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ON SCHWARZ ALTERNATING METHODS FOR NONLINEAR ELLIPTIC PDES*

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Abstract. The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomains.

This paper considers several Schwarz alternating methods for nonlinear elliptic problems. We show that Schwarz alternating methods can be embedded in the framework of common techniques such as Banach and Schauder fixed point methods and global inversion methods used to solve these nonlinear problems.

Key words. domain decomposition, nonlinear elliptic PDE, Schwarz alternating method

AMS subject classifications. 65N55, 65J15

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1. Introduction.

The Schwarz alternating method was devised by H. A. Schwarz more than 100 years ago to solve linear boundary value problems. It has garnered interest recently because of its potential as an efficient algorithm for parallel computers. See the fundamental work of Lions in [11] and [12]. The literature on this method for the linear boundary value problem is huge; see the recent reviews of Chan and Mathew [6] and Le Tallec [19] and the book of Smith, Bjorstad, and Gropp [14]. The literature for nonlinear problems is rather sparse. Besides Lions's works, see also Cai and Dryja [3], Tai [15], Xu [20], Dryja and Hackbusch [7], Tai and Espedal [16], Tai and Xu [18], Tai and Tseng [17], and references therein. The effectiveness of Schwarz methods for nonlinear problems has been demonstrated in many papers. See the proceedings of the annual domain decomposition conferences beginning with [10]. We mention in particular the Newton–Krylov–Schwarz framework adopted in [4] and [5]. In this paper, we prove the convergence of the Schwarz sequence for some nonlinear elliptic PDEs. We do not attempt to define the largest possible class of problems or give the weakest condition under which the Schwarz alternating method converges. The main aim is rather to illustrate that this remarkable method works for a wide variety of nonlinear elliptic PDEs.

This paper is mostly concerned with multiplicative nonlinear Schwarz methods for two subdomains and they are discussed in the next section. In this class of methods, a nonlinear problem is solved in the first subdomain followed by a nonlinear problem in the second subdomain. This is repeated until convergence to a desired accuracy has been reached. We shall consider three types of nonlinear elliptic PDEs corresponding to three theories used to show existence of solutions to these PDEs. The three theories are the Banach fixed point theory, the Schauder fixed point theory, and the theory of global inversion. They are among the most well-known tools in nonlinear analysis. Nonlinear PDEs are difficult to handle because of the infinite variety of nonlinearities

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and the possibility of an arbitrary number (including zero) of solutions. In addition, existence of a solution on the entire domain does not guarantee that the same PDE on a subdomain with a general boundary condition has a solution. There will probably be no single technique that can show existence/uniqueness for all nonlinear PDEs. It is indeed remarkable that the Schwarz method is applicable to all the different types of PDEs mentioned above.

In section 3, we shall discuss three practical variants of Schwarz methods. The first variant considers a sequence of functions resulting from the solution of linear versions of the given PDE while the second produces an "additive" Schwarz sequence which is suitable for parallel computation. The third variant is yet another Schwarz sequence where the subdomain problems can be computed in parallel. These three variants are applied to PDEs of the first type, that is, those whose solution is a fixed point of a contracting operator.

The first Schwarz method for nonlinear problems is due to Lions [11]. He considers a functional $I \in C^1(H_0^1(\Omega), \mathbb{R})$ which is coercive, weakly lower semicontinuous, uniformly convex, and bounded below. By making a correction alternately in each subdomain which minimizes the functional, he shows that the sequence converges to the unique minimizer of the functional. Dryja and Hackbusch [7] study convergence of nonlinear subspace iterations for abstract nonlinear equations. They show that under weak assumptions, the nonlinear iteration converges locally with the same asymptotic speed as the corresponding linear iteration applied to the linearized problem. The paper of Tai and Espedal [16] considers monotone operators and proves the convergence of additive and multiplicative Schwarz sequences. Xu [20] gives convergence estimates for multigrid methods for nonlinear elliptic PDEs discretized by finite elements.

We conclude this introduction with some notation. Let Ω be a bounded, connected domain in $\mathbb{R}^N$ with a smooth boundary. Suppose $\Omega = \Omega_1 \cup \Omega_2$, where the subdomains Ω_i are connected, have smooth boundaries, and are overlapping. Let (u, v) denote the usual $L^2(\Omega)$ inner product and $\|u\|^2 = (u, u)$. Denote the energy inner product in the Sobolev space $H_0^1(\Omega)$ by $[u, v] = \int_\Omega \nabla u \cdot \nabla v$ and let $\|u\|_1 = [u, u]^{1/2}$. Denote the norm on $H^{-1}(\Omega)$ by $\|\cdot\|_{-1}$ with

$$\|u\|_{-1} = \sup_{v \in H_0^1(\Omega),\ \|v\|_1 = 1} |[u, v]|.$$

We also use $(.,.)$ to denote a duality pairing. Let $\triangle_i$ be the Laplacian operator considered as an operator from $H_0^1(\Omega_i)$ onto $H^{-1}(\Omega_i)$, $i = 1, 2$. The smallest eigenvalue of $-\triangle$ on Ω is denoted by λ_1 while the smallest eigenvalue of $-\triangle_i$ is denoted by $\lambda_1(\Omega_i)$, $i = 1, 2$. The collection of eigenvalues on Ω is denoted by $\{\lambda_j\}_{j=1}^{\infty}$. For notational convenience, we define $\lambda_0 = -\infty$. We take overlapping to mean that $H_0^1(\Omega) = H_0^1(\Omega_1) + H_0^1(\Omega_2)$. In this paper, a function in $H_0^1(\Omega_i)$ is considered as a function defined on the whole domain by extension by zero. Let P_i denote the orthogonal (with respect to the energy inner product) projection onto $H_0^1(\Omega_i)$, $i = 1, 2$. It is well known that

$$d \equiv \max(\|(I - P_2)(I - P_1)\|_1, \|(I - P_1)(I - P_2)\|_1) < 1.$$

See Lions [11] and Bramble et al. [2]. Throughout this paper, C will denote a positive constant which may not be the same in different occurrences.

2. Nonlinear Schwarz method. In this section, we use the Schwarz method in conjunction with the methods of Banach and Schauder fixed points and of global inversion to solve some nonlinear PDEs. Each subdomain problem is nonlinear.

2.1. Banach fixed point. The first result is an adaptation of the variational approach of Lions [11] for linear problems to nonlinear problems. We assume the nonlinearity satisfies a certain Lipschitz condition with a sufficiently small Lipschitz constant so that the method of proof for the linear problem still applies. We first prove an existence and uniqueness result using the Banach fixed point theorem.

PROPOSITION 1. *Consider the equation*

$$-\triangle u = f(x,u,\nabla u) \text{ on } \Omega$$

with homogeneous Dirichlet boundary conditions. Assume $f(x,u(x),\nabla u(x)) \in H^{-1}(\Omega)$ *for* $x \in \Omega$ *and* $u \in H_0^1(\Omega)$ *and satisfies the condition*

$$\|f(x,u,\nabla u) - f(x,v,\nabla v)\|_{-1} \le c\,\|u-v\|_1$$

for $u,v \in H_0^1(\Omega)$ *and some constant* $c<1$*. Then, the equation has a unique solution in* $H_0^1(\Omega)$*.*

Proof. Define the operator $A : H_0^1(\Omega) \to H_0^1(\Omega)$ by $A(u) = -\triangle^{-1} f(x,u,\nabla u)$. Then the equation is solved by finding a fixed point of A. For $u,v \in H_0^1(\Omega)$,

$$\begin{aligned}\|A(u)-A(v)\|_1 &= \|\triangle^{-1}(f(x,u,\nabla u) - f(x,v,\nabla v))\|_1 \\ &= \|f(x,u,\nabla u) - f(x,v,\nabla v)\|_{-1} \\ &\le c\,\|u-v\|_1.\end{aligned}$$

By the Banach fixed point theorem, A has a unique fixed point in $H_0^1(\Omega)$. ∎

We remark that the Lipschitz constant above is optimal. For example, the equation

$$-\triangle u = \lambda_1 u + g,$$

where g is not orthogonal in the L^2 sense to an eigenfunction associated with λ_1, has no solution.

