

THE CONTINUOUS NEWTON–RAPHSON METHOD CAN LOOK AHEAD*

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Abstract. This paper is about an intriguing property of the continuous Newton–Raphson method for the minimization of a continuous objective function f : if x is a point in the domain of attraction of a strict local minimizer x^* , then the flux line of the Newton–Raphson flow that starts in x approaches x^* from a direction that depends only on the behavior of f in arbitrarily small neighborhoods around x and x^* . In fact, if \tilde{f} is a sufficiently benign perturbation of f on an open region D not containing x , then the two flux lines through x defined by the Newton–Raphson vector fields that correspond to f and \tilde{f} differ from one another only within D .

Key words. continuous Newton method, Newton–Raphson method, Newton flow, qualitative analysis

AMS subject classifications. 65K10, 65H20, 58C15

DOI. 10.1137/S1052623403432633

1. Introduction. The continuous Newton method for the solution of nonlinear systems of equations arises from Newton’s method when damping with infinitesimal step sizes is applied. To numerical analysts this method is of theoretical interest because the topological behavior of the continuous dynamical system corresponding to a Newton vector field is better understood than the discrete dynamical systems arising from Newton’s method with finite damping; see, for example, the papers of Smale [9], Branin [4], Janovsky and Seige [5], Jongen, Jonker, and Twilt [6], and Neuberger [7]. Such results proved to be useful in the analysis of solutions to systems of PDEs using Sobolev gradient methods where an insufficient number of boundary conditions may not directly guarantee the existence of a unique solution; see, for example, Ball [2] and Neuberger [8]. Anstreicher [1] investigated the continuous Newton–Raphson method in the context of barrier minimization in linear programming. This is the same as the continuous Newton method applied to the zero-finding problem defined by the barrier gradient field.

Applied to the problem of minimizing an arbitrary nonlinear function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the Newton–Raphson flow equations are defined by

$$(1.1) \quad \frac{\partial}{\partial t} \varphi(x_0, t) = -H(\varphi(x_0, t))^{-1} \nabla f(\varphi(x_0, t)), \quad \varphi(x_0, 0) = x_0,$$

where

$$\begin{aligned} \nabla f(y) &:= D_x f(x)^T|_{x=y} \in \mathbb{R}^n, \\ H(y) &:= D_x^2 f(x)|_{x=y} \in \mathbb{R}^{n \times n}. \end{aligned}$$

*Received by the editors August 4, 2003; accepted for publication (in revised form) June 15, 2004; published electronically April 22, 2005.

<http://www.siam.org/journals/siopt/15-3/43263.html>

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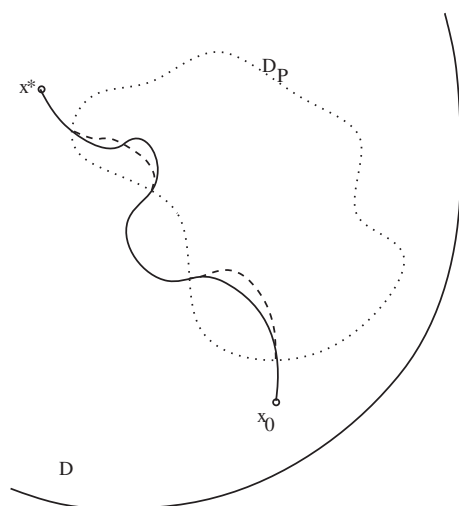


FIG. 1. Flux lines of the original function f (solid line) and of the function \hat{f} which is perturbed in the dotted region \mathcal{D}_P (dashed line).

The continuous Newton–Raphson method then consists of choosing a starting point x_0 and following the flux line $t \mapsto \varphi(x_0, t)$, obtained by integrating the ODE (1.1) to a stationary point $x^* = \lim_{t \rightarrow \infty} \varphi(x_0, t)$ of ∇f .

Janovsky and Seige [5] showed that, subject to certain regularity conditions, the topological behavior of the Newton flow associated with the continuous Newton method for solving systems of equations is invariant under small perturbations of the system. In this paper we consider a more restrictive setting and prove a similar but stronger property: we consider the continuous Newton–Raphson method to minimize f applied at a point x_0 in the domain of attraction of a strict local minimizer x^* . Under these conditions, if \hat{f} is a benign perturbation of f in an open domain \mathcal{D}_P that excludes a small neighborhood of x_0 , and if $\hat{\varphi}(x_0, t)$ denotes the flux line flowing through x_0 under the perturbed dynamical system

$$(1.2) \quad \frac{\partial}{\partial t} \hat{\varphi}(x_0, t) = -\hat{H}(\hat{\varphi}(x_0, t))^{-1} \nabla \hat{f}(\hat{\varphi}(x_0, t)), \quad \hat{\varphi}(x_0, 0) = x_0,$$

with $\nabla \hat{f} = D_x \hat{f}$ and $\hat{H} = D_x^2 \hat{f}$, then $\varphi(x_0, t)$ and $\hat{\varphi}(x_0, t)$ differ only for values of t for which $\varphi(x_0, t), \hat{\varphi}(x_0, t) \in \mathcal{D}_P$. In other words, the dynamics of φ and $\hat{\varphi}$ not only are topologically equivalent, but the flux lines actually coincide outside of the domain of perturbation; see Figure 1.

