**Book:**

# Thomas' Calculus

**Authors:**

**George Thomas Thomas, J., Massachusetts Institute of Technology**

**Maurice D Weir, Naval Postgraduate School**

**Joel R. Hass, University of California, Davis**

**Frank R. Giordano, Naval Postgraduate School**

# Preliminaries

# Real Numbers and the Real Line

## Real Numbers

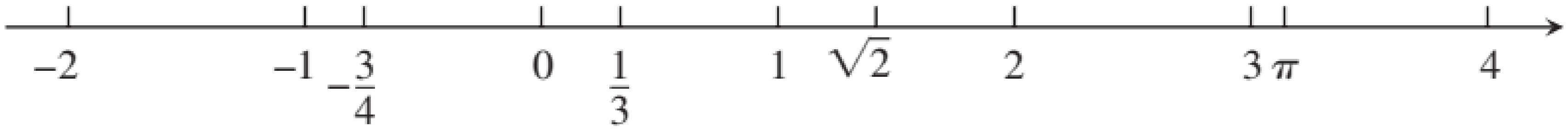
Much of calculus is based on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, such as

$$-\frac{3}{4} = -0.75000\ldots$$

$$\frac{1}{3} = 0.33333\ldots$$

$$\sqrt{2} = 1.4142\ldots$$

The real numbers can be represented geometrically as points on a number line called the **real line**.




The symbol $\mathbb{R}$ denotes either the real number system or, equivalently, the real line.

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. The **algebraic properties** say that the real numbers can be added, subtracted, multiplied, and divided (except by 0) to produce more real numbers under the usual rules of arithmetic. *You can never divide by* 0.

### Rules for Inequalities

If $a$, $b$, and $c$ are real numbers, then:

**1.** $a < b \Rightarrow a + c < b + c$

**2.** $a < b \Rightarrow a - c < b - c$

**3.** $a < b$ and $c > 0 \Rightarrow ac < bc$

**4.** $a < b$ and $c < 0 \Rightarrow bc < ac$
Special case: $a < b \Rightarrow -b < -a$

**5.** $a > 0 \Rightarrow \frac{1}{a} > 0$

**6.** If $a$ and $b$ are both positive or both negative, then $a < b \Rightarrow \frac{1}{b} < \frac{1}{a}$

We distinguish three special subsets of real numbers.

1. The **natural numbers**, namely $1, 2, 3, 4, \ldots$
2. The **integers**, namely $0, \pm 1, \pm 2, \pm 3, \ldots$
3. The **rational numbers**, namely the numbers that can be expressed in the form of a fraction $m/n$, where $m$ and $n$ are integers and $n \neq 0$. Examples are

$$\frac{1}{3}, \quad -\frac{4}{9} = \frac{-4}{9} = \frac{4}{-9}, \quad \frac{200}{13}, \quad \text{and} \quad 57 = \frac{57}{1}.$$

Real numbers that are not rational are called **irrational numbers**.

## Intervals

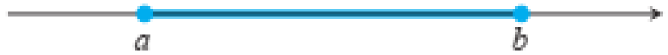
A subset of the real line is called an **interval** if it contains at least two numbers and contains all the real numbers lying between any two of its elements. For example, the set of all real numbers $x$ such that $x > 6$ is an interval, as is the set of all $x$ such that $-2 \le x \le 5$.

The set of all nonzero real numbers is not an interval; since 0 is absent, the set fails to contain every real number between $-1$ and 1 (for example).

Geometrically, intervals correspond to rays and line segments on the real line, along with the real line itself. Intervals of numbers corresponding to line segments are **finite intervals**; intervals corresponding to rays and the real line are **infinite intervals**.

A finite interval is said to be **closed** if it contains both of its endpoints, **half-open** if it contains one endpoint but not the other, and **open** if it contains neither endpoint. The endpoints are also called **boundary points**; they make up the interval's **boundary**. The remaining points of the interval are **interior points**

**TABLE 1.1** Types of intervals

| | Notation | Set description | Type | Picture |
|---|---|---|---|---|
| **Finite:** | $(a, b)$ | $\{x \mid a < x < b\}$ | Open |  |
| | $[a, b]$ | $\{x \mid a \leq x \leq b\}$ | Closed |  |
| | $[a, b)$ | $\{x \mid a \leq x < b\}$ | Half-open |  |
| | $(a, b]$ | $\{x \mid a < x \leq b\}$ | Half-open |  |
| **Infinite:** | $(a, \infty)$ | $\{x \mid x > a\}$ | Open |  |
| | $[a, \infty)$ | $\{x \mid x \geq a\}$ | Closed |  |
| | $(-\infty, b)$ | $\{x \mid x < b\}$ | Open |  |
| | $(-\infty, b]$ | $\{x \mid x \leq b\}$ | Closed |  |
| | $(-\infty, \infty)$ | $\mathbb{R}$ (set of all real numbers) | Both open and closed |  |

**EXAMPLE 1** Solve the following inequalities and show their solution sets on the real line.

**(a)** $2x - 1 < x + 3$ **(b)** $-\frac{x}{3} < 2x + 1$ **(c)** $\frac{6}{x - 1} \geq 5$

**Solution**

**(a)**

$$\begin{aligned} 2x - 1 &< x + 3 \\ 2x &< x + 4 && \text{Add 1 to both sides.} \\ x &< 4 && \text{Subtract } x \text{ from both sides.} \end{aligned}$$

The solution set is the open interval $(-\infty, 4)$

**(b)**

$$\begin{aligned} -\frac{x}{3} &< 2x + 1 \\ -x &< 6x + 3 && \text{Multiply both sides by 3.} \\ 0 &< 7x + 3 && \text{Add } x \text{ to both sides.} \\ -3 &< 7x && \text{Subtract 3 from both sides.} \\ -\frac{3}{7} &< x && \text{Divide by 7.} \end{aligned}$$

The solution set is the open interval $(-3/7, \infty)$

**(c)** The inequality $6/(x-1) \geq 5$ can hold only if $x > 1$, because otherwise $6/(x-1)$ is undefined or negative. Therefore, $(x-1)$ is positive and the inequality will be preserved if we multiply both sides by $(x-1)$, and we have

$$\frac{6}{x-1} \geq 5$$

$$6 \geq 5x - 5 \qquad \text{Multiply both sides by } (x-1).$$

$$11 \geq 5x \qquad \text{Add 5 to both sides.}$$

$$\frac{11}{5} \geq x. \qquad \text{Or } x \leq \frac{11}{5}.$$

The solution set is the half-open interval $(1, 11/5]$

## Absolute Value

The **absolute value** of a number $x$, denoted by $|x|$, is defined by the formula

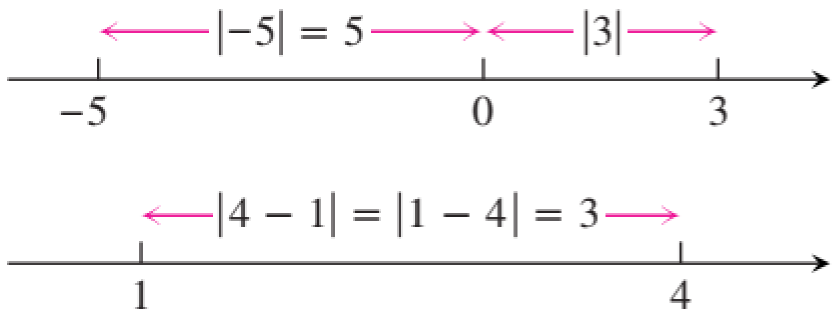
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

**EXAMPLE 2** Finding Absolute Values

$$|3| = 3, \quad |0| = 0, \quad |-5| = -(-5) = 5, \quad |-|a|| = |a|$$ ■

Geometrically, the absolute value of $x$ is the distance from $x$ to 0 on the real number line.

