

Volumes by Cylindrical Shells

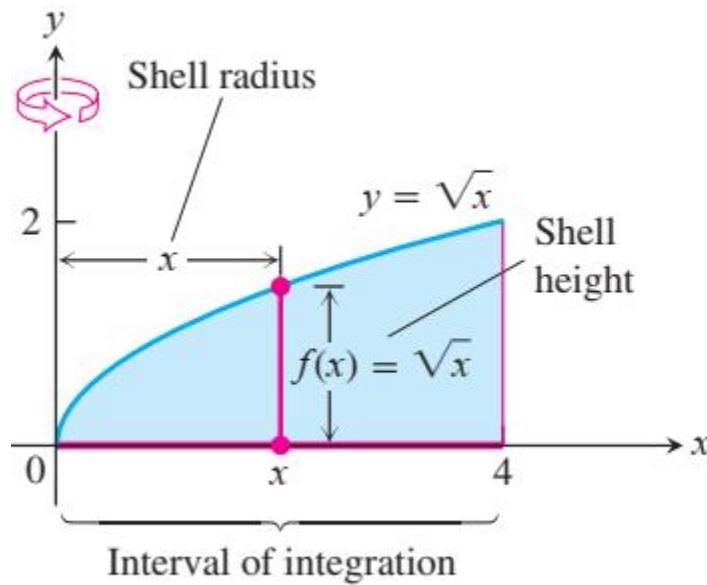
Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x -axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

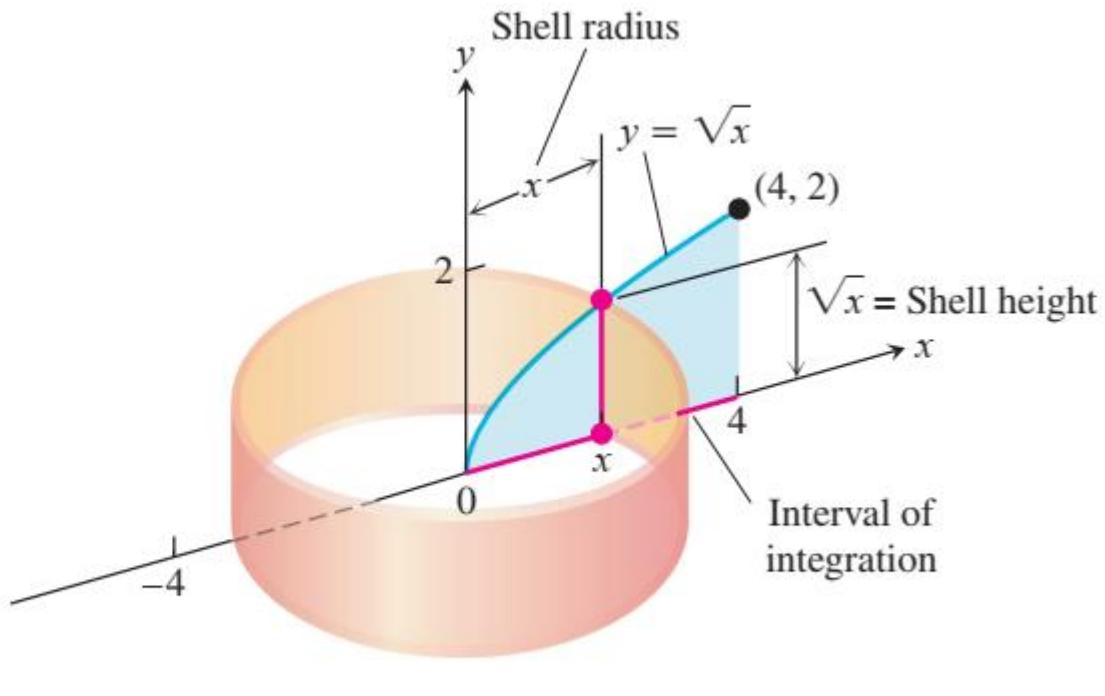
$$V = \int_a^b 2\pi \left(\begin{matrix} \text{shell} \\ \text{radius} \end{matrix} \right) \left(\begin{matrix} \text{shell} \\ \text{height} \end{matrix} \right) dx.$$

EXAMPLE

The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the y -axis to generate a solid. Find the volume of the solid.



