-\equiv 0$. Hence $w\geqslant 0$ on $\bar{Y}_T$. □

We now show convergence of a (multiplicative) Schwarz sequence for the PDE (2.1) for the two-subdomain case. For convenience, we suppress the dependence of f on space and time. Note that each subdomain problem is a linear one.

Theorem 2.1. *Let $u^{(0)}=u^{(-1/2)}=\underline{u}$ on $\bar{D}$ with $\underline{u}=h$ on S. Define the Schwarz sequence by ($n\geqslant 0$)*

$$(\partial_t-\Delta+\underline{c})u^{(n+1/2)}=f(u^{(n-1/2)})+\underline{c}u^{(n-1/2)} \text{ on } D_1,$$

$$u^{(n+1/2)}=u^{(n)} \text{ on } S_1,$$

$$u^{(n+1/2)}(0)=U \text{ on } \Omega_1$$

and

$$(\partial_t-\Delta+\underline{c})u^{(n+1)}=f(u^{(n)})+\underline{c}u^{(n)} \text{ on } D_2,$$

$$u^{(n+1)}=u^{(n+1/2)} \text{ on } S_2,$$

$$u^{(n+1)}(0)=U \text{ on } \Omega_2.$$

Here, $u^{(n+1/2)}$ is defined as $u^{(n)}$ on $\bar{D}\backslash\bar{D}_1$ and $u^{(n+1)}$ is defined as $u^{(n+1/2)}$ on $\bar{D}\backslash\bar{D}_2$. Then $u^{(n+1/2)}\to u$ (pointwise) on D_i, $i=1,2$, where u is the solution of (2.1) in $\mathscr{A}$.

Proof. The proof can be divided into three steps. First, we demonstrate that the sequence is monotone:

$$\underline{u}\leqslant u^{(n-1/2)}\leqslant u^{(n)}\leqslant u^{(n+1/2)}\leqslant \bar{u} \text{ on } \bar{D}, \quad n\geqslant 0. \tag{2.4}$$

Since the sequences are bounded above, the following limits are well defined on $\bar{D}$

$$\lim_{n\to\infty}u^{(n+1/2)}=u_1, \qquad \lim_{n\to\infty}u^{(n)}=u_2.$$

In the second step, we prove that the function u_i satisfies the same PDE on D_i using the argument in [22, p. 64]. In the third step, we prove that $u_1=u_2$ on $\bar{D}$ which follows directly from (2.4). Thus $u_1=u_2=u$ is the solution of (2.1).

The details of the proof of the first step by induction are now given. On D_1,

$$(\partial_t-\Delta+\underline{c})u^{(1/2)}=f(u^{(0)})+\underline{c}u^{(0)} \quad\text{and}\quad (\partial_t-\Delta+\underline{c})u^{(0)}\leqslant f(u^{(0)})+\underline{c}u^{(0)}.$$

On subtraction of these two results, we have

$$(u^{(1/2)}-u^{(0)})_t-\Delta(u^{(1/2)}-u^{(0)})+\underline{c}(u^{(1/2)}-u^{(0)})\geqslant 0 \text{ on } D_1.$$

Since $u^{(1/2)} - u^{(0)} = 0$ on S_1 and $u^{(1/2)}(0) - u^{(0)}(0) = U - u^{(0)}(0) \geqslant 0$ on Ω_1, we conclude that $u^{(1/2)} \geqslant u^{(0)}$ on $\bar{D}_1$ by the maximum principle. This inequality also holds on $\bar{D}$. In a similar manner, we show that $u^{(1/2)} \leqslant \bar{u}$ on $\bar{D}$. Hence (2.4) holds for $n = 0$.

We now show that

$$u^{(n-1/2)} \leqslant u^{(n+1/2)} \text{ on } D_1 \quad \text{and} \quad u^{(n)} \leqslant u^{(n+1)} \text{ on } D_2 \tag{2.5}$$

by induction. This result will be needed in the proof of the induction step of (2.4) later. From the defining equation of $u^{(1)}$ and the fact that $u^{(0)}$ is a subsolution,

$$(u^{(1)} - u^{(0)})_t - \Delta(u^{(1)} - u^{(0)}) + \underline{c}(u^{(1)} - u^{(0)}) \geqslant 0 \text{ on } D_2.$$

Since $u^{(1)} - u^{(0)} = u^{(1/2)} - u^{(0)} \geqslant 0$ on S_2 and $u^{(1)}(0) - u^{(0)}(0) = U - u^{(0)}(0) \geqslant 0$ on Ω_2, it follows from the maximum principle that $u^{(1)} \geqslant u^{(0)}$ on $\bar{D}_2$. Since $u^{(-1/2)} = u^{(0)} \leqslant u^{(1/2)}$ on $\bar{D}_1$ has already been established, (2.5) holds for $n = 0$. Assume (2.5) holds for some n. We shall prove (2.5) with n replaced by $n + 1$. Subtracting the defining equations for $u^{(n+3/2)}$ and $u^{(n+1/2)}$ on D_1, we obtain

$$\begin{aligned}(\partial_t - \Delta + \underline{c})(u^{(n+3/2)} - u^{(n+1/2)}) &= f(u^{(n+1/2)}) - f(u^{(n-1/2)}) + \underline{c}(u^{(n+1/2)} - u^{(n-1/2)}) \\ &\geqslant 0.\end{aligned}$$

The last inequality holds because of the induction hypothesis and (2.2). Now $u^{(n+3/2)} = u^{(n+1)} \geqslant u^{(n)} = u^{(n+1/2)}$ on S_1 by the induction hypothesis and $u^{(n+3/2)}(0) - u^{(n+1/2)}(0) = 0$ on Ω_1. By the maximum principle, $u^{(n+1/2)} \leqslant u^{(n+3/2)}$ on $\bar{D}_1$. Similarly, $u^{(n+2)} \geqslant u^{(n+1)}$ on $\bar{D}_2$ and this finishes the proof of (2.5).

Suppose (2.4) holds. We show that $u^{(n+1/2)} \leqslant u^{(n+1)} \leqslant u^{(n+3/2)} \leqslant \bar{u}$ on $\bar{D}$. On $D_1 \setminus D_2$, $u^{(n+1/2)} = u^{(n+1)}$ by definition. On $D_{12} \equiv D_1 \cap D_2$, subtract the defining equations for $u^{(n+1/2)}$ and $u^{(n+1)}$ to obtain

$$(\partial_t - \Delta + \underline{c})(u^{(n+1)} - u^{(n+1/2)}) = f(u^{(n)}) - f(u^{(n-1/2)}) + \underline{c}(u^{(n)} - u^{(n-1/2)}) \geqslant 0$$

with the latter inequality due to the induction hypothesis and (2.2). In case D_{12} is not smooth, we can multiply the above inequality by a nonnegative $\phi \in H^1(0, T; H_0^1(\Omega_{12}))$ and use integration by parts to get

$$\int_{D_{12}} (u^{(n+1)} - u^{(n+1/2)})_t \phi + \nabla(u^{(n+1)} - u^{(n+1/2)}) \cdot \nabla\phi + \underline{c}(u^{(n+1)} - u^{(n+1/2)})\phi \geqslant 0.$$

Along $S_1 \cap D_2$, $u^{(n+1)} \geqslant u^{(n)} = u^{(n+1/2)}$ by (2.5) while along $S_2 \cap D_1$, $u^{(n+1)} = u^{(n+1/2)}$. Hence $u^{(n+1)} - u^{(n+1/2)} \geqslant 0$ on $D \cap (S_1 \cup S_2)$. Also, $u^{(n+1)}(0) - u^{(n+1/2)}(0) = 0$ on $\Omega_1 \cap \Omega_2$. By the generalized maximum principle, $u^{(n+1)} \geqslant u^{(n+1/2)}$ on $\bar{D}_{12}$. Since $u^{(n+1/2)} = u^{(n)} \leqslant u^{(n+1)}$ on $D_2 \setminus D_1$, $u^{(n+1)} \geqslant u^{(n+1/2)}$ on $\bar{D}$. In the same way, we can show that $u^{(n+3/2)} \geqslant u^{(n+1)}$ on $\bar{D}$.