Also, $|x - y| =$ the distance between $x$ and $y$



## Absolute Value Properties

| | | |
|---|---|---|
| **1.** | $\lvert -a \rvert = \lvert a \rvert$ | A number and its additive inverse or negative have the same absolute value. |
| **2.** | $\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$ | The absolute value of a product is the product of the absolute values. |
| **3.** | $\left\lvert \dfrac{a}{b} \right\rvert = \dfrac{\lvert a \rvert}{\lvert b \rvert}$ | The absolute value of a quotient is the quotient of the absolute values. |
| **4.** | $\lvert a + b \rvert \le \lvert a \rvert + \lvert b \rvert$ | The **triangle inequality**. The absolute value of the sum of two numbers is less than or equal to the sum of their absolute values. |

## Absolute Values and Intervals

If $a$ is any positive number, then

**5.** $|x| = a$ if and only if $x = \pm a$

**6.** $|x| < a$ if and only if $-a < x < a$

**7.** $|x| > a$ if and only if $x > a$ or $x < -a$

**8.** $|x| \le a$ if and only if $-a \le x \le a$

**9.** $|x| \ge a$ if and only if $x \ge a$ or $x \le -a$

### EXAMPLE 4 Solving an Equation with Absolute Values

Solve the equation $|2x - 3| = 7$.

**Solution** By Property 5, $2x - 3 = \pm 7$, so there are two possibilities:

$$\begin{aligned} 2x - 3 &= 7 & \qquad 2x - 3 &= -7 & &\text{Equivalent equations without absolute values} \\ 2x &= 10 & 2x &= -4 & &\text{Solve as usual.} \\ x &= 5 & x &= -2 & & \end{aligned}$$

The solutions of $|2x - 3| = 7$ are $x = 5$ and $x = -2$. ■

## EXAMPLE 5 Solving an Inequality Involving Absolute Values

Solve the inequality $\left|5 - \frac{2}{x}\right| < 1$.

**Solution** We have

$$\begin{aligned}
\left|5 - \frac{2}{x}\right| < 1 &\Leftrightarrow -1 < 5 - \frac{2}{x} < 1 && \text{Property 6} \\
&\Leftrightarrow -6 < -\frac{2}{x} < -4 && \text{Subtract 5.} \\
&\Leftrightarrow 3 > \frac{1}{x} > 2 && \text{Multiply by } -\frac{1}{2}. \\
&\Leftrightarrow \frac{1}{3} < x < \frac{1}{2}. && \text{Take reciprocals.}
\end{aligned}$$

The solution set is the open interval $(1/3, 1/2)$.

**EXAMPLE 6** Solve the inequality and show the solution set on the real line:

**(a)** $|2x - 3| \leq 1$ **(b)** $|2x - 3| \geq 1$

**Solution**

**(a)**

$$\begin{aligned} |2x - 3| &\leq 1 && \\ -1 \leq 2x - 3 &\leq 1 && \text{Property 8} \\ 2 \leq 2x &\leq 4 && \text{Add 3.} \\ 1 \leq x &\leq 2 && \text{Divide by 2.} \end{aligned}$$

The solution set is the closed interval [1, 2] (Figure 1.4a).

