

DIFFERENTIATION

The Derivative as a Function

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

If f' exists at a particular x , we say that f is **differentiable (has a derivative)** at x . If f' exists at every point in the domain of f , we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x .

Derivative of f at x is

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} \\ &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \end{aligned}$$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}.$$

EXAMPLE 1 Applying the Definition

Differentiate $f(x) = \frac{x}{x-1}$.

Solution Here we have $f(x) = \frac{x}{x-1}$

and

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \frac{\frac{x+h}{x+h-1} - \frac{x}{x-1}}{h} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \\
&= \lim_{h \rightarrow 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}.
\end{aligned}$$

EXAMPLE 2 Derivative of the Square Root Function

- (a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.
- (b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

(a) We use the equivalent form to calculate f' :

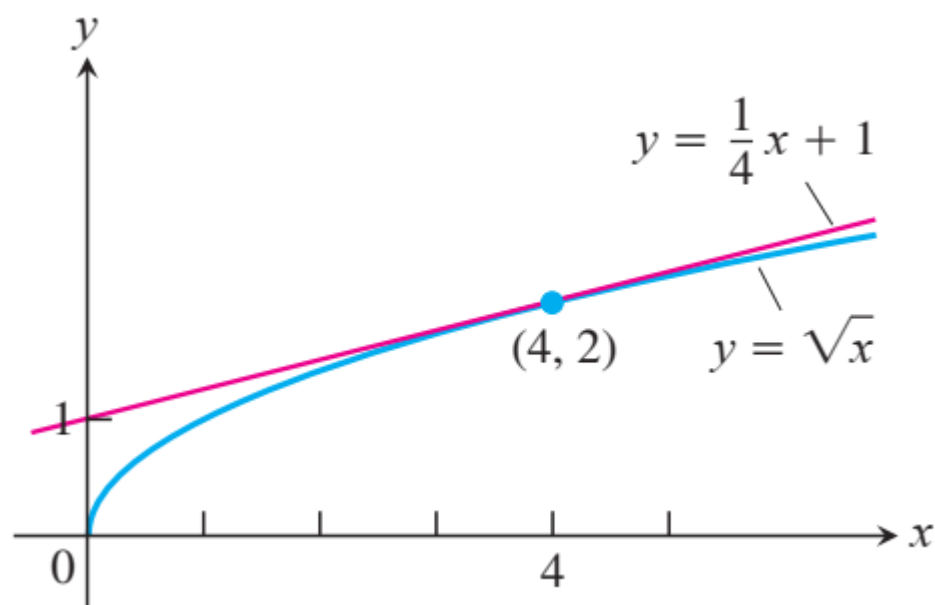
$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z \rightarrow x} \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z \rightarrow x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point $(4, 2)$ with slope $1/4$ (Figure 3.2):

$$y = 2 + \frac{1}{4}(x - 4)$$



Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \rightarrow 0^+} \frac{f(a + h) - f(a)}{h} \quad \textbf{Right-hand derivative at } a$$

$$\lim_{h \rightarrow 0^-} \frac{f(b + h) - f(b)}{h} \quad \textbf{Left-hand derivative at } b$$

exist at the endpoints

Note that a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal

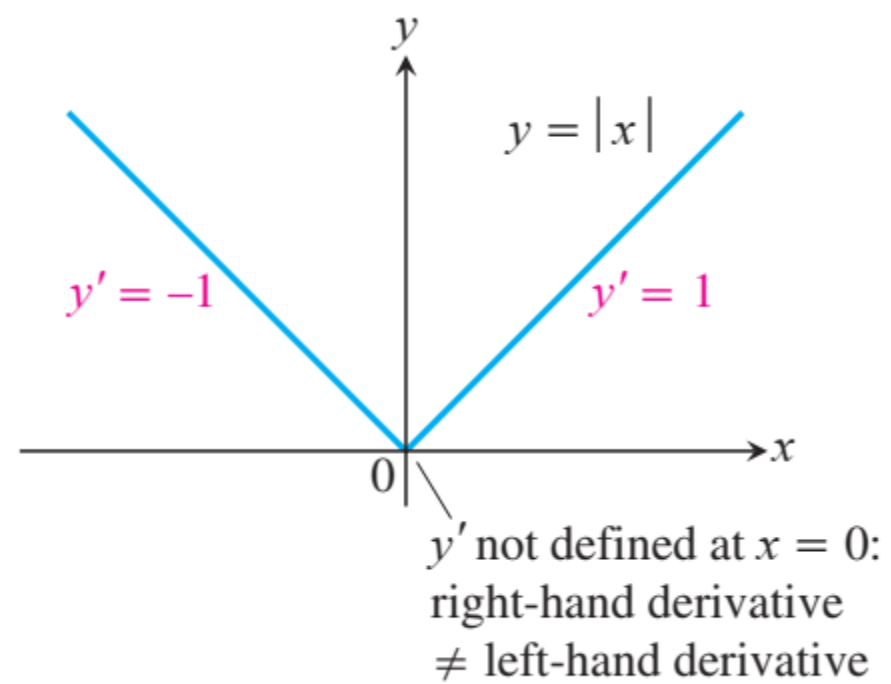
EXAMPLE 5 $y = |x|$ Is Not Differentiable at the Origin

Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Note that at $x=0$, RHD is not equal to LHD. Thus the function is not differentiable at $x=0$.

$$\begin{aligned}\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^+} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{h}{h} \quad |h| = h \text{ when } h > 0. \\ &= \lim_{h \rightarrow 0^+} 1 = 1\end{aligned}$$

$$\begin{aligned}\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h \rightarrow 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \rightarrow 0^-} \frac{|h|}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} \quad |h| = -h \text{ when } h < 0. \\ &= \lim_{h \rightarrow 0^-} -1 = -1.\end{aligned}$$



EXAMPLE 6 $y = \sqrt{x}$ Is Not Differentiable at $x = 0$

In Example 2 we found that for $x > 0$,

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}.$$

We apply the definition to examine if the derivative exists at $x = 0$:

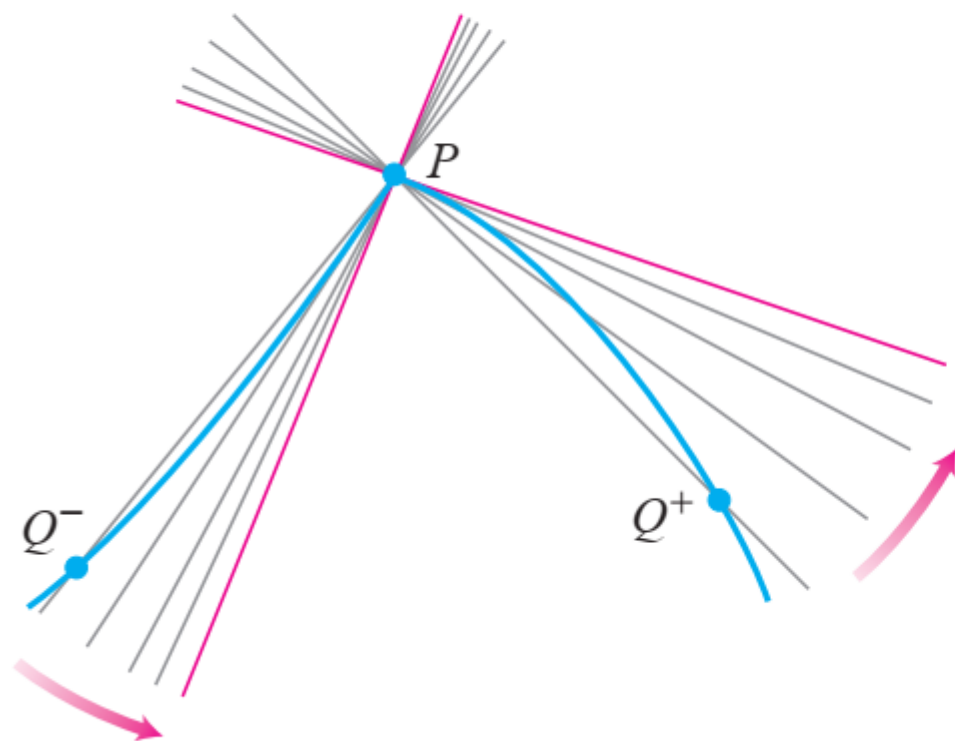
$$\lim_{h \rightarrow 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h \rightarrow 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$.