This property is intriguing, because it seems counterintuitive: $\varphi(x_0, t)$ and $\hat{\varphi}(x_0, t)$ clearly must coincide for values of t that occur before first entering \mathcal{D}_P , because (1.1) and (1.2) coincide on this range. However, one might expect that $\hat{\varphi}$ starts to differ from φ inside \mathcal{D}_P and leads to an exit point $\hat{y} = \hat{\varphi}(x_0, \hat{\tau}) \in \partial \mathcal{D}_P$ from \mathcal{D}_P which is different from the point $y = \varphi(x_0, \tau) \in \partial \mathcal{D}_P$ where φ exits. If that was the case, then the remaining parts of the flux lines

$$\begin{aligned} \varphi(x_0, t) &= \varphi(y, t - \tau), & (t > \tau), \\ \hat{\varphi}(x_0, t) &= \hat{\varphi}(\hat{y}, t - \hat{\tau}), & (t > \hat{\tau}), \end{aligned}$$

would differ from one another.

This intuitive picture is wrong, because a simple lemma shows that the normalized gradient

$$\mathbf{n}(\nabla f(y)) := \frac{\nabla f(y)}{\|\nabla f(y)\|}$$

is an invariant for points y that lie along the flux line $\varphi(x_0, t)$; see Lemma 2.1. The same of course holds true for $\mathbf{n}(\nabla \hat{f})$. This mechanism forces the exit points x and \hat{x} and the exit times τ and $\hat{\tau}$ to coincide, which in turn implies that φ and $\hat{\varphi}$ will perfectly coincide after exiting \mathcal{D}_P , until they next enter the domain of perturbation.

If \mathcal{D}_P furthermore excludes a small neighborhood of x^* , then $\varphi(x_0, t)$ and $\hat{\varphi}(x_0, t)$ coincide for all t large enough. Thus, the continuous Newton–Raphson method “knows” how it will approach x^* solely on the basis of information about f taken from neighborhoods around x_0 and x^* ; the function values of f at points that are positioned along the way have no bearing on this. For example, the limiting direction at which $\varphi(x_0, t)$ approaches x^* is given by

$$\lim_{t \rightarrow +\infty} e^t \frac{\partial}{\partial t} \varphi(x_0, t) = -H(x^*)^{-1} \nabla f(x_0);$$

see Theorem 3.1. Thus, in this specific sense the continuous Newton–Raphson method can look ahead: it knows where it is going and can only be distracted but not ultimately deviated from its predetermined path by local perturbations.

For the purposes of our analysis we will assume throughout that x^* is a point such that $\nabla f(x^*) = 0$ and the Hessian at x^* is positive definite ($H(x^*) \succ 0$) implying that x^* is a strict local minimizer of f , and that $f \in \mathcal{C}^3(\mathcal{D}_3)$ in some open neighborhood \mathcal{D}_3 of x^* . The domain of attraction of x^* is the set of points $x \in \mathbb{R}$ such that $\lim_{t \rightarrow \infty} \varphi(x, t) = x^*$. It is well known that the assumption $H(x^*) \succ 0$ implies that the domain of attraction of x^* contains an open neighborhood \mathcal{D}_a of x^* such that $H(x) \succ 0$ for all $x \in \mathcal{D}_a$. We will use the notation $\text{int}A$, ∂A , and $\text{cl}A$ for the interior, boundary, and closure of a set A , respectively. Additionally, for convex set A we will use $\text{ext}A$ to denote the set of extreme points of A , that is to say, the set of all points $x \in A$ such that $A \setminus \{x\}$ is convex.

2. Characterization of flux lines by an invariance property. A simple but powerful property of the Newton–Raphson flow is that $\mathbf{n}(\nabla f)$ is invariant along flux lines. This claim, proved in the next Lemma, has the interesting consequences that we will formalize in the main theorems.