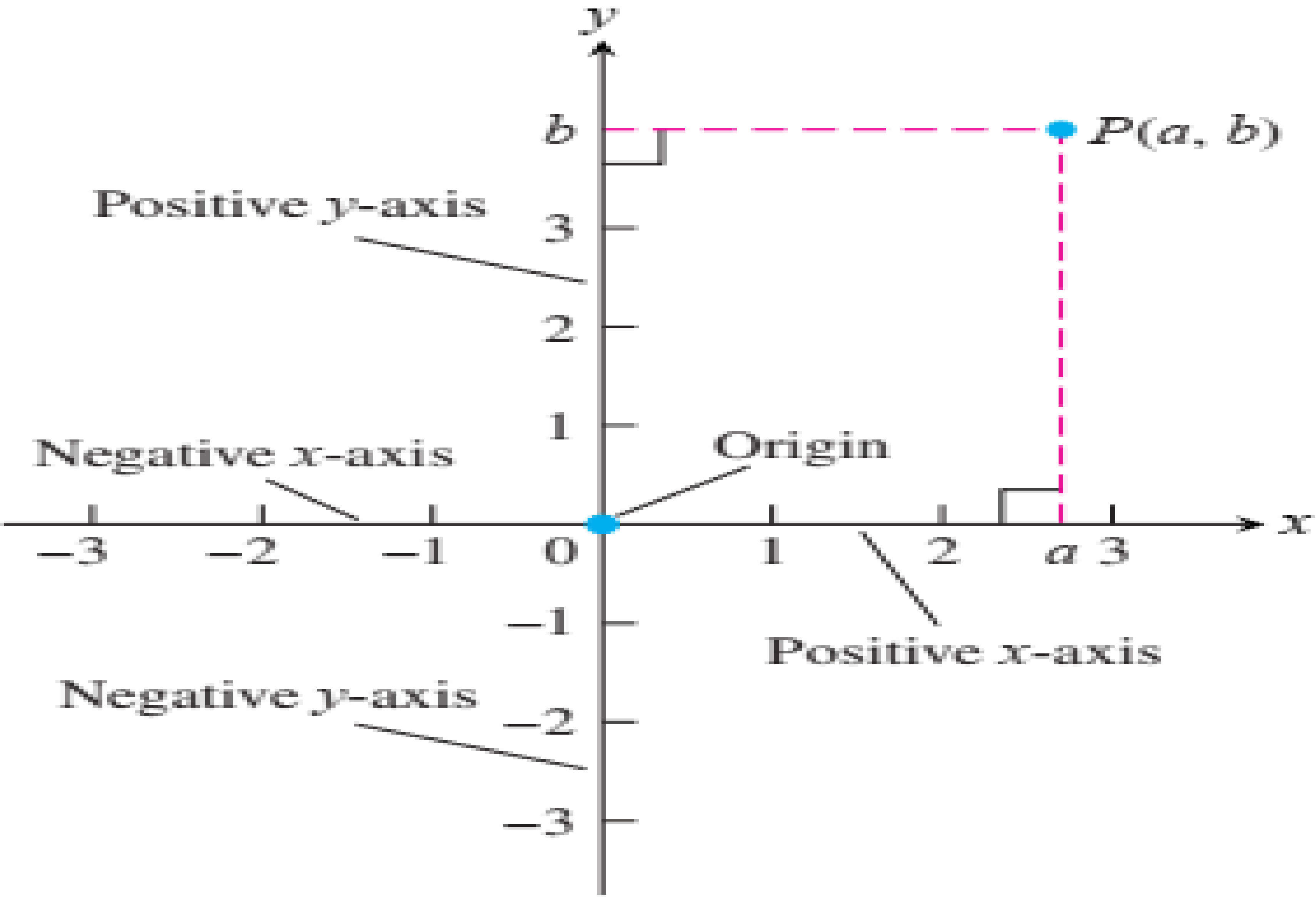
**(b)**

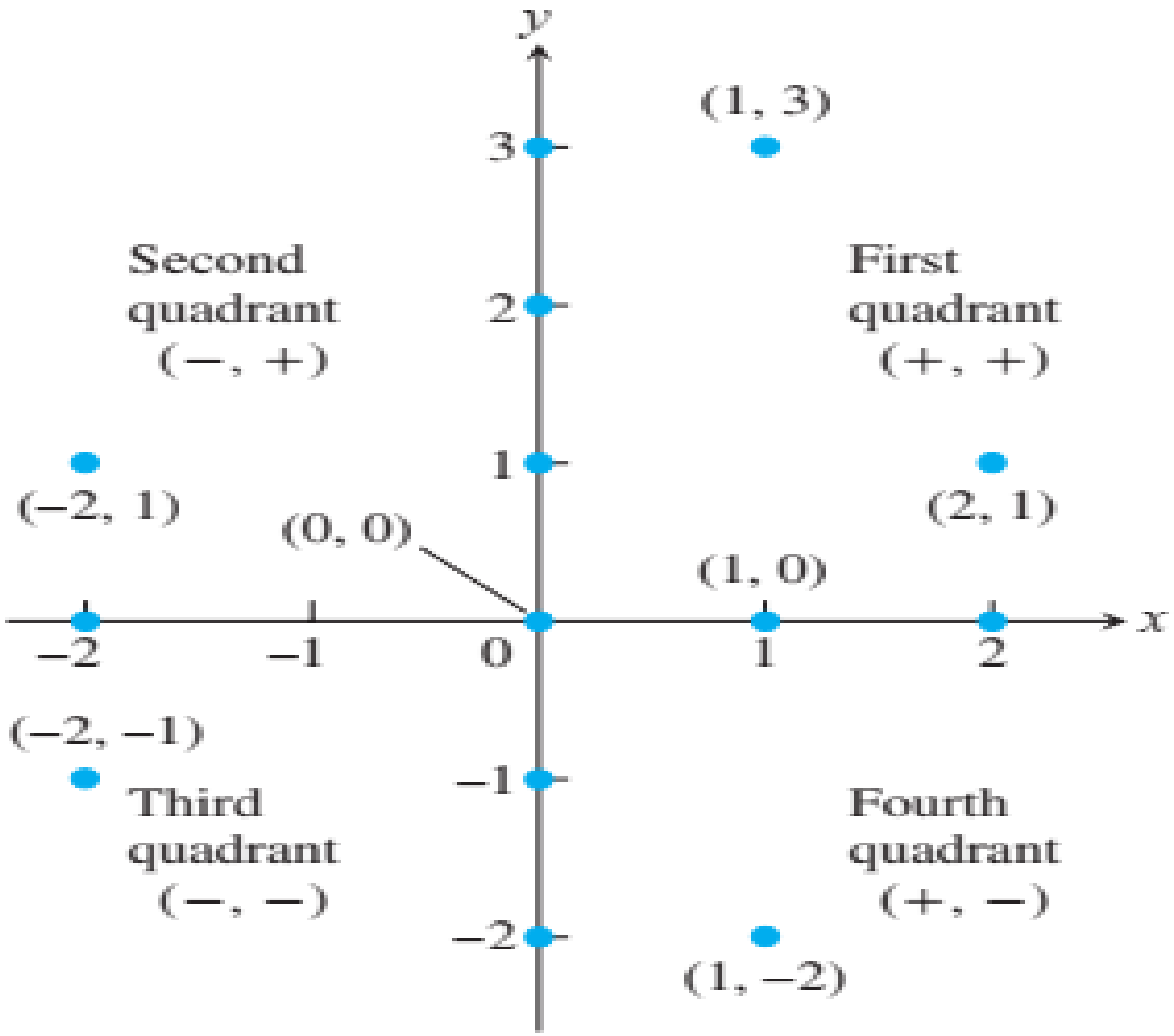
$$|2x - 3| \geq 1$$

$$\begin{aligned} 2x - 3 \geq 1 \quad &\text{or} \quad 2x - 3 \leq -1 && \text{Property 9} \\ x - \frac{3}{2} \geq \frac{1}{2} \quad &\text{or} \quad x - \frac{3}{2} \leq -\frac{1}{2} && \text{Divide by 2.} \\ x \geq 2 \quad &\text{or} \quad x \leq 1 && \text{Add } \frac{3}{2}. \end{aligned}$$

The solution set is $(-\infty, 1] \cup [2, \infty)$ (Figure 1.4b). ■

# Cartesian coordinate system




y
(1, 3)
3
Second
quadrant
(−, +)
2
First
quadrant
(+, +)
1
(−2, 1)
(2, 1)
(0, 0)
(1, 0)
x
−2
−1
0
1
2
(−2, −1)
−1
Third
quadrant
(−, −)
Fourth
quadrant
(+, −)
−2
(1, −2)

**DEFINITION Function**

A **function** from a set $D$ to a set $Y$ is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

The set $D$ of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as $x$ varies throughout $D$ is called the **range** of the function. The range may not include every element in the set $Y$.

The domain and range of a function can be any sets of objects, but often in calculus they are sets of real numbers.

When the range of a function is a set of real numbers, the function is said to be **real-valued**.

Verify the domains and ranges of these functions.

| **Function** | **Domain ($x$)** | **Range ($y$)** |
|---|---|---|
| $y = x^2$ | $(-\infty, \infty)$ | $[0, \infty)$ |
| $y = 1/x$ | $(-\infty, 0) \cup (0, \infty)$ | $(-\infty, 0) \cup (0, \infty)$ |
| $y = \sqrt{x}$ | $[0, \infty)$ | $[0, \infty)$ |
| $y = \sqrt{4 - x}$ | $(-\infty, 4]$ | $[0, \infty)$ |
| $y = \sqrt{1 - x^2}$ | $[-1, 1]$ | $[0, 1]$ |

## Graphs of Functions

Another way to visualize a function is its graph. If $f$ is a function with domain $D$, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for $f$. In set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}.$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates $(x, y)$ for which $y = x + 2$. Its graph is sketched in Figure 1.24.

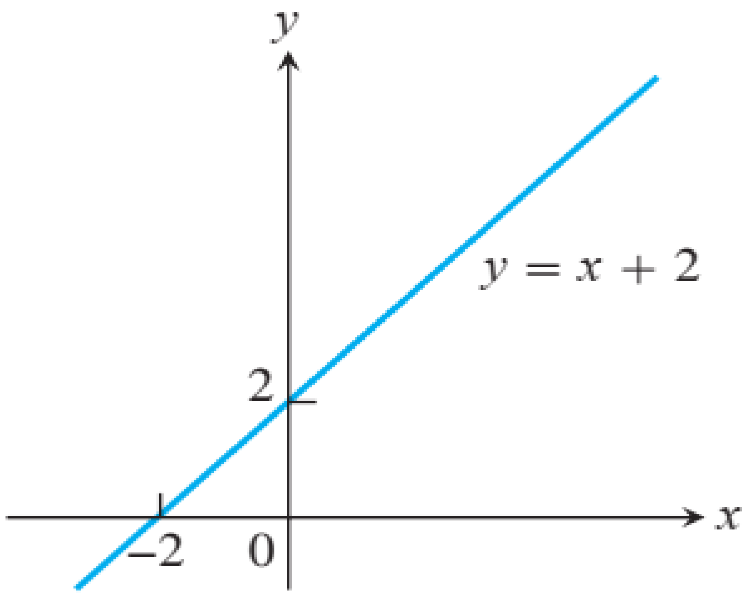
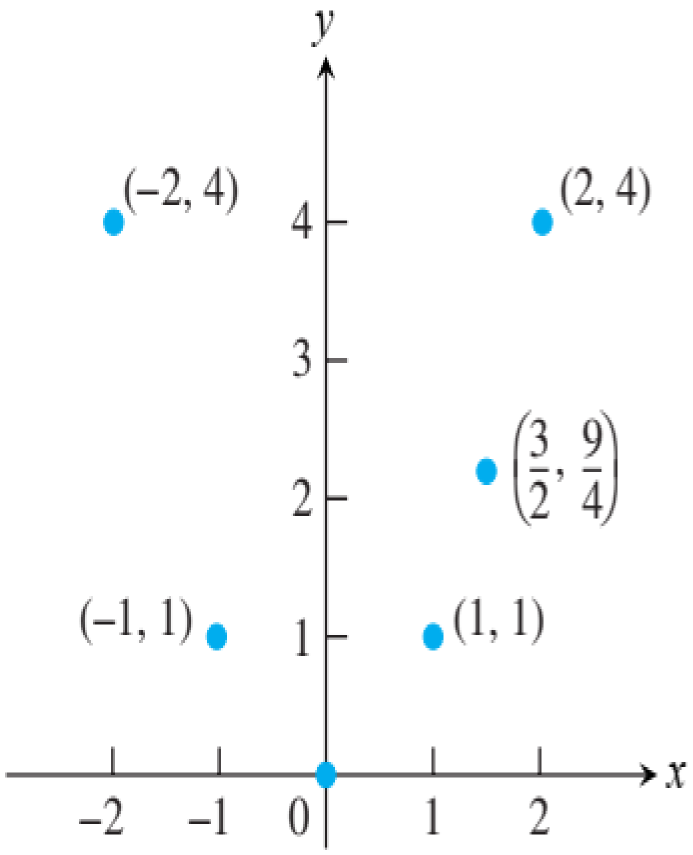


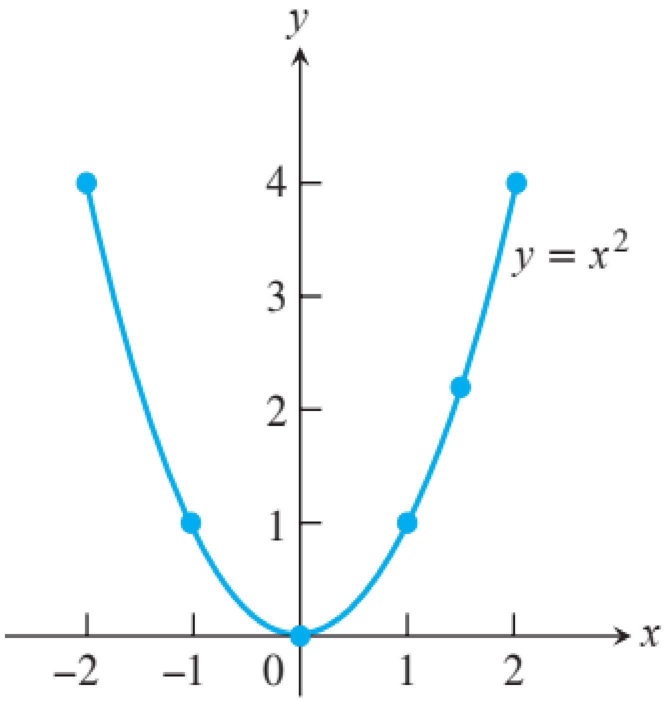

**FIGURE 1.24** The graph of $f(x) = x + 2$ is the set of points $(x, y)$ for which $y$ has the value $x + 2$.

## EXAMPLE 2 Sketching a Graph

Graph the function $y = x^2$ over the interval $[-2, 2]$.