From the defining equation of $u^{(n+3/2)}$ and that $\bar{u}$ is a supersolution, we can apply the maximum principle to show that $\bar{u} \geqslant u^{(n+3/2)}$ on $\bar{D}_1$. This inequality also holds on $\bar{D}$ by the definition of $u^{(n+3/2)}$. This completes the proof of (2.4). □

Note that we can also start the sequence with the supersolution $\bar{u}$ (assuming that $\bar{u} = h$ on S) in which case the sequence will converge to the solution from above.

The above (multiplicative) Schwarz iteration is an adaptation of the classical Schwarz iteration for Poisson's equation to solve a nonlinear equation. The next Schwarz method is called an additive Schwarz method. It generalizes the additive method for linear PDEs first introduced in [9]. It is sometimes preferable to the (multiplicative) Schwarz method above because the subdomain PDEs are independent and hence can be solved in parallel. We consider the general m-subdomain case.

Theorem 2.2. *Let $u^{(0)} = u_i^{(0)} = \underline{u}$ on $\bar{D}$, $i = 1,\ldots,m$ with $\underline{u} = h$ on S. Define the additive Schwarz sequence by* $(n \geqslant 1)$

$$(\partial_t - \Delta + \underline{c})u_i^{(n)} = f(u_i^{(n-1)}) + \underline{c}u_i^{(n-1)} \quad \text{on } D_i,$$

$$u_i^{(n)} = u^{(n-1)} \text{ on } S_i, \quad u_i^{(n)}(0) = U \text{ on } \Omega_i, \quad i = 1,\ldots,m.$$

Here, $u_i^{(n)}$ is defined as $u^{(n-1)}$ on $\bar{D} \setminus \bar{D}_i$ and

$$u^{(n)}(t,x) = \max_{1 \leqslant i \leqslant m} u_i^{(n)}(t,x), \quad (t,x) \in \bar{D}.$$

Then $u_i^{(n)} \to u$ on D_i, $i = 1,\ldots,m$ where u is the solution of (2.1) in $\mathscr{A}$.

Proof. The details of this proof are quite similar to those of the last proof. The following monotone properties hold:

$$\underline{u} \leqslant u_i^{(n)} \leqslant u_i^{(n+1)} \leqslant \bar{u} \text{ on } D_i, \quad \underline{u} \leqslant u^{(n)} \leqslant u^{(n+1)} \leqslant \bar{u} \text{ on } \bar{D}, \tag{2.6}$$

$$u^{(n)} \leqslant u_i^{(n+1)} \text{ on } \bar{D}, \quad i = 1,\ldots,m. \tag{2.7}$$

The inequalities in (2.6) can be shown in a straightforward manner by induction using the maximum principle. To show the second set of inequalities in (2.6), take a fixed n and $(t,x) \in \bar{D}$. Then there is some integer i in between 1 and m inclusive so that $u^{(n)}(t,x) = u_i^{(n)}(t,x) \leqslant u_i^{(n+1)}(t,x) \leqslant u^{(n+1)}(t,x)$.

Inequality (2.7) can also be shown by induction. The result certainly holds when $n = 0$. Suppose (2.7) holds with n replaced by $n-1$. We claim that

$$\int_{D_i} (\nabla u^{(n)} \cdot \nabla \phi + u_t^{(n)} \phi + \underline{c} u^{(n)} \phi) \leqslant \int_{D_i} (f(u^{(n-1)}) + \underline{c} u^{(n-1)})\phi \tag{2.8}$$

for all nonnegative $\phi \in H^1(0,T;H_0^1(\Omega_i))$. This inequality, which will be proved later, says that $u^{(n)}$ is a subsolution in some weak sense.

Now multiply the defining equation for $u_i^{(n+1)}$ by any nonnegative $\phi \in H^1(0,T;H_0^1(\Omega_i))$ and then integrate by parts to obtain

$$\int_{D_i} (\nabla u_i^{(n+1)} \cdot \nabla \phi + (\partial_t u_i^{(n+1)})\phi + \underline{c} u_i^{(n+1)} \phi) = \int_{\Omega_i} (f(u_i^{(n)}) + \underline{c} u_i^{(n)})\phi.$$

Subtract (2.8) from this equation to get

$$\int_{D_i} (\nabla(u_i^{(n+1)} - u^{(n)}) \cdot \nabla\phi + (u_i^{(n+1)} - u^{(n)})_t\phi + \underline{c}(u_i^{(n+1)} - u^{(n)})\phi)$$
$$\geqslant \int_{D_i} (f(u_i^{(n)}) - f(u^{(n-1)}) + \underline{c}(u_i^{(n)} - u^{(n-1)}))\phi$$
$$\geqslant 0$$

by the induction hypothesis $u_i^{(n)} \geqslant u^{(n-1)}$ and (2.2). Since $u_i^{(n+1)} = u^{(n)}$ on S_i and $u_i^{(n+1)}(0) - u^{(n)}(0) = U - u^{(n)}(0) \geqslant 0$ on Ω_i, we can conclude that $u_i^{(n+1)} \geqslant u^{(n)}$ on $\bar{D}_i$ by the generalized maximum principle. Of course this inequality also holds on $\bar{D}$. This completes the proof of (2.7).

Next, we define on $\bar{D}$, for $i = 1, \ldots, m$,

$$\lim_{n\to\infty} u_i^{(n)} = u_i, \qquad \lim_{n\to\infty} u^{(n)} = u_0$$

and show that the limit u_i satisfies the same PDE on D_i, $i = 1, \ldots, m$. We have $u_i \leqslant u_0$ on $\bar{D}$, $i = 1, \ldots, m$. By (2.7), we have for any j, $u_0 \leqslant u_j \leqslant u_0 \leqslant u_i$. From these inequalities, we conclude that $u_i = u_j = u_0$, $1 \leqslant i, j \leqslant m$. Define u to be this common function which must be the solution of (2.1) in $\mathscr{A}$.

To complete the proof, we need to show (2.8). First note that $u^{(n)} \in H^1(0, T; H^1(\Omega) \cap C(\bar{\Omega}))$. To show this, first look at the case $m = 2$. The result certainly holds when $n = 0$ by definition. For a positive n, since $u_i^{(n)} \in H^1(0, T; H^1(\Omega) \cap C(\bar{\Omega}))$,

$$u^{(n)}(t,x) = \max(u_1^{(n)}(t,x), u_2^{(n)}(t,x)) = u_1^{(n)}(t,x) + (u_2^{(n)}(t,x) - u_1^{(n)}(t,x))^+$$

which implies the desired result. The case of m subdomains follows by induction.