THEOREM 1. *Consider the equation*

$$-\triangle u = f(x,u,\nabla u) \text{ on } \Omega \tag{1}$$

with homogeneous Dirichlet boundary conditions. Assume $f(x,u(x),\nabla u(x)) \in H^{-1}(\Omega)$ *for* $x \in \Omega$ *and* $u \in H_0^1(\Omega)$ *and satisfies the condition*

$$\|f(x,u,\nabla u) - f(x,v,\nabla v)\|_{-1} \le c\,\|u-v\|_1,$$

where $u,v \in H_0^1(\Omega)$*,* c *is a constant such that* $c<1$*, and*

$$d < \sqrt{1-c^2} - c. \tag{2}$$

For $n = 0,1,2,\ldots$ *and some* $u^{(0)} \in H_0^1(\Omega)$*, define the nonlinear Schwarz sequence as*

$$-\triangle u^{(n+\frac12)} = f(x,u^{(n+\frac12)},\nabla u^{(n+\frac12)}) \text{ on } \Omega_1, \quad u^{(n+\frac12)} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = f(x,u^{(n+1)},\nabla u^{(n+1)}) \text{ on } \Omega_2, \quad u^{(n+1)} = u^{(n+\frac12)} \text{ on } \partial\Omega_2.$$

Then, the Schwarz sequence converges geometrically to the solution of (1) *in the energy norm. Here,* $u^{(n+\frac12)}$ *is considered as a function in* $H_0^1(\Omega)$ *by defining it to be* $u^{(n)}$ *on* $\Omega\setminus\Omega_1$ *and* $u^{(n+1)}$ *is defined as* $u^{(n+\frac12)}$ *on* $\Omega\setminus\Omega_2$*.*

Proof. By Proposition 1, (1) has a unique solution u.

Next, we show that the Schwarz sequence is well defined. Let $u^{(n+\frac{1}{2})} = u^{(n)} + v_1$, where $v_1 \in H_0^1(\Omega_1)$. The defining equation for $u^{(n+\frac{1}{2})}$ can be written as

$$-\triangle v_1 = \triangle u^{(n)} + h(x, v_1, \nabla v_1) \quad \text{on } \Omega_1, \tag{3}$$

where $h(x, v_1, \nabla v_1) = f(x, u^{(n)} + v_1, \nabla u^{(n)} + \nabla v_1)$. Assume $u^{(n)} \in H_0^1(\Omega)$. For $v, w \in H_0^1(\Omega_1)$,

$$\begin{aligned}
&\|h(x, v, \nabla v) - h(x, w, \nabla w)\|_{H^{-1}(\Omega_1)} \\
&\qquad = \|f(x, u^{(n)} + v, \nabla u^{(n)} + \nabla v) - f(x, u^{(n)} + w, \nabla u^{(n)} + \nabla w)\|_{H^{-1}(\Omega_1)} \\
&\qquad \le c\,\|v - w\|_1.
\end{aligned}$$

By Proposition 1 (applied to Ω_1), (3) has a unique solution in $H_0^1(\Omega_1)$ and thus $u^{(n+\frac{1}{2})}$ exists and is unique. This holds similarly for $u^{(n+1)}$.

We are now ready to show convergence of the Schwarz sequence. For any $v_1 \in H_0^1(\Omega_1)$,

$$\begin{aligned}
[u^{(n+\frac{1}{2})} - u^{(n)}, v_1] &= (f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}), v_1) - [u^{(n)}, v_1] \\
&= (f(x, u, \nabla u), v_1) \\
&\quad + \Big(f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) - f(x, u, \nabla u), v_1\Big) - [u^{(n)}, v_1] \\
&= [u, v_1] + \Big(f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) - f(x, u, \nabla u), v_1\Big) - [u^{(n)}, v_1] \\
&= [P_1(u - u^{(n)}), v_1] + \Big(f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) - f(x, u, \nabla u), v_1\Big).
\end{aligned}$$

Noting that $u^{(n+\frac{1}{2})} - u^{(n)} \in H_0^1(\Omega_1)$, the last equation can also be written as

$$u^{(n+\frac{1}{2})} - u^{(n)} = P_1(u - u^{(n)}) - \triangle_1^{-1}\Big(f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) - f(x, u, \nabla u)\Big)$$

or

$$e^{(n+\frac{1}{2})} = (I - P_1)e^{(n)} - \triangle_1^{-1}\Big(f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) - f(x, u, \nabla u)\Big), \tag{4}$$

where $e^{(n)} = u^{(n)} - u$. For $g \in H^{-1}(\Omega)$, Δ_1^{-1} is shorthand for $\Delta_1^{-1}R_1 g$, where $R_1 = H^{-1}(\Omega) \to H^{-1}(\Omega_1)$ is defined by $(R_1 g, \phi) = (g, \phi)\ \ \forall \phi \in H_0^1(\Omega_1)$. Similarly,

$$e^{(n+1)} = (I - P_2)e^{(n+\frac{1}{2})} - \triangle_2^{-1}\left(f(x, u^{(n+1)}, \nabla u^{(n+1)}) - f(x, u, \nabla u)\right).$$

We have

$$\begin{aligned}
e^{(n+1)} &= (I - P_2)(I - P_1)e^{(n)} - (I - P_2)\triangle_1^{-1}\Big(f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) - f(x, u, \nabla u)\Big) \\
&\quad - \triangle_2^{-1}\left(f(x, u^{(n+1)}, \nabla u^{(n+1)}) - f(x, u, \nabla u)\right).
\end{aligned} \tag{5}$$

From (5), we obtain

$$\|P_2 e^{(n+1)}\|_1 = \|\triangle_2^{-1}\left(f(x, u^{(n+1)}, \nabla u^{(n+1)}) - f(x, u, \nabla u)\right)\|_1 \le c\|e^{(n+1)}\|_1.$$

Noting that $(I - P_1)e^{(n)} = (I - P_1)e^{(n+\frac{1}{2})}$ from (4), we get, upon applying $(I - P_2)$ to (5),

$$\|(I - P_2)e^{(n+1)}\|_1 \le (\|(I - P_2)(I - P_1)\|_1 + c)\,\|e^{(n+\frac{1}{2})}\|_1.$$

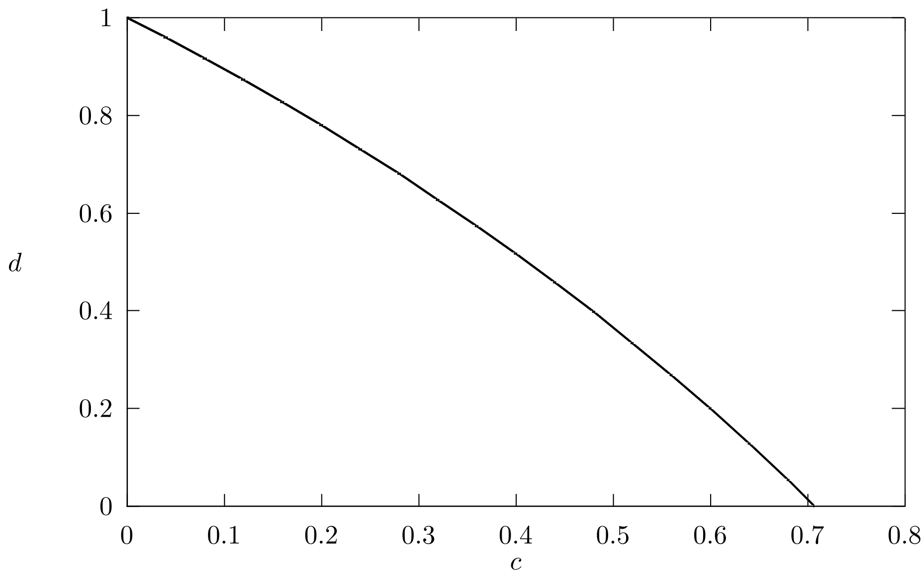

FIG. 1. *The boundary of the region given by* (2).

Thus

$$\begin{aligned}\|e^{(n+1)}\|_1^2 &= \|P_2 e^{(n+1)}\|_1^2 + \|(I-P_2)e^{(n+1)}\|_1^2 \\ &\le c^2\|e^{(n+1)}\|_1^2 + \left(\|(I-P_2)(I-P_1)\|_1 + c\right)^2 \|e^{(n+\frac{1}{2})}\|_1^2\end{aligned}$$

or

$$\|e^{(n+1)}\|_1 \le p\|e^{(n+\frac{1}{2})}\|_1,$$

where

$$p = \frac{c+d}{\sqrt{1-c^2}}.$$

An expression similar to (5) is

$$\begin{aligned}e^{(n+\frac{1}{2})} = (I-P_1)(I-P_2)e^{(n-\frac{1}{2})} - (I-P_1)\triangle_2^{-1}\left(f(x,u^{(n)},\nabla u^{(n)}) - f(x,u,\nabla u)\right) \\ -\triangle_1^{-1}\left(f(x,u^{(n+\frac{1}{2})},\nabla u^{(n+\frac{1}{2})}) - f(x,u,\nabla u)\right).\end{aligned}$$

From this we obtain

$$\|e^{(n+\frac{1}{2})}\|_1 \le p\|e^{(n)}\|_1$$

and hence

$$\|e^{(n+1)}\|_1 \le p^2\|e^{(n)}\|_1.$$

Thus the Schwarz sequence converges geometrically if $p < 1$ or, equivalently, $d < \sqrt{1-c^2} - c$. $\square$

In Figure 1, the region bounded by the curve and the coordinate axes denotes the region in the (c,d)-plane in which the Schwarz sequence converges geometrically. It is an open problem whether the Schwarz sequence converges geometrically with just the condition $c < 1$.

