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1 author:

Shaun Lui
University of Manitoba

ON LINEAR MONOTONE ITERATION AND SCHWARZ METHODS FOR NONLINEAR ELLIPTIC PDES

S. H. LUI *

Abstract. The Schwarz Alternating Method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomains.

In this paper, proofs of convergence of some Schwarz Alternating Methods for nonlinear elliptic problems which are known to have solutions by the monotone method (also known as the method of subsolutions and supersolutions) are given. In particular, an additive Schwarz method for scalar as well some coupled nonlinear PDEs are shown to converge to some solution on finitely many subdomains, even when multiple solutions are possible. In the coupled system case, each subdomain PDE is linear, decoupled and can be solved concurrently with other subdomain PDEs. These results are applicable to several models in population biology.

Key words. domain decomposition, nonlinear elliptic PDE, Schwarz alternating method, monotone methods, subsolution, supersolution

AMS subject classifications. 65N55, 65J15

1. Introduction.

The Schwarz Alternating Method was devised by H. A. Schwarz more than one hundred years ago to solve linear boundary value problems. It has garnered interest recently because of its potential as an efficient algorithm for parallel computers. See the fundamental work of Lions in [14] and [15]. The literature on this method for the linear boundary value problem is huge, see the recent reviews of Chan and Mathew [5], Le Tallec [27], and Xu and Zou [29], and the books of Smith, Bjorstad and Gropp [23] and Quarteroni and Valli [21]. The literature for nonlinear problems is rather sparse. Besides Lions' works, see also Badea [2], Zou and Huang [30], Cai and Dryja [3], Tai [24], Pao [20], Xu [28], Dryja and Hackbusch [8], Tai and Espedal [25], Tai and Xu [26], Lui [16], Lui [18], Lui [17] and references therein. See also Chen et al [6, 7] for a monotone iteration scheme in the boundary element context. The effectiveness of Schwarz methods for nonlinear problems (especially those in fluid mechanics) has been demonstrated in many papers. See proceedings of the annual domain decomposition conferences beginning with [10].

*Hong Kong University of Science & Technology, Department of Mathematics, Clear Water Bay, Kowloon, Hong Kong (shlui@ust.hk). This work was in part supported by a grant from the RGC of HKSAR, China (HKUST6171/99P).

This paper is a continuation of previous works by this author attempting to survey various classes of nonlinear elliptic PDEs for which Schwarz methods are applicable. We consider elliptic PDEs amenable to analysis by the monotone method (also known as the method of subsolutions and supersolutions). The paper of Keller and Cohen [13] was among the first to employ such method to solve boundary value problems. Subsequent works by these two authors as well as by Sattinger [22], Amann [1], and many others have made this method into one of the important tools in nonlinear analysis. See Pao [19] for a very complete reference with many applications as well as a good bibliography. See also Kaushik, Keyes and Smith [12] and Cai and Keyes [4] and the references therein for additional related works.

Lions [15] shows the convergence of a multiplicative Schwarz method for the Poisson's equation using the monotone method. In Lui [17], convergence of some Schwarz methods for scalar as well as coupled systems of nonlinear elliptic PDEs were given. There, the subdomain problems are still nonlinear. Here, we improve on these results in two ways. First, the subdomain problems are now linear. Second, the hypotheses can be weakened so that a larger class of PDEs can be handled. Our results on coupled systems can be applied to the three types of Lotka-Volterra models in population biology: competition, cooperation and predator-prey.

In section 2, we prove convergence of two Schwarz methods for a class of scalar nonlinear elliptic PDEs. In section 3, we treat the so-called quasi-monotone non-increasing case of a coupled system of PDEs, giving a proof of convergence of an additive Schwarz method on finitely many subdomains. The other two cases (quasi-monotone non-decreasing and mixed quasi-monotone) will be discussed in section 4. Some simple numerical will be given in section 5 to be followed by a conclusion in the final section. In the remaining part of this introduction, we set some notations.

Let Ω be a bounded domain in $\mathbf{R}^N$ with a smooth boundary. Suppose Ω is composed of $m \geq 2$ subdomains, that is, $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$. The boundary of each subdomain is also assumed to be smooth. Let $X = C^\alpha(\overline{\Omega}) \cap C^2(\Omega)$ for some $0 < \alpha < 1$. We shall look for solutions of PDEs lying in this space. Let λ_1 be the principal eigenvalue of $-\triangle$ on Ω with homogeneous Dirichlet boundary condition, and ψ_1 be its corresponding (positive) eigenfunction.

2. Scalar Equations. Consider the PDE

$$-\triangle u = f(x,u) \text{ on } \Omega, \qquad u = h \text{ on } \partial\Omega. \tag{2.1}$$

A function $\underline{u} \in X$ is a *subsolution* of the above PDE if

$$-\triangle \underline{u} - f(x,\underline{u}) \leq 0 \text{ on } \Omega \quad \text{and} \quad \underline{u} \leq h \text{ on } \partial\Omega.$$

Similarly, a *supersolution* is one which satisfies the above with both inequalities reversed.

Let us now record the assumptions for the above PDE. Suppose that it has a subsolution $\underline{u}$ and a supersolution $\overline{u}$ which satisfy $\underline{u} \leq \overline{u}$ on Ω. Define the sector of smooth functions

$$\mathcal{A} \equiv \{u \in X, \ \underline{u} \leq u \leq \overline{u} \text{ on } \overline{\Omega}\}.$$

Assume f is a Holder continuous function defined on $\overline{\Omega} \times \mathcal{A}$ and $h \in C^{2+\alpha}(\partial\Omega)$. In addition, suppose there exists some non-negative function $c \in C^\alpha(\overline{\Omega})$ so that

$$-c(x)(u-v) \leq f(x,u) - f(x,v), \qquad x \in \Omega, \quad v \leq u \in \mathcal{A}. \tag{2.2}$$

(Note that in ([17]), we only consider positive solutions and that $f(x,u)/u$ is assumed to be decreasing in u. These assumptions are not necessary in the present case.) With these assumptions, it is known (section 3.2 in Pao [19]) that the PDE has a (not necessarily unique) solution in $\mathcal{A}$. Such PDEs have applications in population biology, chemical kinetics, etc.

The fundamental tool for monotone iteration schemes is the (weak) maximum principle. One useful form is the following.

LEMMA 1. *Let* $w \in C^2(\Omega) \cap C(\overline{\Omega})$ *satisfy*

$$-\triangle w + cw \geq 0 \text{ on } \Omega, \qquad w \geq 0 \text{ on } \partial\Omega.$$

Then $w \geq 0$ *on* $\overline{\Omega}$.

We shall need an extension of the above result to functions which are less smooth and on domains with a possibly non-smooth boundary. We shall call the following the generalized maximum principle.

LEMMA 2. *Suppose* S *is an open set. Let* $w \in H^1(S) \cap C(\overline{S})$ *satisfy*

$$\int_S (\nabla w \cdot \nabla \phi + cw\phi) \geq 0, \qquad \forall \text{ non-negative } \phi \in H_0^1(S) \tag{2.3}$$

and $w \geq 0$ *on* ∂S. *Then* $w \geq 0$ *on* $\overline{S}$.

Proof: Let $w^+(x) = \max(w(x), 0)$, $w^-(x) = \max(-w(x), 0)$. Note that $w = w^+ - w^-$ and that w^- is a non-negative function in $H_0^1(S)$. Taking $\phi = w^-$ in (2.3), we obtain

$$-\int_S \left(|\nabla w^-|^2 + c(w^-)^2\right) \geq 0$$

which is possible only if $w^- \equiv 0$. Hence $w \geq 0$ on $\overline{S}$.

We now show convergence of a (multiplicative) Schwarz sequence for the PDE (2.1) for the two-subdomain case. For convenience, we suppress the dependence of f on $x \in \Omega$. Note that each subdomain problem is a linear one.

THEOREM 2.1. *Let* $u^{(0)} = u^{(-\frac{1}{2})} = \underline{u}$ *on* $\overline{\Omega}$ *with* $\underline{u} = h$ *on* $\partial\Omega$. *Define the Schwarz sequence by* $(n \geq 0)$

$$-\triangle u^{(n+\frac{1}{2})} + cu^{(n+\frac{1}{2})} = f(u^{(n-\frac{1}{2})}) + cu^{(n-\frac{1}{2})} \text{ on } \Omega_1, \qquad u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} + cu^{(n+1)} = f(u^{(n)}) + cu^{(n)} \text{ on } \Omega_2, \qquad u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Here, $u^{(n+\frac{1}{2})}$ *is defined as* $u^{(n)}$ *on* $\overline{\Omega} \setminus \overline{\Omega}_1$ *and* $u^{(n+1)}$ *is defined as* $u^{(n+\frac{1}{2})}$ *on* $\overline{\Omega} \setminus \overline{\Omega}_2$. *Then* $u^{(n+\frac{i}{2})} \to u$ *in* $C^2(\overline{\Omega}_i)$, $i = 1, 2$, *where* u *is a solution of (2.1) in* $\mathcal{A}$. *If* v *is any solution in* $\mathcal{A}$, *then* $u \leq v$ *on* $\overline{\Omega}$.

If $u^{(0)} = u^{(-\frac{1}{2})} = \overline{u}$ on $\overline{\Omega}$ with $\overline{u} = h$ on $\partial\Omega$ instead, then the same conclusion holds except that $u \geq v$ on $\overline{\Omega}$.

