

Areas of Surfaces of Revolution

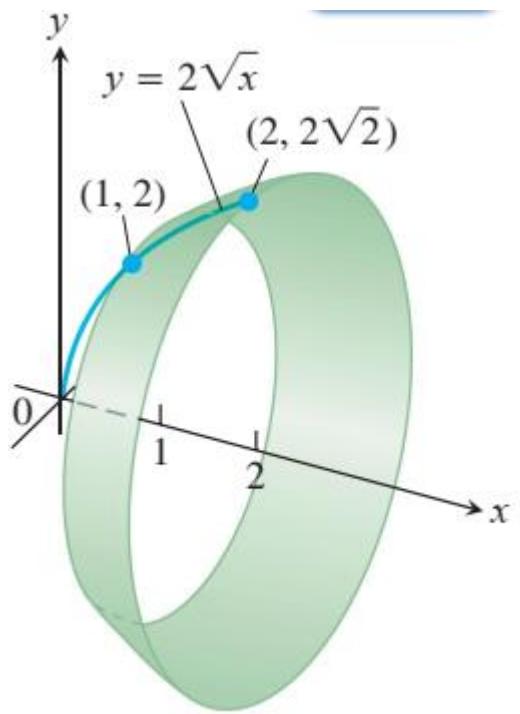
DEFINITION Surface Area for Revolution About the x -Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x -axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx. \quad (3)$$

EXAMPLE

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x -axis (Figure 6.48).



Solution

We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\begin{aligned} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} \\ &= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}. \end{aligned}$$

With these substitutions,

$$\begin{aligned} S &= \int_1^2 2\pi \cdot 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx \\ &= 4\pi \cdot \frac{2}{3} (x+1)^{3/2} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}). \end{aligned}$$

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Revolution About the y -Axis

For revolution about the y -axis, we interchange x and y in Equation (3).

Surface Area for Revolution About the y -Axis

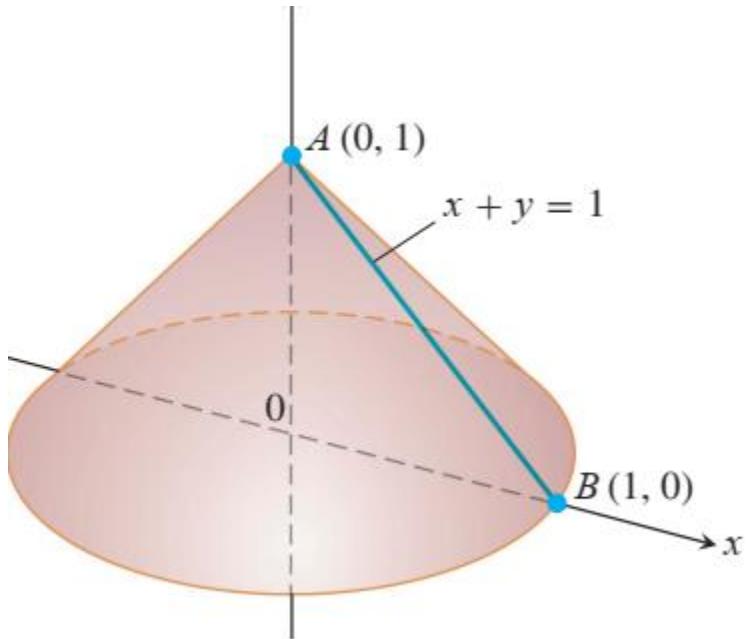
If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y -axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy. \quad (4)$$

EXAMPLE

The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y -axis to generate the cone in Figure 6.49. Find its lateral surface area (which excludes the base area).

FIGURE 6.49 Revolving line segment AB about the y -axis generates a cone whose



Solution

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 2\pi(1 - y)\sqrt{2} dy \\ &= 2\pi\sqrt{2} \left[y - \frac{y^2}{2} \right]_0^1 = 2\pi\sqrt{2} \left(1 - \frac{1}{2} \right) \\ &= \pi\sqrt{2}. \end{aligned}$$

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \leq t \leq b$, is traversed exactly once as t increases from a to b , then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x -axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (5)$$

2. Revolution about the y -axis ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (6)$$

EXAMPLE

The standard parametrization of the circle of radius 1 centered at the point $(0, 1)$ in the xy -plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi.$$

