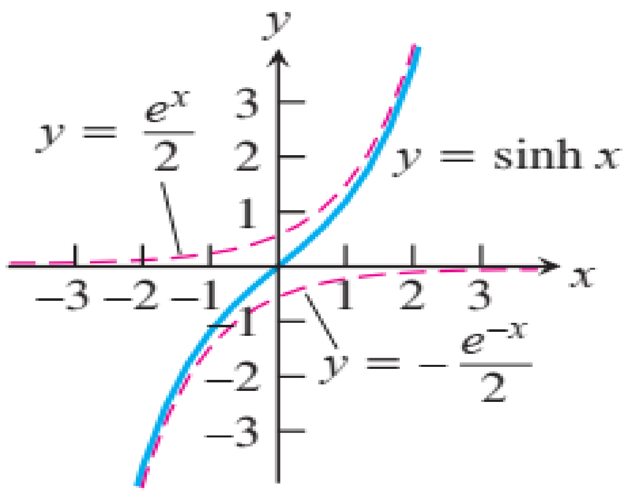
Hyperbolic Functions

TABLE 7.5 The six basic hyperbolic functions

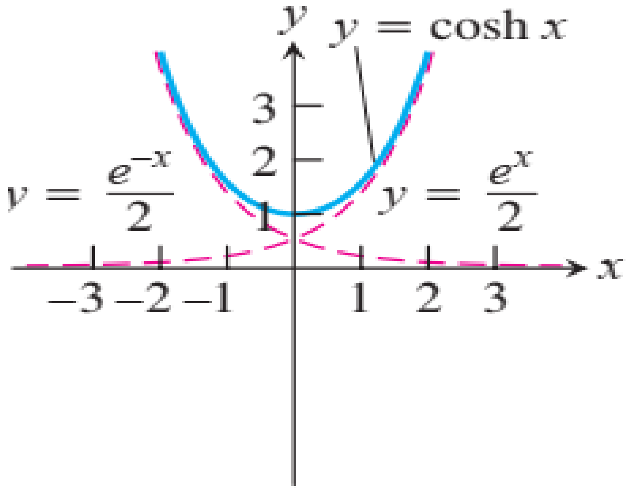
FIGURE 7.31

Hyperbolic sine of x: $\sinh x = \dfrac{e^x - e^{-x}}{2}$



(a)

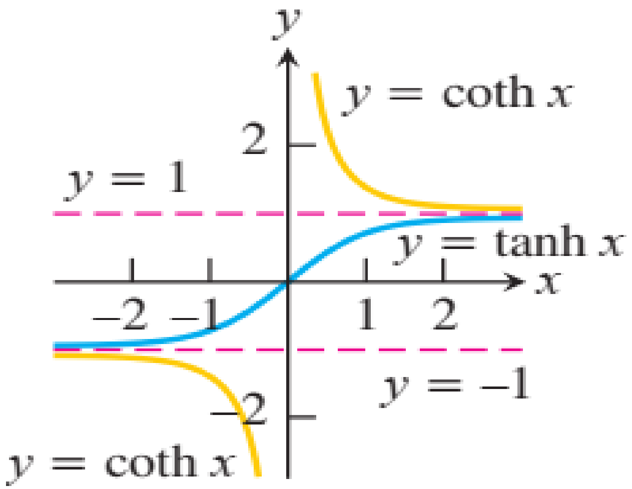
Hyperbolic cosine of x: $\cosh x = \dfrac{e^x + e^{-x}}{2}$



(b)

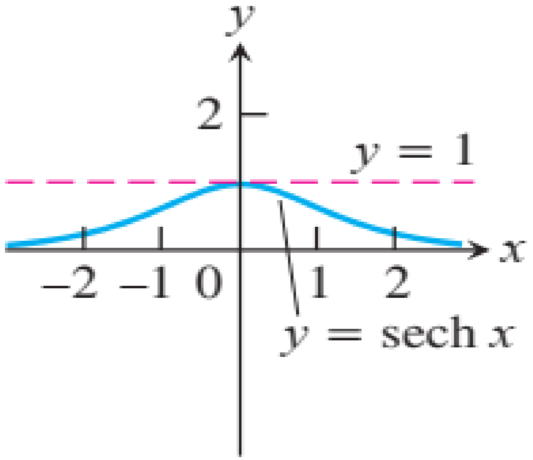
Hyperbolic tangent: $$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent: $$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