Proof: We only consider the case $u^{(0)} = \underline{u}$ with $\underline{u} = h$ on $\partial\Omega$. The proof can be divided into four steps. First, we demonstrate that the sequence is monotone:

$$\underline{u} \leq u^{(n-\frac{1}{2})} \leq u^{(n)} \leq u^{(n+\frac{1}{2})} \leq \overline{u} \text{ on } \overline{\Omega}, \qquad n \geq 0. \tag{2.4}$$

Since the sequences are bounded above, the following limits are well defined on $\overline{\Omega}$

$$\lim_{n\to\infty} u^{(n+\frac{1}{2})} = u_1, \qquad \lim_{n\to\infty} u^{(n)} = u_2.$$

In the second step, we prove that the function u_i satisfies the same PDE on Ω_i using an elliptic regularity argument (see p. 102 in [19] or [17]). We can also infer that the convergence to u_i is in the sense of $C^2(\overline{\Omega}_i)$. In the third step, we prove that $u_1 = u_2$ on $\overline{\Omega}$ which follows directly from (2.4). Define $u = u_1$. Then u is a solution of (2.1). Finally, if v is any other solution in $\mathcal{A}$, replace $\overline{u}$ by v in the above steps to obtain $u \leq v$ on $\overline{\Omega}$.

The details of the proof of the first step by induction are now given. On Ω_1,

$$-\triangle u^{(\frac{1}{2})} + cu^{(\frac{1}{2})} = f(u^{(0)}) + cu^{(0)} \quad \text{and} \quad -\triangle u^{(0)} + cu^{(0)} \leq f(u^{(0)}) + cu^{(0)}.$$

On subtraction of these two results, we have

$$-\triangle(u^{(\frac{1}{2})} - u^{(0)}) + c(u^{(\frac{1}{2})} - u^{(0)}) \geq 0 \text{ on } \Omega_1.$$

Since $u^{(\frac{1}{2})} - u^{(0)} = 0$ on $\partial\Omega_1$, we conclude that $u^{(\frac{1}{2})} \geq u^{(0)}$ on Ω_1 by the maximum principle. This inequality also holds on $\overline{\Omega}$. In a similar manner, we show that $u^{(\frac{1}{2})} \leq \overline{u}$ on $\overline{\Omega}$. Hence (2.4) holds for $n = 0$.

We now show that

$$u^{(n-\frac{1}{2})} \leq u^{(n+\frac{1}{2})} \text{ on } \overline{\Omega}_1 \;\text{ and }\; u^{(n)} \leq u^{(n+1)} \text{ on } \overline{\Omega}_2 \tag{2.5}$$

by induction. This result will be needed in the proof of the induction step of (2.4) later. From the defining equation of $u^{(1)}$ and the fact that $u^{(0)}$ is a subsolution,

$$-\triangle(u^{(1)} - u^{(0)}) + c(u^{(1)} - u^{(0)}) \geq 0 \text{ on } \Omega_2.$$

Since $u^{(1)} - u^{(0)} = u^{(\frac{1}{2})} - u^{(0)} \geq 0$ on $\partial\Omega_2$, it follows from the maximum principle that $u^{(1)} \geq u^{(0)}$ on $\overline{\Omega}_2$. Since $u^{(-\frac{1}{2})} = u^{(0)} \leq u^{(\frac{1}{2})}$ on $\overline{\Omega}_1$ has already been established, (2.5) holds for $n = 0$. Assume (2.5) holds for some n. We shall prove (2.5) with n replaced by $n+1$. Subtracting the defining equations for $u^{(n+\frac{3}{2})}$ and $u^{(n+\frac{1}{2})}$ on Ω_1, we obtain

$$\begin{aligned}(-\triangle + c)(u^{(n+\frac{3}{2})} - u^{(n+\frac{1}{2})}) &= f(u^{(n+\frac{1}{2})}) - f(u^{(n-\frac{1}{2})}) + c(u^{(n+\frac{1}{2})} - u^{(n-\frac{1}{2})}) \\ &\geq 0.\end{aligned}$$

The last inequality holds because of the induction hypothesis and (2.2). Now $u^{(n+\frac{3}{2})} = u^{(n+1)} \geq u^{(n)} = u^{(n+\frac{1}{2})}$ on $\partial\Omega_1$ by the induction hypothesis. By the maximum principle, $u^{(n+\frac{1}{2})} \leq u^{(n+\frac{3}{2})}$ on $\overline{\Omega}_1$. Similarly, $u^{(n+2)} \geq u^{(n+1)}$ on $\overline{\Omega}_2$ and this finishes the proof of (2.5).

Suppose (2.4) holds. We show that $u^{(n+\frac{1}{2})} \leq u^{(n+1)} \leq u^{(n+\frac{3}{2})} \leq \overline{u}$ on $\overline{\Omega}$. On $\Omega_1 \setminus \Omega_2$, $u^{(n+\frac{1}{2})} = u^{(n+1)}$ by definition. On $\Omega_{12} \equiv \Omega_1 \cap \Omega_2$, subtract the defining equations for $u^{(n+\frac{1}{2})}$ and $u^{(n+1)}$ to obtain

$$(-\triangle + c)(u^{(n+1)} - u^{(n+\frac{1}{2})}) = f(u^{(n)}) - f(u^{(n-\frac{1}{2})}) + c(u^{(n)} - u^{(n-\frac{1}{2})}) \geq 0$$

with the latter inequality due to the induction hypothesis and (2.2). In case $\partial\Omega_{12}$ is not smooth, we can multiply the above inequality by a non-negative $\phi \in H_0^1(\Omega_{12})$ and use integration by parts to get

$$\int_{\Omega_{12}} \left(\nabla(u^{(n+1)} - u^{(n+\frac{1}{2})}) \cdot \nabla\phi + c(u^{(n+1)} - u^{(n+\frac{1}{2})})\phi\right) \geq 0.$$

Along $\partial\Omega_1 \cap \Omega_2$, $u^{(n+1)} \geq u^{(n)} = u^{(n+\frac{1}{2})}$ by (2.5) while along $\partial\Omega_2 \cap \Omega_1$, $u^{(n+1)} = u^{(n+\frac{1}{2})}$. Hence $u^{(n+1)} - u^{(n+\frac{1}{2})} \geq 0$ on $\partial\Omega_{12}$. By the generalized maximum principle, $u^{(n+1)} \geq u^{(n+\frac{1}{2})}$ on $\overline{\Omega}_{12}$. Since $u^{(n+\frac{1}{2})} = u^{(n)} \leq u^{(n+1)}$ on $\Omega_2 \setminus \Omega_1$, $u^{(n+1)} \geq u^{(n+\frac{1}{2})}$ on $\overline{\Omega}$. In the same way, we can show that $u^{(n+\frac{3}{2})} \geq u^{(n+1)}$ on $\overline{\Omega}$.

From the defining equation of $u^{(n+\frac{3}{2})}$ and that $\overline{u}$ is a supersolution, we can apply the maximum principle to show that $\overline{u} \geq u^{(n+\frac{3}{2})}$ on $\overline{\Omega}_1$. This inequality also holds on $\overline{\Omega}$ by the definition of $u^{(n+\frac{3}{2})}$. This completes the proof of (2.4).

The above (multiplicative) Schwarz iteration is an adaptation of the classical Schwarz iteration for Poisson's equation to solve a nonlinear equation. The next Schwarz method is called an additive Schwarz method. It generalizes the additive method for linear PDEs first introduced by Dryja and Widlund [9]. It is sometimes preferable to the (multiplicative) Schwarz method above because the subdomain PDEs are independent and hence can be solved in parallel. We consider the general m-subdomain case.

THEOREM 2.2. *Let $u^{(0)} = u_i^{(0)} = \underline{u}$ on $\overline{\Omega}$, $i = 1, \cdots, m$ with $\underline{u} = h$ on $\partial\Omega$. Define the additive Schwarz sequence by ($n \geq 1$)*

$$-\triangle u_i^{(n)} + cu_i^{(n)} = f(u_i^{(n-1)}) + cu_i^{(n-1)} \text{ on } \Omega_i, \qquad u_i^{(n)} = u^{(n-1)} \text{ on } \partial\Omega_i, \quad i = 1, \cdots, m.$$

Here, $u_i^{(n)}$ is defined as $u^{(n-1)}$ on $\overline{\Omega} \setminus \overline{\Omega}_i$ and

$$u^{(n)}(x) = \max_{1 \leq i \leq m} u_i^{(n)}(x), \qquad x \in \overline{\Omega}.$$

Then $u_i^{(n)} \to u$ in $C^2(\Omega_i)$, $i = 1, \cdots, m$ where u is a solution of (2.1) in $\mathcal{A}$. If v is any solution in $\mathcal{A}$, then $u \leq v$ on $\overline{\Omega}$.

If $u^{(0)} = u_i^{(0)} = \overline{u}$ on $\overline{\Omega}$ with $\overline{u} = h$ on $\partial\Omega$ instead, then the same conclusion holds except that $u \geq v$ on $\overline{\Omega}$.

Proof: The details of this proof are quite similar to those of the last proof. Assume $u^{(0)} = \underline{u}$. The following monotone properties hold:

$$\underline{u} \leq u_i^{(n)} \leq u_i^{(n+1)} \leq \overline{u} \text{ on } \overline{\Omega}_i, \qquad \underline{u} \leq u^{(n)} \leq u^{(n+1)} \leq \overline{u} \text{ on } \overline{\Omega}, \tag{2.6}$$

$$u^{(n)} \leq u_i^{(n+1)} \text{ on } \overline{\Omega}, \qquad i = 1, \cdots, m. \tag{2.7}$$

The inequalities in (2.6) can be shown in a straightforward manner by induction using the maximum principle. To show the second set of inequalities in (2.6), take a fixed n and $x \in \Omega$. Then there is some integer i in between 1 and m inclusive so that $u^{(n)}(x) = u_i^{(n)}(x) \leq u_i^{(n+1)}(x) \leq u^{(n+1)}(x)$.