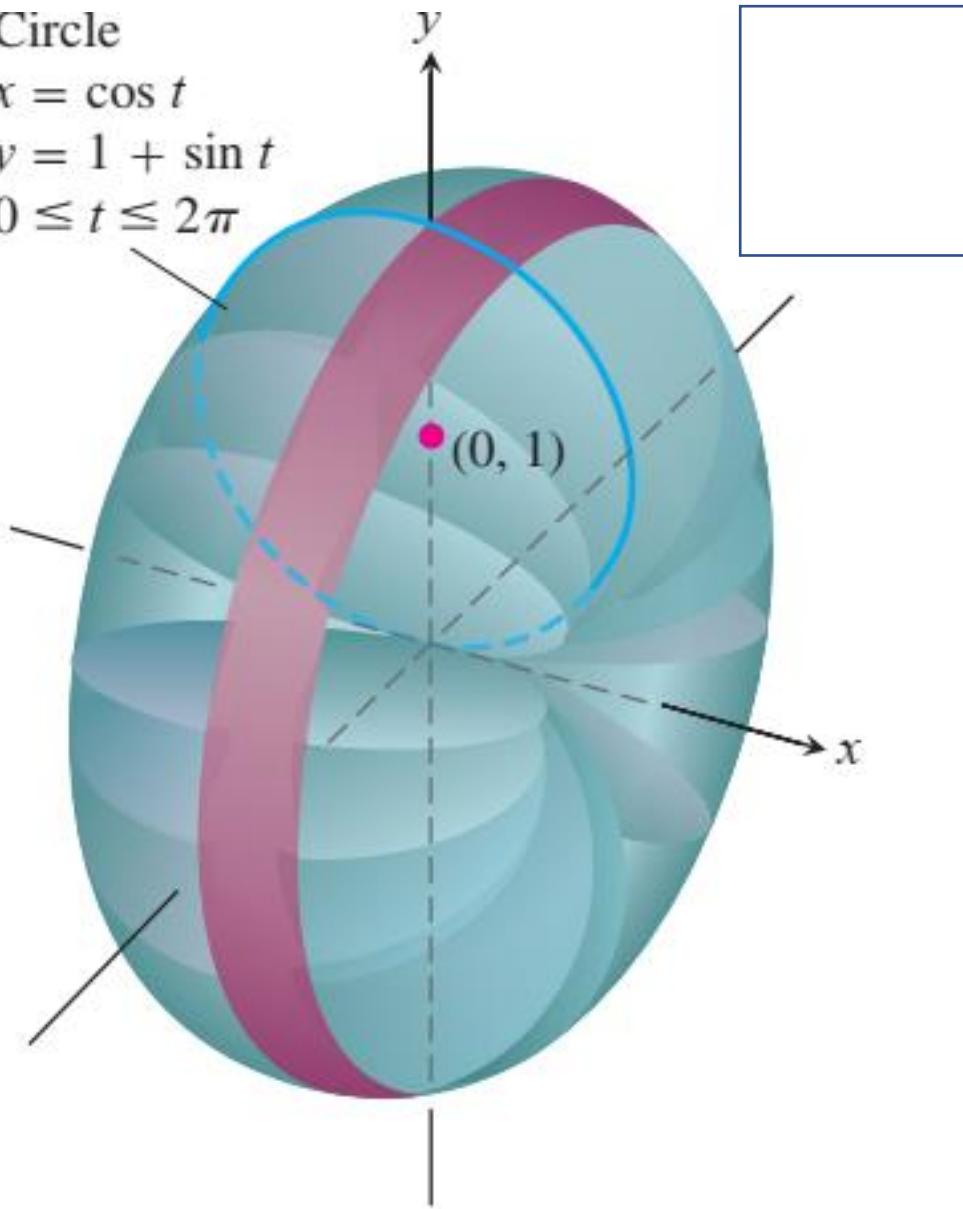
Use this parametrization to find the area of the surface swept out by revolving the circle about the x -axis (Figure 6.50).

Circle

$$x = \cos t$$

$$y = 1 + \sin t$$

$$0 \leq t \leq 2\pi$$



Solution

We evaluate the formula

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{2\pi} 2\pi(1 + \sin t) \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_1} dt \\ &= 2\pi \int_0^{2\pi} (1 + \sin t) dt \\ &= 2\pi [t - \cos t]_0^{2\pi} = 4\pi^2. \end{aligned}$$

Eq. (5) for revolution
about the x -axis;
 $y = 1 + \sin t > 0$



