REALIZATION OF A VARIATION OF THE RELAXATION METHOD

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A method of equation solving based on the minimization of functionals measuring the accuracy of the solution of one of the equations is investigated. Minimization is carried out along a curve which is a solution of a Cauchy problem formulated so that its solution, defined on a finite interval $t_0 \leq t < 0$ (in contrast to solutions of Cauchy problems which are continuous analogs of iterative methods [2, 4]) such that when $t \to 0$, a root of the original equations is obtained and the functional is minimized (at the minimum point the functional is not assumed to be differentiable).

The application of approximate iterative methods to Cauchy problems with the above properties yields iterative methods for the solution of the equations.

The method we propose is somewhat complicated by the fact that it involves the solution of a system of ordinary differential equations. However it has definite advantages over the method of differentiation with respect to a parameter [7, 8, 12] since it yields a system of first-order differential equations solved for the derivatives and since the iterations do not require matrix inversions.

§ 1. Consider the (possibly nonlinear) equation system

$$f(x) = 0, \tag{1}$$

and let the vector a (x) and the functional φ (x), depend on the n-dimensional vector $x = (x_1, x_2, \ldots, x_n)$ where $f = (f_1, f_2, \ldots, f_n)$ and $a = (a_1, a_2, \ldots, a_n)$ are vector functions.

Assume that Eq. (1) satisfies the following conditions in a bounded open set ω containing one root $x = \alpha$ of the equation:

a) The functional $\varphi(x)$ is single-valued, continuous, and $\varphi(x) \geq 0$ with equality occurring only for $x = \alpha$;

b) a, $\varphi_x = \operatorname{grad} \varphi$, $\partial a_i / \partial x_j$, $\partial^2 \varphi / \partial x_i \partial x_j$, i, j = 1, 2, ..., n, are continuous except possibly at the point $x = \alpha$*;

c) the scalar product (φ_x, a) is different from zero except possibly for $x = \alpha$.

Under these conditions we have the following:

THEOREM. There is a closed sphere $\rho \subset \omega$ with center $x = \alpha$ such that the solution x (t) of the differential equation

$$x'(t) = -\frac{a}{(\varphi_x, a)} \tag{2}$$

with initial values $t_0 = -\varphi(x_0)$, $x_0 \in \rho$, $x_0 \neq \alpha$ tends to α when $t \to 0$

$$\lim_{t \to 0} x(t) = \alpha.$$

Proof. Let $\tilde{\omega}$ be the set of all points of ω except the point $x = \alpha$. The conditions of the theorem imply that to every point $x_0 \in \tilde{\omega}$ and $t_0 = -\varphi(x_0)$ corresponds to a unique noncontinuable solution x (t) of Eq. (2) [11].

The relation $\varphi(x) + t = c = \text{const}$ yields a first integral of the system (2) in $\tilde{\omega}$ and the solutions under consideration correspond to c = 0.

*The vector functions a and φ_x and the scalar product (φ_x, a) can be continuous for $x = \alpha$.

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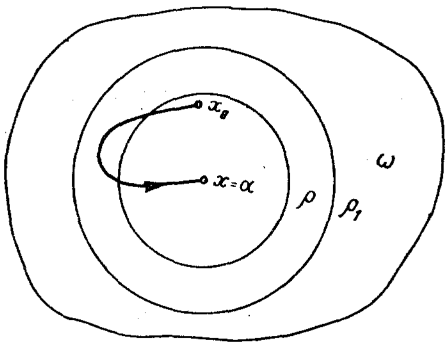

Fig. 1

Consider a closed sphere $\rho_1 \subset \omega$ with center at $x = \alpha$ and let $l = \inf \varphi(x)$ on ρ_1.

Let $x(t)$ be a noncontinuable solution of (2) whose initial value $x_0 \neq \alpha$ is in the closed sphere ρ with center at $x = \alpha$ and radius smaller than the radius of ρ_1 and small enough so that $\varphi(x) < l$ for all $x \in \rho$.

Such a sphere exists since $\varphi(x)$ is continuous and $\varphi(\alpha) = 0$; $x(t)$ is defined for $m_1 < t < m_2$.

Let $m_2 > 0$. Since $-t = \varphi(x(t))$ there is a T such that $\varphi(x(T)) < 0$.