The inequality (2.7) can also be shown by induction. The result certainly holds when $n = 0$. Suppose (2.7) holds with n replaced by $n-1$. We claim that

$$\int_{\Omega_i} (\nabla u^{(n)} \cdot \nabla \phi + c u^{(n)} \phi) \leq \int_{\Omega_i} \Big(f(u^{(n-1)}) + c u^{(n-1)} \Big) \phi, \qquad \forall \text{ non-negative } \phi \in H_0^1(\Omega_i). \tag{2.8}$$

This inequality, which will be proved later, says that $u^{(n)}$ is a subsolution in some weak sense.

Now multiply the defining equation for $u_i^{(n+1)}$ by any non-negative $\phi \in H_0^1(\Omega_i)$ and then integrate by parts to obtain

$$\int_{\Omega_i} (\nabla u_i^{(n+1)} \cdot \nabla \phi + c u_i^{(n+1)} \phi) = \int_{\Omega_i} \Big(f(u_i^{(n)}) + c u_i^{(n)} \Big) \phi.$$

Subtract (2.8) from this equation to get

$$\int_{\Omega_i} \Big(\nabla (u_i^{(n+1)} - u^{(n)}) \cdot \nabla \phi + c(u_i^{(n+1)} - u^{(n)}) \phi \Big) \geq \int_{\Omega_i} \Big(f(u_i^{(n)}) - f(u^{(n-1)}) + c(u_i^{(n)} - u^{(n-1)}) \Big) \phi$$
$$\geq 0$$

by the induction hypothesis $u_i^{(n)} \geq u^{(n-1)}$ and (2.2). Since $u_i^{(n+1)} = u^{(n)}$ on $\partial \Omega_i$, we can conclude that $u_i^{(n+1)} \geq u^{(n)}$ on $\overline{\Omega}_i$ by the generalized maximum principle. Of course this inequality also holds on $\overline{\Omega}$. This completes the proof of (2.7).

Next, we define on $\overline{\Omega}$, for $i = 1, \cdots, m$,

$$\lim_{n \to \infty} u_i^{(n)} = u_i, \qquad \lim_{n \to \infty} u^{(n)} = u_0$$

and show using elliptic regularity theory that the limit u_i satisfies the same PDE on Ω_i, $i = 1, \cdots, m$ and that the convergence to u_i is in the sense of $C^2(\Omega_i)$. (In the previous theorem, convergence is in the sense of $C^2(\overline{\Omega}_i)$ while here, it is in a weaker sense because $u_i^{(n)} = u^{(n)}$ on $\partial \Omega_i$ where $u^{(n+1)}$ is not necessarily a function in $C^{2+\alpha}(\partial \Omega_i)$.) We have $u_i \leq u_0$ on $\overline{\Omega}$, $i = 1, \cdots, m$. By (2.7), we have for any j, $u_0 \leq u_j \leq u_0 \leq u_i$. From these inequalities, we conclude that $u_i = u_j = u_0$, $1 \leq i, j \leq m$. Define u to be this common function which must be a solution of (2.1) in $\mathcal{A}$. The proof of $u \leq v$ for any solution of (2.1) in $\mathcal{A}$ is the same as before.

To complete the proof, we need to show (2.8). First note that $u^{(n)} \in H^1(\Omega) \cap C(\overline{\Omega})$. To show this, first look at the case $m = 2$. The result certainly holds when $n = 0$ by definition. For a positive n, since $u_i^{(n)} \in H^1(\Omega) \cap C(\overline{\Omega})$,

$$u^{(n)}(x) = \max\left(u_1^{(n)}(x), u_2^{(n)}(x)\right) = u_1^{(n)}(x) + \left(u_2^{(n)}(x) - u_1^{(n)}(x)\right)^+$$

which implies the desired result. The case of m subdomains follows by induction.

Consider the PDE

$$(-\triangle + c)h^{(n)} = f(u^{(n-1)}) + cu^{(n-1)} \text{ on } \Omega_i, \qquad h^{(n)} = u^{(n)} \text{ on } \partial\Omega_i.$$

Here, $n \geq 0$ and we define $u^{(-1)} = u^{(0)}$. We claim (to be proved later) that

$$h^{(n+1)} \geq h^{(n)} \text{ on } \overline{\Omega}_i \tag{2.9}$$

and

$$h^{(n)} \geq u^{(n)} \text{ on } \overline{\Omega}_i. \tag{2.10}$$

Suppose there exist some non-negative $\phi \in H_0^1(\Omega_i)$ such that

$$\int_{\Omega_i} \nabla u^{(n)} \cdot \nabla\phi + cu^{(n)}\phi > \int_{\Omega_i} \Big(f(u^{(n-1)}) + cu^{(n-1)}\Big)\phi. \tag{2.11}$$

Multiply the defining equation for $h^{(n)}$ by ϕ and then apply integration by parts. Subtract the resulting equation from (2.11) to obtain

$$\int_{\Omega_i} \nabla(u^{(n)} - h^{(n)}) \cdot \nabla\phi + c(u^{(n)} - h^{(n)})\phi > 0. \tag{2.12}$$

Since $u^{(n)} - h^{(n)} = 0$ on $\partial\Omega_i$, the generalized maximum principle states that $u^{(n)} \geq h^{(n)}$ on $\overline{\Omega}_i$. In fact, since the inequality in (2.12) is strict, it is easy to see that $u^{(n)}(x_0) > h^{(n)}(x_0)$ for some $x_0 \in \Omega_i$. However, this contradicts (2.10). Hence the assumption (2.11) is false and this implies the truth of (2.8).

Let us now show (2.9). On Ω_i,

$$(-\triangle + c)(h^{(n+1)} - h^{(n)}) = f(u^{(n)}) - f(u^{(n-1)}) + c(u^{(n)} - u^{(n-1)}) \geq 0$$

by the monotonicity of the sequence $u^{(n)}$. Since $h^{(n+1)} - h^{(n)} = u^{(n+1)} - u^{(n)} \geq 0$ on $\partial\Omega_i$, (2.9) follows by the maximum principle.

Finally, we show (2.10) by induction. The base case $n = 0$ can easily be shown using the definition of $h^{(0)}$, the fact that $u^{(0)}$ is a subsolution and the maximum principle. Suppose (2.10) holds for n. We show $h^{(n+1)} \geq u^{(n+1)}$ on $\overline{\Omega}_i$ by showing that $h^{(n+1)} \geq u_j^{(n+1)}$ on $\overline{\Omega}_i$ for $1 \leq j \leq m$. When $j = i$, subtract the defining equations of $u_i^{(n+1)}$ and $h^{(n+1)}$ to obtain

$$(-\triangle + c)(h^{(n+1)} - u_i^{(n+1)}) = f(u^{(n)}) - f(u_i^{(n)}) + c(u^{(n)} - u_i^{(n)}) \geq 0 \text{ on } \Omega_i.$$

Since $h^{(n+1)} - u_i^{(n+1)} = u^{(n+1)} - u^{(n)} \geq 0$ on $\partial\Omega_i$, the maximum principle implies that $h^{(n+1)} \geq u_i^{(n+1)}$ on $\overline{\Omega}_i$.

Now suppose $j \neq i$. Note

$$u_j^{(n+1)} = u^{(n)} \leq h^{(n)} \leq h^{(n+1)} \text{ on } \overline{\Omega}_i \setminus \Omega_j$$

by (2.9) and the induction hypothesis. Next, subtract the definitions of $h^{(n+1)}$ and $u_j^{(n+1)}$ on $\Omega_i \cap \Omega_j$ to obtain

$$(-\triangle + c)(h^{(n+1)} - u_j^{(n+1)}) = f(u^{(n)}) - f(u_j^{(n)}) + c(u^{(n)} - u_j^{(n)}) \geq 0$$

or the weak form

$$\int_{\Omega_i \cap \Omega_j} \nabla(h^{(n+1)} - u_j^{(n+1)}) \cdot \nabla\phi + c(h^{(n+1)} - u_j^{(n+1)})\phi \geq 0, \qquad \text{non-negative } \phi \in H_0^1(\Omega_i \cap \Omega_j).$$

Observe that

$$h^{(n+1)} - u_j^{(n+1)} = \begin{cases} h^{(n+1)} - u^{(n)} \geq h^{(n)} - u^{(n)} \geq 0, & \partial\Omega_j \cap \Omega_i; \\ u^{(n+1)} - u_j^{(n+1)} \geq 0, & \partial\Omega_i \cap \Omega_j \end{cases}$$

by (2.9) and the induction hypothesis. Hence $h^{(n+1)} \geq u_j^{(n+1)}$ on $\overline{\Omega_i \cap \Omega_j}$ by the generalized maximum principle. Combining with the result at the beginning of this paragraph, we conclude that $h^{(n+1)} \geq u_j^{(n+1)}$ on $\overline{\Omega}_i$. This completes the proof.

In practice, $u^{(n)}$ need not be computed globally but rather only along the subdomain boundaries. In the next two sections, we consider some coupled systems of nonlinear elliptic PDEs and their solution by an additive Schwarz method.