Consider the PDE

$$(\partial_t - \Delta + \underline{c})h^{(n)} = f(u^{(n-1)}) + \underline{c}u^{(n-1)} \text{ on } D_i,$$

$$h^{(n)} = u^{(n)} \text{ on } S_i, \qquad h^{(n)}(0) = u^{(n)}(0) \text{ on } \Omega_i.$$

Here, $n \geqslant 0$ and we define $u^{(-1)} = u^{(0)}$. We claim (to be proved later) that

$$h^{(n+1)} \geqslant h^{(n)} \text{ on } D_i \tag{2.9}$$

and

$$h^{(n)} \geqslant u^{(n)} \text{ on } \bar{D}_i. \tag{2.10}$$

Suppose there exist some nonnegative $\phi \in H^1(0, T; H_0^1(\Omega_i))$ such that

$$\int_{D_i} \nabla u^{(n)} \cdot \nabla\phi + u_t^{(n)}\phi + \underline{c}u^{(n)}\phi > \int_{D_i} (f(u^{(n-1)}) + \underline{c}u^{(n-1)})\phi. \tag{2.11}$$

Multiply the defining equation for $h^{(n)}$ by ϕ and then apply integration by parts. Subtract the resulting equation from (2.11) to obtain

$$\int_{D_i} \nabla(u^{(n)} - h^{(n)}) \cdot \nabla\phi + (u^{(n)} - h^{(n)})_t\phi + \underline{c}(u^{(n)} - h^{(n)})\phi > 0. \tag{2.12}$$

Since $u^{(n)} - h^{(n)} = 0$ on S_i and $u^{(n)}(0) - h^{(n)}(0) = 0$ on Ω_i, the generalized maximum principle states that $u^{(n)} \geqslant h^{(n)}$ on $\bar{D}_i$. In fact, since the inequality in (2.12) is strict, it is easy to see that $u^{(n)}(t_0,x_0) > h^{(n)}(t_0,x_0)$ for some $(t_0,x_0) \in D_i$. However, this contradicts (2.10). Hence assumption (2.11) is false and this implies the truth of (2.8).

Let us now show (2.9). On D_i,

$$(\partial_t - \Delta + \underline{c})(h^{(n+1)} - h^{(n)}) = f(u^{(n)}) - f(u^{(n-1)}) + \underline{c}(u^{(n)} - u^{(n-1)}) \geqslant 0$$

by the monotonicity of the sequence $u^{(n)}$. Since $h^{(n+1)} - h^{(n)} = u^{(n+1)} - u^{(n)} \geqslant 0$ on S_i and $h^{(n+1)}(0) - h^{(n)}(0) = u^{(n+1)}(0) - u^{(n)}(0) \geqslant 0$ on Ω_i, (2.9) follows by the maximum principle.

Finally, we show (2.10) by induction. The base case $n=0$ can easily be shown using the definition of $h^{(0)}$, the fact that $u^{(0)}$ is a subsolution and the maximum principle. Suppose (2.10) holds for n. We show $h^{(n+1)} \geqslant u^{(n+1)}$ on $\bar{D}_i$ by showing that $h^{(n+1)} \geqslant u_j^{(n+1)}$ on $\bar{D}_i$ for $1 \leqslant j \leqslant m$. When $j = i$, subtract the defining equations of $u_i^{(n+1)}$ and $h^{(n+1)}$ to obtain

$$(\partial_t - \Delta + \underline{c})(h^{(n+1)} - u_i^{(n+1)}) = f(u^{(n)}) - f(u_i^{(n)}) + \underline{c}(u^{(n)} - u_i^{(n)}) \geqslant 0 \text{ on } D_i.$$

Since $h^{(n+1)} - u_i^{(n+1)} = u^{(n+1)} - u^{(n)} \geqslant 0$ on S_i and $h^{(n+1)}(0) - u_i^{(n+1)}(0) = u^{(n+1)}(0) - U = 0$ on Ω_i, the maximum principle implies that $h^{(n+1)} \geqslant u_i^{(n+1)}$ on $\bar{D}_i$.

Now suppose $j \neq i$. Note

$$u_j^{(n+1)} = u^{(n)} \leqslant h^{(n)} \leqslant h^{(n+1)} \text{ on } \bar{D}_i \setminus \bar{D}_j$$

by (2.9) and the induction hypothesis. Next, subtract the definitions of $h^{(n+1)}$ and $u_j^{(n+1)}$ on $D_i \cap D_j$ to obtain

$$(\partial_t - \Delta + \underline{c})(h^{(n+1)} - u_j^{(n+1)}) = f(u^{(n)}) - f(u_j^{(n)}) + \underline{c}(u^{(n)} - u_j^{(n)}) \geqslant 0$$

or the weak form

$$\int_{D_i \cap D_j} \nabla(h^{(n+1)} - u_j^{(n+1)}) \cdot \nabla\phi + (h^{(n+1)} - u_j^{(n+1)})_t\phi + \underline{c}(h^{(n+1)} - u_j^{(n+1)})\phi \geqslant 0$$

for all nonnegative $\phi \in H^1(0,T;H_0^1(\Omega_i \cap \Omega_j))$. Observe that

$$h^{(n+1)} - u_j^{(n+1)} = \begin{cases} h^{(n+1)} - u^{(n)} \geqslant h^{(n)} - u^{(n)} \geqslant 0, & S_j \cap D_i, \\ u^{(n+1)} - u_j^{(n+1)} \geqslant 0, & S_i \cap D_j \end{cases}$$

by (2.9) and the induction hypothesis. Since $h^{(n+1)}(0) - u_j^{(n+1)}(0) = 0$ on $\Omega_i \cap \Omega_j$, we may conclude that $h^{(n+1)} \geqslant u_j^{(n+1)}$ on $\overline{D_i \cap D_j}$ by the generalized maximum principle. Combining with the result at the beginning of this paragraph, we conclude that $h^{(n+1)} \geqslant u_j^{(n+1)}$ on $\bar{D}_i$. This completes the proof. □

In practice, $u^{(n)}$ need not be computed globally but rather only along the subdomain boundaries over the time interval $(0,T)$. In the next two sections, we consider some coupled systems of nonlinear parabolic PDEs and their solution by an additive Schwarz method.

3. Quasi-monotone nonincreasing coupled systems

Consider the system

$$u_t - \Delta u = f(t,x,u,v), \quad v_t - \Delta v = g(t,x,u,v) \quad \text{on } D, \tag{3.1}$$

$$u = r, \quad v = s \quad \text{on } S,$$

$$u(0) = U, \quad v(0) = V \quad \text{on } \Omega.$$

We shall suppress the dependence of f and g on $(t,x)\in D$ for convenience. It is always assumed that $r(0)=U$ and $s(0)=V$ on $\partial\Omega$. The pairs of smooth functions $(\underline{u},\underline{v})$ and $(\bar{u},\bar{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$\underline{u}_t - \Delta\underline{u} - f(\underline{u},\bar{v}) \leqslant 0 \leqslant \bar{u}_t - \Delta\bar{u} - f(\bar{u},\underline{v}) \text{ on } D,$$

$$\underline{v}_t - \Delta\underline{v} - g(\bar{u},\underline{v}) \leqslant 0 \leqslant \bar{v}_t - \Delta\bar{v} - g(\underline{u},\bar{v}) \text{ on } D,$$

$$\underline{u} \leqslant r \leqslant \bar{u}, \quad \underline{v} \leqslant s \leqslant \bar{v} \quad \text{on } S$$

and

$$\underline{u}(0) \leqslant U \leqslant \bar{u}(0), \quad \underline{v}(0) \leqslant V \leqslant \bar{v}(0) \quad \text{on } \Omega.$$

Furthermore, they are said to be *ordered* if

$$\underline{u} \leqslant \bar{u}, \quad \underline{v} \leqslant \bar{v} \quad \text{on } \bar{D}.$$

Define the sector

$$\mathscr{A} \equiv \left\{ \begin{bmatrix} u \\ v \end{bmatrix}, u,v\in X,\ \underline{u} \leqslant u \leqslant \bar{u},\ \underline{v} \leqslant v \leqslant \bar{v} \text{ on } \bar{D} \right\}.$$

Suppose $f,g\in C^1(\bar{D}\times\mathscr{A})$. Our system of PDEs is called *quasi-monotone nonincreasing* if

$$\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \leqslant 0 \text{ on } \bar{D}\times\mathscr{A}.$$

Note that the definition of subsolution and supersolution depends on the assumptions on the nonlinearities. Later on, this definition changes for a different set of assumptions.