This result contradicts condition a) and so $m_2 \leq 0$. We will show that $m_2 = 0$. Assume that $m_2 < 0$. Since $-t = \varphi(x(t))$ is monotonically decreasing for $t > t_0$ we have

$$0 < \varphi(x(t)) \leqslant \varphi(x_0) < l \tag{3}$$

for $t > t_0$.

It follows from (3) that $x(t) \in \rho_1$ and, since $\lim\limits_{t \to m_2} \varphi(x(t)) = -m_2$, $\lim\limits_{t \to m_2} x(t) = \xi$ exists and $\xi \in \rho_1$ (because ρ_1 is closed).

But since $m_2 \neq 0$ we have $\xi \neq \alpha$ and $\xi \in \tilde{\omega}$. Hence the solution $x(t)$ can be continued if $m_2 < 0$. This result contradicts the condition that $x(t)$ cannot be continued to the right. Hence $m_2 = 0$ and there are solutions $x(t)$ for $x_0 \in \rho$, $x_0 \neq \alpha$, on the interval $-\varphi(x_0) \leq t < 0$. When $t \to 0$ along these solutions $\varphi(x) \to 0$. From condition a) this can occur if and only if

$$\lim_{t \to 0} x(t) = \alpha$$

and the desired result is proved.

Equation (2) is nonlinear and so in most cases it cannot be solved in closed form.

The approximate integration of Eq. (2) yields an approximate solution $\tilde{x}(t)$ which, for $t = 0$, gives an approximate value x_1 of the root $x = \alpha$.

If $x_1 \in \rho$ we can solve Eq. (2) approximately with initial values $t_1 = -\varphi(x_1)$ and x_1 obtained a second approximation x_2 for the root $x = \alpha$. Repetition of this process will eventually yield a good approximation for the root.

If the same approximate method is used each time to solve Eq. (2), we have a stationary iterative process.

For example if Eq. (2) is always solved by the Runge-Kutta method [3] with interval $h_n = \varphi(x_n)$, which has the order of approximation h_n^4 (Eq. (2) is solved with initial values $t_n = -\varphi(x_n)$ and x_n and it is assumed that a and φ possess derivatives of a sufficiently high order in $\tilde{\omega}$), we have the iteration

$$x_{n+1} = x_n + \frac{1}{6}(k_n^1 + 4k_n^2 + k_n^3), \qquad n = 0, 1, 2, \ldots,$$

where

$$k_n^1 = -\frac{a(x_n)\varphi(x_n)}{(\varphi_x(x_n), a(x_n))}; \qquad k_n^2 = -\frac{a(x_n + 0{,}5k_n^1)\varphi(x_n)}{(\varphi_x(x_n + 0{,}5k_n^1), a(x_n + 0{,}5k_n^1))};$$

$$k_n^3 = -\frac{a(x_n - k_n^1 + 2k_n^2)\varphi(x_n)}{(\varphi_x(x_n - k_n^1 + 2k_n^2), a(x_n - k_n^1 + 2k_n^2))}.$$

If we set $a = \varphi_x$ we have

$$x'(t) = -\frac{\varphi_x}{(\varphi_x, \varphi_x)}, \tag{4}$$

and this equation is solved by the method of steepest descent [2, 4, 13].

In this case the iteration method is related to gradient methods for the solution of Eq. (1) in the vector field formed by the right-hand side of Eq. (4) and is a quasi-potential method.

§ 2. We now use the theorem we have proved to construct some concrete examples of equations of the form (2).

Let $a = f$, $a = J^* f$, and $a = J^{-1} f$ and let $\varphi = \sqrt{(f, f)}$ (we assume that f can be differentiated a sufficient number of times in $\tilde{\omega}$). Our differential equations are

$$x'(t) = -\frac{f\sqrt{(f, f)}}{(Jf, f)}; \tag{5}$$

$$x'(t) = -\frac{J^* f\sqrt{(f, f)}}{(J^* f, J^* f)}; \tag{6}$$

$$x'(t) = -\frac{J^{-1} f}{\sqrt{(f, f)}} \tag{7}$$

with the initial values $t_0 = -\sqrt{(f_0, f_0)} = -\sqrt{(f(x_0)}$ and x_0. Here J is the Jacobian, J^* is the transpose of the Jacobian, and J^{-1} is the inverse of the Jacobian.