3. Quasi-monotone non-increasing coupled systems. Consider the system

$$-\triangle u = f(x,u,v), \qquad -\triangle v = g(x,u,v) \qquad \text{on } \Omega, \tag{3.1}$$

$$u = r, \qquad v = s \qquad \text{on } \partial\Omega.$$

We shall suppress the dependence of f and g on $x \in \Omega$ for convenience. The pairs of smooth functions $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$-\triangle\underline{u} - f(\underline{u},\overline{v}) \leq 0 \leq -\triangle\overline{u} - f(\overline{u},\underline{v}) \text{ on } \Omega,$$

$$-\triangle\underline{v} - g(\overline{u},\underline{v}) \leq 0 \leq -\triangle\overline{v} - g(\underline{u},\overline{v}) \text{ on } \Omega, \text{ and}$$

$$\underline{u} \leq r \leq \overline{u}, \qquad \underline{v} \leq s \leq \overline{v} \quad \text{on } \partial\Omega.$$

Furthermore, they are said to be *ordered* if

$$\underline{u} \leq \overline{u}, \qquad \underline{v} \leq \overline{v} \quad \text{on } \overline{\Omega}.$$

Define the sector

$$\mathcal{A} \equiv \left\{ \left[\begin{array}{c} u \\ v \end{array} \right],\ u,v \in X,\ \underline{u} \leq u \leq \overline{u},\ \underline{v} \leq v \leq \overline{v} \text{ on } \overline{\Omega} \right\}.$$

Suppose $f,g \in C^1(\mathcal{A})$. Our system of PDEs is called *quasi-monotone non-increasing* if

$$\frac{\partial f}{\partial v},\ \frac{\partial g}{\partial u} \leq 0 \text{ on } \mathcal{A}.$$

Note that the definition of subsolution and supersolution depends on the assumptions on the nonlinearities. Later on, this definition changes for a different set of assumptions.

Suppose our system of PDEs is quasi-monotone non-increasing. Then it can be shown (section 8.4 in Pao [19]) that it has a solution (u,v) in $\mathcal{A}$. Without further assumptions, it may have more than one solution. Despite this, the following additive Schwarz sequence converges for an appropriately chosen initial guess. Note that the subdomain problems at each iteration are linear, independent and are decoupled.

THEOREM 3.1. *Suppose the system (3.1) is quasi-monotone non-increasing and let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be ordered subsolution and supersolution pairs. Consider any non-negative functions* $c, d \in C^{\alpha}(\overline{\Omega})$ *so that*

$$\frac{\partial f(u,v)}{\partial u} \geq -c, \quad \frac{\partial g(u,v)}{\partial v} \geq -d, \qquad (u,v) \in \mathcal{A}. \tag{3.2}$$

For $i = 1, \cdots, m$, *let*

$$u^{(0)} = u_i^{(0)} = \underline{u} \text{ and } v^{(0)} = v_i^{(0)} = \overline{v} \text{ on } \overline{\Omega} \text{ with } \underline{u} = r \text{ and } \overline{v} = s \text{ on } \partial\Omega. \tag{3.3}$$

Define the Schwarz sequence for $i = 1, \cdots, m$ *and* $n \geq 1$

$$-\triangle u_i^{(n)} + cu_i^{(n)} = f(u_i^{(n-1)}, v_i^{(n-1)}) + cu_i^{(n-1)} \text{ on } \Omega_i, \qquad u_i^{(n)} = u^{(n-1)} \text{ on } \partial\Omega_i$$

$$-\triangle v_i^{(n)} + dv_i^{(n)} = g(u_i^{(n-1)}, v_i^{(n-1)}) + dv_i^{(n-1)} \text{ on } \Omega_i, \qquad v_i^{(n)} = v^{(n-1)} \text{ on } \partial\Omega_i.$$

Here, $u_i^{(n)}$ *and* $v_i^{(n)}$ *are defined as* $u^{(n-1)}$ *and* $v^{(n-1)}$, *respectively, on* $\overline{\Omega} \setminus \overline{\Omega}_i$ *while*

$$u^{(n)}(x) = \max_{1 \leq i \leq m} u_i^{(n)}(x), \quad v^{(n)}(x) = \min_{1 \leq i \leq m} v_i^{(n)}(x) \qquad \text{on } \overline{\Omega}.$$

Then $u_i^{(n)} \to \underline{u}_0$ *and* $v_i^{(n)} \to \overline{v}_0$ *in* $C^2(\Omega_i)$, $i = 1, \cdots, m$, *where* $(\underline{u}_0, \overline{v}_0)$ *is a solution of (3.1) in* $\mathcal{A}$. *If* (u, v) *is any solution in* $\mathcal{A}$, *then* $\underline{u}_0 \leq u$ *and* $v \leq \overline{v}_0$.

If $u^{(0)} = u_i^{(0)} = \overline{u}$ *and* $v^{(0)} = v_i^{(0)} = \underline{v}$ *on* $\overline{\Omega}$ *with* $\overline{u} = r$ *and* $\underline{v} = s$ *on* $\partial\Omega$ *replace the assumption (3.3), then the above Schwarz sequence satisfies* $u_i^{(n)} \to \overline{u}_0$ *and* $v_i^{(n)} \to \underline{v}_0$ *in* $C^2(\Omega_i)$, $i = 1, \cdots, m$, *where* $(\overline{u}_0, \underline{v}_0)$ *is also a solution of (3.1) in* $\mathcal{A}$. *If* (u, v) *is any solution in* $\mathcal{A}$, *then* $u \leq \overline{u}_0$ *and* $v \geq \underline{v}_0$.

Proof: We only consider the case where $u^{(0)} = \underline{u}$ and $v^{(0)} = \overline{v}$. The proof can be divided into four steps. We first show that the following monotone properties hold on $\overline{\Omega}$ for $i = 1, \cdots, m$,

$$\underline{u} \leq u_i^{(n)} \leq u_i^{(n+1)} \leq \overline{u}, \qquad u^{(n)} \leq u^{(n+1)}, \qquad u^{(n)} \leq u_i^{(n+1)} \tag{3.4}$$

and

$$\underline{v} \leq v_i^{(n+1)} \leq v_i^{(n)} \leq \overline{v}, \qquad v^{(n+1)} \leq v^{(n)}, \qquad v_i^{(n+1)} \leq v^{(n)}. \tag{3.5}$$

Since the sequences are bounded, the following limits on $\overline{\Omega}$ are well defined

$$\lim_{n\to\infty} u_i^{(n)} = \underline{u}_i, \qquad \lim_{n\to\infty} v_i^{(n)} = \overline{v}_i \qquad i = 1, \cdots, m,$$

and

$$\lim_{n\to\infty} u^{(n)} = \underline{u}_0, \qquad \lim_{n\to\infty} v^{(n)} = \overline{v}_0.$$

In the second step, we prove, using a similar elliptic regularity argument as before, that the limit functions satisfy the following PDEs on Ω_i:

$$-\triangle \underline{u}_i = f(\underline{u}_i, \overline{v}_i), \qquad -\triangle \overline{v}_i = g(\underline{u}_i, \overline{v}_i), \qquad i = 1, \cdots, m, \tag{3.6}$$

and that convergence to $\underline{u}_i$ and to $\overline{v}_i$ is in the sense of $C^2(\Omega_i)$. Third, we demonstrate that the functions $\underline{u}_i$ are identical. This follows because from (3.4) and the definition of $u^{(n)}$,

$$u_i^{(n)} \leq u^{(n)} \leq u_j^{(n+1)} \leq u^{(n+1)} \leq u_i^{(n+2)}, \qquad 1 \leq i, j \leq m.$$

Take the limit to obtain $\underline{u}_i = \underline{u}_j = \underline{u}_0$ on $\overline{\Omega}$. Similarly, we use (3.5) to show $\overline{v}_i = \overline{v}_j = \overline{v}_0$ on $\overline{\Omega}$ for $1 \leq i, j \leq m$. From (3.6), it follows that $(\underline{u}_0, \overline{v}_0)$ is a solution of (3.1).

Fourth, we prove that any solution (u, v) of (3.1) in $\mathcal{A}$ must satisfy

$$\underline{u}_0 \leq u \quad \text{and} \quad v \leq \overline{v}_0 \text{ on } \Omega_i. \tag{3.7}$$

This follows from the observation that $(\underline{u}, v)$ and $(u, \overline{v})$ form subsolution and supersolution pairs. Apply the above result to establish (3.7).

The details of the proof of step one (the inequalities (3.4) and (3.5)) by induction are now given. For the case $n = 0$, use the defining equation of $u_i^{(1)}$ and the fact that $\underline{u}$ is a subsolution to obtain

$$-\triangle(u_i^{(1)} - \underline{u}) + c(u_i^{(1)} - \underline{u}) \geq 0 \text{ on } \Omega_i.$$

Since $u_i^{(1)} - \underline{u} = 0$ on $\partial\Omega_i$, an application of the maximum principle yields $\underline{u} \leq u_i^{(1)}$ on $\overline{\Omega}_i$. The same inequality also holds on $\overline{\Omega}$.

From the defining equation of $u_i^{(1)}$ and the fact that $\overline{u}$ is a supersolution, we get on Ω_i,

$$
\begin{aligned}
-\triangle(\overline{u} - u_i^{(1)}) + c(\overline{u} - u_i^{(1)}) &\geq f(\overline{u}, \underline{v}) - f(\underline{u}, \overline{v}) + c(\overline{u} - \underline{u}) \\
&= (f_u(*) + c)(\overline{u} - \underline{u}) + f_v(*)(\underline{v} - \overline{v}) \\
&\geq 0
\end{aligned}
$$

since the system is quasi-monotone non-increasing. Here, $*$ represents an argument between $(\underline{u}, \overline{v})$ and $(\overline{u}, \underline{v})$ (necessarily lying in $\mathcal{A}$) given by the mean value theorem. We shall use this same notation later without further comment. Of course $*$ will denote different quantities at different occurrences. Along $\partial\Omega_i$, $\overline{u} - u_i^{(1)} = \overline{u} - \underline{u} \geq 0$. By the maximum principle, $u_i^{(1)} \leq \overline{u}$ on $\overline{\Omega}_i$. This inequality can be extended to $\overline{\Omega}$ by the definition of $u_i^{(1)}$. Trivially, we also have $u^{(0)} \leq u^{(1)}$. The base case $n = 0$ for (3.5) can be shown similarly.