Suppose our system of PDEs is quasi-monotone nonincreasing. Assume the given functions occurring in the boundary and initial conditions are smooth with $r(0)=U$ and $s(0)=V$ on $\partial\Omega$. Then it can be shown (Section 8.3 in [22]) that it has a unique solution (u,v) in $\mathscr{A}$. The following additive

Schwarz sequence converges to this solution for an appropriately chosen initial guess. Note that the subdomain problems at each iteration are linear, independent and are decoupled.

Theorem 3.1. *Suppose system (3.1) is quasi-monotone nonincreasing and let $(\underline{u},\underline{v})$ and $(\bar{u},\bar{v})$ be ordered subsolution and supersolution pairs. Consider any nonnegative functions $c,d\in C^{\alpha}(\bar{D})$ for some $0<\alpha<1$ so that*

$$\frac{\partial f}{\partial u}\geqslant -c,\quad \frac{\partial g}{\partial v}\geqslant -d \text{ on } \bar{D}\times\mathscr{A}. \tag{3.2}$$

For $i=1,\ldots,m$, let

$$u^{(0)}=u_i^{(0)}=\underline{u}\quad\text{and}\quad v^{(0)}=v_i^{(0)}=\bar{v}\text{ on }\bar{D}\text{ with }\underline{u}=r\quad\text{and}\quad\bar{v}=s\text{ on }S. \tag{3.3}$$

Define the Schwarz sequence for $i=1,\ldots,m$ and $n\geqslant 1$

$$(\partial_t-\Delta+c)u_i^{(n)}=f(u_i^{(n-1)},v_i^{(n-1)})+cu_i^{(n-1)}\text{ on }D_i,$$

$$u_i^{(n)}=u^{(n-1)}\text{ on }S_i,\qquad u_i^{(n)}(0)=U\text{ on }\Omega_i,$$

$$(\partial_t-\Delta+d)v_i^{(n)}=g(u_i^{(n-1)},v_i^{(n-1)})+dv_i^{(n-1)}\text{ on }D_i,$$

$$v_i^{(n)}=v^{(n-1)}\text{ on }S_i,\qquad v_i^{(n)}(0)=V\text{ on }\Omega_i.$$

Here, $u_i^{(n)}$ and $v_i^{(n)}$ are defined as $u^{(n-1)}$ and $v^{(n-1)}$, respectively, on $\bar{D}\setminus\bar{D}_i$ while

$$u^{(n)}(t,x)=\max_{1\leqslant i\leqslant m}u_i^{(n)}(t,x),\quad v^{(n)}(t,x)=\min_{1\leqslant i\leqslant m}v_i^{(n)}(t,x),\quad (t,x)\in\bar{D}.$$

Then $(u_i^{(n)},v_i^{(n)})\to(u,v)$, $i=1,\ldots,m$, where (u,v) is the solution of (3.1) in $\mathscr{A}$.

Proof. The proof can be divided into three steps. We first show that the following monotone properties hold on $\bar{D}$ for $i=1,\ldots,m$,

$$\underline{u}\leqslant u_i^{(n)}\leqslant u_i^{(n+1)}\leqslant\bar{u},\quad u^{(n)}\leqslant u^{(n+1)},\quad u^{(n)}\leqslant u_i^{(n+1)} \tag{3.4}$$

and

$$\underline{v}\leqslant v_i^{(n+1)}\leqslant v_i^{(n)}\leqslant\bar{v},\quad v^{(n+1)}\leqslant v^{(n)},\quad v_i^{(n+1)}\leqslant v^{(n)}. \tag{3.5}$$

Since the sequences are bounded, the following limits on $\bar{D}$ are well defined

$$\lim_{n\to\infty}u_i^{(n)}=\underline{u}_i,\quad \lim_{n\to\infty}v_i^{(n)}=\bar{v}_i,\quad i=1,\ldots,m$$

and

$$\lim_{n\to\infty}u^{(n)}=\underline{u}_0,\quad \lim_{n\to\infty}v^{(n)}=\bar{v}_0.$$

In the second step, we prove, using a similar argument as in [22], that the limit functions satisfy the following PDEs on D_i:

$$\partial_t \underline{u}_i - \Delta \underline{u}_i = f(\underline{u}_i, \bar{v}_i), \quad \partial_t \bar{v}_t - \Delta \bar{v}_i = g(\underline{u}_i, \bar{v}_i), \quad i = 1, \ldots, m. \tag{3.6}$$

Third, we demonstrate that the functions $\underline{u}_i$ are identical. This follows because from (3.4) and the definition of $u^{(n)}$,

$$u_i^{(n)} \leqslant u^{(n)} \leqslant u_j^{(n+1)} \leqslant u^{(n+1)} \leqslant u_i^{(n+2)}, \quad 1 \leqslant i, j \leqslant m.$$

Take the limit to obtain $\underline{u}_i = \underline{u}_j = \underline{u}_0$ on $\bar{D}$. Similarly, we use (3.5) to show $\bar{v}_i = \bar{v}_j = \bar{v}_0$ on $\bar{D}$ for $1 \leqslant i, j \leqslant m$. From (3.6), it follows that $(\underline{u}_0, \bar{v}_0)$ is a solution of (3.1) and thus $(\underline{u}_0, \bar{v}_0) = (u, v)$.

The details of the proof of step one (inequalities (3.4) and (3.5)) by induction are now given. For the case $n = 0$, use the defining equation of $u_i^{(1)}$ and the fact that $\underline{u}$ is a subsolution to obtain

$$(\partial_t - \Delta + c)(u_i^{(1)} - \underline{u}) \geqslant 0 \text{ on } D_i.$$

Since $u_i^{(1)} - \underline{u} = 0$ on S_i and $u_i^{(1)}(0) - \underline{u}(0) = U - \underline{u}(0) \geqslant 0$ on Ω_i, an application of the maximum principle yields $\underline{u} \leqslant u_i^{(1)}$ on $\bar{D}_i$. The same inequality also holds on $\bar{D}$.

From the defining equation of $u_i^{(1)}$ and the fact that $\bar{u}$ is a supersolution, we get on D_i,

$$\begin{aligned}(\partial_t - \Delta + c)(\bar{u} - u_i^{(1)}) &\geqslant f(\bar{u}, \underline{v}) - f(\underline{u}, \bar{v}) + c(\bar{u} - \underline{u}) \\ &= (f_u(*) + c)(\bar{u} - \underline{u}) + f_v(*)(\underline{v} - \bar{v}) \\ &\geqslant 0\end{aligned}$$

since the system is quasi-monotone nonincreasing. Here, $*$ represents an argument between $(\underline{u}, \bar{v})$ and $(\bar{u}, \underline{v})$ (necessarily lying in $\mathscr{A}$) given by the mean value theorem. We shall use this same notation later without further comment. Of course $*$ will denote different quantities at different occurrences. Along S_i, $\bar{u} - u_i^{(1)} = \bar{u} - \underline{u} \geqslant 0$ while on Ω, $\bar{u}(0) - u_i^{(1)}(0) = \bar{u}(0) - U \geqslant 0$. By the maximum principle, $u_i^{(1)} \leqslant \bar{u}$ on $\bar{D}_i$. This inequality can be extended to $\bar{D}$ by the definition of $u_i^{(1)}$. Trivially, we also have $u^{(0)} \leqslant u^{(1)}$. The base case $n = 0$ for (3.5) can be shown similarly.