| $x$ | $y = x^2$ |
|---|---|
| $-2$ | $4$ |
| $-1$ | $1$ |
| $0$ | $0$ |
| $1$ | $1$ |
| $\frac{3}{2}$ | $\frac{9}{4}$ |
| $2$ | $4$ |



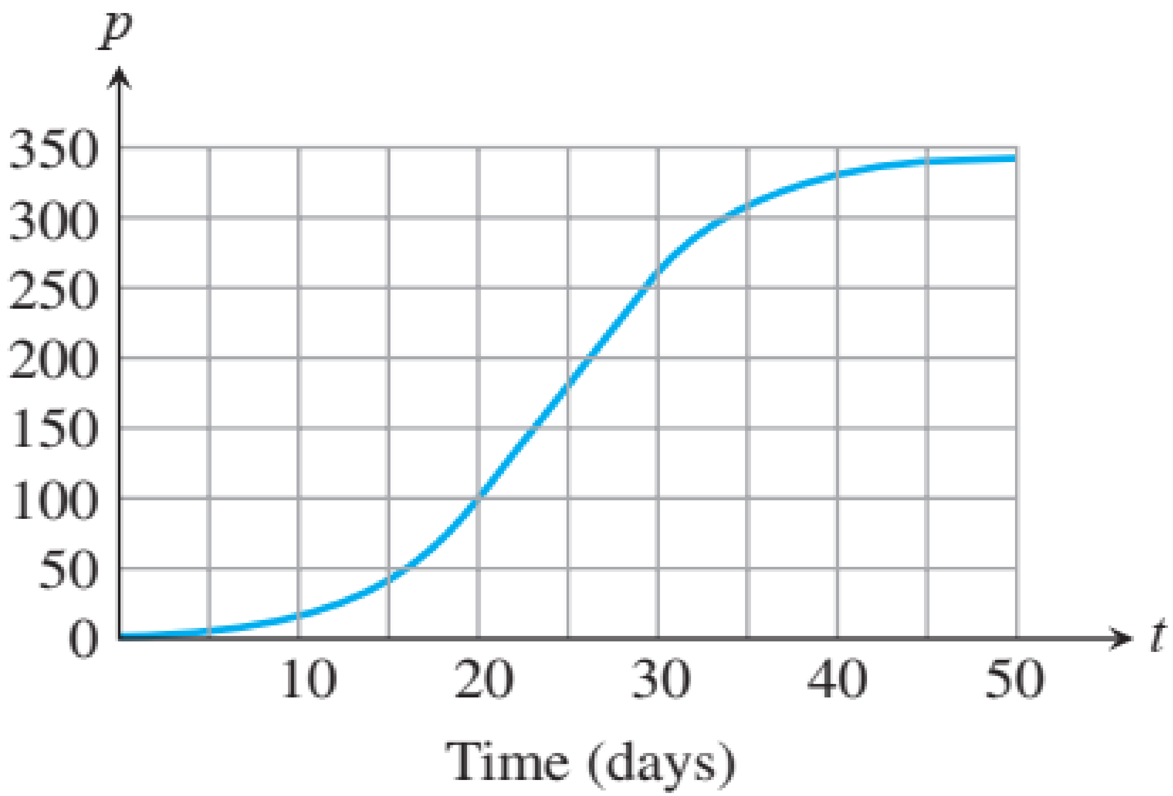



## EXAMPLE 3 Evaluating a Function from Its Graph

The graph of a fruit fly population $p$ is shown in Figure 1.26.

**(a)** Find the populations after 20 and 45 days.

**(b)** What is the (approximate) range of the population function over the time interval $0 \leq t \leq 50$?



**Solution**

**(a)** We see from Figure 1.26 that the point (20, 100) lies on the graph, so the value of the population $p$ at 20 is $p(20) = 100$. Likewise, $p(45)$ is about 340.

**(b)** The range of the population function over $0 \leq t \leq 50$ is approximately [0, 345]. We also observe that the population appears to get closer and closer to the value $p = 350$ as time advances. ■

## The Vertical Line Test

Not every curve you draw is the graph of a function. A function $f$ can have only one value $f(x)$ for each $x$ in its domain, so no *vertical line* can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function since some vertical lines intersect the circle twice (Figure 1.28a). If $a$ is in the domain of a function $f$, then the vertical line $x = a$ will intersect the graph of $f$ in the single point $(a, f(a))$.

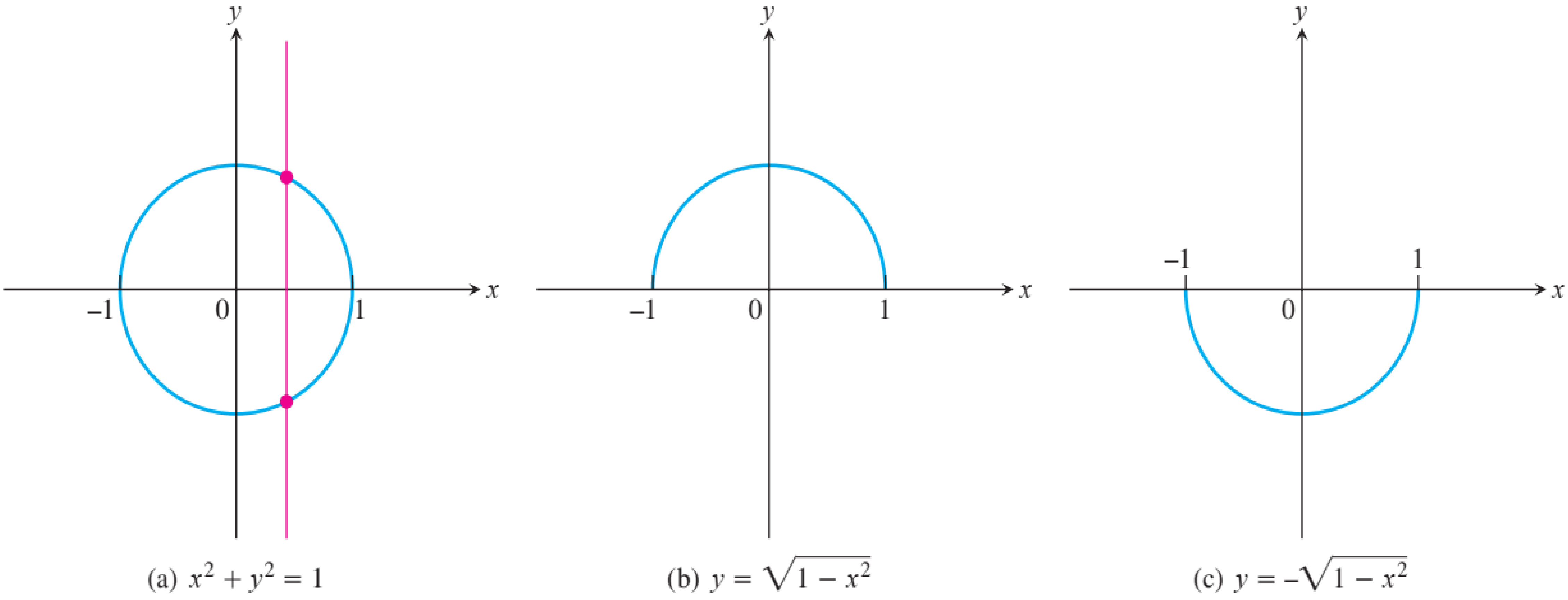



**FIGURE 1.28** (a) The circle is not the graph of a function; it fails the vertical line test. (b) The upper semicircle is the graph of a function $f(x) = \sqrt{1 - x^2}$. (c) The lower semicircle is the graph of a function $g(x) = -\sqrt{1 - x^2}$.

## Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function**

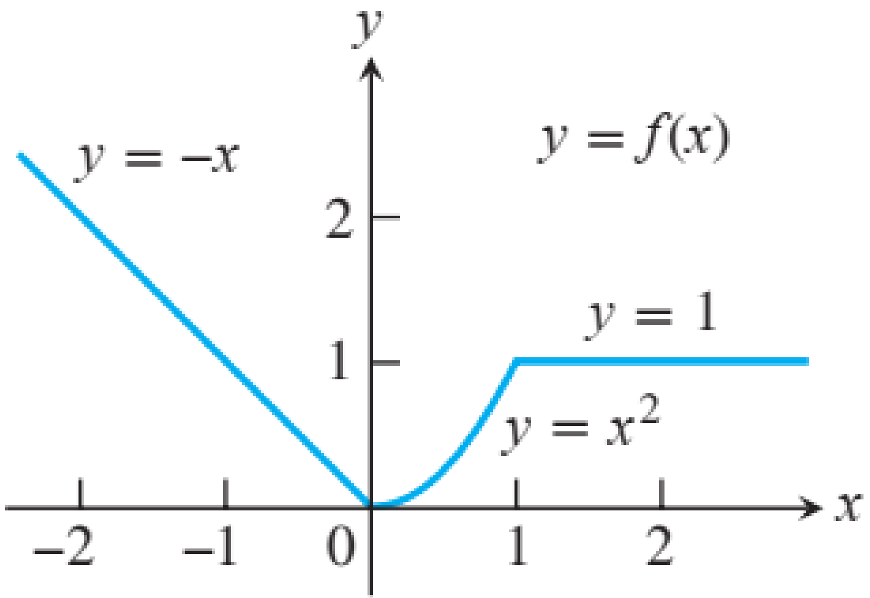
$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

**EXAMPLE 5** Graphing Piecewise-Defined Functions

The function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

is defined on the entire real line but has values given by different formulas depending on the position of $x$. The values of $f$ are given by: $y = -x$ when $x < 0, y = x^2$ when $0 \leq x \leq 1$, and $y = 1$ when $x > 1$. The function, however, is *just one function* whose domain is the entire set of real numbers (Figure 1.30). ■




**EXAMPLE 6** The Greatest Integer Function

The function whose value at any number $x$ is the *greatest integer less than or equal to* $x$ is called the **greatest integer function** or the **integer floor function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$ or int $x$. Figure 1.31 shows the graph. Observe that

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$

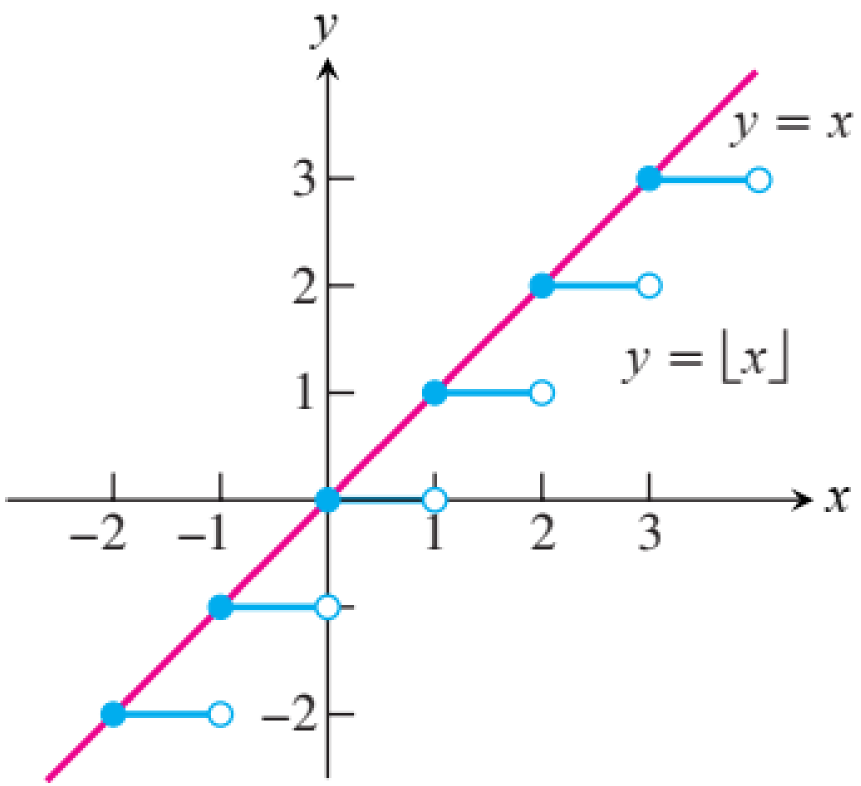



**FIGURE 1.31** The graph of the greatest integer function $y = \lfloor x \rfloor$ lies on or below the line $y = x$, so it provides an integer floor for $x$ (Example 6).

## EXAMPLE 7 The Least Integer Function

The function whose value at any number $x$ is the *smallest integer greater than or equal to* $x$ is called the **least integer function** or the **integer ceiling function**. It is denoted $\lceil x \rceil$.

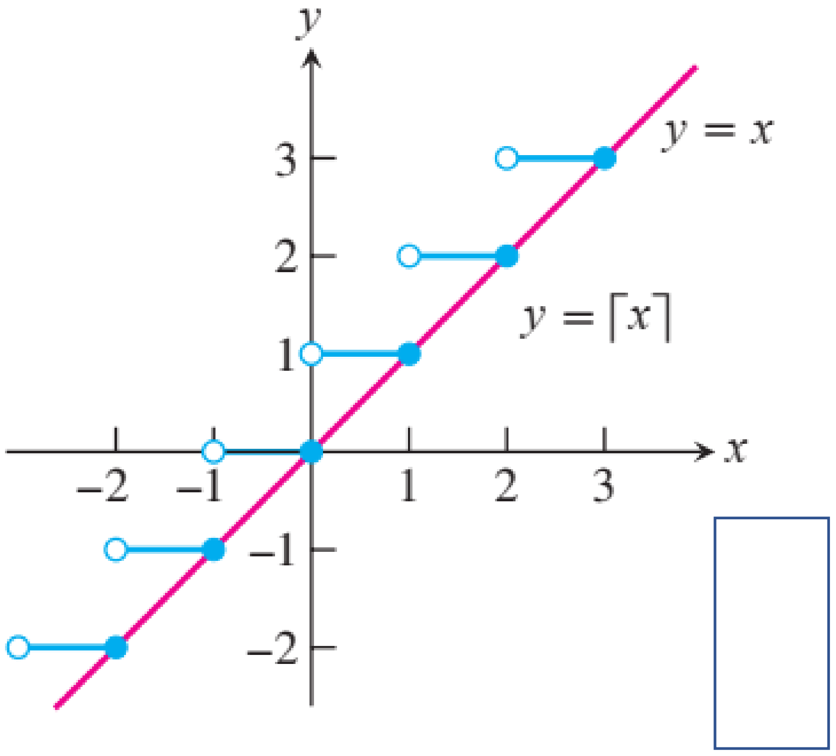
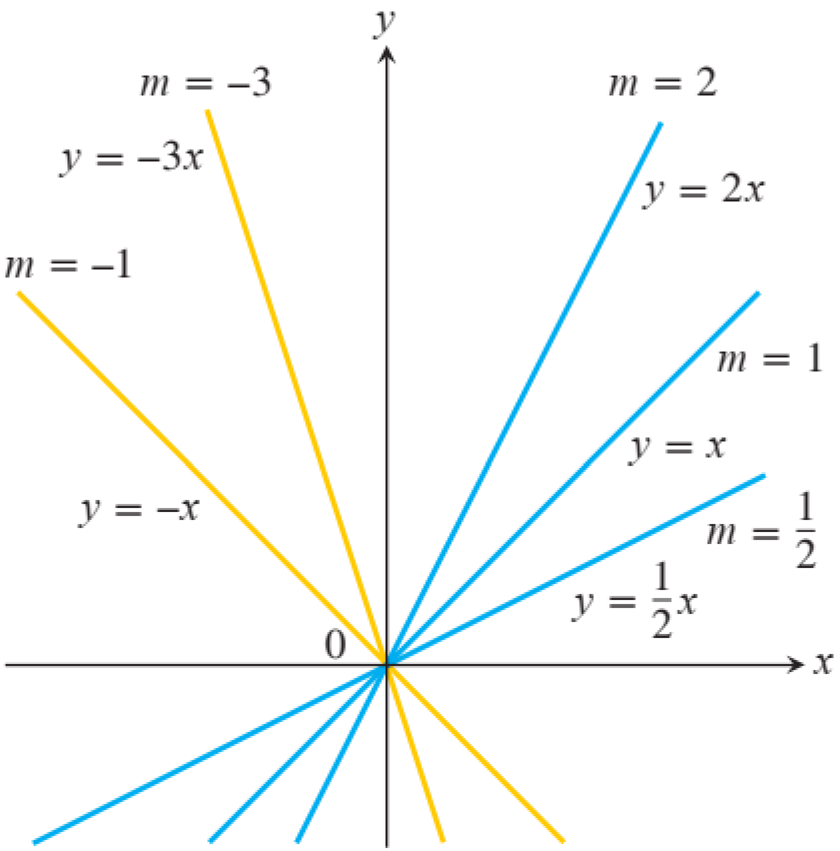
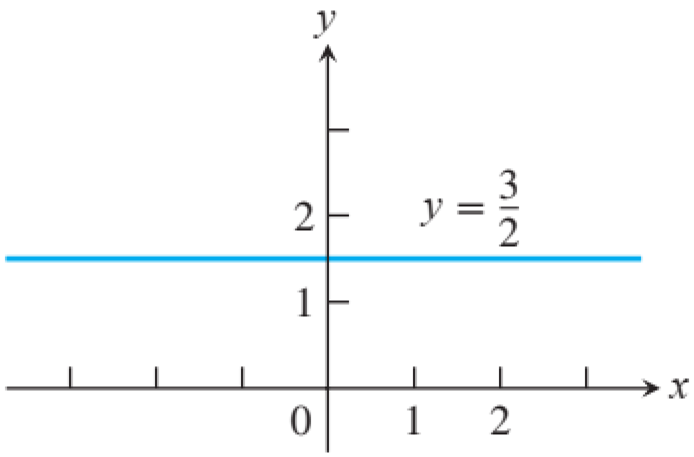