Now suppose (3.4) and (3.5) hold with n replaced by $n-1$. From the defining equations for $u_i^{(n)}$ and $u_i^{(n+1)}$, we obtain

$$
\begin{aligned}
(-\triangle + c)(u_i^{(n+1)} - u_i^{(n)}) &= f(u_i^{(n)}, v_i^{(n)}) - f(u_i^{(n-1)}, v_i^{(n-1)}) + c(u_i^{(n)} - u_i^{(n-1)}) \\
&= (f_u(*) + c)(u_i^{(n)} - u_i^{(n-1)}) + f_v(*)(v_i^{(n)} - v_i^{(n-1)}) \\
&\geq 0
\end{aligned}
$$

where the induction hypothesis and the fact that the system is quasi-monotone non-increasing have been used. Since $u_i^{(n+1)} - u_i^{(n)} = u^{(n)} - u^{(n-1)} \geq 0$ on $\partial\Omega_i$ by the induction hypothesis, an application of the maximum principle proves that $u_i^{(n)} \leq u_i^{(n+1)}$ on $\overline{\Omega}_i$ and thus also on $\overline{\Omega}$.

To show that $u^{(n)} \leq u_i^{(n+1)}$ on Ω_i by induction, we first argue as in (2.8) in

Theorem 2.2 that

$$\int_{\Omega_i} \nabla u^{(n)} \cdot \nabla \phi + cu^{(n)}\phi \le \int_{\Omega_i} \Big(f(u^{(n-1)}, v^{(n-1)}) + cu^{(n-1)} \Big) \phi$$

where ϕ is a non-negative function in $H_0^1(\Omega_i)$. Multiply the defining equation for $u_i^{(n+1)}$ by this function ϕ and integrate by parts to obtain

$$\int_{\Omega_i} \nabla u_i^{(n+1)} \cdot \nabla \phi + cu_i^{(n+1)}\phi = \int_{\Omega_i} \Big(f(u_i^{(n)}, v_i^{(n)}) + cu_i^{(n)} \Big) \phi.$$

Subtract these two results to get

$$\begin{aligned}
\int_{\Omega_i} (\nabla u_i^{(n+1)} - u^{(n)}) \cdot \nabla \phi + c(u_i^{(n+1)} - u^{(n)})\phi &\ge \int_{\Omega_i} \Big(f(u_i^{(n)}, v_i^{(n)}) - f(u^{(n-1)}, v^{(n-1)}) + c(u_i^{(n)} - u^{(n-1)}) \Big) \phi \\
&= \int_{\Omega_i} \Big((f_u(*) + c)(u_i^{(n)} - u^{(n-1)}) + f_v(*)(v_i^{(n)} - v^{(n-1)}) \Big) \phi \\
&\ge 0
\end{aligned}$$

by the induction hypothesis and the fact that the system is quasi-monotone non-increasing. Since $u_i^{(n+1)} - u^{(n)} = 0$ on $\partial\Omega_i$, we can conclude that $u^{(n)} \le u_i^{(n+1)}$ on $\overline{\Omega}_i$ by the generalized maximum principle. The same inequality also holds on $\overline{\Omega}$. Other inequalities in (3.4) and (3.5) can similarly be shown. We shall omit their proof. This completes the proof of the theorem.

One example where a quasi-monotone non-increasing system occurs is the Lotka-Volterra competition model

$$-\triangle u = u(a_1 - b_1 u - c_1 v), \qquad -\triangle v = v(a_2 - b_2 u - c_2 v).$$

Here u, v stand for the population of two species competing for the same food sources and/or territories and all other variables are positive constants. See Gui and Lou [11].

4. Other coupled systems. In this section, we also consider solutions of the system (3.1) with two other sets of assumptions on the nonlinearities. For the first of these, the pairs of smooth functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$-\triangle \underline{u} - f(\underline{u}, \underline{v}) \le 0 \le -\triangle \overline{u} - f(\overline{u}, \overline{v}) \text{ on } \Omega,$$

$$-\triangle \underline{v} - g(\underline{u}, \underline{v}) \le 0 \le -\triangle \overline{v} - g(\overline{u}, \overline{v}) \text{ on } \Omega, \text{ and}$$

$$\underline{u} \le r \le \overline{u}, \qquad \underline{v} \le s \le \overline{v} \quad \text{ on } \partial\Omega.$$

Assuming that the subsolution-supersolution pairs are ordered, our system of PDEs is called *quasi-monotone non-decreasing* if

$$\frac{\partial f}{\partial v}, \ \frac{\partial g}{\partial u} \ge 0 \text{ on } \mathcal{A},$$

where $\mathcal{A}$ is defined as above.

Suppose our system of PDEs is quasi-monotone non-decreasing. Then it can be shown (section 8.4 in Pao [19]) that it has a solution (u, v) in $\mathcal{A}$. Without further assumptions, it may have more than one solution.

THEOREM 4.1. *Suppose the system (3.1) is quasi-monotone non-decreasing and let $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ be ordered subsolution and supersolution pairs. Consider any non-negative functions $c, d \in C^\alpha(\overline{\Omega})$ so that*

$$\frac{\partial f(u, v)}{\partial u} \ge -c, \quad \frac{\partial g(u, v)}{\partial v} \ge -d, \qquad (u, v) \in \mathcal{A}.$$

For $i = 1, \cdots, m$, let

$$u^{(0)} = u_i^{(0)} = \underline{u} \text{ and } v^{(0)} = v_i^{(0)} = \underline{v} \text{ on } \overline{\Omega} \text{ with } \underline{u} = r \text{ and } \underline{v} = s \text{ on } \partial\Omega. \tag{4.1}$$

Define the Schwarz sequence for $i = 1, \cdots, m$ and $n \ge 1$

$$-\triangle u_i^{(n)} + c u_i^{(n)} = f(u_i^{(n-1)}, v_i^{(n-1)}) + c u_i^{(n-1)} \text{ on } \Omega_i, \qquad u_i^{(n)} = u^{(n-1)} \text{ on } \partial\Omega_i$$

$$-\triangle v_i^{(n)} + d v_i^{(n)} = g(u_i^{(n-1)}, v_i^{(n-1)}) + d v_i^{(n-1)} \text{ on } \Omega_i, \qquad v_i^{(n)} = v^{(n-1)} \text{ on } \partial\Omega_i.$$

Here, $u_i^{(n)}$ and $v_i^{(n)}$ are defined as $u^{(n-1)}$ and $v^{(n-1)}$, respectively, on $\overline{\Omega} \setminus \overline{\Omega}_i$ while

$$u^{(n)}(x) = \max_{1 \le i \le m} u_i^{(n)}(x), \quad v^{(n)}(x) = \max_{1 \le i \le m} v_i^{(n)}(x) \qquad \text{on } \overline{\Omega}.$$

Then $u_i^{(n)} \to \underline{u}_0$ and $v_i^{(n)} \to \underline{v}_0$ in $C^2(\Omega_i)$, $i = 1, \cdots, m$, where $(\underline{u}_0, \underline{v}_0)$ is a solution of (3.1) in $\mathcal{A}$. If (u, v) is any solution in $\mathcal{A}$, then $\underline{u}_0 \le u$ and $\underline{v}_0 \le v$.

If $u^{(0)} = u_i^{(0)} = \overline{u}$ and $v^{(0)} = v_i^{(0)} = \overline{v}$ on $\overline{\Omega}$ with $\overline{u} = r$ and $\overline{v} = s$ on $\partial\Omega$ replace the assumption (4.1), then the above Schwarz sequence satisfies $u_i^{(n)} \to \overline{u}_0$ and $v_i^{(n)} \to \overline{v}_0$ in $C^2(\Omega_i)$, $i = 1, \cdots, m$, where $(\overline{u}_0, \overline{v}_0)$ is also a solution of (3.1) in $\mathcal{A}$. If (u, v) is any solution in $\mathcal{A}$, then $u \le \overline{u}_0$ and $v \le \overline{v}_0$.

Proof: The proof is similar to the previous one and thus we only give a sketch. Suppose $u^{(0)} = \underline{u}$ and $v^{(0)} = \underline{v}$. The first step is to show that the following monotone properties hold on $\overline{\Omega}$ for $i = 1, \cdots, m$,

$$\underline{u} \le u_i^{(n)} \le u_i^{(n+1)} \le \overline{u}, \qquad u^{(n)} \le u^{(n+1)}, \qquad u^{(n)} \le u_i^{(n+1)}$$

and

$$\underline{v} \le v_i^{(n)} \le v_i^{(n+1)} \le \overline{v}, \qquad v^{(n)} \le v^{(n+1)}, \qquad v^{(n)} \le v_i^{(n+1)}.$$

Since the sequences are bounded above, the following limits on $\overline{\Omega}$ are well defined

$$\lim_{n\to\infty} u_i^{(n)} = \underline{u}_i, \qquad \lim_{n\to\infty} v_i^{(n)} = \underline{v}_i, \qquad i = 1, \cdots, m$$

and

$$\lim_{n\to\infty} u^{(n)} = \underline{u}_0, \qquad \lim_{n\to\infty} v^{(n)} = \underline{v}_0.$$

In the second step, we prove using elliptic regularity theory that the limit functions satisfy the following PDEs on Ω_i:

$$-\triangle \underline{u}_i = f(\underline{u}_i, \underline{v}_i), \qquad -\triangle \underline{v}_i = g(\underline{u}_i, \underline{v}_i), \qquad i = 1, \cdots, m$$

and that convergence to $\underline{u}_i$ and to $\underline{v}_i$ is in the sense of $C^2(\Omega_i)$. In the third step, we use the monotone property of the sequences to show that $\underline{u}_i = \underline{u}_j = \underline{u}_0$ and $\underline{v}_i = \underline{v}_j = \underline{v}_0$ on $\overline{\Omega}$, $i, j = 1, \cdots, m$. This demonstrates that $(\underline{u}_0, \underline{v}_0)$ is a solution of (3.1). Finally, let (u, v) be a solution of (3.1) in $\mathcal{A}$. Since $(\underline{u}, \underline{v})$ and (u, v) form subsolution and supersolution pairs, the above result demonstrates that $\underline{u}_0 \le u$ and $\underline{v}_0 \le v$ on $\overline{\Omega}$. This completes the sketch of the proof.

