The Derivative as a Rate of Change

DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

EXAMPLE 1 How a Circle's Area Changes with Its Diameter

The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing at rate $(\pi/2)10 = 5\pi\ \mathrm{m}^2/\mathrm{m}$. ■

DEFINITION Velocity

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left| \frac{ds}{dt} \right|$$

DEFINITIONS Acceleration, Jerk

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

EXAMPLE 4 Modeling Free Fall

Figure 3.16 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration then?

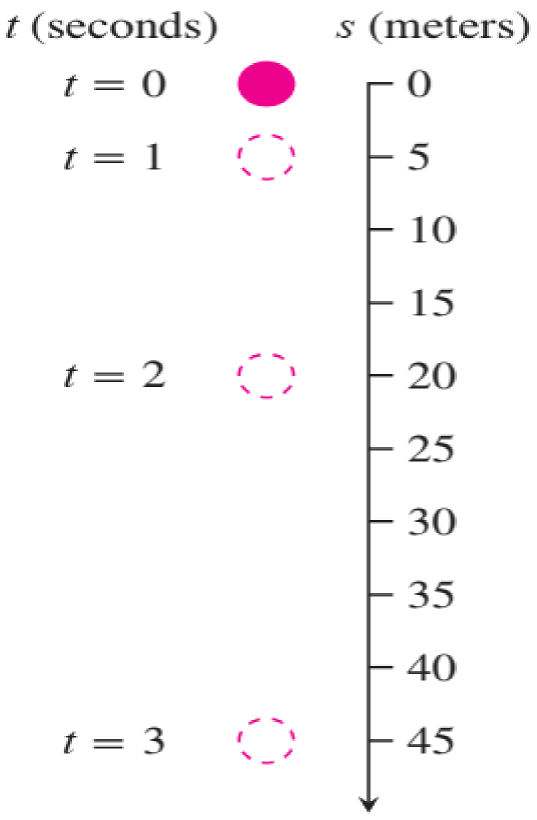


FIGURE 3.16 A ball bearing falling from rest (Example 4).

Solution

(a) The metric free-fall equation is $s = 4.9t^2$. During the first 2 sec, the ball falls

$$s(2) = 4.9(2)^2 = 19.6 \text{ m}.$$

(b) At any time t, *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

At $t = 2$, the velocity is

$$v(2) = 19.6 \text{ m/sec}$$

in the downward (increasing s) direction. The *speed* at $t = 2$ is

$$\text{Speed} = |v(2)| = 19.6 \text{ m/sec}.$$

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/sec}^2.$$

At $t = 2$, the acceleration is 9.8 m/sec^2.

EXAMPLE 5 Modeling Vertical Motion

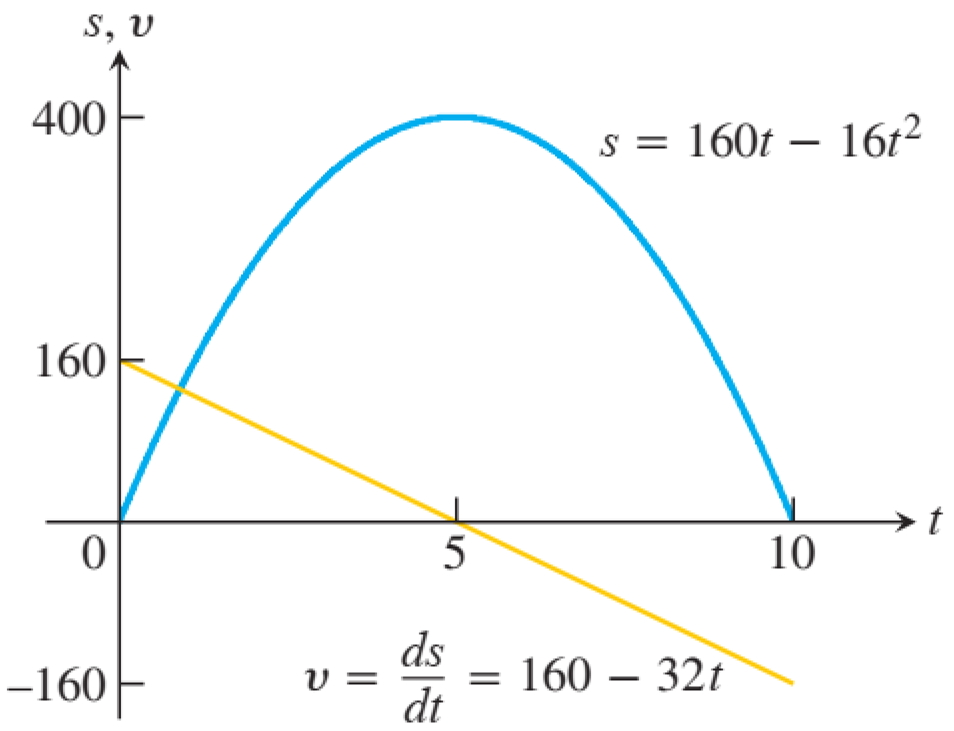
A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.17a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?