LEMMA 2.1. *The direction of the gradient of the function f is constant along flux lines defined by its Newton–Raphson vector field.*

Proof. Let $\varphi(x_0, t)$ denote the flux line of the Newton–Raphson vector field defined by f passing through x_0 . Then

$$\begin{aligned} \frac{\partial}{\partial t} \nabla f(\varphi(x_0, t)) &= H(\varphi(x_0, t)) \frac{\partial}{\partial t} \varphi(x_0, t) \\ &= H(\varphi(x_0, t)) (-H(\varphi(x_0, t))^{-1}) \nabla f(\varphi(x_0, t)) \\ &= -\nabla f(\varphi(x_0, t)). \end{aligned}$$

Therefore,

$$(2.1) \quad \nabla f(\varphi(x_0, t)) = e^{-t} \nabla f(\varphi(x_0, 0))$$

and $\|\nabla f(\varphi(x_0, t))\| = e^{-t} \|\nabla f(\varphi(x_0, 0))\|$, that is,

$$\mathbf{n}(\nabla f(\varphi(x_0, t))) = \mathbf{n}(\nabla f(\varphi(x_0, 0))),$$

which proves the claim. \square

3. Limiting behavior of flux lines. As a direct consequence of Lemma 2.1 we now obtain the following result.

THEOREM 3.1. *Let $x_0 \in \mathcal{D}$. Then the limiting direction of the flux line $\varphi(x_0, t)$ is the normalization of the vector*

$$\lim_{t \rightarrow +\infty} e^t \frac{\partial}{\partial t} \varphi(x_0, t) = -H(x^*)^{-1} \nabla f(x_0).$$

Proof. Using (2.1) we obtain

$$\lim_{t \rightarrow +\infty} e^t \frac{\partial}{\partial t} \varphi(x_0, t) = - \lim_{t \rightarrow +\infty} H(\varphi(x_0, t))^{-1} \nabla f(x_0) = -H(x^*)^{-1} \nabla f(x_0),$$

where the last equality follows from the fact that flux line $\varphi(x_0, t)$ converges to x^* as $t \rightarrow \infty$ and the assumption that the Hessian is a continuous function. \square

Theorem 3.1 shows that the direction at which the flux line $\varphi(x_0, t)$ approaches x^* depends only on the gradient of f at x_0 and the Hessian at x^* . In other words, knowing only the behavior of the function f in a small neighborhood around x_0 and x^* , one can determine the limiting direction of a flux line; see Figure 2. Since the limiting direction does not depend on the value of f at intermediate points along the flux line $\varphi(x_0, t)$, does this mean that if we perturb f in a region that lies in its path, the behavior of $\varphi(x_0, t)$ around x^* will be the same? If the perturbation is sufficiently benign, the answer is yes. As a matter of fact, a much stronger claim is true: the flux lines of f and its perturbation differ only in the region of perturbation! This is the subject of the next section.

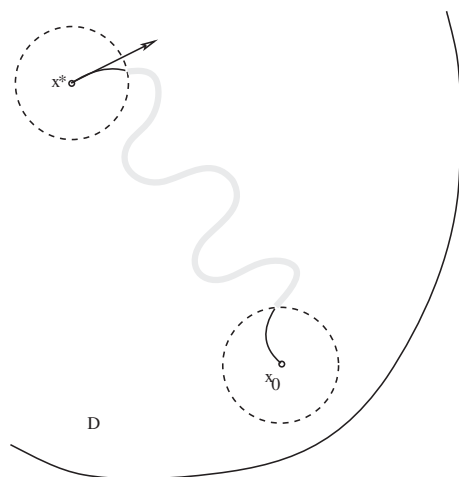


FIG. 2. The limiting direction of the flux line through x_0 depends only on information in small neighborhoods of x_0 and x^* .

4. Change of flux lines under perturbations of function f . In this section we will prove the main theorem of this paper, which we described in the introduction as counterintuitive. Thinking back to Lemma 2.1 and its use in the proof of Theorem 3.1, the result we claim here is now more plausible: the normalized gradient of the f along $\varphi(x_0, t)$ takes the constant value $\mathbf{n}(\nabla f(x_0))$. Moreover, under the regularity assumption we have made the set of points where $\mathbf{n}(\nabla f) = \mathbf{n}(\nabla f(x_0))$ is one-dimensional. Since f and its perturbation \hat{f} are identical in the unperturbed region, the loci of the points on the flux lines $\varphi(x_0, t)$ and $\hat{\varphi}(x_0, t)$ have to lie in the same one-dimensional set, forcing them to coincide.

Before formalizing and proving this result, we need the following definitions and auxiliary results.

DEFINITION 4.1. Let $C \subset \mathbb{R}^n$ and $x \in \partial C$. A supporting hyperplane of C at x is $S_C(x) = \{y \in \mathbb{R}^n : \langle a; y - x \rangle = 0\}$, where $0 \neq a \in \mathbb{R}^n$ is a vector such that for all $y \in C$, $\langle a; y - x \rangle \leq 0$.

In other words, a supporting hyperplane of a set C at x is a hyperplane containing x such that the whole set C lies in one closed halfspace defined by the hyperplane. It is well known (see, e.g., [3]) that if C is convex, then a supporting hyperplane exists for every $x \in \partial C$.