One example where a quasi-monotone non-decreasing system occurs is the Lotka-Volterra cooperating model

$$-\triangle u = u(a_1 - b_1 u + c_1 v), \qquad -\triangle v = v(a_2 + b_2 u - c_2 v).$$

Here u, v stand for the population of two species which have a symbiotic relationship and all other variables are positive constants.

Finally, we consider a third class of coupled systems. The pairs of smooth functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ are called *subsolution and supersolution pairs* if they satisfy

$$-\triangle \underline{u} - f(\underline{u}, \overline{v}) \leq 0 \leq -\triangle \overline{u} - f(\overline{u}, \underline{v}) \text{ on } \Omega,$$

$$-\triangle \underline{v} - g(\underline{u}, \underline{v}) \leq 0 \leq -\triangle \overline{v} - g(\overline{u}, \overline{v}) \text{ on } \Omega, \text{ and}$$

$$\underline{u} \leq r \leq \overline{u}, \qquad \underline{v} \leq s \leq \overline{v} \quad \text{ on } \partial\Omega.$$

In case the subsolution-supersolution pairs are ordered, our system of PDEs is called *mixed quasi-monotone* if

$$\frac{\partial f}{\partial v} \leq 0 \text{ and } \frac{\partial g}{\partial u} \geq 0 \text{ on } \mathcal{A}.$$

Suppose our system of PDEs is mixed quasi-monotone and either

$$\frac{\partial f}{\partial u} > \lambda_1 \text{ or } \frac{\partial g}{\partial v} > \lambda_1 \text{ on } \mathcal{A} \tag{4.2}$$

holds. Then it can be shown (section 8.5 in Pao [19]) that it has a solution (u, v) in $\mathcal{A}$. Without further assumptions, it may have more than one solution.

We need the following lemma.

LEMMA 4.2. *Let ϕ be any non-negative function in $H_0^1(\Omega)$. Then there exist non-negative $\chi_j \in H_0^1(\Omega_j)$ such that*

$$\phi = \chi_1 + \cdots + \chi_m \text{ on } \overline{\Omega}.$$

Proof: Take the case $m = 2$. Since the subdomains are overlapping, $H_0^1(\Omega) = H_0^1(\Omega_1) + H_0^1(\Omega_2)$ and thus $\phi = \phi_1 + \phi_2$ for some $\phi_j \in H_0^1(\Omega_j)$. An alternate decomposition is

$$\phi = (\phi_1^+ - \phi_2^-) + (\phi_2^+ - \phi_1^-) \equiv \chi_1 + \chi_2.$$

The support of ϕ_1^- and of ϕ_2^- must be a subset of $\overline{\Omega_1 \cap \Omega_2}$ since ϕ is non-negative. Thus $\chi_i \in H_0^1(\Omega_i),\ i = 1, 2$. We now show that they are non-negative.

If $\phi_2(x) < 0$ for some $x \in \Omega_1 \cap \Omega_2$, then $\phi_1(x) > 0$. Thus $0 \leq \phi(x) = \chi_1(x)$ while $\chi_2(x) = 0$. If $\phi_2(x) \geq 0$, then $\phi(x) = \phi_1^+(x) + (\phi_2^+(x) - \phi_1^-(x)) \geq 0$. We consider two cases. If $\phi_1(x) < 0$, then $\chi_1(x) = 0$ and $0 \leq \phi(x) = \chi_2(x)$. In the second case $\phi_1(x) \geq 0$, then $\chi_1(x) = \phi_1^+(x) \geq 0$ and $\chi_2(x) = \phi_2^+(x) \geq 0$. Hence the lemma holds for $m = 2$. The general case holds by induction. This completes the proof of the lemma.

THEOREM 4.3. *Suppose the system (3.1) is mixed quasi-monotone and let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be ordered subsolution and supersolution pairs. Consider any non-negative functions* $c, d \in C^\alpha(\overline{\Omega})$ *so that*

$$\frac{\partial f(u,v)}{\partial u} \geq -c, \quad \frac{\partial g(u,v)}{\partial v} \geq -d, \qquad (u,v) \in \mathcal{A}.$$

Suppose (4.2) holds. For $i = 1, \cdots, m$, *let* $\underline{u}^{(0)} = \underline{u}_i^{(0)} = \underline{u}$, $\overline{u}^{(0)} = \overline{u}_i^{(0)} = \overline{u}$, $\underline{v}^{(0)} = \underline{v}_i^{(0)} = \underline{v}$, *and* $\overline{v}^{(0)} = \overline{v}_i^{(0)} = \overline{v}$ *on* $\overline{\Omega}$ *with* $\underline{u} = \overline{u} = r$ *and* $\underline{v} = \overline{v} = s$ *on* $\partial\Omega$. *Define the Schwarz sequences, for* $i = 1, \cdots, m$ *and* $n \geq 1$

$$-\triangle \underline{u}_i^{(n)} + c\underline{u}_i^{(n)} = f(\underline{u}_i^{(n-1)}, \overline{v}_i^{(n-1)}) + c\underline{u}_i^{(n-1)} \ \text{on } \Omega_i, \qquad \underline{u}_i^{(n)} = \underline{u}^{(n-1)} \ \text{on } \partial\Omega_i,$$

$$-\triangle \overline{u}_i^{(n)} + c\overline{u}_i^{(n)} = f(\overline{u}_i^{(n-1)}, \underline{v}_i^{(n-1)}) + c\overline{u}_i^{(n-1)} \ \text{on } \Omega_i, \qquad \overline{u}_i^{(n)} = \overline{u}^{(n-1)} \ \text{on } \partial\Omega_i,$$

$$-\triangle \underline{v}_i^{(n)} + d\underline{v}_i^{(n)} = g(\underline{u}_i^{(n-1)}, \underline{v}_i^{(n-1)}) + d\underline{v}_i^{(n-1)} \ \text{on } \Omega_i, \qquad \underline{v}_i^{(n)} = \underline{v}^{(n-1)} \ \text{on } \partial\Omega_i,$$

$$-\triangle \overline{v}_i^{(n)} + d\overline{v}_i^{(n)} = g(\overline{u}_i^{(n-1)}, \overline{v}_i^{(n-1)}) + d\overline{v}_i^{(n-1)} \ \text{on } \Omega_i, \qquad \overline{v}_i^{(n)} = \overline{v}^{(n-1)} \ \text{on } \partial\Omega_i.$$

Here, $\underline{u}_i^{(n)}$ and $\overline{u}_i^{(n)}$ are defined as $\underline{u}^{(n-1)}$ and $\overline{u}^{(n-1)}$, respectively, on $\overline{\Omega} \setminus \overline{\Omega}_i$ and

$$\underline{u}^{(n)} = \max_{1 \le i \le m} \underline{u}_i^{(n)}, \qquad \overline{u}^{(n)} = \min_{1 \le i \le m} \overline{u}_i^{(n)} \text{ on } \overline{\Omega}.$$

The other functions $\underline{v}_i^{(n)}, \overline{v}_i^{(n)}, \underline{v}^{(n)}, \overline{v}^{(n)}$ are similarly defined. Then $\underline{u}_i^{(n)} \to \underline{u}_0$, $\overline{u}_i^{(n)} \to \overline{u}_0$, $\underline{v}_i^{(n)} \to \underline{v}_0$, and $\overline{v}_i^{(n)} \to \overline{v}_0$ in $C^2(\Omega_i)$, $i = 1, \cdots, m$, where $(\underline{u}_0, \underline{v}_0)$ and $(\overline{u}_0, \overline{v}_0)$ are solutions of (3.1) in $\mathcal{A}$.

Furthermore, if (u, v) is any solution in $\mathcal{A}$, then $\underline{u}_0 \le u \le \overline{u}_0$ and $\underline{v}_0 \le v \le \overline{v}_0$.

Proof: The current case is slightly more complicated than the previous two cases because the four pairs of Schwarz iterates are somehow related. However, the ideas and techniques of the proof are essentially the same. Hence we only give a sketch of the proof.

We first show that the sequences obey the following monotone properties on $\overline{\Omega}$,

$$\underline{u} \le \underline{u}_i^{(n)} \le \underline{u}_i^{(n+1)} \le \overline{u}_i^{(n+1)} \le \overline{u}_i^{(n)} \le \overline{u}, \qquad \underline{v} \le \underline{v}_i^{(n)} \le \underline{v}_i^{(n+1)} \le \overline{v}_i^{(n+1)} \le \overline{v}_i^{(n)} \le \overline{v}$$

and

$$\underline{u}^{(n)} \le \underline{u}_i^{(n+1)}, \quad \overline{u}^{(n)} \ge \overline{u}_i^{(n+1)}, \qquad \underline{v}^{(n)} \le \underline{v}_i^{(n+1)}, \quad \overline{v}^{(n)} \ge \overline{v}_i^{(n+1)}$$

for $i = 1, \cdots, m$ and

$$\underline{u}^{(n)} \le \underline{u}^{(n+1)} \le \overline{u}^{(n+1)} \le \overline{u}^{(n)}, \qquad \underline{v}^{(n)} \le \underline{v}^{(n+1)} \le \overline{v}^{(n+1)} \le \overline{v}^{(n)}.$$