2.2. Schauder fixed point. Next, we give a similar result for an equation whose solution is shown to exist by the Schauder–Schaeffer fixed point theorem (see Evans [9], for example). The idea here is that if the nonlinear term can be controlled by the gradient of the solution, then an a priori bound on the solution can be obtained. While the Schauder fixed point theorem guarantees the existence of a solution, we also need a maximum principle (see Evans [9, p. 327–333]) to guarantee uniqueness.

MAXIMUM PRINCIPLE. *Suppose* $w \in C^2(\Omega) \cap C(\overline{\Omega})$ *and* $Lw = -\triangle w + b \cdot \nabla w + cw$ *where* b *and* c *are continuous. Then*

(1) *If* $c \equiv 0$ *and* $Lw \leq 0$ *on* Ω*, then* $\max_{\overline{\Omega}} w = \max_{\partial\Omega} w$*.*

(2) *If* $c \geq 0$ *and* $Lw = 0$ *on* Ω*, then* $\max_{\overline{\Omega}} |w| = \max_{\partial\Omega} |w|$*.*

(3) *If* $c \geq 0$ *and* $Lw \leq 0$ *on* Ω *and* w *attains a nonnegative maximum over* $\overline{\Omega}$ *at an interior point, then* w *is constant on* Ω*.*

The third statement is also known as the strong maximum principle. We now recall an existence theorem, a similar version of which can be found in Nirenberg [13]. We give a proof because it is short and ideas from the proof are useful later on.

PROPOSITION 2. *Consider the equation*

$$-\triangle u = f(x, u, \nabla u) \text{ on } \Omega \tag{6}$$

with homogeneous Dirichlet boundary conditions. Assume that $f \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$ *and for every* $x \in \Omega,\ v \in H_0^1(\Omega),\ \partial f(x, v(x), \nabla v(x))/\partial u \leq 0$*, and* $|f(x, v(x), \nabla v(x))| \leq C(1 + |\nabla v(x)|^\gamma)$*, where* C, γ *are positive constants with* $\gamma < 1$*. This elliptic equation has a unique classical solution.*

Proof. Define $A : H_0^1(\Omega) \to H_0^1(\Omega)$ by $A(u) = -\triangle^{-1} f(x, u, \nabla u)$. The problem reduces to showing that A has a fixed point. This is accomplished by using the Schauder–Schaeffer fixed point theorem. We are required to show that A is compact and continuous and the set $S = \{u \in H_0^1(\Omega), u = \lambda A(u) \text{ for some } 0 \leq \lambda \leq 1\}$ is bounded.

To show compactness, let $\|u_n\|_1 = 1$. We show that $A(u_n)$ has a convergent subsequence. By the growth condition on f, $\|f(x, u_n, \nabla u_n)\| \leq C(1 + \|u_n\|_1^\gamma) \leq C$ and hence $\{\|A(u_n)\|_{H^2}\}$ is bounded. Thus $A(u_n)$ has a convergent subsequence in $H_0^1(\Omega)$.

The continuity of A follows from the continuity and growth condition of f and that $\triangle^{-1}$ is a bounded linear operator. (Let $u_n \to u$ in $H_0^1(\Omega)$. Then $u_{n_i} \to u$ almost everywhere (a.e.) and $\nabla u_{n_i} \to \nabla u$ a.e. for some subsequence labeled by n_i. By the continuity of $f, f(x, u_{n_i}, \nabla u_{n_i}) \to f(x, u, \nabla u)$ a.e. By the growth condition on f, we can use the Lebesgue dominated convergence theorem to conclude that $\|f(x, u_{n_i}, \nabla u_{n_i}) - f(x, u, \nabla u)\| \to 0$. Hence $A(u_{n_i}) \to A(u)$ in H^1 and thus $A(u_n) \to A(u)$.) For any $u \in S$, again by the growth condition on f, we have $\|u\|_1 \leq C(1 + \|u\|_1^\gamma)$. Hence S must be bounded.

To show uniqueness, let u, v be two solutions. Assuming that $\partial\Omega$ is smooth, then u, v are twice continuously differentiable functions by elliptic regularity. Let $w = u - v \in H_0^1(\Omega) \cap C^2(\Omega) \cap C(\overline{\Omega})$. Now $-\triangle(u - v) = f(x, u, \nabla u) - f(x, v, \nabla v)$ or

$$Lw \equiv -\triangle w - \frac{\partial f}{\partial \nabla u}(x, z, \nabla z) \cdot \nabla w - \frac{\partial f}{\partial u}(x, z, \nabla z) w = 0,$$

where z lies on a line in between u and v. By the maximum principle,

$$\max_{\overline{\Omega}} |w| = \max_{\partial\Omega} |w| = 0,$$

which means that $w \equiv 0$. ∎

THEOREM 2. *Consider the equation*

$$-\triangle u = f(x, u, \nabla u) \ \text{on } \Omega \tag{7}$$

with homogeneous Dirichlet boundary conditions. Assume that $f \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$ *and for every* $x \in \Omega$, $v \in H_0^1(\Omega)$, $\partial f(x, v(x), \nabla v(x))/\partial u \le 0$, *and* $|f(x, v(x), \nabla v(x))| \le C(1 + |\nabla v(x)|^\gamma)$, *where* C, γ *are positive constants with* $\gamma < 1$. *Assume* $u^{(0)} \in H_0^1(\Omega)$. *For* $n = 0, 1, 2, \ldots,$ *define the Schwarz sequence as*

$$-\triangle u^{(n+\frac{1}{2})} = f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})}) \ \text{on } \Omega_1, \quad u^{(n+\frac{1}{2})} = u^{(n)} \ \text{on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = f(x, u^{(n+1)}, \nabla u^{(n+1)}) \ \text{on } \Omega_2, \quad u^{(n+1)} = u^{(n+\frac{1}{2})} \ \text{on } \partial\Omega_2.$$

Then, the Schwarz sequence converges geometrically to the solution of (7) *in the* L^∞ *norm.*

Proof. By the above theorem, (7) has a unique classical solution u.

Since f satisfies a certain growth condition and $\partial f/\partial u \le 0$ on Ω, these properties still hold on each subdomain. Hence by the same fixed point theorem, $u^{(n+\frac{1}{2})}$ and $u^{(n+1)}$ exist and are unique.

In the first part of the proof, we show that the Schwarz sequence is bounded. For any $v_1 \in H_0^1(\Omega_1)$,

$$[u^{(n+\frac{1}{2})}, v_1] = (f^{(n+\frac{1}{2})}, v_1),$$

where $f^{(n+\frac{1}{2})} = f(x, u^{(n+\frac{1}{2})}, \nabla u^{(n+\frac{1}{2})})$ and hence

$$u^{(n+\frac{1}{2})} = -\triangle_1^{-1}(f^{(n+\frac{1}{2})}) + (I - P_1)u^{(n)}. \tag{8}$$

Note $P_1 u^{(n+\frac{1}{2})} = -\triangle_1^{-1}(f^{(n+\frac{1}{2})})$ and $(I - P_1)u^{(n+\frac{1}{2})} = (I - P_1)u^{(n)}$. Similarly,

$$u^{(n+1)} = -\triangle_2^{-1}(f^{(n+1)}) + (I - P_2)u^{(n+\frac{1}{2})}. \tag{9}$$

Suppose $\{\|u^{(n+\frac{1}{2})}\|_1\}$ is unbounded. Then there is some subsequence which we label by $n_j + \frac{1}{2}$ such that $\|u^{(n_j+\frac{1}{2})}\|_1 \to \infty$ as $j \to \infty$. Now

$$\begin{aligned}\|u^{(n_j+\frac{1}{2})}\|_1^2 &= \|P_1 u^{(n_j+\frac{1}{2})}\|_1^2 + \|(I - P_1)u^{(n_j+\frac{1}{2})}\|_1^2 \\ &\le C\left(1 + \|u^{(n_j+\frac{1}{2})}\|_1^\gamma\right)^2 + \|(I - P_1)u^{(n_j)}\|_1^2\end{aligned}$$

or

$$1 \le o(1) + \frac{\|(I - P_1)u^{(n_j)}\|_1^2}{\|u^{(n_j+\frac{1}{2})}\|_1^2}.$$

Hence, $\{\|u^{(n_j)}\|_1\}$ goes to infinity as $j \to \infty$. By a technical lemma (to be stated and proved after this proof), the entire sequence (not just a subsequence) must go to infinity:

$$\lim_{n\to\infty} \|u^{(n+\frac{1}{2})}\|_1 = \infty, \qquad \lim_{n\to\infty} \|u^{(n)}\|_1 = \infty.$$

Combining (8) and (9), we obtain

$$u^{(n+1)} = -\triangle_2^{-1}(f^{(n+1)}) + (I - P_2)(I - P_1)u^{(n)} - (I - P_2)\triangle_1^{-1}(f^{(n+\frac{1}{2})}) \text{ and}$$
$$u^{(n+\frac{1}{2})} = -\triangle_1^{-1}(f^{(n+\frac{1}{2})}) + (I - P_1)(I - P_2)u^{(n-\frac{1}{2})} - (I - P_1)\triangle_2^{-1}(f^{(n)}).$$

We can obtain the estimates

$$\|u^{(n+1)}\|_1 \leq o(1)\|u^{(n+1)}\|_1 + (d + o(1))\|u^{(n+\frac{1}{2})}\|_1$$
$$\|u^{(n+\frac{1}{2})}\|_1 \leq o(1)\|u^{(n+\frac{1}{2})}\|_1 + (d + o(1))\|u^{(n)}\|_1.$$

Consequently,

$$\|u^{(n+\frac{1}{2})}\|_1 \leq (d^2 + o(1))\|u^{(n-\frac{1}{2})}\|_1,$$

contradicting the assumption that $\{\|u^{(n+\frac{1}{2})}\|_1\}$ is unbounded. Hence we conclude that $\{\|u^{(n+\frac{1}{2})}\|_1\}$ is bounded.

