Continuity

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point *c*** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint *a*** or is **continuous at a right endpoint *b*** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

If a function f is not continuous at a point c, we say that f is **discontinuous** at c and c is a **point of discontinuity** of f.

A function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x\to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x\to c^-} f(x) = f(c)$.

EXAMPLE 2 A Function Continuous Throughout Its Domain

The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$ (Figure 2.52), including $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ■

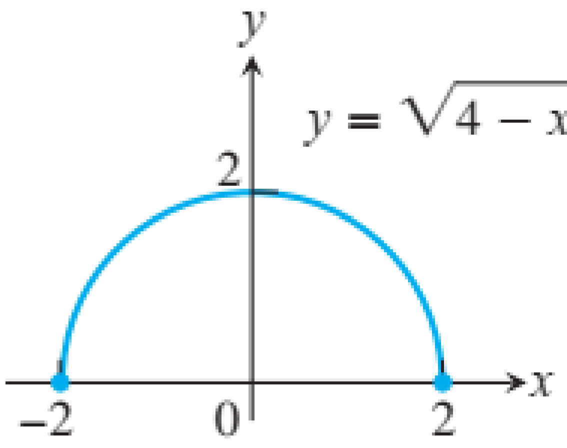


FIGURE 2.52 A function that is continuous at every domain point (Example 2).

EXAMPLE 3 The Unit Step Function Has a Jump Discontinuity

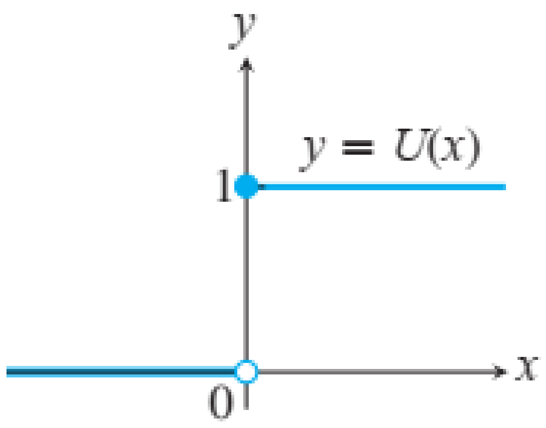


FIGURE 2.53 A function that is right-continuous, but not left-continuous, at the origin. It has a jump discontinuity there

The unit step function $U(x)$, graphed in Figure 2.53, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

Right conti. at x=0, because right hand limit at 0 is equal to function value at 0, f(0)=1

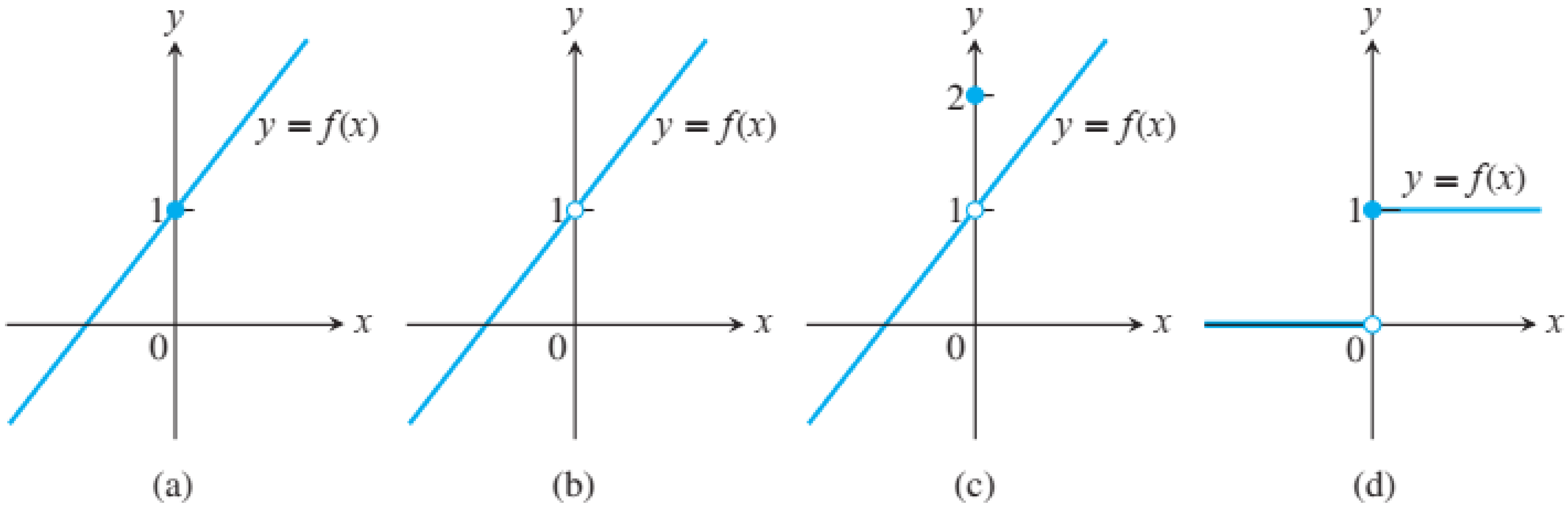
Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (*c* lies in the domain of *f*)
2. $\lim_{x\to c} f(x)$ exists (*f* has a limit as $x \to c$)
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value)