Since the sequences are bounded, the following limits are well defined on $\overline{\Omega}$ for $i = 1, \cdots, m$

$$\lim_{n\to\infty} \underline{u}_i^{(n)} = \underline{u}_i, \quad \lim_{n\to\infty} \overline{u}_i^{(n)} = \overline{u}_i, \quad \lim_{n\to\infty} \underline{v}_i^{(n)} = \underline{v}_i, \quad \lim_{n\to\infty} \overline{v}_i^{(n)} = \overline{v}_i$$

and

$$\lim_{n\to\infty} \underline{u}^{(n)} = \underline{u}_0, \quad \lim_{n\to\infty} \overline{u}^{(n)} = \overline{u}_0, \quad \lim_{n\to\infty} \underline{v}^{(n)} = \underline{v}_0, \quad \lim_{n\to\infty} \overline{v}^{(n)} = \overline{v}_0.$$

In the second step, we prove using elliptic regularity theory that the limit functions satisfy the following PDEs on Ω_i, $i = 1, \cdots, m$:

$$-\triangle \underline{u}_i = f(\underline{u}_i, \overline{v}_i), \qquad -\triangle \overline{u}_i = f(\overline{u}_i, \underline{v}_i),$$

and

$$-\triangle \underline{v}_i = g(\underline{u}_i, \underline{v}_i), \qquad -\triangle \overline{v}_i = g(\overline{u}_i, \overline{v}_i).$$

Furthermore, convergence to these limits is in the sense of $C^2(\Omega_i)$. Next, we use the monotone properties to prove that $\underline{u}_i = \underline{u}_j = \underline{u}_0$, $\overline{u}_i = \overline{u}_j = \overline{u}_0$, $\underline{v}_i = \underline{v}_j = \underline{v}_0$ and $\overline{v}_i = \overline{v}_j = \overline{v}_0$ on $\overline{\Omega}$, $i, j = 1, \cdots, m$. Then we demonstrate that $(\underline{u}_0, \underline{v}_0)$ and $(\overline{u}_0, \overline{v}_0)$ are solutions of (3.1). In the final step, we show that any solution (u, v) in $\mathcal{A}$ of (3.1) must satisfy $\underline{u}_0 \le u \le \overline{u}_0$ and $\underline{v}_0 \le v \le \overline{v}_0$ on $\overline{\Omega}$.

Step one can be shown by induction as before. We show $\underline{u}^{(1)} \le \overline{u}^{(1)}$ as an illustration. Arguing as in (2.8) in Theorem 2.2, for any i so that $1 \le i \le m$,

$$\int_{\Omega_i} \nabla \underline{u}^{(1)} \cdot \nabla \phi_i + c\underline{u}^{(1)} \phi_i \le \int_{\Omega_i} \Big(f(\underline{u}, \overline{v}) + c\underline{u} \Big) \phi_i \tag{4.3}$$

and

$$\int_{\Omega_i} \nabla \overline{u}^{(1)} \cdot \nabla \phi_i + c\overline{u}^{(1)} \phi_i \ge \int_{\Omega_i} \Big(f(\overline{u}, \underline{v}) + c\overline{u} \Big) \phi_i,$$

where ϕ_i is a non-negative function in $H_0^1(\Omega_i)$. Subtract these equations to obtain

$$\begin{aligned}
\int_{\Omega_i} \nabla(\overline{u}^{(1)} - \underline{u}^{(1)}) \cdot \nabla \phi_i + c(\overline{u}^{(1)} - \underline{u}^{(1)}) \phi_i &\ge \int_{\Omega_i} \Big(f(\overline{u}, \underline{v}) - f(\underline{u}, \overline{v}) + c(\overline{u} - \underline{u}) \Big) \phi_i \\
&= \int_{\Omega_i} \Big((f_u(*) + c)(\overline{u} - \underline{u}) + f_v(*)(\underline{v} - \overline{v}) \Big) \phi_i \\
&\ge 0.
\end{aligned}$$

Now let ϕ be any non-negative function in $H_0^1(\Omega)$. By the above lemma, there are non-negative functions $\phi_j \in H_0^1(\Omega_j)$ such that $\phi = \phi_1 + \cdots + \phi_m$. Now

$$\begin{aligned}
\int_{\Omega} \nabla(\overline{u}^{(1)} - \underline{u}^{(1)}) \cdot \nabla \phi + c(\overline{u}^{(1)} - \underline{u}^{(1)}) \phi &= \sum_{i=1}^{m} \int_{\Omega_i} \nabla(\overline{u}^{(1)} - \underline{u}^{(1)}) \cdot \nabla \phi_i + c(\overline{u}^{(1)} - \underline{u}^{(1)}) \phi_i \\
&\ge 0.
\end{aligned}$$

Since $\overline{u}^{(1)} - \underline{u}^{(1)} = 0$ on $\partial\Omega$, we conclude that $\overline{u}^{(1)} \ge \underline{u}^{(1)}$ on $\overline{\Omega}$ by the generalized maximum principle.

We now provide the details for the penultimate step for the case $f_u > \lambda_1$ on $\mathcal{A}$. On Ω, multiply each of the equations

$$-\triangle \overline{u}_0 = f(\overline{u}_0, \underline{v}_0), \qquad -\triangle \underline{u}_0 = f(\underline{u}_0, \overline{v}_0)$$

by ψ_1 and then subtract followed by an integration over Ω to obtain

$$\int_{\partial\Omega} \frac{\partial \psi_1}{\partial n}(\overline{u}_0 - \underline{u}_0) + \lambda_1 \int_{\Omega} \psi_1 (\overline{u}_0 - \underline{u}_0) = \int_{\Omega} \left[\frac{\partial f(*)}{\partial u}(\overline{u}_0 - \underline{u}_0) + \frac{\partial f(*)}{\partial v}(\underline{v}_0 - \overline{v}_0) \right] \psi_1$$

or

$$0 \geq \int_{\Omega} \left(\frac{\partial f(*)}{\partial u} - \lambda_1 \right) (\overline{u}_0 - \underline{u}_0) \psi_1$$

which is possible only if $\underline{u}_0 \equiv \overline{u}_0$ on Ω. (Note that from the monotone property $\overline{u}^{(n)} \geq \underline{u}^{(n)}$, we obtain $\overline{u}_0 \geq \underline{u}_0$.) Hence on Ω,

$$-\triangle \underline{u}_0 = f(\underline{u}_0, \underline{v}_0), \qquad -\triangle \underline{v}_0 = g(\underline{u}_0, \underline{v}_0),$$

$$-\triangle \underline{u}_0 = f(\underline{u}_0, \overline{v}_0), \qquad -\triangle \overline{v}_0 = g(\underline{u}_0, \overline{v}_0),$$

which means that $(\underline{u}_0, \underline{v}_0)$ and $(\underline{u}_0, \overline{v}_0)$ are solutions to (3.1).

For the final step, first use induction to show $\underline{u}^{(n)} \leq u \leq \overline{u}^{(n)}$ and $\underline{v}^{(n)} \leq v \leq \overline{v}^{(n)}$ on $\overline{\Omega}$. Then $\underline{u}_0 \leq u \leq \overline{u}_0$ and $\underline{v}_0 \leq v \leq \overline{v}_0$ result after taking limits. As an illustration, we show $u \geq \underline{u}^{(1)}$. Use (4.3) in conjuction with the above lemma and the fact that (u, v) is a solution of (3.1) to get

$$\begin{aligned} \int_{\Omega} \nabla (u - \underline{u}^{(1)}) \cdot \nabla \phi + c(u - \underline{u}^{(1)})\phi &\geq \int_{\Omega} \Big((f_u(*) + c)(u - \underline{u}) + f_v(*)(v - \overline{v}) \Big) \phi \\ &\geq 0 \end{aligned}$$

for any non-negative $\phi \in H_0^1(\Omega)$. Since $u - \underline{u}^{(1)} = 0$ on $\partial\Omega$, an application of the generalized maximum principle yields $\underline{u}^{(1)} \leq u$ on $\overline{\Omega}$. This completes the sketch of the proof.

Note that a sufficient condition for a unique solution to the mixed quasi-monotone system (3.1) in $\mathcal{A}$ is that both conditions in (4.2) hold.

One example where a mixed quasi-monotone system occurs is the Lotka-Volterra predator-prey model

$$-\triangle u = u(a_1 - b_1 u - c_1 v), \qquad -\triangle v = v(a_2 + b_2 u - c_2 v).$$

Here u stands for the population of a prey while v denotes the population of a predator and all other variables are positive constants.

5. Numerical Results. We present some simple numerical experiments using the MATLAB PDE Toolbox. This toolbox provides a convenient environment to solve some classes of linear and nonlinear PDEs using finite elements. In the first example, the PDE is

$$-\triangle u = u(1-u) + 1$$

with u vanishing on the boundary of the domain which is the unit square. The domain is subdivided into four overlapping squares each of side length 0.6 and occupying a corner position of the original square. The subsolution is the zero function and $c = 3$. We used the additive Schwarz method, employing 1000 triangles in each subdomain. The plot of the the relative error versus Schwarz iteration is given in Figure 1a. Here, relative error is defined as

$$\frac{|u^{(n)} - u|_\infty}{|u|_\infty}$$

where u is the 'exact' solution computed using the nonlinear solver using a finer mesh.

In the second experiment, we solve the quasi-monotone non-increasing system

$$-\triangle u = u(1-u-v) + 1$$
$$-\triangle v = v(1-u-v) + xy$$

on the same domain as in the previous example with zero boundary conditions. Using $c = d = 5$ with all other specifications the same as above, we obtain the plot Figure 1b.

The third PDE is a singular perturbation problem

$$-.01\ \triangle u = f(x, y, u) = e^u[u - 10xy(1-y)]^2$$

with $u = 0$ on the boundary of the domain which is again the unit square. The solution has a boundary layer near $x = 1$. The two overlapping subdomains used are $[0, .8] \times [0, 1]$ and $[.75, 1] \times [0, 1]$. The number of triangles used are 2000 for each subdomain. With a finer mesh in the boundary layer, we are able to resolve the solution in this region. Using $c = 2$ and $\underline{u} = 0$, we obtain the plot Figure 1c. We remark that had we used Newton's method to solve this problem, we would have encountered an additional difficulty in solving the resulting linear equation

$$-.01\ \triangle\phi - f_u(u^{(n)})\phi = -f(u^{(n)}).$$

In the boundary layer, $f_u(u^{(n)})$ is positive and of order one. This is not unlike the high frequency Helmholtz problem for which standard iterative solvers will either not converge or converge extremely slowly.