DEFINITION 4.2. Let $C \subset \mathbb{R}^n$ be a convex set and $x \in C$. The normal cone of C at x is $N_C(x) = \{d \in \mathbb{R}^n : \langle d; y - x \rangle \leq 0 \ \forall y \in C\}$. The tangent cone of C at x is $T_C(x) = \{\alpha(c - x) : \alpha \in \mathbb{R}_+, c \in C\}$.

DEFINITION 4.3. Let $A \subset \mathbb{R}^n$. The polar cone of A is

$$A^- = \{y \in \mathbb{R}^n : \langle y; x \rangle \leq 0 \ \forall x \in A\}.$$

It is not difficult to prove that $N_C(x) = T_C(x)^-$ and $T_C(x) = N_C(x)^-$; see, e.g., [3].

Using the notions defined above, we can now show that for a function with a strictly convex level set L_k , that is, $\partial L_k = \text{ext}(L_k)$, there exists a unique point in ∂L_k where ∇f has a given direction.

LEMMA 4.4. Let $f \in C^2$ be a function with positive definite Hessian $H \succ 0$ for every x in some neighborhood \mathcal{D} of a strict local minimizer x^* , and let g be a given unit vector. Then for any fixed level $k > f(x^*)$ such that $L_k^f := \{x : f(x) \leq k\} \subseteq \mathcal{D}$ there exists a unique point $x_k \in \partial L_k$ such that $\mathbf{n}(\nabla f(x_k)) = g$.

Proof. It is easy to prove that the condition $H(x) \succ 0$ on \mathcal{D} implies that all level sets $L_k^f = \{x : f(x) \leq k\}$ such that $L_k \subseteq \mathcal{D}$ are strictly convex, that is to say, $\partial L_k^f = \text{ext} L_k^f$.

Existence. Let us first find a point $x_k \in \partial L_k^f$, where g defines a supporting hyperplane. Since we want $\langle g; y - x_k \rangle = \langle g; y \rangle - \langle g; x_k \rangle \leq 0$ to hold for every $y \in L_k^f$, let us choose $x_k \in \arg\max\{\langle g; x \rangle : x \in L_k^f\}$. We will prove that $\mathbf{n}(\nabla f(x_k)) = g$. It is easy to see that $g \in N_{L_k^f}(x_k)$ and that $\nabla f(x_k) \neq 0$. By the implicit function theorem, ∂L_k^f is a \mathcal{C}^1 manifold and hence $T_{\partial L_k^f}(x_k)$ is well defined and of dimension $n - 1$. Therefore $T_{L_k^f}(x_k)$ is a linear halfspace and $N_{L_k^f}(x_k) = T_{L_k^f}(x_k)^-$ is a ray (a one-dimensional pointed cone). Since $\nabla f(x_k) \in N_{L_k^f}(x_k)$, this shows that $\mathbf{n}(\nabla f(x_k)) = g$; see Figure 3.

Uniqueness. Let us now assume that there exist two points x_k and y_k satisfying the claim. Then, both the supporting hyperplanes for x_k and y_k are defined by g . The definition of $S_{L_k^f}(x_k)$ implies that $\langle g; y_k - x_k \rangle \leq 0$, while $\langle g; x_k - y_k \rangle \leq 0$ follows

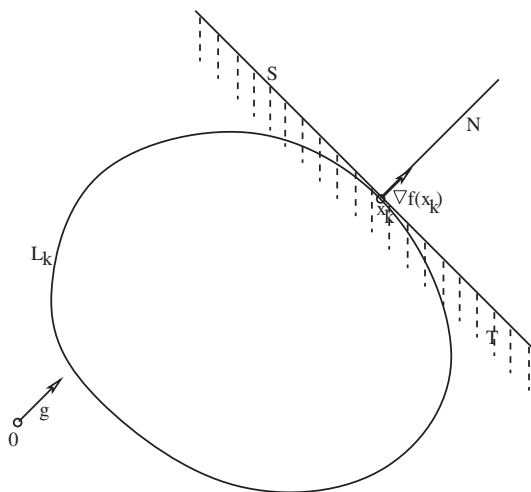


FIG. 3. Supporting hyperplane S defined by the vector g , normal cone N , and tangent cone T of level set L_k at x_k

from the definition of $S_{L_k^f}(y_k)$. Hence $\langle g; x_k - y_k \rangle = 0$ and $y_k \in S_{L_k^f}(x_k)$. But this implies that the entire line segment defined by x_k and y_k belongs to ∂L_k^f , so for every $\lambda \in (0, 1)$ we have that $\lambda x_k + (1 - \lambda)y_k \in \partial L_k^f \setminus \text{ext} L_k^f$, which contradicts the assumption that L_k^f is strictly convex. \square

Hereafter we will say that the function \hat{f} is a *regular perturbation* of $f \in \mathcal{C}^3$ with local minimizer x^* if the following conditions are satisfied:

- (i) $\hat{f} \in \mathcal{C}^2$ and $H_{\hat{f}} \succ 0 \forall x \in \mathcal{D}$;
- (ii) $f(x) - \hat{f}(x) \neq 0 \Rightarrow x \in \mathcal{D}_P \subset \mathcal{D}$, where \mathcal{D}_P is an open set.