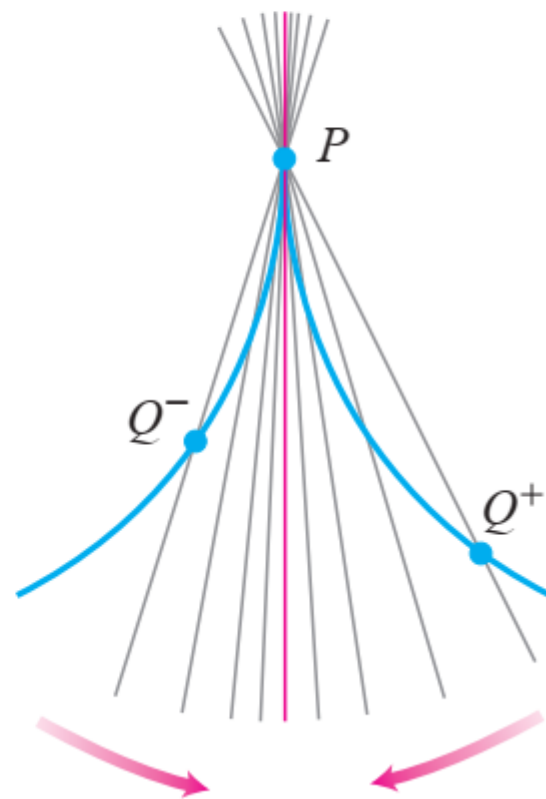
When Does a Function *Not* Have a Derivative at a Point?

A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a limit as Q approaches P . Whenever the secants fail to take up a limiting position or become vertical as Q approaches P , the derivative does not exist. Thus differentiability is a “smoothness” condition on the graph of f . A function whose graph is otherwise smooth will fail to have a derivative at a point for several reasons, such as at points where the graph has

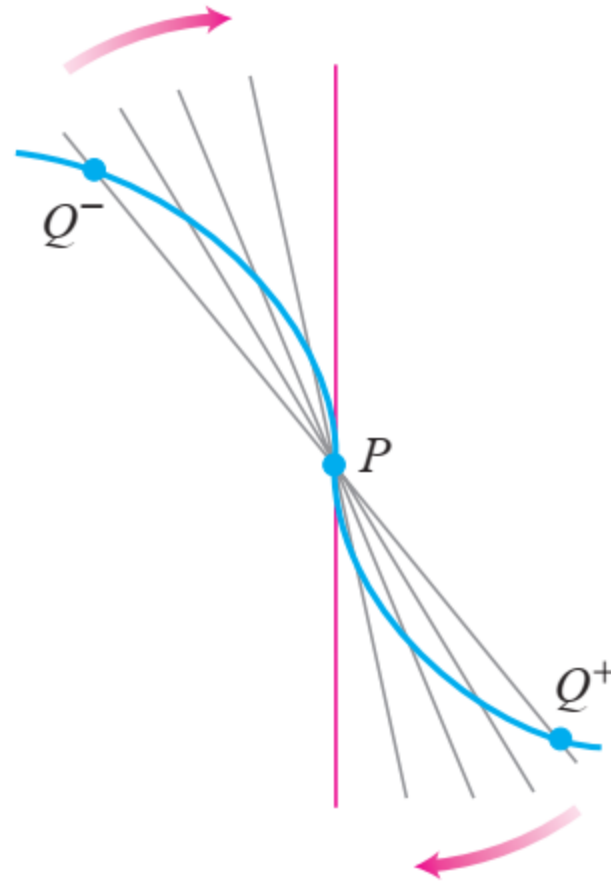
1. a *corner*, where the one-sided derivatives differ.