**FIGURE 1.32** The graph of the least integer function $y = \lceil x \rceil$ lies on or above the line $y = x$, so it provides an integer ceiling for $x$ (Example 7).

# Identifying Functions;

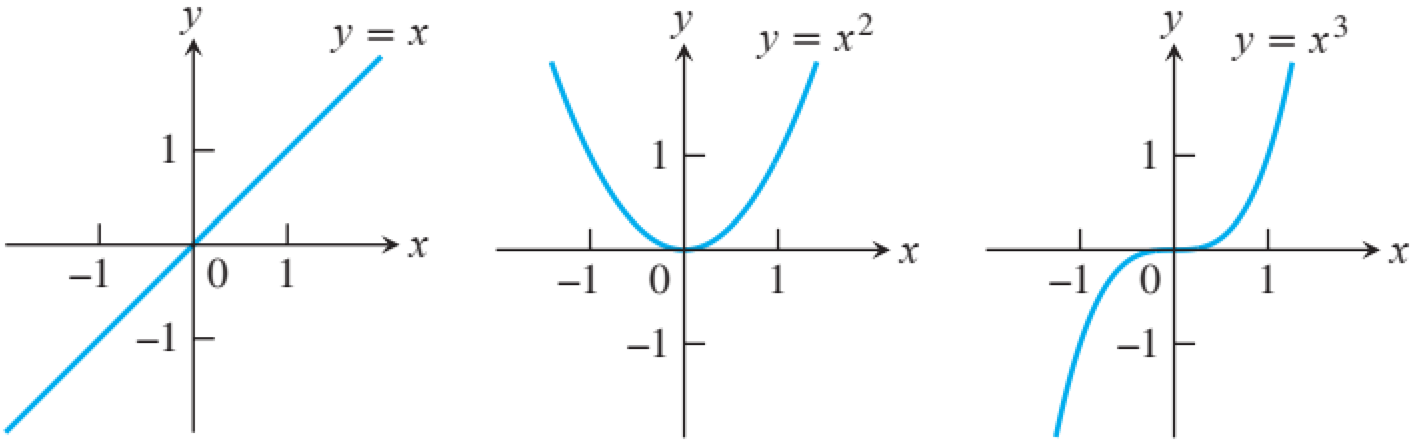
**Linear Functions** A function of the form $f(x) = mx + b$, for constants $m$ and $b$, is called a **linear function**.



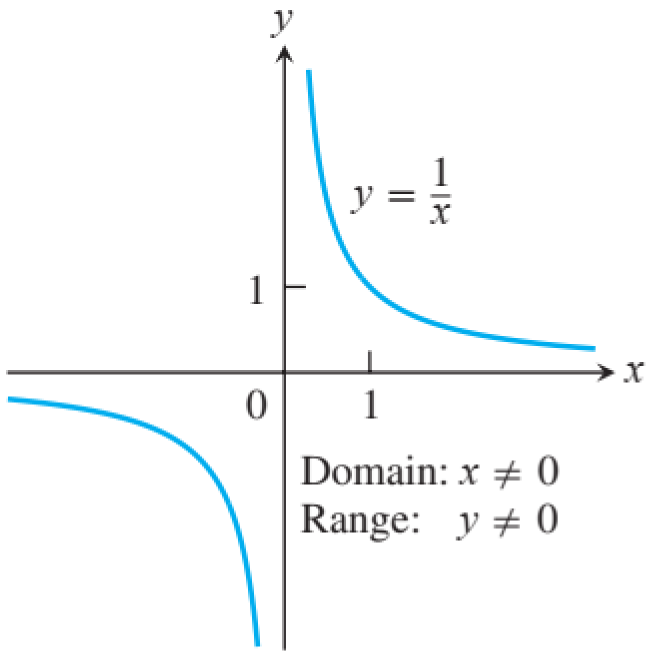



**Power Functions** A function $f(x) = x^a$, where $a$ is a constant, is called a **power function**. There are several important cases to consider.

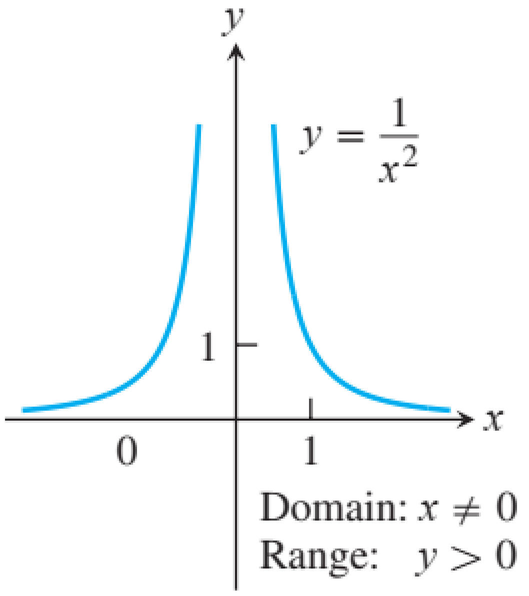
The graphs of $f(x) = x^n$, for $n = 1, 2, 3$