Now suppose (3.4) and (3.5) hold with n replaced by $n - 1$. From the defining equations for $u_i^{(n)}$ and $u_i^{(n+1)}$, we obtain

$$\begin{aligned}(\partial_t - \Delta + c)(u_i^{(n+1)} - u_i^{(n)}) &= f(u_i^{(n)}, v_i^{(n)}) - f(u_i^{(n-1)}, v_i^{(n-1)}) + c(u_i^{(n)} - u_i^{(n-1)}) \\ &= (f_u(*) + c)(u_i^{(n)} - u_i^{(n-1)}) + f_v(*)(v_i^{(n)} - v_i^{(n-1)}) \\ &\geqslant 0,\end{aligned}$$

where the induction hypothesis and the fact that the system is quasi-monotone nonincreasing have been used. Since $u_i^{(n+1)} - u_i^{(n)} = u^{(n)} - u^{(n-1)} \geqslant 0$ on S_i by the induction hypothesis and $u_i^{(n+1)}(0) - u_i^{(n)}(0) = 0$ on Ω_i, an application of the maximum principle proves that $u_i^{(n)} \leqslant u_i^{(n+1)}$ on $\bar{D}_i$ and thus also on $\bar{D}$.

To show that $u^{(n)} \leqslant u_i^{(n+1)}$ on D_i by induction, we first argue as in (2.8) in Theorem 2.2 that

$$\int_{D_i} \nabla u^{(n)} \cdot \nabla \phi + u_t^{(n)} \phi + c u^{(n)} \phi \leqslant \int_{\Omega_i} (f(u^{(n-1)}, v^{(n-1)}) + c u^{(n-1)}) \phi,$$

where ϕ is a nonnegative function in $H^1(0,T;H_0^1(\Omega_i))$. Multiply the defining equation for $u_i^{(n+1)}$ by this function ϕ and integrate by parts to obtain

$$\int_{D_i} \nabla u_i^{(n+1)} \cdot \nabla\phi + (\partial_t u_i^{(n+1)})\phi + cu_i^{(n+1)}\phi = \int_{D_i} (f(u_i^{(n)}, v_i^{(n)}) + cu_i^{(n)})\phi.$$

Subtract these two results to get

$$\begin{aligned}
&\int_{D_i} (\nabla u_i^{(n+1)} - u^{(n)}) \cdot \nabla\phi + (u_i^{(n+1)} - u^{(n)})_t\phi + c(u_i^{(n+1)} - u^{(n)})\phi \\
&\quad \geqslant \int_{D_i} (f(u_i^{(n)}, v_i^{(n)}) - f(u^{(n-1)}, v^{(n-1)}) + c(u_i^{(n)} - u^{(n-1)}))\phi \\
&\quad = \int_{D_i} ((f_u(*) + c)(u_i^{(n)} - u^{(n-1)}) + f_v(*)(v_i^{(n)} - v^{(n-1)}))\phi \\
&\quad \geqslant 0
\end{aligned}$$

by the induction hypothesis and the fact that the system is quasi-monotone nonincreasing. Since $u_i^{(n+1)} - u^{(n)} = 0$ on S_i and $u_i^{(n+1)}(0) - u^{(n)}(0) = U - u^{(n)}(0) \geqslant 0$ on Ω_i, we can conclude that $u^{(n)} \leqslant u_i^{(n+1)}$ on $\bar{D}_i$ by the generalized maximum principle. The same inequality also holds on $\bar{D}$. Other inequalities in (3.4) and (3.5) can similarly be shown. We shall omit their proof. This completes the proof of the theorem. □

One example where a quasi-monotone nonincreasing system occurs is the Lotka–Volterra competition model

$$-\Delta u = u(a_1 - b_1 u - c_1 v), \qquad -\Delta v = v(a_2 - b_2 u - c_2 v).$$

Here u, v stand for the population of two species competing for the same food sources and/or territories and all other variables are positive constants.

4. Other coupled systems

In this section, we also consider solutions of system (3.1) with two other sets of assumptions on the nonlinearities. For the first of these, the pairs of smooth functions $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$\underline{u}_t - \Delta\underline{u} - f(\underline{u}, \underline{v}) \leqslant 0 \leqslant \bar{u}_t - \Delta\bar{u} - f(\bar{u}, \bar{v}) \text{ on } D,$$

$$\underline{v}_t - \Delta\underline{v} - g(\underline{u}, \underline{v}) \leqslant 0 \leqslant \bar{v}_t - \Delta\bar{v} - g(\bar{u}, \bar{v}) \text{ on } D,$$

$$\underline{u} \leqslant r \leqslant \bar{u}, \quad \underline{v} \leqslant s \leqslant \bar{v} \text{ on } S$$

and

$$\underline{u}(0) \leqslant U \leqslant \bar{u}(0) \qquad \underline{v}(0) \leqslant V \leqslant \bar{v}(0) \text{ on } \Omega.$$

Assuming that the subsolution–supersolution pairs are ordered, our system of PDEs is called *quasi-monotone nondecreasing* if

$$\frac{\partial f}{\partial v}, \frac{\partial g}{\partial u} \geqslant 0 \text{ on } \bar{D} \times \mathscr{A},$$

where $\mathscr{A}$ is defined as above.

Suppose our system of PDEs is quasi-monotone nondecreasing. Then it can be shown in [22, Section 8.e] that it has a unique solution (u, v) in $\mathscr{A}$.

Theorem 4.1. *Suppose system* (3.1) *is quasi-monotone nondecreasing and let* $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ *be ordered subsolution and supersolution pairs. Consider any nonnegative functions* c, $d \in C^{\alpha}(\bar{D})$ *for some* $0 < \alpha < 1$ *so that*

$$\frac{\partial f}{\partial u} \geqslant -c, \quad \frac{\partial g}{\partial v} \geqslant -d \text{ on } \bar{D} \times \mathscr{A}.$$

For $i = 1, \ldots, m$, *let*

$$u^{(0)} = u_i^{(0)} = \underline{u} \text{ and } v^{(0)} = v_i^{(0)} = \underline{v} \text{ on } \bar{D} \text{ with } \underline{u} = r \text{ and } \underline{v} = s \text{ on } S. \tag{4.1}$$

Define the Schwarz sequence for $i = 1, \ldots, m$ *and* $n \geqslant 1$

$$(\partial_t - \Delta + c)u_i^{(n)} = f(u_i^{(n-1)}, v_i^{(n-1)}) + cu_i^{(n-1)} \text{ on } D_i$$

$$u_i^{(n)} = u^{(n-1)} \text{ on } S_i, \qquad u_i^{(n)}(0) = U \text{ on } \Omega,$$

$$-\Delta v_i^{(n)} + dv_i^{(n)} = g(u_i^{(n-1)}, v_i^{(n-1)}) + dv_i^{(n-1)} \text{ on } D_i$$

$$v_i^{(n)} = v^{(n-1)} \text{ on } S_i, \qquad v_i^{(n)}(0) = V \text{ on } \Omega.$$

Here, $u_i^{(n)}$ *and* $v_i^{(n)}$ *are defined as* $u^{(n-1)}$ *and* $v^{(n-1)}$, *respectively, on* $\bar{D} \setminus \bar{D}_i$ *while*

$$u^{(n)}(t, x) = \max_{1 \leqslant i \leqslant m} u_i^{(n)}(t, x), \quad v^{(n)}(t, x) = \max_{1 \leqslant i \leqslant m} v_i^{(n)}(t, x), \quad (t, x) \in \bar{D}.$$

Then $(u_i^{(n)}, v_i^{(n)}) \to (u, v)$, $i = 1, \ldots, m$, *where* (u, v) *is the solution of* (3.1) *in* $\mathscr{A}$.