(a)



(b)

The shell thickness variable is x , so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is then

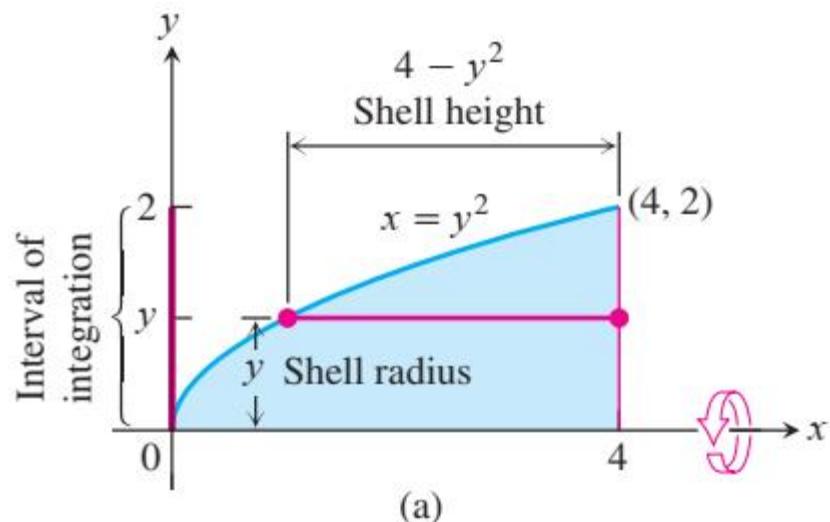
$$\begin{aligned} V &= \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) dx \\ &= \int_0^4 2\pi(x) (\sqrt{x}) dx \\ &= 2\pi \int_0^4 x^{3/2} dx = 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$

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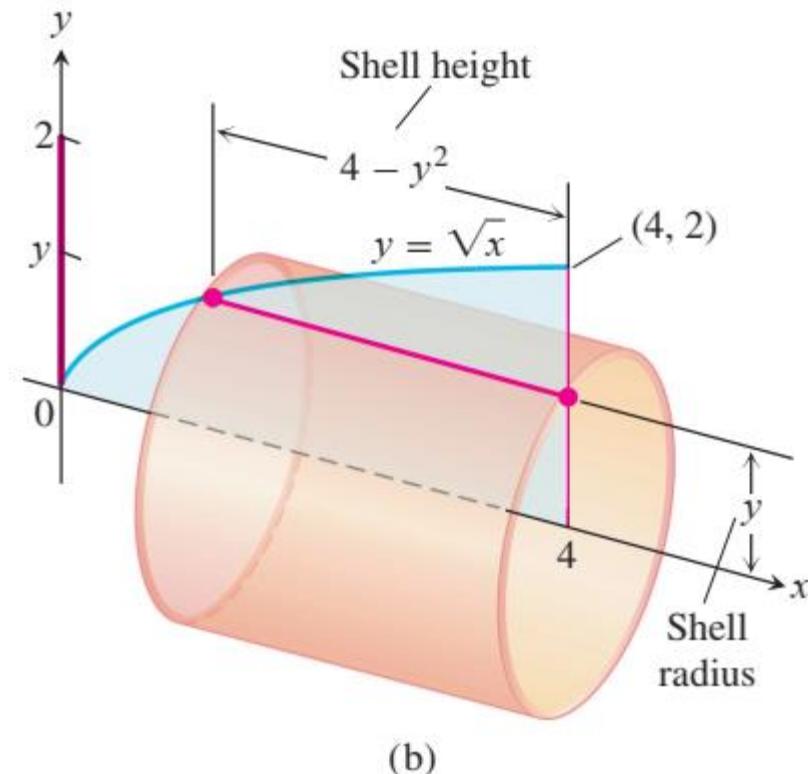
EXAMPLE

The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the x -axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.22a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.22b, but you need not do that.)



(a)



(b)

In this case, the shell thickness variable is y , so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y -axis in Figure 6.22). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) dy \\ &= \int_0^2 2\pi(y)(4 - y^2) dy \\ &= \int_0^2 2\pi(4y - y^3) dy \\ &= 2\pi \left[2y^2 - \frac{y^4}{4} \right]_0^2 = 8\pi. \end{aligned}$$

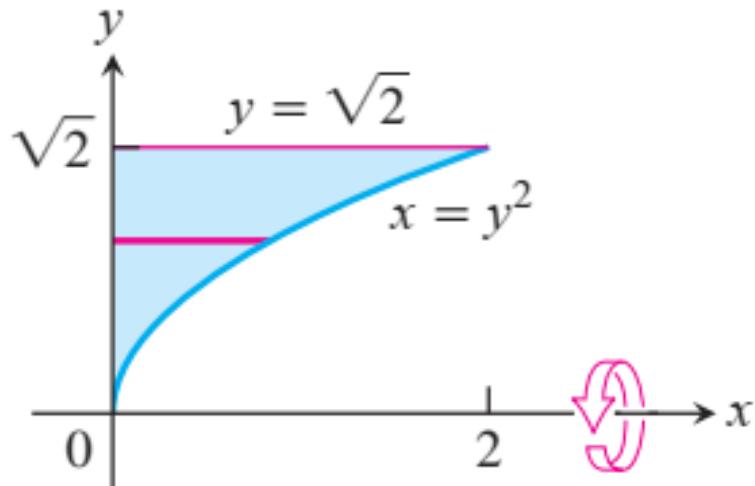


Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the segment's height or length (shell height) and distance from the axis of revolution (shell radius).*
2. *Find the limits of integration for the thickness variable.*
3. *Integrate the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.*