Now if $\{\|u^{(n)}\|_1\}$ is unbounded, then

$$\|u^{(n+1)}\|_1 \leq \|P_2u^{(n+1)}\|_1 + \|(I-P_2)u^{(n+1)}\|_1 \leq C(1+\|u^{(n+1)}\|_1^{\gamma}) + \|(I-P_2)u^{(n+\frac{1}{2})}\|_1.$$

Hence,

$$1 \leq o(1) + \frac{\|(I - P_2)u^{(n+\frac{1}{2})}\|_1}{\|u^{(n+1)}\|_1},$$

which is a contradiction since $\{\|u^{(n+\frac{1}{2})}\|_1\}$ is bounded. This shows that $\{\|u^{(n)}\|_1\}$ must also be bounded.

Since the Schwarz sequences are bounded in $\|\cdot\|_1$, there exist $u_0, u_1, u_2 \in H_0^1(\Omega)$ and a subsequence labeled by n_j such that $u^{(n_j)} \rightharpoonup u_0$, $u^{(n_j+\frac{1}{2})} \rightharpoonup u_1$, and $u^{(n_j+1)} \rightharpoonup u_2$ (weak convergence in the energy norm). We now show that the subsequences actually converge strongly in the energy norm. By compactness of the restriction map $H^1(\Omega_1) \hookrightarrow L^2(\partial\Omega_1)$, we have $u^{(n_j+\frac{1}{2})} \to u_1$ in $L^2(\partial\Omega_1)$. Since $|f^{(n_j+\frac{1}{2})}| \leq C$, $\{u^{(n_j+\frac{1}{2})}\}$ is bounded in $H^2(\Omega_1)$ from the defining equation of $u^{(n_j+\frac{1}{2})}$. By extracting a further subsequence if necessary, $u^{(n_j+\frac{1}{2})}$ converges to u_1 strongly in $H^1(\Omega_1)$. Similarly, $u^{(n_j)} \to u_0$ and $u^{(n_j+1)} \to u_2$ strongly in $H^1(\Omega_2)$. Hence, $u^{(n_j)}, u^{(n_j+\frac{1}{2})}$, and $u^{(n_j+1)}$ converge strongly in $H_0^1(\Omega)$. Note that u_0 and u_2 are weak solutions of (7) on Ω_2 while u_1 is a weak solution on Ω_1.

Let $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$ be nonempty and $\Gamma_2 = \partial\Omega_2 \cap \Omega_1$. Let $e^{(k)} = u^{(k)} - u$. Note that

$$-\triangle e^{(n+\frac{1}{2})} - f_{\nabla u}(x, v, \nabla v) \cdot \nabla e^{(n+\frac{1}{2})} - f_u(x, v, \nabla v)e^{(n+\frac{1}{2})} = 0 \text{ on } \Omega_1$$

for some function v. By the maximum principle,

$$\sup_{\overline{\Omega}_1} |e^{(n+\frac{1}{2})}| = \sup_{\Gamma_1} |e^{(n+\frac{1}{2})}| = \sup_{\Gamma_1} |e^{(n)}| \leq k_n \sup_{\overline{\Omega}_2} |e^{(n)}|$$

for some $0 \leq k_n < 1$. The proof of this latter inequality will be given shortly. Similarly, for some $0 \leq k_{n+\frac{1}{2}} < 1$,

$$\sup_{\overline{\Omega}_2} |e^{(n+1)}| = \sup_{\Gamma_2} |e^{(n+1)}| = \sup_{\Gamma_2} |e^{(n+\frac{1}{2})}| \leq k_{n+\frac{1}{2}} \sup_{\overline{\Omega}_1} |e^{(n+\frac{1}{2})}|.$$

Consequently

$$\sup_{\overline{\Omega}_2} |e^{(m)}| \leq \prod_{n=0}^{m} k_{n+\frac{1}{2}} k_n \sup_{\overline{\Omega}_2} |e^{(0)}|. \tag{10}$$

With $e_2 = u_2 - u$,

$$\sup_{\overline{\Omega}_2} |e_2| \leq \prod_{n=0}^{\infty} k_{n+\frac{1}{2}} k_n \sup_{\overline{\Omega}_2} |e^{(0)}|.$$

We now show that $\{k_{n_j}\}$ is bounded away from one. Because of strong convergence of $u^{(n_j+1)}$, we have

$$Le_2 \equiv -\triangle e_2 - f_{\nabla u}(x, v, \nabla v) \cdot \nabla e_2 - f_u(x, v, \nabla v) e_2 = 0 \text{ on } \Omega_2$$

for some function v and hence $\sup_{\overline{\Omega}_2} |e_2| > \sup_{\Gamma_1} |e_2|$, implying

$$\lim_{j \to \infty} k_{n_j+1} = c$$

for some constant $c < 1$. Thus $e_2 \equiv 0$ and from (10), the entire sequence $e^{(n)} \to 0$ in the L^∞ norm.

Finally, we give the details of the proof of the inequality $k_n < 1$ by the strong maximum principle. Note that if $e^{(n)}$ is constant, then it must be zero from the boundary conditions. If it is not constant, then since $Le^{(n)} \leq 0$, the strong maximum principle implies that $e^{(n)}$ cannot achieve a nonnegative maximum in Ω_2. Also, since $Le^{(n)} \geq 0$, $e^{(n)}$ cannot achieve a nonpositive minimum in Ω_2.

Now suppose $\sup_{\overline{\Omega}_2} |e^{(n)}| = \sup_{\Gamma_1} |e^{(n)}|$. Case I: If $\sup_{\Gamma_1} |e^{(n)}| = \sup_{\Gamma_1} e^{(n)}$, then $e^{(n)}$ achieves a nonnegative maximum in Ω_2, contradicting a statement in the above paragraph. Case II: If $\sup_{\Gamma_1} |e^{(n)}| = -\inf_{\Gamma_1} e^{(n)}$, then $e^{(n)}$ achieves a nonpositive minimum in Ω_2, again contradicting a statement in the above paragraph. Hence $\sup_{\overline{\Omega}_2} |e^{(n)}| > \sup_{\Gamma_1} |e^{(n)}|$. We may define

$$k_n = \frac{\sup_{\Gamma_1} |e^{(n)}|}{\sup_{\overline{\Omega}_2} |e^{(n)}|} < 1. \qquad \square$$

LEMMA 1. *Let $\{x_n\}$ be an unbounded sequence of positive real numbers. Suppose it has the property that if $\{x_{n_j}\}$ is any subsequence such that $x_{n_j} \to \infty$ as $j \to \infty$, then $x_{n_j-1} \to \infty$ as $j \to \infty$. Then*

$$\lim_{n \to \infty} x_n = \infty.$$

Proof. Since the original sequence is unbounded, there must exist some subsequence $\{x_{n_j}\}$ which goes to infinity as $j \to \infty$. Hence $\{x_{n_j-1}\}$ also goes to infinity as $j \to \infty$. By the same property, $x_{n_j-2} \to \infty$ as $j \to \infty$, etc. Given any positive ϵ, there are numbers $N_0, N_1, N_2, \ldots$ such that

$$\begin{aligned} n_i > N_0 &\Rightarrow x_{n_i} > \frac{1}{\epsilon}, \\ n_i - 1 > N_1 &\Rightarrow x_{n_i-1} > \frac{1}{\epsilon}, \\ n_i - 2 > N_2 &\Rightarrow x_{n_i-2} > \frac{1}{\epsilon}, \\ &\vdots \end{aligned}$$

Suppose the conclusion of the lemma is false. Then there is some subsequence $\{x_{k_j}\}$ which is bounded by M, say. Take $\epsilon = M^{-1}$. Then there are some j and l so that $k_j = n_i - l$ with $n_i - l > N_l$. Thus

$$x_{k_j} = x_{n_i - l} > \frac{1}{\epsilon} = M,$$

which is a contradiction. This completes the proof. □

It is natural to inquire whether the rather strong condition on the nonlinearity, $\partial f/\partial u \leq 0$, is really necessary. We believe that any restriction on f leading to a unique solution would also do. However, without any conditions on f, the quasilinear equation may have multiple solutions and some numerical evidence suggests that the Schwarz sequence does not converge. We tried several examples for which there are at least two distinct solutions. We monitor $\|u^{(n+\frac{1}{2})} - u^{(n)}\|$ in $\Omega_1 \cap \Omega_2$ and find that it oscillates and does not seem to converge.