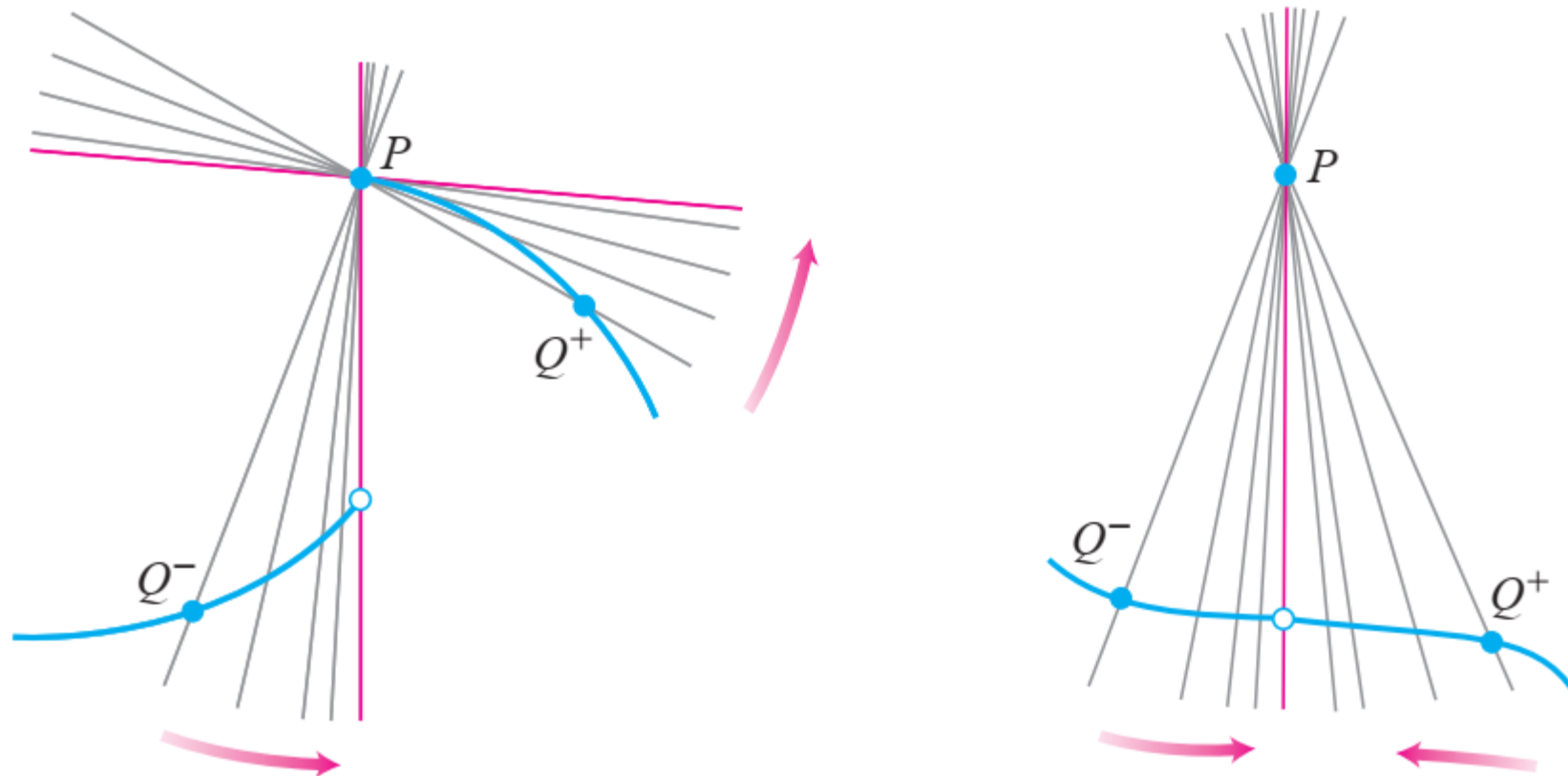
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a discontinuity.



Differentiable Functions Are Continuous

THEOREM 1 **Differentiability Implies Continuity**

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Finding Derivative Functions

1. $f(x) = 4 - x^2$; $f'(-3)$, $f'(0)$, $f'(1)$

Step 1: $f(x) = 4 - x^2$ and $f(x + h) = 4 - (x + h)^2$

Step 2:
$$\frac{f(x+h)-f(x)}{h} = \frac{[4-(x+h)^2] - (4-x^2)}{h} = \frac{(4-x^2-2xh-h^2) - 4 + x^2}{h} = \frac{-2xh-h^2}{h} = \frac{h(-2x-h)}{h}$$
$$= -2x - h$$

Step 3: $f'(x) = \lim_{h \rightarrow 0} (-2x - h) = -2x$; $f'(-3) = 6$, $f'(0) = 0$, $f'(1) = -2$

