Indeterminate Forms and L'Hôpital's Rule

Indeterminate Form $0/0$, ∞/∞, $\infty \cdot 0$, $\infty - \infty$

L'Hôpital's Rule

L'Hospital theorems deal with calculus of various limits of the form:

$$\lim_{x \to a} \frac{f(x)}{g(x)}, \quad \lim_{x \to \infty} \frac{f(x)}{g(x)}, \quad \lim_{x \to -\infty} \frac{f(x)}{g(x)} \tag{9.1}$$

in the "indeterminate" case when: $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = +\infty$, etc.

Under appropriate assumptions we can establish that, for instance

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} \tag{9.2}$$

(the limit on right-hand side will exist and accordingly, the limit on left-hand side will exist too).

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE

(a) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x}-1-x/2}{x^2}$ $\dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2}-1/2}{2x}$$ Still $\dfrac{0}{0}$; differentiate again.

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$ Not $\dfrac{0}{0}$; limit is found.

(b) $\displaystyle\lim_{x\to 0} \frac{x-\sin x}{x^3}$ $\dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2}$$ Still $\dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{\sin x}{6x}$$ Still $\dfrac{0}{0}$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$ Not $\dfrac{0}{0}$; limit is found.   ■

EXAMPLE 3   Incorrectly Applying the Stronger Form of L'Hôpital's Rule

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0 \qquad \text{Not } \frac{0}{0}; \text{ limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is wrong. L'Hôpital's Rule can only be applied to limits which give indeterminate forms, and $0/1$ is not an indeterminate form. ∎

EXAMPLE 5   Working with the Indeterminate Form ∞/∞

Find

(a) $\displaystyle \lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$

(b) $\displaystyle \lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5x}$

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$$\lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x} \qquad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

(b) $\displaystyle \lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x \to \infty} \frac{1 - 4x}{6x + 5} = \lim_{x \to \infty} \frac{-4}{6} = -\frac{2}{3}.$

EXAMPLE 6   Working with the Indeterminate Form $\infty \cdot 0$

Find

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$$

Solution

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right) \qquad \infty \cdot 0$$

$$= \lim_{h \to 0^+} \left(\frac{1}{h} \sin h \right) \qquad \text{Let } h = 1/x.$$

$$= 1$$

First convert to 0/0 or infinity/infinity form

EXAMPLE 7   Working with the Indeterminate Form $\infty - \infty$

Find

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

Solution   If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \qquad \text{Common denominator is } x \sin x$$

Then apply l'Hôpital's Rule to the result:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.$$

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