Figure 2.55 is a catalog of discontinuity types. The function in Figure 2.55a is continuous at $x = 0$. The function in Figure 2.55b would be continuous if it had $f(0) = 1$. The function in Figure 2.55c would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in Figure 2.55b and c are **removable**. Each function has a limit as $x \rightarrow 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

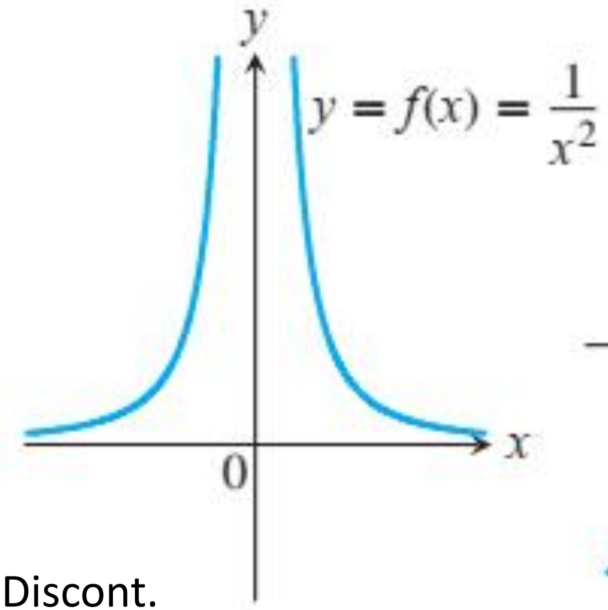
The discontinuities in Figure 2.55d through f are more serious: $\lim_{x \rightarrow 0} f(x)$ does not exist, and there is no way to improve the situation by changing f at 0. The step function in Figure 2.55d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.55e has an **infinite discontinuity**. The function in Figure 2.55f has an **oscillating discontinuity**: It oscillates too much to have a limit as $x \rightarrow 0$.