Proof. The proof is similar to the previous one and thus we only give a sketch. The first step is to show that the following monotone properties hold on $\bar{D}$ for $i = 1, \ldots, m$,

$$\underline{u} \leqslant u_i^{(n)} \leqslant u_i^{(n+1)} \leqslant \bar{u}, \qquad u^{(n)} \leqslant u^{(n+1)}, \qquad u^{(n)} \leqslant u_i^{(n+1)}$$

and

$$\underline{v} \leqslant v_i^{(n)} \leqslant v_i^{(n+1)} \leqslant \bar{v}, \qquad v^{(n)} \leqslant v^{(n+1)}, \qquad v^{(n)} \leqslant v_i^{(n+1)}.$$

Since the sequences are bounded above, the following limits on $\bar{D}$ are well defined

$$\lim_{n \to \infty} u_i^{(n)} = \underline{u}_i, \quad \lim_{n \to \infty} v_i^{(n)} = \underline{v}_i, \quad i = 1, \ldots, m$$

and

$$\lim_{n \to \infty} u^{(n)} = \underline{u}_0, \qquad \lim_{n \to \infty} v^{(n)} = \underline{v}_0.$$

In the second step, we prove that the limit functions satisfy the following PDEs on D_i:

$$\partial_t \underline{u}_i - \Delta \underline{u}_i = f(\underline{u}_i, \underline{v}_i), \quad \partial_t \underline{v}_i - \Delta \underline{v}_i = g(\underline{u}_i, \underline{v}_i), \quad i = 1, \ldots, m.$$

In the third step, we use the monotone property of the sequences to show that $\underline{u}_i = \underline{u}_j = \underline{u}_0$ and $\underline{v}_i = \underline{v}_j = \underline{v}_0$ on $\bar{D}$, $i, j = 1, \ldots, m$. This demonstrates that $(\underline{u}_0, \underline{v}_0)$ is a solution of (3.1) in $\mathscr{A}$ and thus $(\underline{u}_0, \underline{v}_0) = (u, v)$. This completes the sketch of the proof. $\square$

One example where a quasi-monotone nondecreasing system occurs is the Lotka–Volterra cooperating model

$$-\Delta u = u(a_1 - b_1 u + c_1 v), \qquad -\Delta v = v(a_2 + b_2 u - c_2 v).$$

Here u, v stand for the population of two species which have a symbiotic relationship and all other variables are positive constants.

Finally, we consider a third class of coupled systems. The pairs of smooth functions $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$\underline{u}_t - \Delta \underline{u} - f(\underline{u}, \bar{v}) \leqslant 0 \leqslant \bar{u}_t - \Delta \bar{u} - f(\bar{u}, \underline{v}) \text{ on } D,$$

$$\underline{v}_t - \Delta \underline{v} - g(\underline{u}, \underline{v}) \leqslant 0 \leqslant \bar{v}_t - \Delta \bar{v} - g(\bar{u}, \bar{v}) \text{ on } D,$$

$$\underline{u} \leqslant r \leqslant \bar{u}, \quad \underline{v} \leqslant s \leqslant \bar{v} \quad \text{on } S$$

and

$$\underline{u}(0) \leqslant U \leqslant \bar{u}(0), \quad \underline{v}(0) \leqslant V \leqslant \bar{v}(0) \quad \text{on } \Omega.$$

In case the subsolution–supersolution pairs are ordered, our system of PDEs is called *mixed quasi-monotone* if

$$\frac{\partial f}{\partial v} \leqslant 0 \quad \text{and} \quad \frac{\partial g}{\partial u} \geqslant 0 \text{ on } \bar{D} \times \mathscr{A}.$$

Suppose our system of PDEs is mixed quasi-monotone. Then it can be shown in [22, Section 8.3] that it has a unique solution (u, v) in $\mathscr{A}$.

We need the following lemma.

Lemma 4.2. *Let ϕ be any nonnegative function in $H^1(0, T; H_0^1(\Omega))$. Then there exist nonnegative $\chi_j \in H^1(0, T; H_0^1(\Omega_j))$ such that*

$$\phi = \chi_1 + \cdots + \chi_m \text{ on } \bar{D}.$$

Proof. Take the case $m = 2$. Since the subdomains are overlapping, $H_0^1(\Omega) = H_0^1(\Omega_1) + H_0^1(\Omega_2)$ and thus $\phi = \phi_1 + \phi_2$ for some $\phi_j \in H^1(0, T; H_0^1(\Omega_j))$. An alternate decomposition is

$$\phi = (\phi_1^+ - \phi_2^-) + (\phi_2^+ - \phi_1^-) \equiv \chi_1 + \chi_2.$$

For a fixed t, the support of ϕ_1^- and of ϕ_2^- must be a subset of $(0, T) \times \overline{\Omega_1 \cap \Omega_2}$ since ϕ is nonnegative. Thus $\chi_i \in H^1(0, T; H_0^1(\Omega_i))$, $i = 1, 2$. We now show that they are nonnegative.

If $\phi_2(t, x) < 0$ for some $(t, x) \in (0, T) \times \Omega_1 \cap \Omega_2$, then $\phi_1(t, x) > 0$. Thus $0 \leqslant \phi(t, x) = \chi_1(t, x)$ while $\chi_2(t, x) = 0$. If $\phi_2(t, x) \geqslant 0$, then $\phi(t, x) = \phi_1^+(t, x) + (\phi_2^+(t, x) - \phi_1^-(t, x)) \geqslant 0$. We consider two cases. If $\phi_1(t,x) < 0$, then $\chi_1(t,x)=0$ and $0 \leqslant \phi(t,x)=\chi_2(t,x)$. In the second case $\phi_1(t,x) \geqslant 0$, then $\chi_1(t,x) = \phi_1^+(t,x) \geqslant 0$ and $\chi_2(t,x) = \phi_2^+(t,x) \geqslant 0$. Hence the lemma holds for $m=2$. The general case holds by induction. This completes the proof of the lemma. □

Theorem 4.3. *Suppose system* (3.1) *is mixed quasi-monotone and let* $(\underline{u},\underline{v})$ *and* $(\bar{u},\bar{v})$ *be ordered subsolution and supersolution pairs. Consider any nonnegative functions* $c, d \in C^{\alpha}(\bar{D})$ *for some* $0<\alpha<1$ *so that*

$$\frac{\partial f}{\partial u} \geqslant -c, \quad \frac{\partial g}{\partial v} \geqslant -d \text{ on } \bar{D}\times\mathscr{A}.$$