Note that the condition (i) ensures that the Newton–Raphson vector field is well defined and no bifurcation of flux lines occurs. We will use the notation

$$\Gamma_{\varphi}(x_0) = \{x : x = \varphi(x_0, t) \text{ for some } t \geq 0\}$$

to denote the flux line $\varphi(x_0, t)$ as a set of points, and we write $A \triangle B$ for the symmetric difference between two sets A and B .

All the tools are now available to formalize and prove the main result of this paper: the Newton–Raphson–flux lines defined by a regular perturbation \hat{f} of f differ from the Newton–Raphson–flux lines defined by f only in the region of perturbation \mathcal{D}_P .

THEOREM 4.5. *Let f be a \mathcal{C}^2 function with $H_f \succ 0$ for all $x \in \mathcal{D}$, and let $x^* \in \mathcal{D}$ be a strict local minimizer of f . Let \hat{f} be a regular perturbation of f , and let $\varphi(x_0, t)$ and $\hat{\varphi}(x_0, t)$ be the Newton–Raphson flows defined by f and \hat{f} . Let x_0 be a point in $\mathcal{D} \setminus \text{cl}(\mathcal{D}_P)$ such that $L_{f(x_0)}^f \subset \mathcal{D}$ and $L_{\hat{f}(x_0)}^{\hat{f}} \subset \mathcal{D}$. Then*

$$\Gamma_{\varphi}(x_0) \triangle \Gamma_{\hat{\varphi}}(x_0) \subset \mathcal{D}_P$$

and $\varphi(x_0, t) = \hat{\varphi}(x_0, t)$ for all $t \in \mathbb{R}$ such that $\varphi(x_0, t) \notin \mathcal{D}_P$.

Proof. Let $g = \mathbf{n}(\nabla f(x_0)) = \mathbf{n}(\nabla \hat{f}(x_0))$, where the latter equality holds because $f = \hat{f}$ in a neighborhood of x_0 . We claim that

$$\begin{aligned}\Gamma_\varphi(x_0) &= \{x \in \mathcal{D} : \mathbf{n}(\nabla f(x)) = g, f(x) \leq f(x_0)\} =: \Upsilon_g(x_0), \\ \Gamma_{\hat{\varphi}}(x_0) &= \{x \in \mathcal{D} : \mathbf{n}(\nabla \hat{f}(x)) = g, \hat{f}(x) \leq \hat{f}(x_0)\} =: \hat{\Upsilon}_g(x_0).\end{aligned}$$

Since $H_f \succ 0$ and $H_{\hat{f}} \succ 0$, level sets L_k^f and $L_k^{\hat{f}}$ are strictly convex in \mathcal{D} for every $k \leq f(x_0)$. Therefore $\Gamma_\varphi(x_0) \subseteq \Upsilon_g(x_0)$ by Lemma 2.1. On the other hand, assume that there exists a point $y \in \Upsilon_g(x_0) \setminus \Gamma_\varphi(x_0)$. We know that $x_0 \in \Gamma_\varphi(x_0)$ so $y \neq x_0$. Since f is continuous and strictly decreasing along $\varphi(x_0, t)$ and since $\varphi(x_0, t)$ converges to x^* , there exists a point $x_1 \in \Gamma_\varphi(x_0)$ such that $f(x_1) = f(y) < f(x_0)$ and $\mathbf{n}(\nabla f(x_1)) = \mathbf{n}(\nabla f(x_0)) = \mathbf{n}(\nabla f(y))$, where the first equality follows from the Lemma 2.1 and the second from the fact that $y \in \Upsilon_g(x_0)$. This implies that there are two distinct points in the level set $L_{f(y)}^f$ with the same gradient direction, in contradiction to Lemma 4.4. Therefore, $\Gamma_\varphi(x_0) = \Upsilon_g(x_0)$. The result for the function \hat{f} can be derived in a similar fashion. Now, since $f(x) = \hat{f}(x)$ for $x \in \mathcal{D} \setminus \mathcal{D}_P$ and in particular $f(x_0) = \hat{f}(x_0)$, it is easy to see that $\Upsilon_g(x_0)$ and $\hat{\Upsilon}_g(x_0)$ are the same in the unperturbed region, i.e., $\Upsilon_g(x_0) \cap (\mathcal{D} \setminus \mathcal{D}_P) = \hat{\Upsilon}_g(x_0) \cap (\mathcal{D} \setminus \mathcal{D}_P)$. In other words, $\Gamma_\varphi(x_0) \cap (\mathcal{D} \setminus \mathcal{D}_P) = \Gamma_{\hat{\varphi}}(x_0) \cap (\mathcal{D} \setminus \mathcal{D}_P)$, showing the first claim. The last claim now follows from (2.1). \square