2.3. Global inversion. Next, we show that the Schwarz method can be applied to a certain class of semilinear elliptic problem whose solution can be shown to be unique using the global inversion theorem. Previous theories are not applicable because the equations have a term linear in the unknown. A strong assumption on the nonlinearity can imply existence and uniqueness of the solution. We shall consider two cases corresponding to resonance and nonresonance.

For completeness, we first prove the existence and uniqueness result for the nonresonance case. A similar version can be found in Ambrosetti and Prodi [1].

PROPOSITION 3. *Consider the semilinear equation*

$$-\triangle u = \lambda u + f(x,u) + g \text{ on } \Omega \tag{11}$$

with homogeneous Dirichlet boundary conditions. Here $\lambda \in \mathbb{R}$ is given with $\lambda \neq \lambda_j \ \forall\, j$ and $f \in C^1(\overline{\Omega}, \mathbb{R})$ and satisfies the conditions

$$\frac{\|f(x,v_n)\|_{-1}}{\|v_n\|_1} \to 0 \quad \text{whenever } \|v_n\|_1 \to \infty \tag{12}$$

and

$$\lambda_{k-1} < \lambda + f_u(x,t) < \lambda_k \tag{13}$$

for every $x \in \Omega$ and $t \in \mathbb{R}$ and for some $k \in \mathbb{N}$. The function g is assumed to be in $L^2(\Omega)$. Then (11) *has a unique solution in $H_0^1(\Omega)$.*

Proof. We use the global inversion theorem (see [1] for instance) to show this result. Let $F : H_0^1(\Omega) \to H_0^1(\Omega)$ with $F(u) = u + \lambda\triangle^{-1}u + \triangle^{-1}f(x,u)$ for $u \in H_0^1(\Omega)$. Now F is continuous and we need to show that it is proper and locally invertible. For the former, suppose $h_n \in H_0^1(\Omega)$ and $u_n \in H_0^1(\Omega)$ with $F(u_n) = h_n \to h \in H_0^1(\Omega)$. We need to show that u_n has a convergent subsequence. Now suppose $\{\|u_n\|_1\}$ is unbounded. Then there is some subsequence which we still label by n such that $\|u_n\|_1 \to \infty$ as $n \to \infty$. Let $z_n = u_n/\|u_n\|_1$. Then

$$z_n + \lambda\triangle^{-1}z_n = \frac{h_n}{\|u_n\|_1} - \frac{\triangle^{-1}f(x,u_n)}{\|u_n\|_1}.$$

Certainly, the right-hand side of the above goes to zero in the L^2 norm as $n \to \infty$. Thus $\{z_n\}$ is bounded in H^2 and must have a convergent subsequence converging to some nonzero z strongly in the energy norm. From the above equation, we obtain the contradiction that z is an eigenfunction of $-\triangle$ with corresponding eigenvalue λ. Thus $\{\|u_n\|_1\}$ must be bounded. This implies that $\{\lambda\triangle^{-1}u_n = h_n - u_n - \triangle^{-1}f(x,u_n)\}$ is a bounded sequence in $L^2(\Omega)$. Thus $\{u_n\}$ is bounded in H^2 and hence must have a strongly convergent subsequence in $H_0^1(\Omega)$.

To show that F is locally invertible, we simply note that the linear problem

$$(I + \lambda\triangle^{-1} + \triangle^{-1}f_u(x,v))w = 0, \qquad v, w \in H_0^1(\Omega),$$

rewritten in a more familiar form

$$-\triangle w = (\lambda + f_u(x,v))w, \tag{14}$$

has only the trivial solution because of the assumption on $\lambda + f_u$. The Fredholm alternative implies that $F'(u)$ is invertible and by the inverse function theorem, F is locally invertible. We finally can conclude from the global inversion theorem that the semilinear elliptic equation (11) has a unique solution. ∎

Note that (12) is satisfied when, for instance, f is a bounded function. Note also that if f also depends on ∇u, then (12) contains an extra term involving ∇w and more assumptions are required to conclude that the equation corresponding to (14) has only the trivial solution.

For $j = 1, 2$, let $\|\cdot\|_{H^1(\Omega_j)}$ be the norm induced by the inner product

$$(v,w)_{H^1(\Omega_j)} \equiv \int_{\Omega_j} \nabla v \cdot \nabla w + vw.$$

The following lemma is useful.

Lemma 2. *Let $j = 1$ or 2 and $\lambda \in \mathbb{R},\ \lambda \neq \lambda_i(\Omega_j)\ \forall i$. Then for $w \in H_\Gamma^1(\Omega_j) \equiv \{v \in H^1(\Omega_j),\ v = 0$ on $\partial\Omega_j \cap \partial\Omega\}$,*

$$\|(P_j + \lambda\triangle_j^{-1})w\|_{H^1(\Omega_j)} \geq C\|P_j w\|_{H^1(\Omega_j)}.$$

Here $P_j + \lambda\triangle_j^{-1}$ is considered as an operator from $H_\Gamma^1(\Omega_j)$ onto $H_0^1(\Omega_j)$.

Proof. Let $\{\phi_i\}$ be an orthonormal basis (with respect to the inner product $(\cdot,\cdot)_{H^1(\Omega_j)}$) of eigenfunctions of $-\triangle_j$ with corresponding eigenvalues $\lambda_i(\Omega_j)$. Let

$$w = \sum_{i=1}^{\infty} c_i\phi_i + v,$$

where $c_i \in \mathbb{R}$ and v is in the orthogonal complement of $H_0^1(\Omega_j)$ in $H_\Gamma^1(\Omega_j)$ (i.e., $v \in H_0^1(\Omega_j)^\perp$ where $H_\Gamma^1(\Omega_j) = H_0^1(\Omega_j) \oplus H_0^1(\Omega_j)^\perp$). Note that $\triangle_j^{-1}$ is shorthand for $\triangle_j^{-1}R_j$, where $R_j : H_\Gamma^1(\Omega_j) \to H^{-1}(\Omega_j)$ is the operator defined by

$$(R_j z, y) = (z,y)_{H^1(\Omega_j)}, \qquad z \in H_\Gamma^1(\Omega_j),\ y \in H_0^1(\Omega_j).$$

By definition, $R_j v = 0$. Thus

$$\begin{aligned}
\|(P_j + \lambda\triangle_j^{-1})w\|_{H^1(\Omega_j)}^2 &= \left\| \sum_{i=1}^{\infty} c_i \left(1 - \frac{\lambda}{\lambda_i(\Omega_j)}\right)\phi_i \right\|_{H^1(\Omega_j)} \\
&= \sum_{i=1}^{\infty} c_i^2 \left(1 - \frac{\lambda}{\lambda_i(\Omega_j)}\right)^2 \\
&\geq \left(1 - \frac{\lambda}{\lambda_c(\Omega_j)}\right)^2 \|P_j w\|_{H^1(\Omega_j)}^2,
\end{aligned}$$

where $\lambda_c(\Omega_j)$ is an eigenvalue of $-\triangle_j$ closest to λ in the sense that

$$\left|1-\frac{\lambda}{\lambda_i(\Omega_j)}\right| \geq \left|1-\frac{\lambda}{\lambda_c(\Omega_j)}\right| \quad \forall i. \qquad \square$$

THEOREM 3. *Consider the semilinear elliptic equation as in Proposition* 3 *except that* (13) *is replaced by*

$$\lambda + f_u(x,t) \leq 0, \quad t \in \mathbb{R}. \tag{15}$$

For $n = 0, 1, 2, \ldots$ *and any* $u^{(0)} \in H_0^1(\Omega)$, *define the Schwarz sequence as*

$$-\triangle u^{(n+\frac{1}{2})} = \lambda u^{(n+\frac{1}{2})} + f(x, u^{(n+\frac{1}{2})}) + g \ \textit{on}\ \Omega_1,\ \ u^{(n+\frac{1}{2})} = u^{(n)} \ \textit{on}\ \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = \lambda u^{(n+1)} + f(x, u^{(n+1)}) + g \ \textit{on}\ \Omega_2,\ \ u^{(n+1)} = u^{(n+\frac{1}{2})} \ \textit{on}\ \partial\Omega_2.$$

Then the Schwarz sequence converges geometrically to the unique solution of the semilinear elliptic equation (11) *in the* L^∞ *norm.*

Proof. Because of (15), the Schwarz sequence is well defined by Proposition 3.