4. $k(z) = \frac{1-z}{2z}$; $k'(-1)$, $k'(1)$, $k'(\sqrt{2})$

$$\begin{aligned} k(z) &= \frac{1-z}{2z} \text{ and } k(z+h) = \frac{1-(z+h)}{2(z+h)} \Rightarrow k'(z) = \lim_{h \rightarrow 0} \frac{\left(\frac{1-(z+h)}{2(z+h)} - \frac{1-z}{2z}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1-z-h)z - (1-z)(z+h)}{2(z+h)zh} = \lim_{h \rightarrow 0} \frac{z - z^2 - zh - z - h + z^2 + zh}{2(z+h)zh} = \lim_{h \rightarrow 0} \frac{-h}{2(z+h)zh} = \lim_{h \rightarrow 0} \frac{-1}{2(z+h)z} \\ &= \frac{-1}{2z^2}; k'(-1) = -\frac{1}{2}, k'(1) = -\frac{1}{2}, k'(\sqrt{2}) = -\frac{1}{4} \end{aligned}$$

find the indicated derivatives.

$$\frac{dz}{dw} \quad \text{if} \quad z = \frac{1}{\sqrt{3w-2}}$$

$$\begin{aligned}\frac{dz}{dw} &= \lim_{h \rightarrow 0} \frac{\left(\frac{1}{\sqrt{3(w+h)-2}} - \frac{1}{\sqrt{3w-2}} \right)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{3w-2} - \sqrt{3w+3h-2}}{h\sqrt{3w+3h-2}\sqrt{3w-2}} \\&= \lim_{h \rightarrow 0} \frac{(\sqrt{3w-2} - \sqrt{3w+3h-2})}{h\sqrt{3w+3h-2}\sqrt{3w-2}} \cdot \frac{(\sqrt{3w-2} + \sqrt{3w+3h-2})}{(\sqrt{3w-2} + \sqrt{3w+3h-2})} \\&= \lim_{h \rightarrow 0} \frac{(3w-2) - (3w+3h-2)}{h\sqrt{3w+3h-2}\sqrt{3w-2}(\sqrt{3w-2} + \sqrt{3w+3h-2})} \\&= \lim_{h \rightarrow 0} \frac{-3}{\sqrt{3w+3h-2}\sqrt{3w-2}(\sqrt{3w-2} + \sqrt{3w+3h-2})} = \frac{-3}{\sqrt{3w-2}\sqrt{3w-2}(\sqrt{3w-2} + \sqrt{3w-2})} \\&= \frac{-3}{2(3w-2)\sqrt{3w-2}}\end{aligned}$$

find the values of the derivatives.

$$\left. \frac{dy}{dx} \right|_{x=\sqrt{3}} \quad \text{if } y = 1 - \frac{1}{x}$$

$$\begin{aligned} y = f(x) = 1 - \frac{1}{x} \text{ and } f(x+h) = 1 - \frac{1}{x+h} &\Rightarrow \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\left(1 - \frac{1}{x+h}\right) - \left(1 - \frac{1}{x}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{1}{x} - \frac{1}{x+h}}{h} = \lim_{h \rightarrow 0} \frac{h}{x(x+h)h} = \lim_{h \rightarrow 0} \frac{1}{x(x+h)} = \frac{1}{x^2} \Rightarrow \left. \frac{dy}{dx} \right|_{x=\sqrt{3}} = \frac{1}{3} \end{aligned}$$

Assignment: Exercise 3.1 Question #39-44.

Differentiation Rules

RULE 1 **Derivative of a Constant Function**

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

RULE 2 **Power Rule for Positive Integers**

If n is a positive integer, then

$$\frac{d}{dx} x^n = nx^{n-1}.$$

EXAMPLE 2 Interpreting Rule 2

f	x	x^2	x^3	x^4	\dots
f'	1	$2x$	$3x^2$	$4x^3$	\dots

RULE 3 Constant Multiple Rule

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c \frac{du}{dx}.$$

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 4 Derivative of a Sum

$$y = x^4 + 12x$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \\ &= 4x^3 + 12\end{aligned}$$

Note that:

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

EXAMPLE 5 Derivative of a Polynomial

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0$$

$$= 3x^2 + \frac{8}{3}x - 5$$

EXAMPLE 6 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. We have,

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x :

$$\begin{aligned}4x^3 - 4x &= 0 \\4x(x^2 - 1) &= 0 \\x &= 0, 1, -1.\end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$, and -1 . The corresponding points on the curve are $(0, 2)$, $(1, 1)$ and $(-1, 1)$. See Figure 3.10. ■