(b)

(a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0 \qquad \text{or} \qquad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{\max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

(b) To find the rock's velocity at 256 ft on the way up and again on the way down, we first find the two values of t for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation, we write

$$\begin{aligned} 16t^2 - 160t + 256 &= 0 \\ 16(t^2 - 10t + 16) &= 0 \\ (t - 2)(t - 8) &= 0 \\ t = 2 \text{ sec}, t &= 8 \text{ sec}. \end{aligned}$$

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec}.$$
$$v(8) = 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec}.$$

At both instants, the rock's speed is 96 ft/sec. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ sec; it is moving downward (s is decreasing) at $t = 8$ because $v(8) < 0$.

(c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. As the rock rises, it slows down; as it falls, it speeds up.

(d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returned to the ground 10 sec later. ■

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x.$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

Note that:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \qquad \sec x = \frac{1}{\cos x}, \qquad \text{and} \qquad \csc x = \frac{1}{\sin x}$$

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

$$\cos(x + h) = \cos x \cos h - \sin x \sin h$$

$$\lim_{h \to 0} \frac{\cos h - 1}{h} = 0 \qquad \lim_{h \to 0} \frac{\sin h}{h} = 1$$

Show that $$\frac{d}{dx}(\sin x) = \cos x.$$

If $f(x) = \sin x$, then

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Sine angle sum identity} \\
&= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\
&= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 5(a) and Theorem 7, Section 2.4} \\
&= \cos x.
\end{aligned}
$$

Show that $$\frac{d}{dx}(\cos x) = -\sin x$$

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} && \text{Derivative definition} \\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity} \\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h} \\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos x \cdot 0 - \sin x \cdot 1 && \text{Example 5(a) and Theorem 7, Section 2.4} \\
&= -\sin x.
\end{aligned}$$

EXAMPLE 1 Derivatives Involving the Sine

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x.$$

(b) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + 2x \sin x \qquad \text{Product Rule}$$

$$= x^2 \cos x + 2x \sin x.$$

EXAMPLE 2 Derivatives Involving the Cosine

$$y = \frac{\cos x}{1 - \sin x}:$$

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}.$$

■

EXAMPLE

Find $d(\tan x)/dx$.

Solution

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \dfrac{d}{dx}(\sin x) - \sin x \dfrac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x \qquad \blacksquare$$

EXAMPLE

Find y'' if $y = \sec x$.

Solution

$$
\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) && \text{Product Rule} \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}
$$

Exercise 3.4 Question # 1-16.

The Chain Rule and Parametric Equations

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE

Differentiate $\sin(x^2 + x)$ with respect to x.

Solution

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\substack{\text{inside} \\ \text{left alone}}} \cdot \underbrace{(2x + 1)}_{\substack{\text{derivative of} \\ \text{the inside}}}$$

EXAMPLE

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$
\begin{aligned}
g'(t) &= \frac{d}{dt}\left(\tan\left(5 - \sin 2t\right)\right) \\
&= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}\left(5 - \sin 2t\right) && \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t \\
&= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}\left(2t\right)\right) && \text{Derivative of } 5 - \sin u \text{ with } u = 2t \\
&= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\
&= -2(\cos 2t) \sec^2(5 - \sin 2t).
\end{aligned}
$$

■

Power Chain Rule:

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}\left(u^n\right) = nu^{n-1}$$

f(u)=u^n and u=u(x).