5. Numerical examples.

5.1. A two-dimensional example. Let us consider the quadratic function $f(x) = c + b^T x + \frac{1}{2} x^T B x$ for $c = 2$, $b = (1, 2)^T$, and $B = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$ with global minimizer $x^* = (6/7, -5/7)^T$ and perturb it in an annulus around x^* with inner and outer radii r_a and r_b , respectively. Let us choose a perturbation term given by a polynomial of degree 8 with 4-tuple zeros r_a and r_b : $\hat{f}(x) = f(x) + p(\|x - x^*\|)$, where

$$(5.1) \quad p(s) = \begin{cases} \varepsilon(s - r_a)^4(s - r_b)^4 & \text{for } r_a < s < r_b, \\ 0 & \text{otherwise.} \end{cases}$$

The figures in this section show the results for $r_a = 1$, $r_b = 3$, and $\varepsilon = 1$ obtained in numerical computations performed in MATLAB using the ODE integrator `ode45`. Figure 4 shows a few level sets of the perturbed function \hat{f} , the annulus where the original function was perturbed, and the flux lines φ and $\hat{\varphi}$, respectively, corresponding to f (solid lines) and \hat{f} (dashed curves). It can be clearly seen that the flux lines differ only in the region of perturbation, as proved in Theorem 4.5. In fact, this example shows that the convexity assumptions of Theorem 4.5 are sufficient but not necessary for its claim to hold.

Note that the time scale of the flux lines for f and \hat{f} is in general different in the perturbed region. Therefore, to measure the difference between points on $\Gamma_\varphi(x_0)$ and $\Gamma_{\hat{\varphi}}(x_0)$, we have reparametrized the flux lines in terms of the distance to x^* . Accordingly, Figure 5 shows the norm of the difference of flux lines shown in Figure 4 as functions of $\|x - x^*\|$. The flux lines differ only in the region $1 < \|x - x^*\| < 3$, confirming the claim of Theorem 4.5. Note that $p(s)$ has exactly one local extremum s^* for $r_a < s < r_b$. Therefore, for \bar{x} such that $\|\bar{x} - x^*\| = s^*$ we have $\nabla \hat{f}(\bar{x}) = \nabla f(\bar{x}) + p'(s^*) \frac{d}{dx} \|x - x^*\|_{|x=\bar{x}} = \nabla f(\bar{x})$; so recalling the proof of Theorem 4.5 we see that the perturbed and unperturbed flux lines intersect at such points, which explains the dip observed in Figure 5.

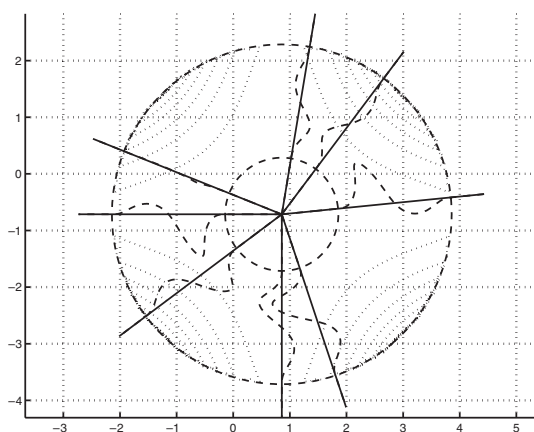


FIG. 4. Perturbed region, level sets of the perturbed function, and flux lines of the original (solid line) and the perturbed (dashed curves) function.

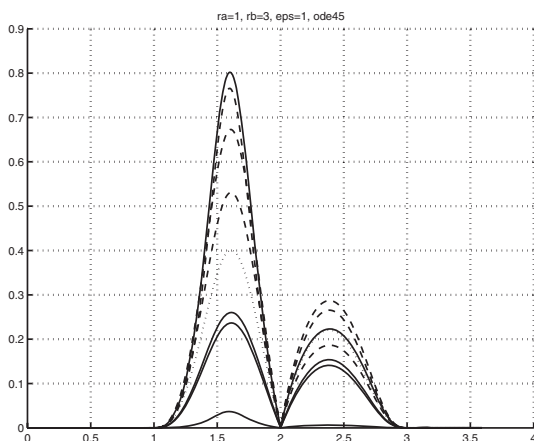


FIG. 5. Norms of the difference of flux lines for f and \hat{f} as functions of $\|x - x^*\|$.