For the conditions of the theorem to be satisfied, Eq. (5) must be used for systems for which J is positive definite in $\tilde{\omega}$ and Eqs. (6) and (7) must be used for systems for which J is not degenerate in $\tilde{\omega}$.

In the one-dimensional case Eqs. (5)-(7) coincide, except for the sign $(-f)$, with the equation considered in [5, 12]. Equation (7) is investigated in [1, 9].

We illustrate our method with some numerical examples. All computations were performed on an M-20 computer and systems of the form (5) and (6) were solved for the roots.

In examples 1-3, Eq. (6) with initial values $t_n = -\sqrt{(f_n, f_n)}$ and x_n was solved by the Runge-Kutta method with error of order h_{nk}^4.

In example 4, Eq. (5) with initial values $t_n = -\sqrt{(f_n, f_n)}$ and x_n was solved by Euler's method and yielded L. V. Kantorovich's method of steepest descent for systems of linear equations [6, 13] and, by the Runge-Kutta method with errors of order h_{nk}^3.

In this example the use of the Runge-Kutta method required only one-sixth the volume of computation involved in L. V. Kantorovich's method.

The interval $h_{nk} = \sqrt{(f_n, f_n)}/k$ was used in all examples, where k is the number of divisions of the interval $(t_n, 0)$ and n is the number of the approximation.

Example 1.

$$\begin{aligned} f_1(x) &= 4x_1^2 + x_2^2 + 2x_1x_2 - x_2 - 2 = 0; \\ f_2(x) &= 2x_1^2 + x_2^2 + 3x_1x_2 - 3 = 0. \end{aligned} \tag{8}$$

A root of this system is $\alpha = (0.5,\ 1)$.

Table 1 gives results of calculations (with $k = 100$) starting with several initial approximations.

If $\varphi(x)$ oscillates with deep minima close to $x = \alpha$, the approximate solution of Eq. (6) can lead to motion not along the "bottom of a ravine" but to motion almost perpendicular to the direction of a "ravine." This leads to slow convergence of the iteration process.

The phenomenon described above is due to the fact that the matrix J^*J is ill-conditioned close to $x = \alpha$ (i.e., the Todd measure of condition is large [10, 13]) and so the family of closed surfaces

$$\varphi(x) = \text{const},$$

containing the root $x = \alpha$ is strongly elongated.

In some cases the convergence of the iteration process close to $x = \alpha$ can be improved by applying a nondegenerate linear transformation

$$x = Ay,$$

such that close to $x = \alpha$ the Jacobian $\tilde{J} = JA$ of the system

$$f(Ay) = 0 \tag{9}$$

is close to being an orthogonal matrix. The root $y = \beta$ of Eq. (9) is calculated with sufficient accuracy and then the root $x = \alpha$ of the original system is given by the formula

$$\alpha = A\beta.$$

In our example the Todd number of the conditioning of the matrix J^*J for $x = \alpha$ is $p \approx 46.7$.

This indicates that $\varphi(x)$ oscillates strongly close to $x = \alpha$.

For comparison, Table 2 shows results of calculations (with $k = 100$) using the linear transformations $x = A_1y$ and $x = A_2y$ possessing the properties described above.

The matrix

$$A_1 = \begin{vmatrix} 0.1904849, & -0.6169644 \\ 0 & 0.6465224 \end{vmatrix}$$

was obtained from the condition of orthogonality of $\tilde{J}$ at the point $x = (0.4, 0.9)$.

The matrix

$$A_2 = \begin{vmatrix} \frac{1}{\sqrt{61}}, & -\frac{59}{22\sqrt{61}} \\ 0, & \frac{\sqrt{61}}{11} \end{vmatrix}$$

was obtained from the condition of orthogonality of $\tilde{J}$ at the point $x = \alpha$.

Example 2.

$$\begin{aligned} f_1(x) &= x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 - 5x_1x_3 - 17 = 0; \\ f_2(x) &= x_1^2 + 2x_3^2 + x_1x_2 - x_1 + x_2 - x_3 - 49 = 0; \\ f_3(x) &= 5x_1^2 + x_2^2 - x_1x_3 + x_1 - 3x_2 + 5x_3 - 24 = 0. \end{aligned} \tag{10}$$

$\alpha = (1, 2, 5)$ is a root of (10).