EXAMPLE Applying the Power Chain Rule

(a)
$$\begin{aligned}\frac{d}{dx}(5x^3 - x^4)^7 &= 7(5x^3 - x^4)^6\frac{d}{dx}\left(5x^3 - x^4\right) \\ &= 7(5x^3 - x^4)^6(5\cdot 3x^2 - 4x^3) \\ &= 7(5x^3 - x^4)^6(15x^2 - 4x^3)\end{aligned}$$

Power Chain Rule with $u = 5x^3 - x^4, n = 7$

(b)
$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x - 2)^{-1} \\ &= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2) \\ &= -1(3x - 2)^{-2}(3) \\ &= -\frac{3}{(3x - 2)^2}\end{aligned}$$

Power Chain Rule with $u = 3x - 2, n = -1$

DEFINITION Parametric Curve

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

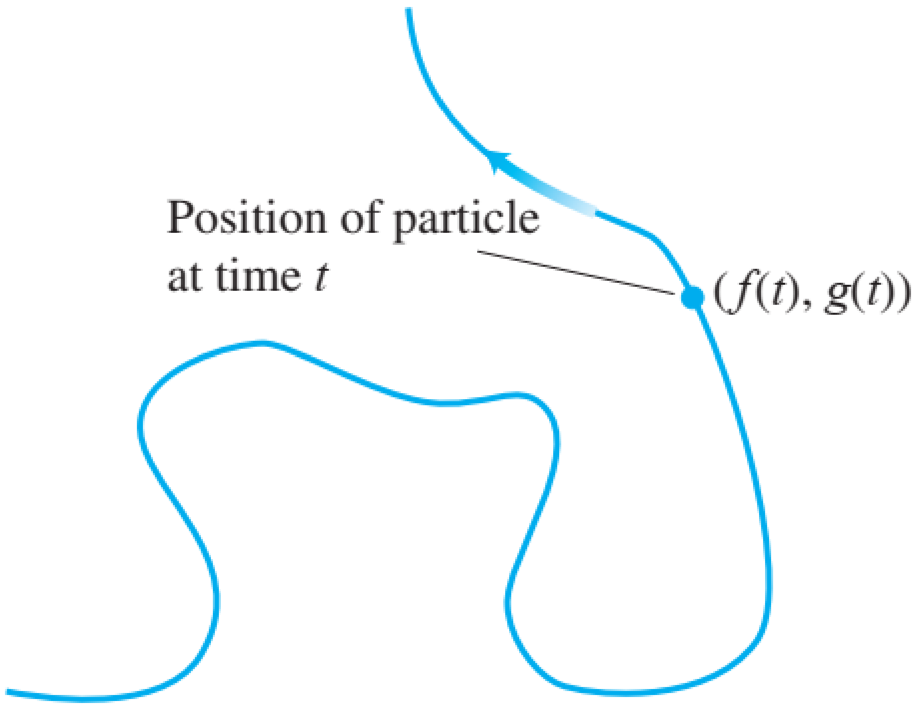


FIGURE 3.29 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

EXAMPLE

Graph the parametric curves

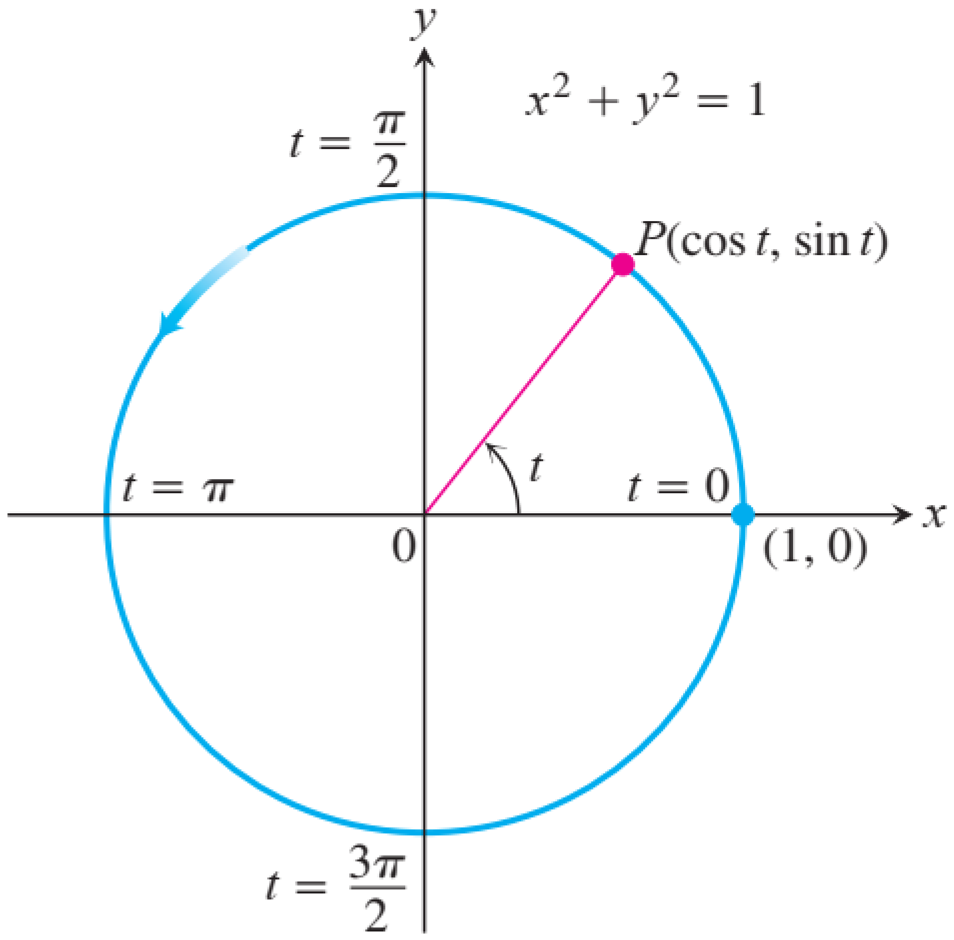
(a) $x = \cos t, \qquad y = \sin t, \qquad 0 \leq t \leq 2\pi.$

(b) $x = a\cos t, \qquad y = a\sin t, \qquad 0 \leq t \leq 2\pi.$

Solution

(a) Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the parametric curve lies along the unit circle $x^2 + y^2 = 1$. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at $(1, 0)$ and traces the entire circle once counterclockwise (Figure 3.30).