Remove Discont.
by taking f(0)=1

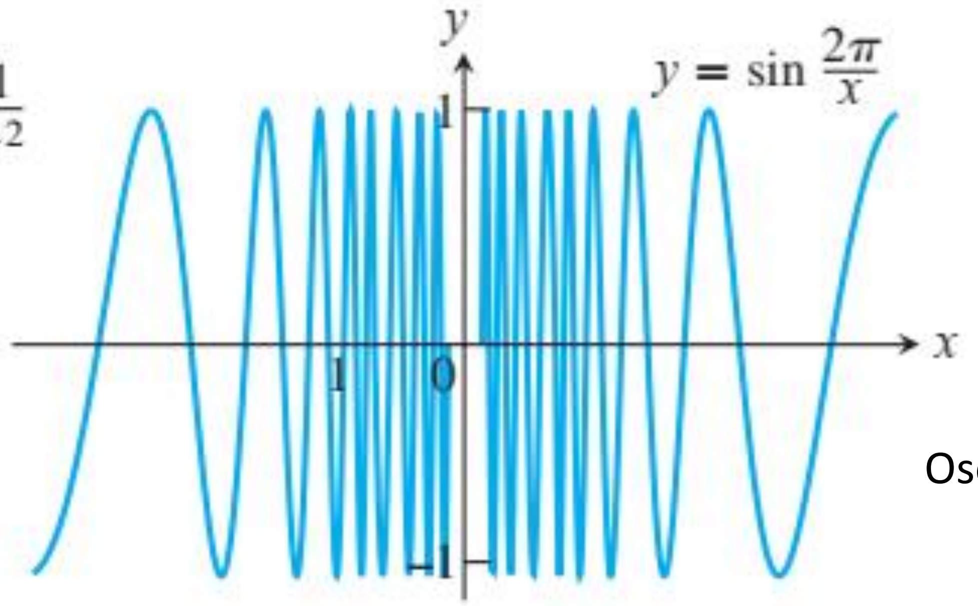
Cont.

J. Discont.



Inf. Discont.

(e)



Osc. Discont.

(f)

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval.

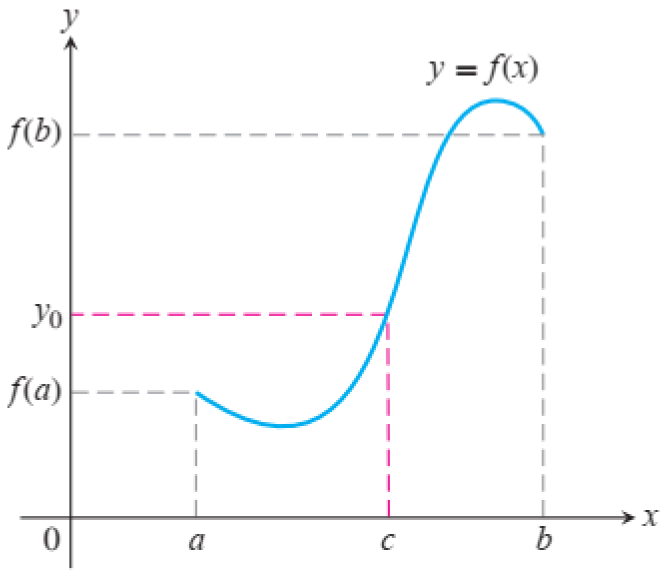
THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

THEOREM 11 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Problem: For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

As defined, $\lim\limits_{x \to 3^-} f(x) = (3)^2 - 1 = 8$ and $\lim\limits_{x \to 3^+} (2a)(3) = 6a$. For f(x) to be continuous we must have $6a = 8 \Rightarrow a = \frac{4}{3}$.