To emphasize the fact that this property is closely related to the Newton–Raphson vector field and does not necessarily hold for all vector fields, the same tests were performed on a negative gradient field, and the results are shown in Figures 6 and 7. It is clear that flux lines for the perturbed function exit the region of perturbation at the points different from exit points of the flux lines for the original function and therefore converge to the solution x^* following different flows.

5.2. A three-dimensional example. The following three-dimensional example was constructed in a similar fashion. Let us again consider the quadratic function $f(x) = c + b^T x + \frac{1}{2} x^T B x$ and its perturbation $\hat{f} = f(x) + p(\|x - x^*\|)$ which differs from f on a sphere of inner radius r_a and outer radius r_b around the exact minimizer x^* . The perturbation term is given by the polynomial $p(s)$ defined in (5.1).

The particular values $r_a = 1$, $r_b = 3$, $\varepsilon = 1$, $c = 2$, $b = (2, -5, 3)^T$, and

$$B = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 4 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

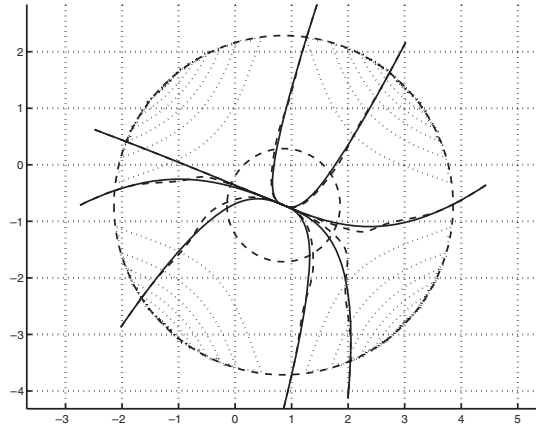


FIG. 6. Flux lines of the negative gradient field for the original (solid line) and the perturbed (dashed curves) function.

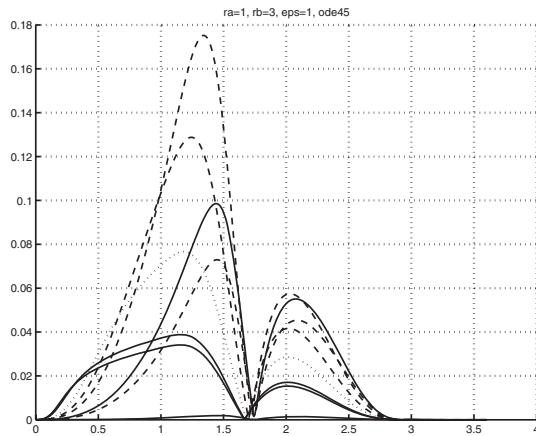


FIG. 7. Norms of the difference of the negative gradient field flux lines for f and \hat{f} as functions of $\|x - x^*\|$.

were used in the numerical experiments. In this case the exact minimizer is $x^* = (1, -2, 1)^T$.

Figures 8 and 9 show that the flux lines for f and \hat{f} again differ only in the region of perturbation.

6. Discussion. The main theorems of this paper highlight an interesting property of the global dynamics of the continuous Newton–Raphson method. For the purposes of a succinct analysis we assumed that f and its perturbation \hat{f} are both strictly convex functions. This assumption is merely a sufficient condition that could be relaxed in many ways. In fact, using the topological results of Janovsky and Seige [5], it is possible in principle to extend the results presented here to the case where perturbations are small and where neither $H_f(x)$ nor $H_{\hat{f}}(x)$ is singular anywhere along $\varphi(x_0, t)$ and $\hat{\varphi}(x_0, t)$, guaranteeing that the flux lines cannot bifurcate. Our numerical experiments confirm this where we have chosen nonconvex examples which tend to make the claim of Theorem 4.5 more clearly visible than convex examples.

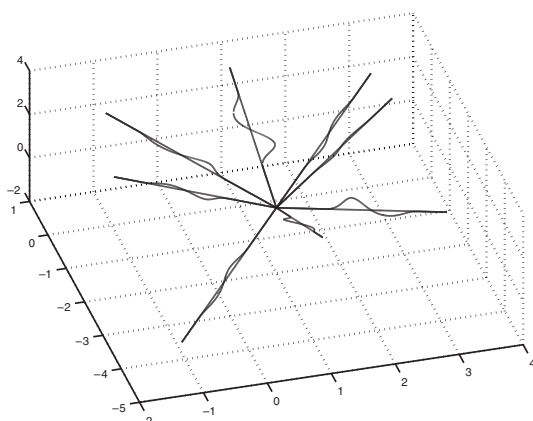
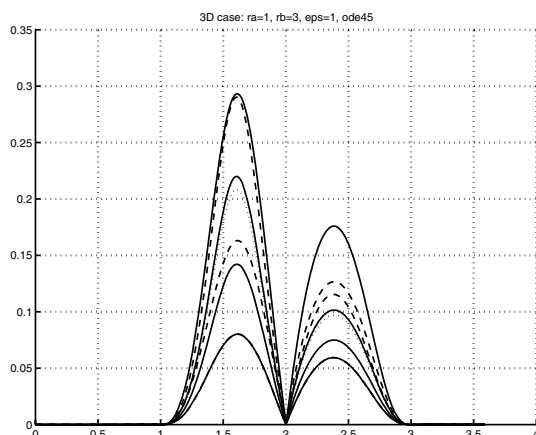