The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown

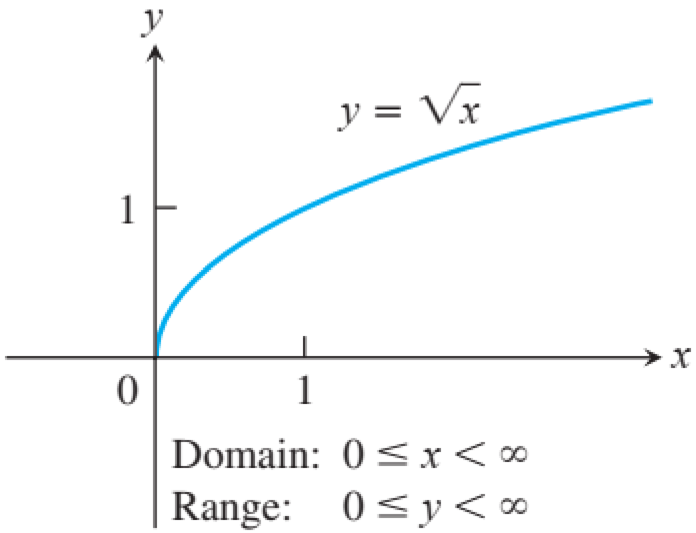


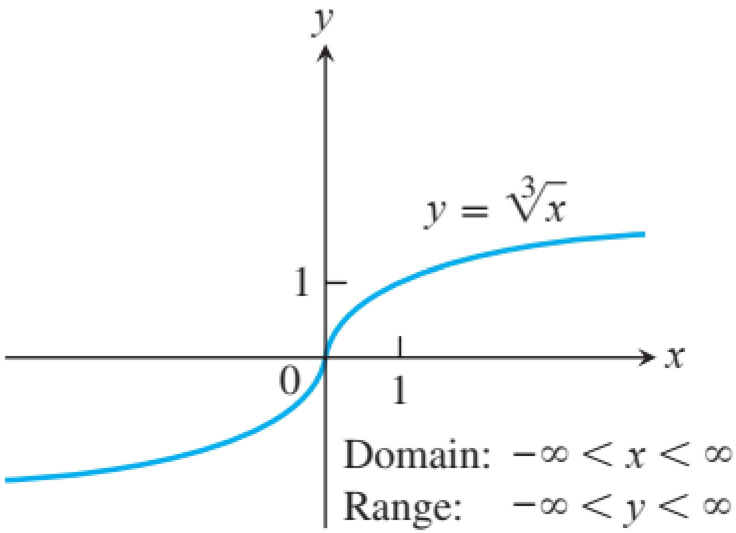

(a)




(b)

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively.




**Polynomials** A function $p$ is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n$ is a nonnegative integer and the numbers $a_0, a_1, a_2, \ldots, a_n$ are real constants (called the **coefficients** of the polynomial). All polynomials have domain $(-\infty, \infty)$.

**Rational Functions** A **rational function** is a quotient or ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p$ and $q$ are polynomials. The domain of a rational function is the set of all real $x$ for which $q(x) \neq 0$. For example, the function

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

is a rational function with domain $\{x \mid x \neq -4/7\}$.

**Algebraic Functions** An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). Rational functions are special cases of algebraic functions.

**Exponential Functions** Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$.

**Logarithmic Functions** These are the functions $f(x) = \log_a x$, where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions

**Transcendental Functions** These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well such as the hyperbolic functions

## EXAMPLE 1 Recognizing Functions

Identify each function given here as one of the types of functions we have discussed. Keep in mind that some functions can fall into more than one category. For example, $f(x) = x^2$ is both a power function and a polynomial of second degree.

**(a)** $f(x) = 1 + x - \frac{1}{2}x^5$ **(b)** $g(x) = 7^x$ **(c)** $h(z) = z^7$

**(d)** $y(t) = \sin\left(t - \frac{\pi}{4}\right)$

### Solution

**(a)** $f(x) = 1 + x - \frac{1}{2}x^5$ is a polynomial of degree 5.

**(b)** $g(x) = 7^x$ is an exponential function with base 7. Notice that the variable $x$ is the exponent.

**(c)** $h(z) = z^7$ is a power function. (The variable $z$ is the base.)

**(d)** $y(t) = \sin\left(t - \frac{\pi}{4}\right)$ is a trigonometric function. ■

## Increasing Versus Decreasing Functions

If the graph of a function *climbs* or *rises* as you move from left to right, we say that the function is *increasing*. If the graph *descends* or *falls* as you move from left to right, the function is *decreasing*.

| **Function** | **Where increasing** | **Where decreasing** |
|---|---|---|
| $y = x^2$ | $0 \leq x < \infty$ | $-\infty < x \leq 0$ |
| $y = x^3$ | $-\infty < x < \infty$ | Nowhere |
| $y = 1/x$ | Nowhere | $-\infty < x < 0$ and $0 < x < \infty$ |
| $y = 1/x^2$ | $-\infty < x < 0$ | $0 < x < \infty$ |

## Even Functions and Odd Functions: Symmetry

**DEFINITIONS** **Even Function, Odd Function**

A function $y = f(x)$ is an

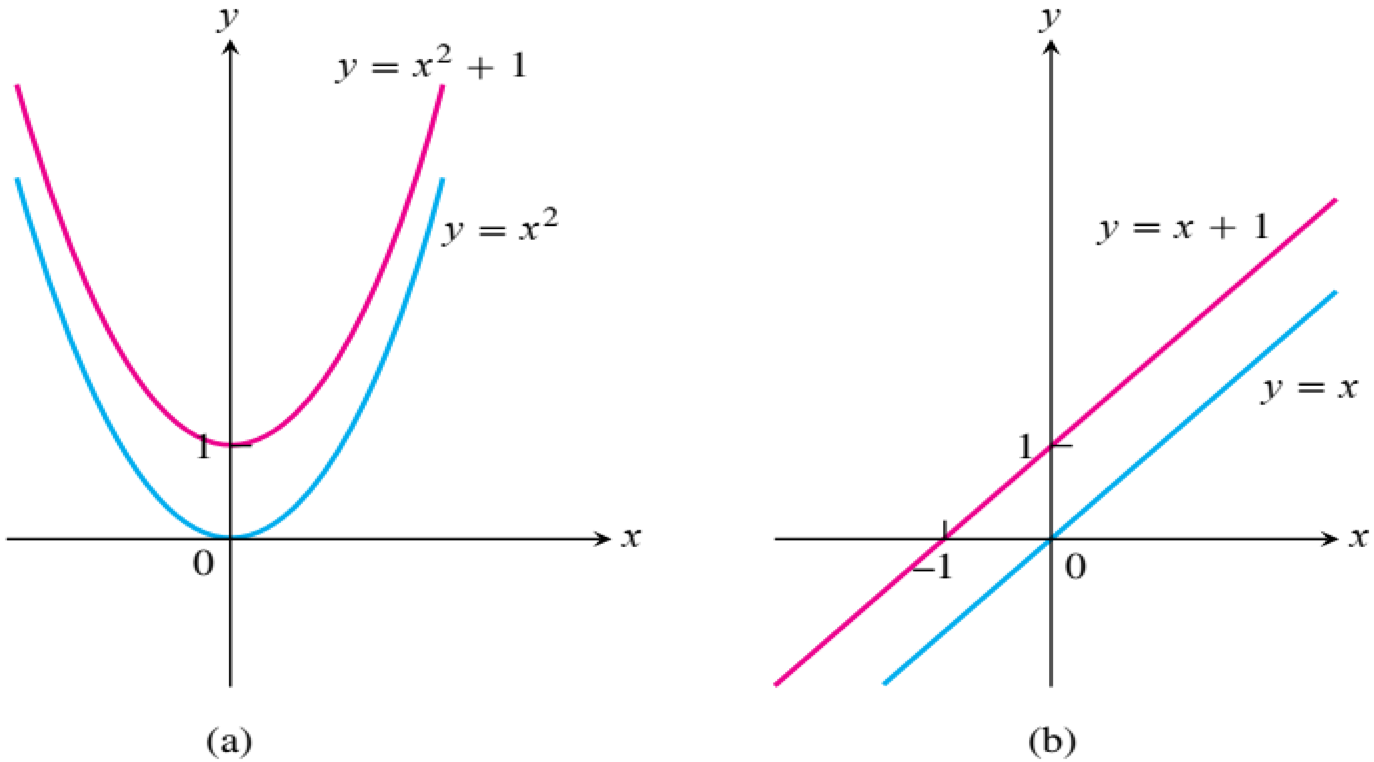
$$\textbf{even function of } \boldsymbol{x} \quad \text{if } f(-x) = f(x),$$
$$\textbf{odd function of } \boldsymbol{x} \quad \text{if } f(-x) = -f(x),$$

for every $x$ in the function's domain.

**EXAMPLE 2** Recognizing Even and Odd Functions

| | |
|---|---|
| $f(x) = x^2$ | Even function: $(-x)^2 = x^2$ for all $x$; symmetry about $y$-axis. |
| $f(x) = x^2 + 1$ | Even function: $(-x)^2 + 1 = x^2 + 1$ for all $x$; symmetry about $y$-axis (Figure 1.47a). |

| | |
|---|---|
| $f(x) = x$ | Odd function: $(-x) = -x$ for all $x$; symmetry about the origin. |
| $f(x) = x + 1$ | Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal.<br>Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$ (Figure 1.47b). |



# Combining Functions; Shifting and Scaling Graphs

## Sums, Differences, Products, and Quotients

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions. If $f$ and $g$ are functions, then for every $x$ that belongs to the domains of both $f$ and $g$ (that is, for $x \in D(f) \cap D(g)$), we define functions $f + g$, $f - g$, and $fg$ by the formulas

$$(f + g)(x) = f(x) + g(x).$$
$$(f - g)(x) = f(x) - g(x).$$
$$(fg)(x) = f(x)g(x).$$

Notice that the $+$ sign on the left-hand side of the first equation represents the operation of addition of *functions*, whereas the $+$ on the right-hand side of the equation means addition of the real numbers $f(x)$ and $g(x)$.