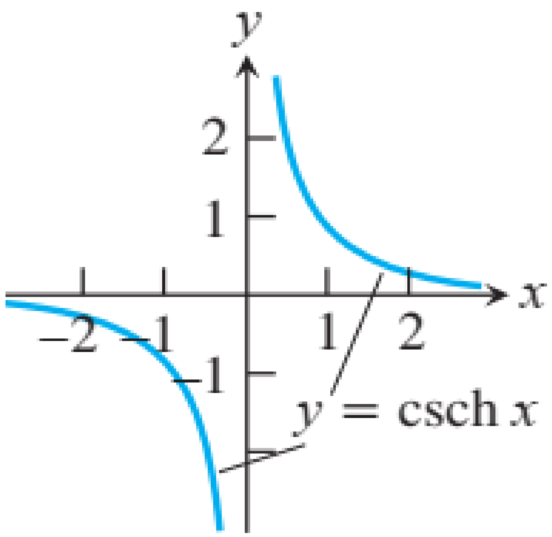
(c)

Hyperbolic secant: $$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(d)

Hyperbolic cosecant: $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^{x} - e^{-x}}$



(e)

TABLE 7.6 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

$$
\begin{aligned}
2 \sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{e^{2x} - e^{-2x}}{2} \\
&= \sinh 2x .
\end{aligned}
$$

TABLE 7.7 Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

The derivative formulas are derived from the derivative of e^u:

$$
\begin{aligned}
\frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of } \sinh u \\
&= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u \\
&= \cosh u \frac{du}{dx} && \text{Definition of } \cosh u
\end{aligned}
$$

This gives the first derivative formula. The calculation

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of csch } u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of csch } u \text{ and coth } u
\end{aligned}$$

EXAMPLE 1 Finding Derivatives and Integrals

(a) $$\frac{d}{dt}\left(\tanh \sqrt{1 + t^2}\right) = \operatorname{sech}^2 \sqrt{1 = t^2} \cdot \frac{d}{dt}\left(\sqrt{1 + t^2}\right)$$

$$= \frac{t}{\sqrt{1 + t^2}} \operatorname{sech}^2 \sqrt{1 + t^2}$$

Inverse Hyperbolic Functions

TABLE 7.9 Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cosh\left(\cosh^{-1}\left(\frac{1}{x}\right)\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x$$

TABLE 7.10 Derivatives of inverse hyperbolic functions

$\dfrac{d(\sinh^{-1} u)}{dx} = \dfrac{1}{\sqrt{1 + u^2}} \dfrac{du}{dx}$	
$\dfrac{d(\cosh^{-1} u)}{dx} = \dfrac{1}{\sqrt{u^2 - 1}} \dfrac{du}{dx},$	$u > 1$
$\dfrac{d(\tanh^{-1} u)}{dx} = \dfrac{1}{1 - u^2} \dfrac{du}{dx},$	$\lvert u \rvert < 1$
$\dfrac{d(\coth^{-1} u)}{dx} = \dfrac{1}{1 - u^2} \dfrac{du}{dx},$	$\lvert u \rvert > 1$
$\dfrac{d(\operatorname{sech}^{-1} u)}{dx} = \dfrac{-du/dx}{u\sqrt{1 - u^2}},$	$0 < u < 1$
$\dfrac{d(\operatorname{csch}^{-1} u)}{dx} = \dfrac{-du/dx}{\lvert u \rvert\sqrt{1 + u^2}},$	$u \neq 0$

EXAMPLE 2 Derivative of the Inverse Hyperbolic Cosine

Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}.$$

Solution First we find the derivative of $y = \cosh^{-1} x$ for $x > 1$ by applying Theorem 1 with $f(x) = \cosh x$ and $f^{-1}(x) = \cosh^{-1} x$. Theorem 1 can be applied because the derivative of $\cosh x$ is positive for $0 < x$.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \qquad \text{Theorem 1}$$

$$= \frac{1}{\sinh(\cosh^{-1} x)} \qquad f'(u) = \sinh u$$

$$= \frac{1}{\sqrt{\cosh^2(\cosh^{-1} x) - 1}} \qquad \cosh^2 u - \sinh^2 u = 1, \quad \sinh u = \sqrt{\cosh^2 u - 1}$$

$$= \frac{1}{\sqrt{x^2 - 1}} \qquad \cosh(\cosh^{-1} x) = x$$

In short,

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

The Chain Rule gives the final result:

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}. \qquad \blacksquare$$

Instead of applying Theorem 1 directly, as in Example 2, we could also find the derivative of $y = \cosh^{-1} x, x > 1$, using implicit differentiation and the Chain Rule:

$$y = \cosh^{-1} x$$

$$x = \cosh y \qquad \text{Equivalent equation}$$

$$1 = \sinh y \frac{dy}{dx} \qquad \text{Implicit differentiation with respect to } x \text{, and the Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} \qquad \text{Since } x > 1, y > 0 \text{ and } \sinh y > 0$$

$$= \frac{1}{\sqrt{x^2 - 1}}. \qquad \cosh y = x$$

Each of Exercises 1–4 gives a value of sinh x or cosh x. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = -\frac{3}{4}$

$\sinh x = -\frac{3}{4} \Rightarrow \cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(-\frac{3}{4}\right)^2} = \sqrt{1+\frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$, $\tanh x = \frac{\sinh x}{\cosh x} = \frac{\left(-\frac{3}{4}\right)}{\left(\frac{5}{4}\right)} = -\frac{3}{5}$,
$\coth x = \frac{1}{\tanh x} = -\frac{5}{3}$, $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{4}{5}$, and $\operatorname{csch} x = \frac{1}{\sin x} = -\frac{4}{3}$

Rewrite the expressions in Exercises 5–10 in terms of exponentials and simplify the results as much as you can.

$$2\cosh(\ln x) = 2\left(\frac{e^{\ln x}+e^{-\ln x}}{2}\right) = e^{\ln x} + \frac{1}{e^{\ln x}} = x + \frac{1}{x}$$

$$\sinh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{2} = \frac{e^{\ln x^2} - e^{\ln x^{-2}}}{2} = \frac{\left(x^2 - \frac{1}{x^2}\right)}{2} = \frac{x^4 - 1}{2x^2}$$

$$\cosh 5x + \sinh 5x = \frac{e^{5x} + e^{-5x}}{2} + \frac{e^{5x} - e^{-5x}}{2} = e^{5x}$$

11. Use the identities

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$
$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

to show that

a. $\sinh 2x = 2 \sinh x \cosh x$

b. $\cosh 2x = \cosh^2 x + \sinh^2 x$.

(a) $\sinh 2x = \sinh(x + x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x$

(b) $\cosh 2x = \cosh(x + x) = \cosh x \cosh x + \sinh x \sin x = \cosh^2 x + \sinh^2 x$

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

$$y = 6 \sinh \tfrac{x}{3} \;\Rightarrow\; \tfrac{dy}{dx} = 6 \left(\cosh \tfrac{x}{3}\right)\left(\tfrac{1}{3}\right) = 2 \cosh \tfrac{x}{3}$$

$$y = \ln \cosh v - \tfrac{1}{2} \tanh^2 v \;\Rightarrow\; \tfrac{dy}{dv} = \tfrac{\sinh v}{\cosh v} - \left(\tfrac{1}{2}\right)(2 \tanh v)\left(\operatorname{sech}^2 v\right) = \tanh v - (\tanh v)\left(\operatorname{sech}^2 v\right)$$
$$= (\tanh v)\left(1 - \operatorname{sech}^2 v\right) = (\tanh v)\left(\tanh^2 v\right) = \tanh^3 v$$

$$y = \ln x + \sqrt{1 - x^2}\,\operatorname{sech}^{-1} x = \ln x + \left(1 - x^2\right)^{1/2} \operatorname{sech}^{-1} x \;\Rightarrow\; \frac{dy}{dx}$$

$$= \frac{1}{x} + \left(1 - x^2\right)^{1/2}\left(\frac{-1}{x\sqrt{1-x^2}}\right) + \left(\frac{1}{2}\right)\left(1 - x^2\right)^{-1/2}(-2x)\operatorname{sech}^{-1} x = \frac{1}{x} - \frac{1}{x} - \frac{x}{\sqrt{1-x^2}}\operatorname{sech}^{-1} x = \frac{-x}{\sqrt{1-x^2}}\operatorname{sech}^{-1} x$$