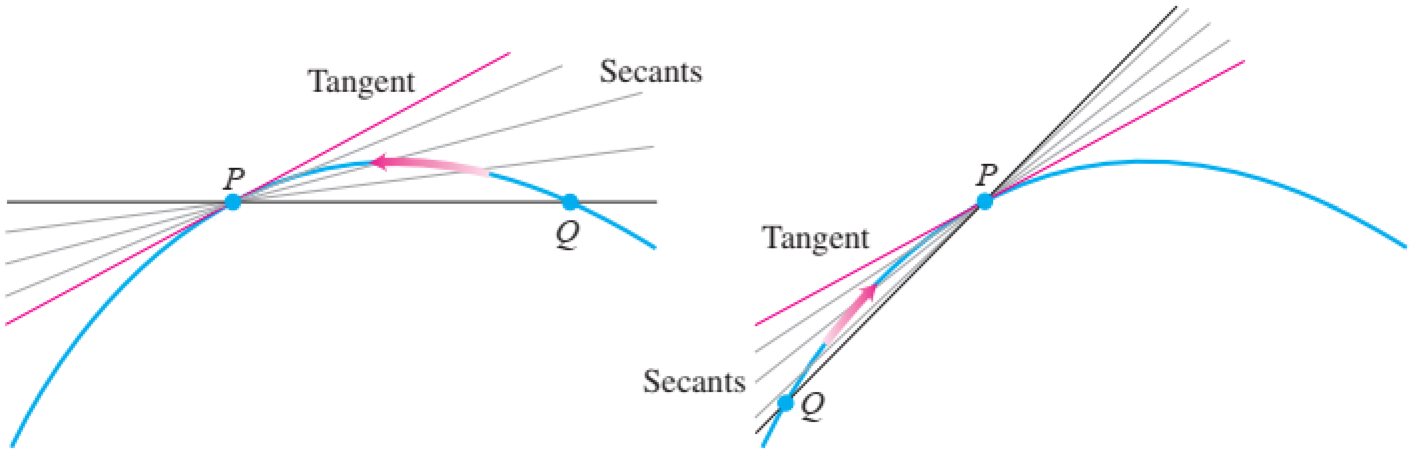
Problem: For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

As defined, $\lim_{x \to -2^-} g(x) = -2$ and $\lim_{x \to -2^+} g(x) = b(-2)^2 = 4b$. For g(x) to be continuous we must have $4b = -2 \Rightarrow b = -\frac{1}{2}$.

Tangents and Derivatives



Tangent line at P is a limiting case of secant line as Q approaches to P

DEFINITIONS Slope, Tangent Line

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

This slope is the derivative of f(x) at x=x_0.

find the slope of the curve at the point indicated.

$y = 5x^2, \quad x = -1$

At $x = -1, y = 5 \Rightarrow m = \lim_{h \to 0} \frac{5(-1+h)^2 - 5}{h} = \lim_{h \to 0} \frac{5(1-2h+h^2)-5}{h} = \lim_{h \to 0} \frac{5h(-2+h)}{h} = -10$, slope

y=f(x) so f(-1)=5, f(-1+h)=5(-1+h)^2

This slope is the derivative of f(x) at x=-1.

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

1. Calculate $f(x_0)$ and $f(x_0 + h)$.

2. Calculate the slope

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

3. If the limit exists, find the tangent line as

$$y = y_0 + m(x - x_0).$$

Find equations of all lines having slope -1 that are tangent to the curve $y = 1/(x - 1)$.

$$-1 = m = \lim_{h \to 0} \frac{\frac{1}{(x+h)-1} - \frac{1}{x-1}}{h} = \lim_{h \to 0} \frac{(x-1)-(x+h-1)}{h(x-1)(x+h-1)} = \lim_{h \to 0} \frac{-h}{h(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$$

$\Rightarrow (x-1)^2 = 1 \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow x = 0$ or $x = 2$. If $x = 0$, then $y = -1$ and $m = -1$ $\Rightarrow y = -1 - (x - 0) = -(x+1)$. If $x = 2$, then $y = 1$ and $m = -1 \Rightarrow y = 1 - (x-2) = -(x-3)$.

Find an equation of the straight line having slope 1/4 that is tangent to the curve $y = \sqrt{x}$.

$$\frac{1}{4} = m = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} = \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$. Thus, $\frac{1}{4} = \frac{1}{2\sqrt{x}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4 \Rightarrow y = 2$. The tangent line is $y = 2 + \frac{1}{4}(x - 4) = \frac{x}{4} + 1$.

Does the graph of

$$g(x) = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a tangent at the origin? Give reasons for your answer.

Slope at origin $= \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim\limits_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \Rightarrow$ yes, f(x) does have a tangent at the origin with slope 0.