FIG. 8. Flux lines of original and perturbed function in the three-dimensional case.

FIG. 9. Norms of a difference of flux lines for f and \hat{f} as functions of $\|x - x^*\|$ in the three-dimensional case.

As pointed out by one of the referees, Theorem 4.5 can be proved without resorting to the tools of convex analysis.

LEMMA 6.1. *Let $f \in C^2$ be a function with positive definite Hessian $H \succ 0$ for every x in some neighborhood \mathcal{D} of a strict local minimizer x^* , and $y, z \in \mathcal{D}$. If $\nabla f(y) = \nabla f(z)$, then $y = z$.*

Proof. First, note that the condition $H(x) \succ 0$ on \mathcal{D} implies that the function f is strictly convex on \mathcal{D} . Let $p := \nabla f(y) = \nabla f(z)$ and let us define a new function $g(x) = f(x) - \langle p; x \rangle$. Since g is obtained by adding a linear term to a strictly convex function f , g is also strictly convex. By straightforward calculation one obtains that $\nabla g(y) = \nabla g(z) = 0$, which, together with strict convexity of g , implies that both y and z are global minimizers of g and therefore $g(y) = g(z)$. If $y \neq z$, this implies that for every point $w = \lambda y + (1 - \lambda)z$, $\lambda \in (0, 1)$ one gets $g(y) = g(w) = g(z)$, which is a contradiction with g being strictly convex. Therefore $y = z$ must hold. \square

This approach yields a slightly more general statement of Theorem 4.5 obtained by excluding the conditions $L_{f(x_0)}^f \subset \mathcal{D}$ and $L_{\hat{f}(x_0)}^{\hat{f}} \subset \mathcal{D}$.

THEOREM 6.2. Let f be a C^2 function with $H_f \succ 0$ for all $x \in \mathcal{D}$, and let $x^* \in \mathcal{D}$ be a strict local minimizer of f . Let \hat{f} be a regular perturbation of f , and let $\varphi(x_0, t)$ and $\hat{\varphi}(x_0, t)$ be the Newton-Raphson-flows defined by f and \hat{f} . Let x_0 be a point in $\mathcal{D} \setminus cl(\mathcal{D}_P)$. Then

$$\Gamma_\varphi(x_0) \Delta \Gamma_{\hat{\varphi}}(x_0) \subset \mathcal{D}_P$$

and $\varphi(x_0, t) = \hat{\varphi}(x_0, t)$ for all $t \in \mathbb{R}$ such that $\varphi(x_0, t) \notin \mathcal{D}_P$ or $\hat{\varphi}(x_0, t) \notin \mathcal{D}_P$.

Proof. Suppose $t > 0$ is such that $\varphi(x_0, t) \notin \mathcal{D}_P$. Then

$$\nabla \hat{f}(\varphi(x_0, t)) = \nabla f(\varphi(x_0, t)) \stackrel{(2.1)}{=} e^{-t} \nabla f(x_0) = e^{-t} \nabla \hat{f}(x_0) \stackrel{(2.1)}{=} \nabla \hat{f}(\hat{\varphi}(x_0, t)),$$

where the first and the third equality hold because $f = \hat{f}$ outside the region of perturbation \mathcal{D}_P . Since \hat{f} is strictly convex and $\nabla \hat{f}(\varphi(x_0, t)) = \nabla \hat{f}(\hat{\varphi}(x_0, t))$, Lemma 6.1 implies that $\varphi(x_0, t) = \hat{\varphi}(x_0, t)$. In a similar fashion, one can show that if $\hat{\varphi}(x_0, t) \notin \mathcal{D}_P$, then $\nabla f(\hat{\varphi}(x_0, t)) = \nabla f(\varphi(x_0, t))$, and therefore strict convexity of f and Lemma 6.1 imply that again $\varphi(x_0, t) = \hat{\varphi}(x_0, t)$. This shows that the flux lines can differ only inside the region of perturbation, which concludes the proof. \square

The authors would like to thank the anonymous referee for this interesting observation.

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