In the first part of the proof, we show that the Schwarz sequence is bounded. For any $v_1 \in H_0^1(\Omega_1)$,

$$[u^{(n+\frac{1}{2})} - u^{(n)}, v_1] = (\lambda u^{(n+\frac{1}{2})} + f^{(n+\frac{1}{2})} + g, v_1) - [u^{(n)}, v_1],$$

where $f^{(n+\frac{1}{2})} = f(x, u^{(n+\frac{1}{2})})$. Noting that $u^{(n+\frac{1}{2})} - u^{(n)} \in H_0^1(\Omega_1)$, we have

$$(I + \lambda\triangle_1^{-1})u^{(n+\frac{1}{2})} = -\triangle_1^{-1}(f^{(n+\frac{1}{2})} + g) + (I - P_1)u^{(n)}$$

or

$$(P_1 + \lambda\triangle_1^{-1})u^{(n+\frac{1}{2})} = -\triangle_1^{-1}(f^{(n+\frac{1}{2})} + g) \ \text{ and } \ (I - P_1)u^{(n+\frac{1}{2})} = (I - P_1)u^{(n)}.$$

Applying Lemma 2 to the first term, noting that $\lambda \neq \lambda_i(\Omega_1)\ \forall i$, we obtain

$$\|P_1 u^{(n+\frac{1}{2})}\|_1 \leq C(1 + \|f^{(n+\frac{1}{2})}\|_{-1}).$$

Assume $\{\|u^{(n+\frac{1}{2})}\|_1\}$ is unbounded. Then there is some subsequence which we label by $n_j + \frac{1}{2}$ such that $\|u^{(n_j+\frac{1}{2})}\|_1 \to \infty$ as $j \to \infty$. Now

$$\begin{aligned} \|u^{(n_j+\frac{1}{2})}\|_1 &\leq \|P_1 u^{(n_j+\frac{1}{2})}\|_1 + \|(I-P_1)u^{(n_j+\frac{1}{2})}\|_1 \\ &\leq C(1 + \|f^{(n_j+\frac{1}{2})}\|_{-1}) + \|(I-P_1)u^{(n_j)}\|_1. \end{aligned} \tag{16}$$

Thus

$$1 \leq o(1) + \frac{\|(I-P_1)u^{(n_j)}\|_1}{\|u^{(n_j+\frac{1}{2})}\|_1}.$$

This shows that $\{\|u^{(n_j)}\|_1\}$ must also be unbounded. By Lemma 1, the entire sequences (not just subsequences) $\{\|u^{(n)}\|_1\}$ and $\{\|u^{(n+\frac{1}{2})}\|_1\}$ go to infinity as $n \to \infty$.

Applying Lemma 2 to

$$(P_2 + \lambda\triangle_2^{-1})u^{(n)} = -\triangle_2^{-1}(f^{(n)} + g),$$

we have

$$\|P_2 u^{(n)}\|_1 \le C(1 + \|f^{(n)}\|_{-1}). \tag{17}$$

Also,

$$\begin{aligned}\|(I-P_1)u^{(n)}\|_1 &\le \|(I-P_1)P_2u^{(n)}\|_1 + \|(I-P_1)(I-P_2)u^{(n)}\|_1\\ &\le C(1+\|f^{(n)}\|_{-1}) + d\|u^{(n)}\|_1.\end{aligned}$$

Using this result in (16), we obtain

$$\frac{\|u^{(n+\frac{1}{2})}\|_1}{\|u^{(n)}\|_1} \le o(1)\frac{\|u^{(n+\frac{1}{2})}\|_1}{\|u^{(n)}\|_1} + o(1) + d$$

or

$$\frac{\|u^{(n+\frac{1}{2})}\|_1}{\|u^{(n)}\|_1} \le d + o(1).$$

In a parallel development, we also have

$$\frac{\|u^{(n+1)}\|_1}{\|u^{(n+\frac{1}{2})}\|_1} \le d + o(1).$$

Combining the above equations, we have

$$\frac{\|u^{(n+1)}\|_1}{\|u^{(n)}\|_1} \le d^2 + o(1),$$

contradicting that $\{\|u^{(n)}\|_1\}$ is unbounded since $d < 1$. Thus $\{\|u^{(n+\frac{1}{2})}\|_1\}$ is bounded. If $\{\|u^{(n)}\|_1\}$ is unbounded, then from (17),

$$\|u^{(n+1)}\|_1 \le \|P_2u^{(n+1)}\|_1 + \|(I-P_2)u^{(n+1)}\|_1 \le C(1+\|f^{(n+1)}\|_{-1}) + \|(I-P_2)u^{(n+\frac{1}{2})}\|_1.$$

Hence,

$$1 \le o(1) + \frac{\|(I-P_2)u^{(n+\frac{1}{2})}\|_1}{\|u^{(n+1)}\|_1}$$

which is a contradiction since $\{\|u^{(n+\frac{1}{2})}\|_1\}$ is bounded. This shows that $\{\|u^{(n)}\|_1\}$ must also be bounded.

Since the Schwarz sequences are bounded in $\|\cdot\|_1$, there exist $u_0, u_1, u_2 \in H_0^1(\Omega)$ and a subsequence labeled by n_j such that $u^{(n_j)} \rightharpoonup u_0$, $u^{(n_j+\frac{1}{2})} \rightharpoonup u_1$, and $u^{(n_j+1)} \rightharpoonup u_2$ (weak convergence in the energy norm). We now show that the subsequences actually converge strongly in the energy norm. By compactness of the restriction map $H^1(\Omega_1) \hookrightarrow L^2(\partial\Omega_1)$, we have $u^{(n_j+\frac{1}{2})} \to u_1$ in $L^2(\partial\Omega_1)$. Since $\{\lambda u^{(n_j+\frac{1}{2})} + f^{(n_j+\frac{1}{2})} + g\}$ is bounded in the L^2 norm, $\{u^{(n_j+\frac{1}{2})}\}$ is bounded in $H^2(\Omega_1)$ from the defining equation of $u^{(n_j+\frac{1}{2})}$. By extracting a further subsequence if necessary, $u^{(n_j+\frac{1}{2})}$ converges to u_1 strongly in $H^1(\Omega_1)$. Similarly, $u^{(n_j)} \to u_0$ and $u^{(n_j+1)} \to u_2$ strongly in $H^1(\Omega_2)$. Hence, $u^{(n_j)}, u^{(n_j+\frac{1}{2})}$, and $u^{(n_j+1)}$ converge strongly in $H_0^1(\Omega)$. Note that u_0 and u_2 are weak solutions of semilinear equation (11) on Ω_2 while u_1 is a weak solution on Ω_1.

Finally, as in the proof of the previous theorem, we apply the strong maximum principle to show convergence in the L^∞ norm of the iterates to the solution to the semilinear equation (11) on Ω. ∎

For the above semilinear equation, we made the strong assumption (15) so that the maximum principle can be applied in the final step of the proof. It is unknown whether the Schwarz iteration with the weaker condition (13) converges.

Next we consider the resonance problem for the above semilinear equation. See [1] for a proof.

PROPOSITION 4. *Consider the semilinear equation*

$$-\triangle u = \lambda_1 u + f(x,u) + g \text{ on } \Omega \tag{18}$$

with homogeneous Dirichlet boundary conditions. Here $f \in C^1(\overline{\Omega}, \mathbb{R})$ *and satisfies the following conditions:*

(1) $\exists M$ *such that* $|f(x,s)| \leq M \ \forall x \in \Omega, \ s \in \mathbb{R}$.

(2) $\lim_{s\to\pm\infty} f(x,s) = f_\pm \in \mathbb{R} \ \forall x \in \Omega$.

(3) $f_- \cdot \int_\Omega \phi_1 < -\int_\Omega g\phi_1 < f_+ \cdot \int_\Omega \phi_1$, *where* ϕ_1 *is the positive eigenfunction of* $-\triangle$ *corresponding to the principal eigenvalue* λ_1.

(4) $f_u(x,s) \neq 0 \ \forall x \in \Omega, \ s \in \mathbb{R}$.

(5) $\lambda_1 + f_u(x,t) < \lambda_2 \ \forall x \in \Omega, \ t \in \mathbb{R}$.

The function g *is assumed to be in* $L^2(\Omega)$. *Then,* (18) *has a unique solution in* $H_0^1(\Omega)$.