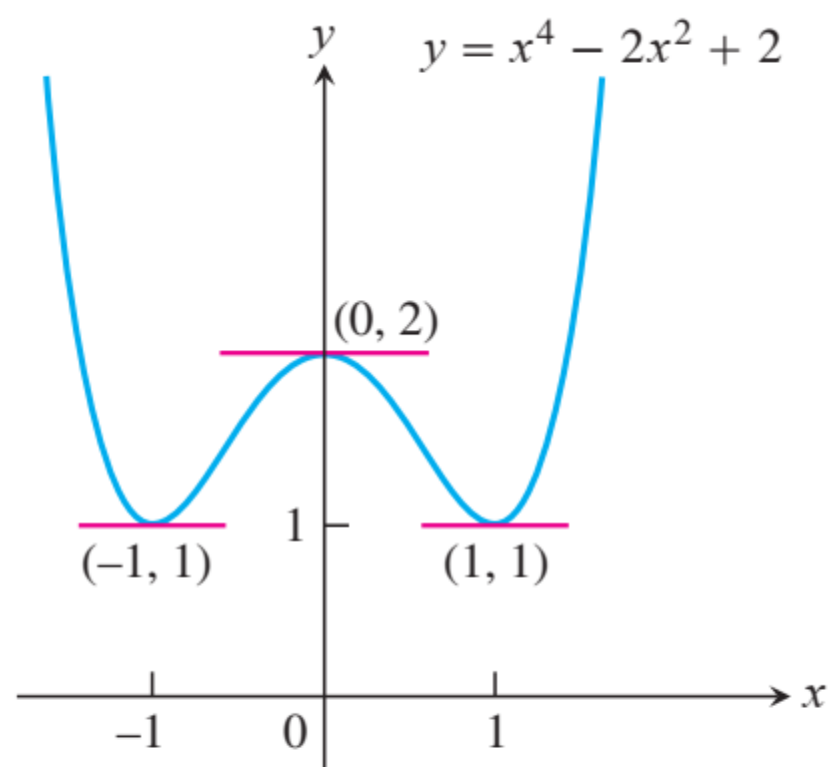


FIGURE 3.10 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

RULE 5 Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$

EXAMPLE 7 Using the Product Rule

Find the derivative of

$$y = \frac{1}{x} \left(x^2 + \frac{1}{x} \right).$$

Solution We apply the Product Rule with $u = 1/x$ and $v = x^2 + (1/x)$:

$$\frac{d}{dx} \left[\frac{1}{x} \left(x^2 + \frac{1}{x} \right) \right] = \frac{1}{x} \left(2x - \frac{1}{x^2} \right) + \left(x^2 + \frac{1}{x} \right) \left(-\frac{1}{x^2} \right)$$

$$\begin{aligned} &= 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3} \\ &= 1 - \frac{2}{x^3}. \end{aligned}$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x , and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}.$$


EXAMPLE 10 Using the Quotient Rule

Find the derivative of

$$y = \frac{t^2 - 1}{t^2 + 1}.$$

Solution

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^2 + 1) \cdot 2t - (t^2 - 1) \cdot 2t}{(t^2 + 1)^2} & \frac{d}{dt} \left(\frac{u}{v} \right) &= \frac{v(du/dt) - u(dv/dt)}{v^2} \\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2 + 1)^2} \\ &= \frac{4t}{(t^2 + 1)^2}.\end{aligned}$$


Proof of Rule 6

$$\begin{aligned}\frac{d}{dx} \left(\frac{u}{v} \right) &= \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx} \left(\frac{u}{v} \right) &= \lim_{h \rightarrow 0} \frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)} \\ &= \lim_{h \rightarrow 0} \frac{v(x) \frac{u(x+h) - u(x)}{h} - u(x) \frac{v(x+h) - v(x)}{h}}{v(x+h)v(x)}.\end{aligned}$$

Taking the limit in the numerator and denominator now gives the Quotient Rule. ■

RULE 7 **Power Rule for Negative Integers**

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 11

$$\text{(a)} \quad \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{d}{dx} (x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

$$\text{(b)} \quad \frac{d}{dx} \left(\frac{4}{x^3} \right) = 4 \frac{d}{dx} (x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

EXAMPLE 13 Choosing Which Rule to Use

Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4},$$

expand the numerator and divide by x^4 :

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\begin{aligned}\frac{dy}{dx} &= -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4} \\ &= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.\end{aligned}$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f'' .

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dy'}{dx}$$

. The function f'' is called the **second derivative** of f

EXAMPLE 14 Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

The function has derivatives of all orders, the fifth and later derivatives all being zero.



Exercise 3.2 Question # 1-36.