For $i=1,\dots,m$, *let* $\underline{u}^{(0)} = \underline{u}_i^{(0)} = \underline{u}$, $\bar{u}^{(0)} = \bar{u}_i^{(0)} = \bar{u}$, $\underline{v}^{(0)} = \underline{v}_i^{(0)} = \underline{v}$, *and* $\bar{v}^{(0)} = \bar{v}_i^{(0)} = \bar{v}$ *on* $\bar{D}$ *with* $\underline{u}=\bar{u}=r$ *and* $\underline{v}=\bar{v}=s$ *on* S. *Define the Schwarz sequences, for* $i=1,\dots,m$ *and* $n \geqslant 1$

$$(\partial_t - \Delta + c)\underline{u}_i^{(n)} = f(\underline{u}_i^{(n-1)}, \bar{v}_i^{(n-1)}) + c\underline{u}_i^{(n-1)} \text{ on } D_i, \quad \underline{u}_i^{(n)} = \underline{u}^{(n-1)} \text{ on } S_i,$$

$$(\partial_t - \Delta + c)\bar{u}_i^{(n)} = f(\bar{u}_i^{(n-1)}, \underline{v}_i^{(n-1)}) + c\bar{u}_i^{(n-1)} \text{ on } D_i, \quad \bar{u}_i^{(n)} = \bar{u}^{(n-1)} \text{ on } S_i,$$

$$(\partial_t - \Delta + d)\underline{v}_i^{(n)} = g(\underline{u}_i^{(n-1)}, \underline{v}_i^{(n-1)}) + d\underline{v}_i^{(n-1)} \text{ on } D_i, \quad \underline{v}_i^{(n)} = \underline{v}^{(n-1)} \text{ on } S_i,$$

$$(\partial_t - \Delta + d)\bar{v}_i^{(n)} = g(\bar{u}_i^{(n-1)}, \bar{v}_i^{(n-1)}) + d\bar{v}_i^{(n-1)} \text{ on } D_i, \quad \bar{v}_i^{(n)} = \bar{v}^{(n-1)} \text{ on } S_i.$$

The initial conditions are $\underline{u}_i^{(n)}(0)=\bar{u}_i^{(n)}(0)=U$ *and* $\underline{v}_i^{(n)}(0)=\bar{v}_i^{(n)}(0)=V$ *on* Ω_i. *Here,* $\underline{u}_i^{(n)}$ *and* $\bar{u}_i^{(n)}$ *are defined as* $\underline{u}^{(n-1)}$ *and* $\bar{u}^{(n-1)}$, *respectively, on* $\bar{D}\setminus\bar{D}_i$ *and*

$$\underline{u}^{(n)}(t,x) = \max_{1\leqslant i\leqslant m} \underline{u}_i^{(n)}(t,x), \quad \bar{u}^{(n)}(t,x) = \min_{1\leqslant i\leqslant m} \bar{u}_i^{(n)}(t,x), \quad (t,x)\in\bar{D}.$$

The other functions $\underline{v}_i^{(n)}$, $\bar{v}_i^{(n)}$, $\underline{v}^{(n)}$, $\bar{v}^{(n)}$ *are similarly defined. Then* $\underline{u}_i^{(n)} \to u$, $\bar{u}_i^{(n)} \to u$, $\underline{v}_i^{(n)} \to v$, *and* $\bar{v}_i^{(n)} \to v$, $i=1,\dots,m$, *where* (u,v) *is the solution of* (3.1) *in* $\mathscr{A}$.

Proof. The current case is slightly more complicated than the previous two cases because the four pairs of Schwarz iterates are somehow related. However, the ideas and techniques of the proof are essentially the same. Hence we only give a sketch of the proof.

We first show that the sequences obey the following monotone properties on $\bar{D}$,

$$\underline{u} \leqslant \underline{u}_i^{(n)} \leqslant \underline{u}_i^{(n+1)} \leqslant \bar{u}_i^{(n+1)} \leqslant \bar{u}_i^{(n)} \leqslant \bar{u}, \quad \underline{v} \leqslant \underline{v}_i^{(n)} \leqslant \underline{v}_i^{(n+1)} \leqslant \bar{v}_i^{(n+1)} \leqslant \bar{v}_i^{(n)} \leqslant \bar{v}$$

and

$$\underline{u}^{(n)} \leqslant \underline{u}_i^{(n+1)}, \quad \bar{u}^{(n)} \geqslant \bar{u}_i^{(n+1)}, \quad \underline{v}^{(n)} \leqslant \underline{v}_i^{(n+1)}, \quad \bar{v}^{(n)} \geqslant \bar{v}_i^{(n+1)}$$

for $i=1,\ldots,m$ and

$$\underline{u}^{(n)} \leqslant \underline{u}^{(n+1)} \leqslant \bar{u}^{(n+1)} \leqslant \bar{u}^{(n)}, \qquad \underline{v}^{(n)} \leqslant \underline{v}^{(n+1)} \leqslant \bar{v}^{(n+1)} \leqslant \bar{v}^{(n)}.$$

Since the sequences are bounded, the following limits are well defined on $\bar{D}$ for $i=1,\ldots,m$

$$\lim_{n\to\infty} \underline{u}_i^{(n)} = \underline{u}_i, \quad \lim_{n\to\infty} \bar{u}_i^{(n)} = \bar{u}_i, \quad \lim_{n\to\infty} \underline{v}_i^{(n)} = \underline{v}_i, \quad \lim_{n\to\infty} \bar{v}_i^{(n)} = \bar{v}_i$$

and

$$\lim_{n\to\infty} \underline{u}^{(n)} = \underline{u}_0, \quad \lim_{n\to\infty} \bar{u}^{(n)} = \bar{u}_0, \quad \lim_{n\to\infty} \underline{v}^{(n)} = \underline{v}_0, \quad \lim_{n\to\infty} \bar{v}^{(n)} = \bar{v}_0.$$

In the second step, we prove that the limit functions satisfy the following PDEs on D_i, $i=1,\ldots,m$:

$$\partial_t \underline{u}_i - \Delta \underline{u}_i = f(\underline{u}_i, \bar{v}_i), \quad \partial_t \bar{u}_i - \Delta \bar{u}_i = f(\bar{u}_i, \underline{v}_i)$$

and

$$\partial_t \underline{v}_i - \Delta \underline{v}_i = g(\underline{u}_i, \underline{v}_i), \quad \partial_t \bar{v}_i - \Delta \bar{v}_i = g(\bar{u}_i, \bar{v}_i).$$

Next, we use the monotone properties to prove that $\underline{u}_i = \underline{u}_j = \underline{u}_0$, $\bar{u}_i = \bar{u}_j = \bar{u}_0$, $\underline{v}_i = \underline{v}_j = \underline{v}_0$ and $\bar{v}_i = \bar{v}_j = \bar{v}_0$ on $\bar{D}$, $i,j=1,\ldots,m$. Then we can apply the same argument as in [22, p. 401] to demonstrate that $(\underline{u}_0, \underline{v}_0) = (\bar{u}_0, \bar{v}_0)$. This guarantees that $(\underline{u}_0, \underline{v}_0) = (u, v)$ is the required solution of (3.1) in $\mathscr{A}$.