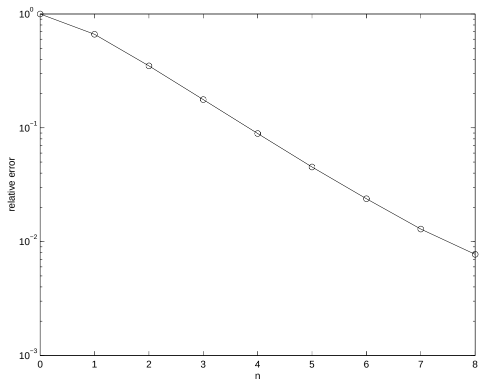
In the final example, we solve

$$-\triangle u = u(1-u) + g$$

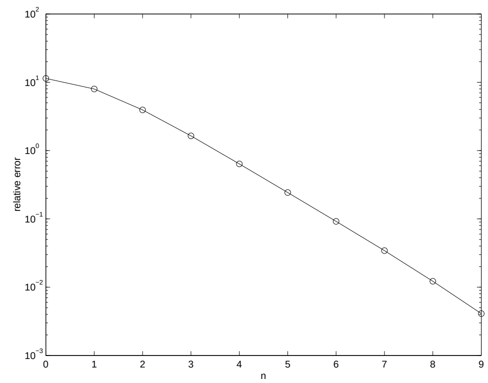
on the 3/4-disk of radius one with center at the origin and zero boundary conditions. The function g depends only on the spatial coordinates and is chosen so that the exact solution of this problem in polar coordinates is

$$u(r, \theta) = (r^{2/3} - r^2)\sin\frac{2\theta}{3}.$$

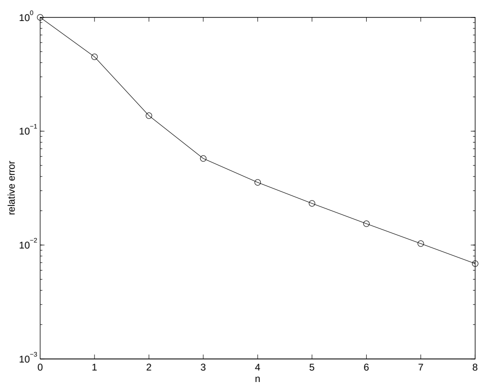
Note that the solution is singular at the origin. The two overlapping subdomains are $0 \leq r \leq .4,\ 0 \leq \theta \leq 3\pi/2$ (with 500 triangles) and $.2 \leq r \leq 1,\ 0 \leq \theta \leq 3\pi/2$ (with 4000 triangles). The value of c is 3. We assume that the nature of the singularity at the origin is known and thus for the first subdomain, we take the unknown function as $v = u - r^{2/3}\sin\dfrac{2\theta}{3}$, which is a smooth function. See Figure 1d for the error



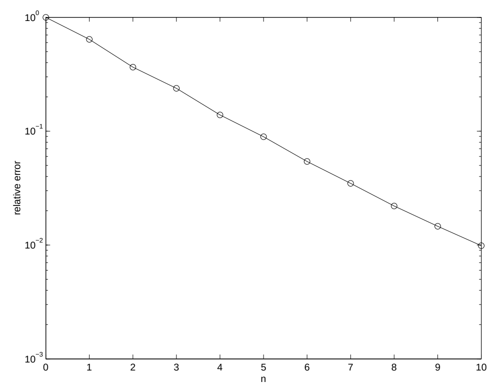
(a) scalar



(b) system



(c) singular perturbation



(d) 3/4-disk

plot. In this rather contrived example, we illustrate that in problems where there is a geometric singularity, if the type of singularity can be determined by, for example, asymptotic analysis, then we can subtract out this singularity for the problem in the subdomain which contains the singularity. Other subdomain problems do not need to know about this at all.

6. Conclusion. In this paper, we have shown convergence of some Schwarz methods for nonlinear PDEs whose solutions can be demonstrated by the monotone method. Our results include a parallel additive Schwarz method for a domain which is decomposed into finitely many subdomains. Both scalar and coupled systems can be handled. For the latter, subdomain problems are linear, independent and decoupled in each iteration. Even in the presence of multiple solutions, the Schwarz iteration converges to a specified solution, namely the maximal or minimal solution depending on whether the starting iterate is a supersolution or a subsolution.

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REFERENCES

[1] H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces. *SIAM Rev.*, 18:620–709, 1976.

[2] L. Badea. On the schwarz alternating method with more than two subdomains for nonlinear monotone problems. *SIAM J. Numer. Anal.*, 28:179–204, 1991.

[3] X. C. Cai and M. Dryja. Domain decomposition methods for monotone nonlinear elliptic problems. In D. Keyes and J. Xu, editors, *Domain decomposition methods in scientific and engineering computing*, pages 335–360, Providence, R.I., 1994. AMS.

[4] X. C. Cai and D. E. Keyes. Nonlinearly preconditioned inexact newton algorithms. *preprint*, 2000.

[5] T. F. Chan and T. P. Mathew. Domain decomposition algorithms. *Acta Numerica*, pages 61–143, 1994.

[6] G. Chen, Y. Deng, W. M. Ni, and J. Zhou. Boundary element monotone iteration scheme for semilinear elliptic partial differential equations. *Math. Comp.*, 65:943–982, 1996.

[7] G. Chen, Y. Deng, W. M. Ni, and J. Zhou. Boundary element monotone iteration scheme for semilinear elliptic partial differential equations part ii. *Math. Comp.*, 69:629–652, 2000.

[8] M. Dryja and W. Hackbusch. On the nonlinear domain decomposition method. *BIT*, pages 296–311, 1997.

[9] M. Dryja and O. B. Widlund. An additive variant of the Schwarz alternating method for the case of many subregions. Technical report, Courant Institute Technical Report 339, 1987.

[10] R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux. *First Int. Symp. on Domain Decomposition Methods*. SIAM, Philadelphia, 1988.

[11] C. Gui and Y. Lou. Uniqueness and nonuniqueness of coexistence states in the lotka-volterra competition model. *CPAM*, 47:1571–1594, 1994.

[12] D. K. Kaushik, D. E. Keyes, and B. F. Smith. On the interaction of architecture and algorithm in the domain-based parallelization of an unstructured-grid incompressible flow code. In J. Mandel, C. Farhat, and X. C. Cai, editors, *Domain Decomposition Methods 10*, pages 287–295, Providence, 1998. AMS.

[13] H.B. Keller and D.S. Cohen. Some positone problems suggested by nonlinear heat generation. *J. Math. Mech.*, 16(12):1361–1376, 1967.

[14] P. L. Lions. On the Schwarz alternating method I. In R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, editors, *First Int. Symp. on Domain Decomposition Methods*, pages 1–42, Philadelphia, 1988. SIAM.

[15] P. L. Lions. On the Schwarz alternating method II. In T. F. Chan, R. Glowinski, J. Periaux, and O. Widlund, editors, *Second Int. Conference on Domain Decomposition Methods*, pages 47–70, Philadelphia, 1989. SIAM.

[16] S. H. Lui. On Schwarz alternating methods for nonlinear elliptic pdes. *SIAM J. Sci. Comput.*, 21:1506–1523, 2000.

[17] S. H. Lui. On monotone and Schwarz alternating methods for nonlinear elliptic pdes. *M2AN*, 2001.

[18] S. H. Lui. On Schwarz alternating methods for the incompressible Navier-Stokes equations. *SIAM J. Sci. Comput.*, To Appear.

[19] C. V. Pao. *Nonlinear Parabolic and Elliptic Equations*. Plenum Press, New York, 1992.

[20] C. V. Pao. Block monotone iterative methods for numerical solutions of nonlinear elliptic equations. *Numer. Math.*, 72:239–262, 1995.

[21] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford University Press, Oxford, 1999.

[22] D. H. Sattinger. Monotone methods in nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.*, 21:979–1000, 1972.

[23] B. F. Smith, P. Bjorstad, and W. D. Gropp. *Domain Decomposition : Parallel Multilevel Algorithms for Elliptic Partial Differential Equations.* Cambridge University Press, New York, 1996.

[24] X. C. Tai. Domain decomposition for linear and nonlinear elliptic problems via function or space decomposition. In D. Keyes and J. Xu, editors, *Domain decomposition methods in scientific and engineering computing*, pages 335–360, Providence, R.I., 1994. AMS.

[25] X. C. Tai and M. Espedal. Rate of convergence of some space decomposition methods for linear and nonlinear problems. *SIAM J. Numer. Anal.*, 35:1558–1570, 1998.

[26] X. C. Tai and J. Xu. Global convergence of subspace correction methods for convex optimization problems. *Math. Comp.*, To appear.

[27] P. Le Tallec. Domain decomposition methods in computational mechanics. *Computational Mechanics Advances*, 1:121–220, 1994.

[28] J. Xu. Two-grid discretization techniques for linear and nonlinear PDEs. *SIAM J. Numer. Anal.*, 33:1759–1777, 1996.

[29] J. Xu and J. Zou. Some nonoverlapping domain decomposition methods. *SIAM Rev.*, 40:857–914, 1998.

[30] J. Zou and H.-C. Huang. Algebraic subproblem decomposition methods and parallel algorithms with monotone convergence. *J. Comput. Math.*, 10:47–59, 1992.