THEOREM 4. *Consider the hypotheses as in the above proposition except that the fifth condition is replaced by* $\forall x \in \Omega$ *and* $s \in \mathbb{R}$, $\lambda_1 + f_u(x,s) \leq 0$. *In addition, assume the subdomains are proper subsets of* Ω *so that* $\lambda_1 < \min(\lambda_1(\Omega_1), \ \lambda_1(\Omega_2))$. *For* $n = 0, 1, 2, \ldots$ *and any* $u^{(0)} \in H_0^1(\Omega)$, *define the Schwarz sequence as*

$$-\triangle u^{(n+\frac{1}{2})} = \lambda_1 u^{(n+\frac{1}{2})} + f(x, u^{(n+\frac{1}{2})}) + g \text{ on } \Omega_1, \ u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = \lambda_1 u^{(n+1)} + f(x, u^{(n+1)}) + g \text{ on } \Omega_2, \ u^{(n+1)} = u^{(n+\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Then the Schwarz sequence converges geometrically to the unique solution of the semilinear elliptic equation (18) *in the* L^∞ *norm.*

Proof. The Schwarz sequence is well defined by Proposition 3 because of the assumption that the subdomains are proper subsets of Ω. The rest of the proof follows exactly as in the proof of Theorem 3. ∎

3. Other Schwarz methods. In the last section, each subdomain problem is nonlinear. We now consider iterations where linear problems are solved in each subdomain. This is of great importance because in practice, we always like to avoid solving nonlinear problems. One way is in the framework of Newton's method. Write a model semilinear problem as $G(u) \equiv u - \triangle^{-1} f(x,u) = 0$ for $u \in H_0^1(\Omega)$. Suppose it has a solution u and suppose that $\|\triangle^{-1} f_u(x,u)\| < 1$; then for initial guess $u^{(0)}$ sufficiently close to u, the Newton iterates $u^{(n)}$ defined by

$$u^{(n+1)} = u^{(n)} - G_u(u^{(n)})^{-1} G(u^{(n)}) \tag{19}$$

converge to u. Note that the assumption means that $G_u = I - \triangle^{-1} f_u$ has a bounded inverse in a neighborhood of u. Now each linear problem (19) can be solved using the classical Schwarz alternating method. We take three different approaches.

In this section, we consider the equation

$$-\triangle u = f(x, u, \nabla u) \text{ on } \Omega \tag{20}$$

with homogeneous Dirichlet boundary conditions. Assume $f(x,u(x),\nabla u(x)) \in H^{-1}(\Omega)$ for $x \in \Omega$ and $u \in H_0^1(\Omega)$ and satisfies the condition

$$\|f(x,u,\nabla u) - f(x,v,\nabla v)\|_{-1} \le c\,\|u-v\|_1,$$

where $u,v \in H_0^1(\Omega)$, c is a constant such that $c<1$.

3.1. Linear Schwarz method. In the linear Schwarz sequence defined below, each subdomain problem is linear.

THEOREM 5. *Assume $d < \sqrt{1-c^2} - c$. For $n = 0,1,2,\ldots$ and any $u^{(0)} \in H_0^1(\Omega)$, define the linear Schwarz sequence by*

$$-\triangle u^{(n+\frac12)} = f(x,u^{(n)},\nabla u^{(n)}) \text{ on } \Omega_1,\ \ u^{(n+\frac12)} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = f(x,u^{(n+\frac12)},\nabla u^{(n+\frac12)}) \text{ on } \Omega_2,\ \ u^{(n+1)} = u^{(n+\frac12)} \text{ on } \partial\Omega_2.$$

Then, the Schwarz sequence converges to the solution of (20) *in the energy norm.*

Proof. In a similar manner as in the proof of Theorem 1, we obtain

$$u^{(n+\frac12)} - u^{(n)} = P_1(u-u^{(n)}) - \triangle_1^{-1}\left(f(x,u^{(n)},\nabla u^{(n)}) - f(x,u,\nabla u)\right) \tag{21}$$

and

$$u^{(n+1)} - u^{(n+\frac12)} = P_2(u-u^{(n+\frac12)}) - \triangle_2^{-1}\left(f(x,u^{(n+\frac12)},\nabla u^{(n+\frac12)}) - f(x,u,\nabla u)\right).$$

Defining $e^{(n)} = u^{(n)} - u$, we have

$$\begin{aligned} e^{(n+1)} = {} & (I-P_2)(I-P_1)e^{(n)} - (I-P_2)\triangle_1^{-1}\left(f(x,u^{(n)},\nabla u^{(n)}) - f(x,u,\nabla u)\right) \\ & -\triangle_2^{-1}\left(f(x,u^{(n+\frac12)},\nabla u^{(n+\frac12)}) - f(x,u,\nabla u)\right). \end{aligned}$$

From (21), $e^{(n+\frac12)} = (I-P_1)e^{(n)} - \triangle_1^{-1}\left(f(x,u^{(n)},\nabla u^{(n)}) - f(x,u,\nabla u)\right)$.

We have the estimate

$$\begin{aligned} \|e^{(n+1)}\|_1^2 &= \|P_2 e^{(n+1)}\|_1^2 + \|(I-P_2)e^{(n+1)}\|_1^2 \\ &\le c^2\|e^{(n+\frac12)}\|_1^2 + \|(I-P_2)e^{(n+\frac12)}\|_1^2 \\ &\le c^2\|e^{(n+\frac12)}\|_1^2 + \|(I-P_2)(I-P_1)e^{(n)} \\ &\quad -(I-P_2)\triangle_1^{-1}\left(f(x,u^{(n)},\nabla u^{(n)}) - f(x,u,\nabla u)\right)\|_1^2 \\ &\le c^2\|e^{(n+\frac12)}\|_1^2 + (d+c)^2\|e^{(n)}\|_1^2. \end{aligned}$$

We rewrite the above as

$$\begin{bmatrix} \|e^{(n+1)}\|_1^2 \\ \|e^{(n+\frac12)}\|_1^2 \end{bmatrix} \le A \begin{bmatrix} \|e^{(n+\frac12)}\|_1^2 \\ \|e^{(n)}\|_1^2 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} c^2 & (c+d)^2 \\ 1 & 0 \end{bmatrix}.$$

Similarly,

$$\begin{bmatrix} \|e^{(n+\frac12)}\|_1^2 \\ \|e^{(n)}\|_1^2 \end{bmatrix} \le A \begin{bmatrix} \|e^{(n)}\|_1^2 \\ \|e^{(n-\frac12)}\|_1^2 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \|e^{(n+1)}\|_1^2 \\ \|e^{(n+\frac12)}\|_1^2 \end{bmatrix} \le A^2 \begin{bmatrix} \|e^{(n)}\|_1^2 \\ \|e^{(n-\frac12)}\|_1^2 \end{bmatrix}.$$

A sufficient condition for the convergence of these vectors to zero is that the eigenvalues of A^2 have magnitude less than one or equivalently that the spectral radius of A is less than one. The eigenvalues of A are

$$\frac{c^2 \pm \sqrt{c^4 + 4(c+d)^2}}{2}.$$

Clearly, the eigenvalue with the larger magnitude is the one with the plus sign in the above expression. Thus the Schwarz sequence converges in the energy norm provided

$$\frac{c^2 + \sqrt{c^4 + 4(c+d)^2}}{2} < 1 \quad \text{or} \quad d < \sqrt{1-c^2} - c. \qquad \square$$

3.2. Additive Schwarz method. One of the main motivations for studying Schwarz methods is their suitability for parallel computers. The algorithms discussed thus far are not ideal for parallel computers because the subdomain problems must be solved sequentially. We now show convergence of an additive Schwarz method in which subdomain problems can be solved concurrently. The additive Schwarz method was first proposed by Dryja and Widlund [8] for linear elliptic PDEs.

THEOREM 6. *For $n = 0, 1, 2, \ldots$ and any $u^{(0)} \in H_0^1(\Omega)$, define the sequences*

$$\begin{aligned} -\triangle d^{(n+\frac{1}{2})} &= f(x, u^{(n)}, \nabla u^{(n)}) + \triangle u^{(n)} \ \text{ on } \Omega_1, \\ -\triangle d^{(n+1)} &= f(x, u^{(n)}, \nabla u^{(n)}) + \triangle u^{(n)} \ \text{ on } \Omega_2, \end{aligned}$$

for $d^{(n+\frac{1}{2})} \in H_0^1(\Omega_1)$ and $d^{(n+1)} \in H_0^1(\Omega_2)$. Define the additive Schwarz sequence by $u^{(n+1)} = u^{(n)} + \omega(d^{(n+\frac{1}{2})} + d^{(n+1)})$, where ω is a relaxation parameter with $0 < \omega < 1/2$. Assume $\|I - \omega(P_1 + P_2)\|_1 + 2\omega c < 1$. Then the additive Schwarz sequence converges geometrically to the solution of (20) *in the energy norm.*

Proof. This analysis is similar to the earlier one and we record only the key equations. From the defining equations of $d^{(n+\frac{1}{2})}$ and $d^{(n+1)}$, we have

$$\begin{aligned} d^{(n+\frac{1}{2})} &= -P_1 e^{(n)} - \triangle_1^{-1}\left(f(x, u^{(n)}, \nabla u^{(n)}) - f(x, u, \nabla u)\right), \\ d^{(n+1)} &= -P_2 e^{(n)} - \triangle_2^{-1}\left(f(x, u^{(n)}, \nabla u^{(n)}) - f(x, u, \nabla u)\right). \end{aligned}$$