At any point of $D(f) \cap D(g)$ at which $g(x) \neq 0$, we can also define the function $f/g$ by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0).$$

Functions can also be multiplied by constants: If $c$ is a real number, then the function $cf$ is defined for all $x$ in the domain of $f$ by

$$(cf)(x) = cf(x).$$

## EXAMPLE 1 Combining Functions Algebraically

The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x},$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function $fg$.

| Function | Formula | Domain |
|---|---|---|
| $f+g$ | $(f+g)(x)=\sqrt{x}+\sqrt{1-x}$ | $[0, 1] = D(f) \cap D(g)$ |
| $f-g$ | $(f-g)(x)=\sqrt{x}-\sqrt{1-x}$ | $[0, 1]$ |
| $g-f$ | $(g-f)(x)=\sqrt{1-x}-\sqrt{x}$ | $[0, 1]$ |
| $f \cdot g$ | $(f \cdot g)(x)=f(x)g(x)=\sqrt{x(1-x)}$ | $[0, 1]$ |
| $f/g$ | $\dfrac{f}{g}(x)=\dfrac{f(x)}{g(x)}=\sqrt{\dfrac{x}{1-x}}$ | $[0, 1)$ ($x = 1$ excluded) |
| $g/f$ | $\dfrac{g}{f}(x)=\dfrac{g(x)}{f(x)}=\sqrt{\dfrac{1-x}{x}}$ | $(0, 1]$ ($x = 0$ excluded) |

**DEFINITION Composition of Functions**

If $f$ and $g$ are functions, the **composite** function $f \circ g$ ("$f$ composed with $g$") is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of the numbers $x$ in the domain of $g$ for which $g(x)$ lies in the domain of $f$.

## EXAMPLE 3 Finding Formulas for Composites

If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

**(a)** $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

### Solution

| | **Composite** | **Domain** |
|---|---|---|
| **(a)** | $(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$ | $[-1, \infty)$ |
| **(b)** | $(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$ | $[0, \infty)$ |
| **(c)** | $(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$ | $[0, \infty)$ |
| **(d)** | $(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$ | $(-\infty, \infty)$ |

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real $x$ but belongs to the domain of $f$ only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

# Shifting a Graph of a Function

## Shift Formulas

**Vertical Shifts**

| | |
|---|---|
| $y = f(x) + k$ | Shifts the graph of $f$ *up* $k$ units if $k > 0$ |
| | Shifts it *down* $\lvert k \rvert$ units if $k < 0$ |

**Horizontal Shifts**

| | |
|---|---|
| $y = f(x + h)$ | Shifts the graph of $f$ *left* $h$ units if $h > 0$ |
| | Shifts it *right* $\lvert h \rvert$ units if $h < 0$ |

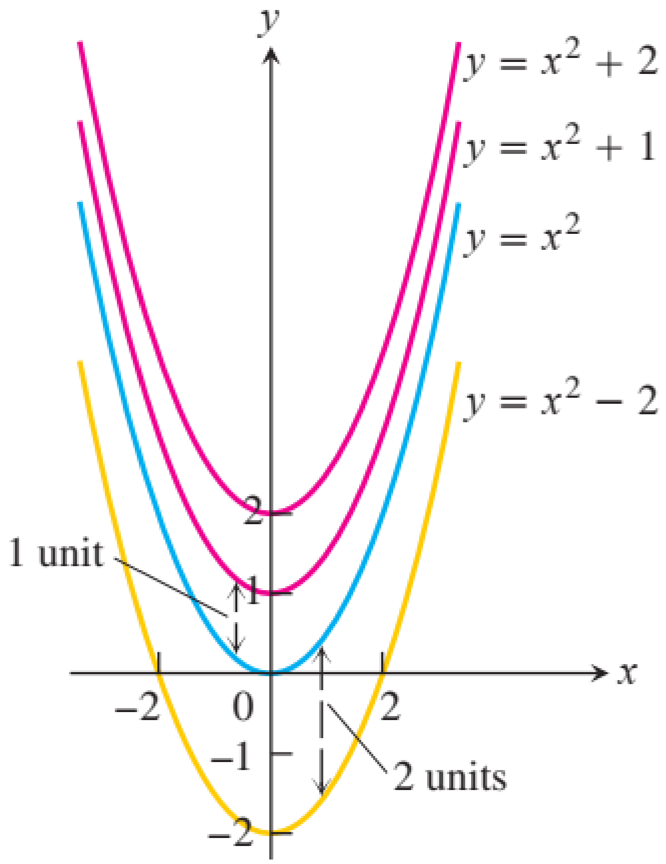



**FIGURE 1.54** To shift the graph of $f(x) = x^2$ up (or down), we add positive (or negative) constants to the formula for $f$

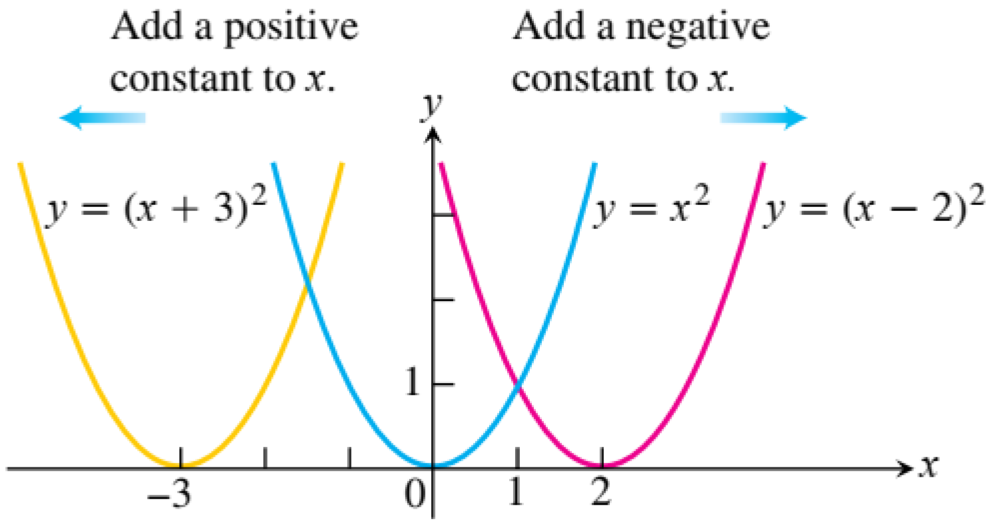



**FIGURE 1.55** To shift the graph of $y = x^2$ to the left, we add a positive constant to $x$. To shift the graph to the right, we add a negative constant to $x$

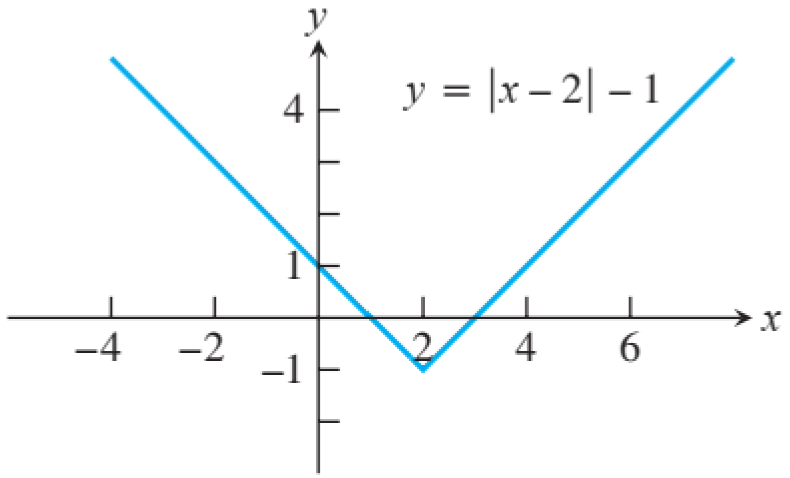



**FIGURE 1.56** Shifting the graph of $y = |x|$ 2 units to the right and 1 unit down