Step one can be shown by induction as before. We show $\underline{u}^{(1)} \leqslant \bar{u}^{(1)}$ as an illustration. Arguing as in (2.8) in Theorem 2.2, for any i so that $1 \leqslant i \leqslant m$,

$$\int_{D_i} \nabla \underline{u}^{(1)} \cdot \nabla \phi_i + \underline{u}_t^{(1)} \phi_i + c\underline{u}^{(1)} \phi_i \leqslant \int_{D_i} (f(\underline{u}, \bar{v}) + c\underline{u})\, \phi_i \tag{4.2}$$

and

$$\int_{D_i} \nabla \bar{u}^{(1)} \cdot \nabla \phi_i + \bar{u}_t^{(1)} \phi_i + c\bar{u}^{(1)} \phi_i \geqslant \int_{D_i} (f(\bar{u}, \underline{v}) + c\bar{u})\, \phi_i,$$

where ϕ_i is a nonnegative function in $H^1(0,T;H_0^1(\Omega_i))$. Subtract these equations to obtain

$$\begin{aligned}
&\int_{D_i} \nabla(\bar{u}^{(1)} - \underline{u}^{(1)}) \cdot \nabla \phi_i + (\bar{u}^{(1)} - \underline{u}^{(1)})_t \phi_i + c(\bar{u}^{(1)} - \underline{u}^{(1)})\phi_i \\
&\quad \geqslant \int_{D_i} (f(\bar{u}, \underline{v}) - f(\underline{u}, \bar{v}) + c(\bar{u} - \underline{u}))\, \phi_i \\
&\quad = \int_{D_i} ((f_u(*) + c)(\bar{u} - \underline{u}) + f_v(*)(\underline{v} - \bar{v}))\, \phi_i \\
&\quad \geqslant 0.
\end{aligned}$$

Now let ϕ be any nonnegative function in $H^1(0,T;H_0^1(\Omega))$. By the above lemma, there are nonnegative functions $\phi_j \in H^1(0,T;H_0^1(\Omega_j))$ such that $\phi = \phi_1 + \cdots + \phi_m$. Now

$$\begin{aligned}&\int_D \nabla(\bar{u}^{(1)} - \underline{u}^{(1)}) \cdot \nabla\phi + (\bar{u}^{(1)} - \underline{u}^{(1)})_t\phi + c(\bar{u}^{(1)} - \underline{u}^{(1)})\phi \\ &\quad = \sum_{i=1}^{m} \int_{D_i} \nabla(\bar{u}^{(1)} - \underline{u}^{(1)}) \cdot \nabla\phi_i + (\bar{u}^{(1)} - \underline{u}^{(1)})_t\phi_i + c(\bar{u}^{(1)} - \underline{u}^{(1)})\phi_i \\ &\quad \geqslant 0.\end{aligned}$$

Since $\bar{u}^{(1)} - \underline{u}^{(1)} = 0$ on S and $\bar{u}^{(1)}(0) - \underline{u}^{(1)}(0) \geqslant 0$ on Ω, we conclude that $\bar{u}^{(1)} \geqslant \underline{u}^{(1)}$ on $\bar{D}$ by the generalized maximum principle. This completes the sketch of the proof. $\square$

One example where a mixed quasi-monotone system occurs is the Lotka–Volterra predator–prey model

$$-\Delta u = u(a_1 - b_1 u - c_1 v), \qquad -\Delta v = v(a_2 + b_2 u - c_2 v).$$

Here u stands for the population of a prey while v denotes the population of a predator and all other variables are positive constants.

5. Numerical results

We present two simple numerical experiments using the MATLAB PDE Toolbox. This toolbox provides a convenient environment to solve some classes of linear and nonlinear PDEs using finite elements. In the first example, the PDE is

$$u_t - \Delta u = u(1-u) + g$$

with u vanishing on the boundary of the domain which is the unit square. Here g is a function depending upon time and space so that the exact solution is $u(x,y,t) = t\sin\pi x \sin\pi y$. The domain is subdivided into two overlapping rectangles $(0,0.6)\times(0,1)$ and $(0.4,1)\times(0,1)$. The subsolution is the zero function and $c=1$. We used the additive Schwarz method, employing 1000 triangles in each subdomain linear solve. The plot of the relative error versus Schwarz iteration at time 1 is given in Fig. 1a. Here, relative error is defined as

$$\frac{|u^{(n)} - u|_\infty}{|u|_\infty}$$

computed at the final time. The Schwarz iteration is stopped when the relative error is smaller than 10^{-2}.

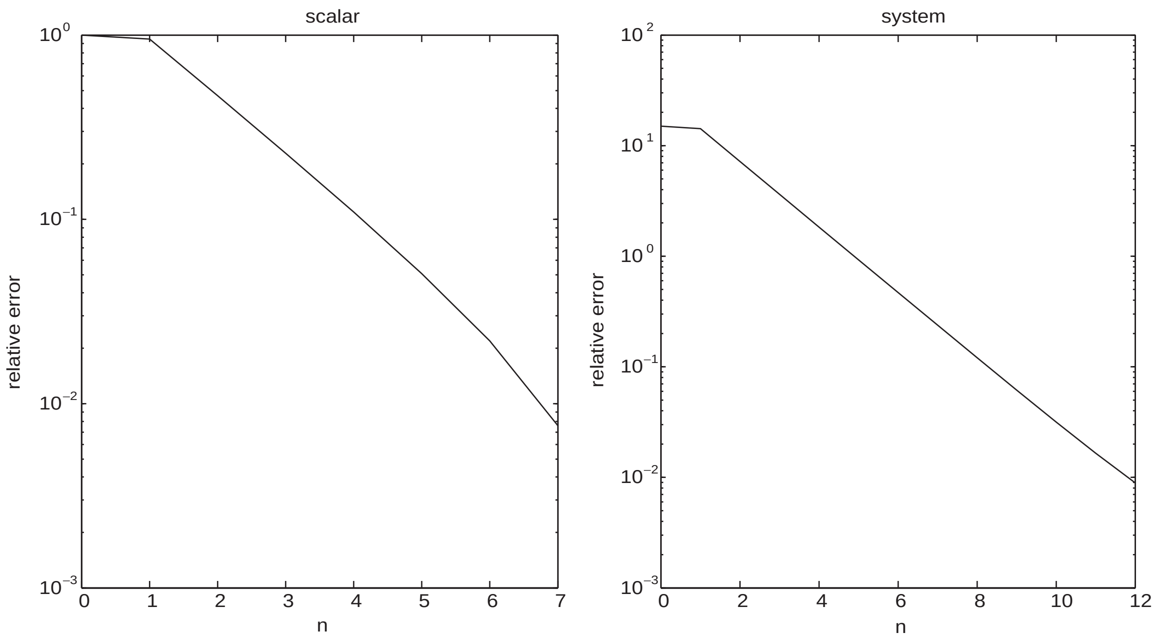

Fig. 1. Convergence of the additive Schwarz scheme for a scalar and coupled system.

In the second experiment, we solve the quasi-monotone nonincreasing system

$$u_t - \Delta u = u(1 - u - v) + g,$$

$$v_t - \Delta v = v(1 - u - v) + h,$$

on the same domain as in the previous example with zero boundary conditions. The functions g and h depend only upon time and space and are chosen so that the exact solution is $u(x, y, t) = t \sin \pi x \sin \pi y$ and $v(x, y, t) = tx(1 - x)y(1 - y)$. Using $c = d = 2$, $\underline{u} = 0$ and $\bar{v} = \sin \pi x \sin \pi y$, we obtain the plot Fig. 1b.

6. Conclusion

In this paper, we have shown convergence of some Schwarz methods for nonlinear PDEs whose solutions can be demonstrated by the monotone method. Our results include a parallel additive Schwarz method for a domain which is decomposed into finitely many subdomains. Both scalar and coupled systems can be handled. For the latter, subdomain problems are linear, independent and decoupled in each iteration.

It remains to be seen whether this waveform relaxation approach is competitive with the more usual approach (implicit time marching). This paper provides a theoretical justification of the method and much more work is needed. An important task is to analyze a discrete approximation of the problem, especially the dependence of the rate of convergence on the discretization parameter and the number of subdomains. Also, it would be desirable to extend the analysis to other classes of nonlinear parabolic equations.

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