Table 1 shows results obtained for $k = 100$.

Example 3.

$$\begin{aligned} f_1(x) &= 5x_1 + 2x_2 + x_3 - 10x_4 - 2 = 0; \\ f_2(x) &= 3x_1 - 2x_2 + 3x_3 + 2x_4 - 28 = 0; \\ f_3(x) &= 10x_1 + 5x_2 - 4x_3 + 5x_4 - 58 = 0; \\ f_4(x) &= x_1 - 4x_2 - 5x_3 - 7x_4 + 92 = 0, \end{aligned} \tag{11}$$

$\alpha = (1, 2, 3, 10)$ is a root of (11).

Table 1 shows results obtained for $k = 100$.

Example 4.

$$f_1(x) = 5x_1 + 7x_2 + 6x_3 + 5x_4 - 48.07 = 0;$$

TABLE 1

Approximation number n	x_1	x_2	x_3	x_4	$t_n=\sqrt{(f_n, f_n)}$
			Example 1		
0	1,2	0,2	—	—	4,12989104
1	0,54123214	0,90331337	—	—	0,13754527
5	0,50000023	0,99999958	—	—	0,00000036
0	0,2	0,7	—	—	2,67824569
1	0,53151852	0,92661673	—	—	0,097300793
5	0,50000012	0,99999979	—	—	0,00000028
0	0,0	1,0	—	—	2,82842712
1	0,44213676	1,11886966	—	—	0,15452976
5	0,49999983	1,00000035	—	—	0,00000042
			Example 2		
0	10,0	5,0	1,0	—	575,45286601
1	1,32418561	2,32692707	4,72977159	—	3,82819903
3	1,00079201	2,00112654	4,99962459	—	0,00017324
			Example 3		
0	0	0	0	0	64,44258031
1	1,03357838	2,55525143	3,27632059	9,55916215	2,32024301
4	1,00000013	2,00000196	3,00000092	9,99999850	0,00000026
			Example 4		
Fuler's method 0	0	0	0	0	118,234500
400	2,86009672	2,87689410	1,81290837	0,547088940	0,0269081
5600	3,74762056	2,34143555	1,59051623	0,67964462	0,0014202
Runge-Kutta method 0	0	0	0	0	118,234500
100	3,04466617	2,76562579	1,36681152	0,574708904	6,7199515
493	3,74844729	2,34093837	1,59031047	0,679769065	0,0013038

TABLE 2

Approximation number n	x_1	x_2	$t_n=\sqrt{(f_n, f_n)}$	$x=A_1y$: x_1	$x=A_1y$: x_2
0	1,2	0,2	4,1298910	1,2	0,
1	0,5412321	0,9033134	0,1375453	0,5337742	0,9134749
2	0,5025320	0,9954211	0,0059619	0,5010726	0,9969977
3	0,50009587	0,9997350	0,0002725	0,5000583	0,9999960
4	0,5000046	0,9999913	0,0000121	—	—

Approximation number n	$x=A_1y$: $t_n=\sqrt{(f_n, f_n)}$	$x=A_2y$: x_1	$x=A_2y$: x_2	$x=A_2y$: $t_n=\sqrt{(f_n, f_n)}$
0	4,1298910	1,2	0,2	4,1298910
1	0,1287874	0,5032026	0,995224	0,0103336
2	0,0043023	0,5000081	0,9999999	0,0000100
3	0,0000112	—	—	—
4	—	—	—	—

$$\begin{aligned} f_2(x) &= 7x_1 + 10x_2 + 8x_3 + 7x_4 - 67.13 = 0; \\ f_3(x) &= 6x_1 + 8x_2 + 10x_3 + 9x_4 - 63.24 = 0; \\ f_4(x) &= 5x_1 + 7x_2 + 9x_3 + 10x_4 - 56.24 = 0, \end{aligned} \tag{12}$$

$\alpha = (3.75, 2.34, 1.59, 0.68)$ is a root of (12) and the Todd conditioning number is $p \approx 2984$.

Table 1 shows results obtained for $k = 1$.

We note in conclusion that our method can be generalized to apply to more general functional equations.

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