Substituting into the definition of $u^{(n+1)}$, we obtain

$$e^{(n+1)} = (I - \omega(P_1 + P_2))e^{(n)} - \omega(\triangle_1^{-1} + \triangle_2^{-1})\left(f(x, u^{(n)}, \nabla u^{(n)}) - f(x, u, \nabla u)\right).$$

Hence

$$\|e^{(n+1)}\|_1 \le \|I - \omega(P_1 + P_2)\|_1 \, \|e^{(n)}\|_1 + 2\omega c \|e^{(n)}\|_1.$$

When $0 < \omega < 1/2$, $\|I - \omega(P_1 + P_2)\|_1 < 1$ and the result follows. $\square$

Note that the subdomain problems are linear and can be solved concurrently. Roughly speaking, $d^{(n+\frac{1}{2})}$ and $d^{(n+1)}$ are corrections to the iterate $u^{(n)}$ in the subdomains Ω_1 and Ω_2, respectively, and the right-hand sides of the defining equations for these corrections are the residuals of $u^{(n)}$ in the subdomains. If f is independent of u, then this reduces to the classical additive Schwarz method with a relaxation parameter.

3.3. Parallel Schwarz method. Other variations are also possible. We give one more which we call the parallel Schwarz method because the subdomain problems can also be solved in parallel. The proof is similar to previous ones and is omitted.

THEOREM 7. *Assume $d < 1 - 2c$. For $n = 0, 1, 2, \ldots$ and any $u^{(0)} \in H_0^1(\Omega)$, define $u^{(-\frac{1}{2})} = u^{(0)}$ and define the parallel Schwarz sequence by*

$$-\triangle u^{(n+\frac{1}{2})} = f(x, u^{(n)}, \nabla u^{(n)}) \text{ on } \Omega_1, \; u^{(n+\frac{1}{2})} = u^{(n)} \text{ on } \partial\Omega_1,$$

$$-\triangle u^{(n+1)} = f(x, u^{(n-\frac{1}{2})}, \nabla u^{(n-\frac{1}{2})}) \text{ on } \Omega_2, \; u^{(n+1)} = u^{(n-\frac{1}{2})} \text{ on } \partial\Omega_2.$$

Then, the parallel Schwarz sequence converges to the solution of (20) *in the energy norm.*

4. Discussions and conclusion. In this paper, we showed how Schwarz alternating methods can be embedded within the framework of Banach and Schauder fixed point theories and global inversion theory to construct solutions of nonlinear elliptic PDEs. We also give other versions of these methods where a linear problem is solved in each subdomain and these linear problems can be computed in parallel.

We treated homogeneous boundary conditions in this paper. For the problems considered here, nonhomogeneous boundary conditions can also be handled. If the boundary condition is $u = h \in H^{1/2}(\partial\Omega)$, then the procedure is to extend h to be a function in $H^1(\Omega)$ in a bounded way and to make a change of variable $w = u - h$ so that w satisfies the perturbed PDE

$$-\triangle w = F(x, w, \nabla w) + \triangle h$$

with homogeneous boundary conditions, where $F(x, w, \nabla w) = f(x, w + h, \nabla(w + h))$. For PDEs of Theorem 1, it is trivial to check that F satisfies the same Lipschitz condition as f. For PDEs of Theorem 2, $\partial F/\partial w = \partial f/\partial w$ and

$$\begin{aligned}
|F(x, w, \nabla w)| &= |f(x, w + h, \nabla(w + h))| \\
&\le C(1 + |\nabla(w + h)|^\gamma) \\
&\le C\left(1 + (|\nabla w| + |\nabla h|)^\gamma\right) \\
&\le C\left[1 + 2^\gamma \left(|\nabla w|^\gamma + |\nabla h|^\gamma\right)\right] \\
&\le C(1 + |\nabla w|^\gamma),
\end{aligned}$$

and hence Theorem 2 can be applied to the perturbed PDE. We can also check that Theorem 3 can be applied to the perturbed PDE. This procedure, of course, does not work for all nonlinear PDEs.

Future work will include some numerical experiments and extending our results to the case of multiple subdomains. For the PDEs of Theorem 1, this is manageable. In particular, for the additive Schwarz sequence, this is trivial. If there are m subdomains, as long as

$$\|I - \omega(P_1 + \cdots + P_m)\|_1 + m\omega c < 1,$$

the sequence converges. Here, $0 < \omega < 1/K$ where K is the minimum number of colors needed to color the subdomains in such a way that overlapping subdomains are assigned different colors. The extension for the other classes of PDEs is not at all obvious mainly because of the difficulty of applying the maximum principle in these cases. It is desirable to use tools other than the maximum principle for these classes of PDEs. Besides possibly weakening the hypotheses required, they may allow extension to multiple subdomains.

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REFERENCES

[1] A. Ambrosetti and G. Prodi, *A Primer of Nonlinear Analysis*, Cambridge University Press, Cambridge, UK, 1993.

[2] J. H. Bramble, J. E. Pasciak, J. Wang, and J. Xu, *Convergence estimates for product iterative methods with applications to domain decomposition*, Math. Comp., 57 (1991), pp. 1–21.

[3] X. C. Cai and M. Dryja, *Domain decomposition methods for monotone nonlinear elliptic problems*, in Domain Decomposition Methods in Scientific and Engineering Computing, D. Keyes and J. Xu, eds., AMS, Providence, RI., 1994, pp. 335–360.

[4] X. C. Cai, W. D. Gropp, D. E. Keyes, R. G. Melvin, and D. P. Young, *Parallel Newton–Krylov–Schwarz algorithms for the transonic full potential equation*, SIAM J. Sci. Comput., 19 (1998), pp. 245–265.

[5] X. C. Cai, D. E. Keyes, and V. Venkatakrishnan, *Newton-Krylov-Schwarz: An implicit solver for cfd*, in Proceedings of the Eighth International Conference on Domain Decomposition Methods in Science and Engineering, R. Glowinski et al., eds., Wiley, New York, 1997.

[6] T. F. Chan and T. P. Mathew, *Domain decomposition algorithms*, Acta Numerica, (1994), pp. 61–143.

[7] M. Dryja and W. Hackbusch, *On the nonlinear domain decomposition method*, BIT, (1997), pp. 296–311.

[8] M. Dryja and O. B. Widlund, *An Additive Variant of the Schwarz Alternating Method for the Case of Many Subregions*, Technical report 339, Courant Institute of Mathematical Sciences, New York University, New York, 1987.

[9] L. C. Evans, *Partial Differential Equations*, AMS, Providence, RI, 1998.

[10] R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, *First International Symposium on Domain Decomposition Methods for Partial Differential Equations*, Proc. Appl. Math. 30, SIAM, Philadelphia, PA, 1988.

[11] P. L. Lions, *On the Schwarz alternating method.* I, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, Proc. Appl. Math. 30, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, eds., SIAM, Philadelphia, PA, 1988, pp. 1–42.

[12] P. L. Lions, *On the Schwarz alternating method.* II, *Stochastic interpretation and order properties*, in Second International Conference on Domain Decomposition Methods, T. F. Chan, R. Glowinski, J. Periaux, and O. Widlund, eds., SIAM, Philadelphia, PA, 1989, pp. 47–70.

[13] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, New York University, New York, 1974.

[14] B. F. Smith, P. E. Bjorstad, and W. D. Gropp, *Domain Decomposition: Parallel Multilevel Algorithms for Elliptic Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 1996.

[15] X.-C. Tai, *Domain decomposition for linear and nonlinear elliptic problems via function or space decomposition*, in Domain Decomposition Methods in Scientific and Engineering Computing, D. Keyes and J. Xu, eds., AMS, Providence, RI, 1994, pp. 335–360.

[16] X.-C. Tai and M. Espedal, *Rate of convergence of some space decomposition methods for linear and nonlinear problems*, SIAM J. Numer. Anal., 35 (1998), pp. 1558–1570.

[17] X.-C. Tai and P. Tseng, *Convergence Rate Analysis of an Asynchronous Space Decomposition Method for Convex Minimization*, Report 120, Department of Mathematics, University of Bergen, Norway, 1998.

[18] X.-C. Tai and J. Xu, *Global Convergence of Subspace Correction Methods for Convex Optimization Problems*, Report 114, Department of Mathematics, University of Bergen, Norway, 1998.

[19] P. Le Tallec, *Domain decomposition methods in computational mechanics*, Comput. Mech. Adv., 1 (1994), pp. 121–220.

[20] J. Xu, *Two-grid discretization techniques for linear and nonlinear PDEs*, SIAM J. Numer. Anal., 33 (1996), pp. 1759–1777.