(b) For $x = a\cos t$, $y = a\sin t$, $0 \leq t \leq 2\pi$, we have $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$. The parametrization describes a motion that begins at the point $(a, 0)$ and traverses the circle $x^2 + y^2 = a^2$ once counterclockwise, returning to $(a, 0)$ at $t = 2\pi$. ■


y
$x^2 + y^2 = 1$
$t = \frac{\pi}{2}$
$P(\cos t, \sin t)$
t
$t = \pi$
$t = 0$
x
0
(1, 0)
$t = \frac{3\pi}{2}$

Slopes of Parametrized Curves

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{2}$$

EXAMPLE Differentiating with a Parameter

If $x = 2t + 3$ and $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

Solution Equation (2) gives dy/dx as a function of t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t = \frac{x-3}{2}.$$

When $t = 6$, $dy/dx = 6$. Notice that we are also able to find the derivative dy/dx as a function of x. ■

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \qquad y' = \frac{dy}{dx} \qquad (3)$$

EXAMPLE

Find d^2y/dx^2 as a function of t if $x = t - t^2,\ y = t - t^3$.

Solution

1. Express $y' = dy/dx$ in terms of t.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

2. Differentiate y' with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2} \qquad \text{Quotient Rule}$$

3. Divide dy'/dt by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3} \qquad \text{Eq. (3)} \qquad \blacksquare$$

Exercise 3.5 Question # 1-10, 61, 62, 71.