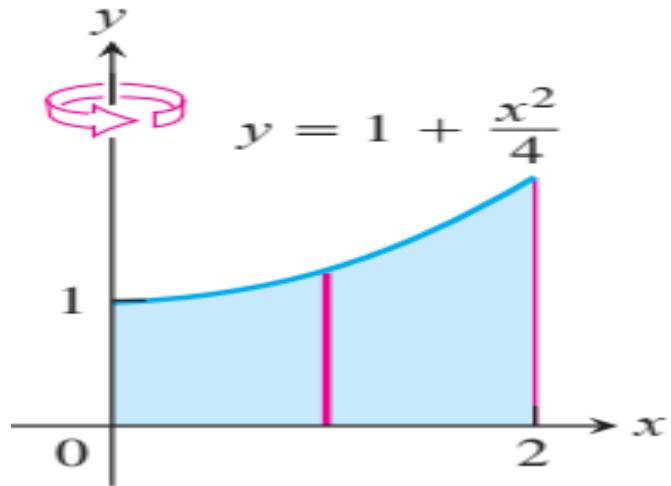
In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis. 3.



For the sketch given, $c = 0$, $d = \sqrt{2}$;

$$V = \int_c^d 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) dy = \int_0^{\sqrt{2}} 2\pi y \cdot (y^2) dy = 2\pi \int_0^{\sqrt{2}} y^3 dy = 2\pi \left[\frac{y^4}{4} \right]_0^{\sqrt{2}} = 2\pi$$

1.



For the sketch given, $a = 0$, $b = 2$;

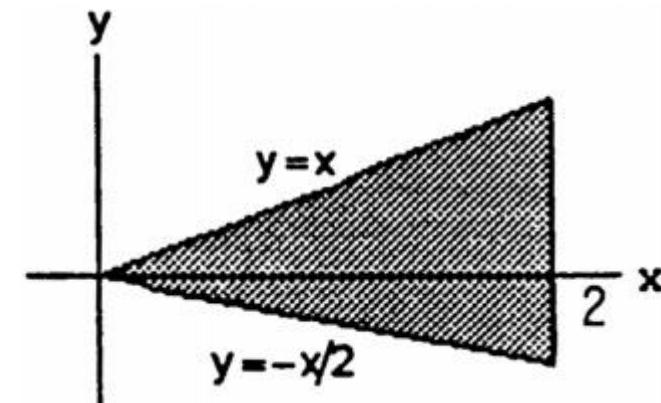
$$\begin{aligned} V &= \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) dx = \int_0^2 2\pi x \left(1 + \frac{x^2}{4} \right) dx \\ &= 2\pi \int_0^2 \left(x + \frac{x^3}{4} \right) dx = 2\pi \left[\frac{x^2}{2} + \frac{x^4}{16} \right]_0^2 = 2\pi \left(\frac{4}{2} + \frac{16}{16} \right) \\ &= 2\pi \cdot 3 = 6\pi \end{aligned}$$

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–14 about the y -axis.

7. $y = x$, $y = -x/2$, $x = 2$

$$a = 0, b = 2;$$

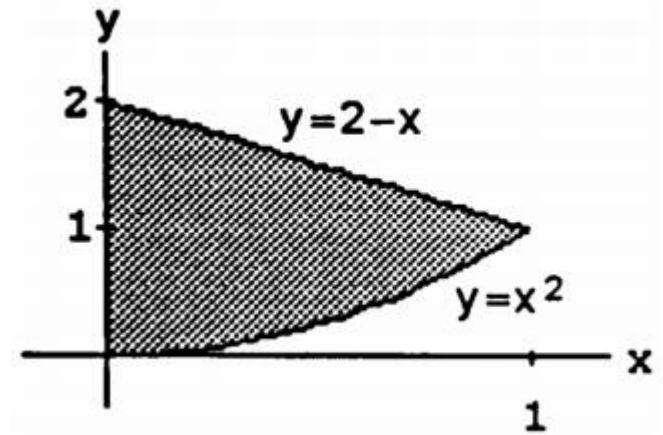
$$\begin{aligned} V &= \int_a^b 2\pi \left(\frac{\text{shell}}{\text{radius}} \right) \left(\frac{\text{shell}}{\text{height}} \right) dx = \int_0^2 2\pi x \left[x - \left(-\frac{x}{2} \right) \right] dx \\ &= \int_0^2 2\pi x^2 \cdot \frac{3}{2} dx = \pi \int_0^2 3x^2 dx = \pi [x^3]_0^2 = 8\pi \end{aligned}$$



9. $y = x^2$, $y = 2 - x$, $x = 0$, for $x \geq 0$

$$a = 0, b = 1;$$

$$\begin{aligned} V &= \int_a^b 2\pi \left(\text{radius} \right) \left(\text{height} \right) dx = \int_0^1 2\pi x [(2-x) - x^2] dx \\ &= 2\pi \int_0^1 (2x - x^2 - x^3) dx = 2\pi \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 \\ &= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4} \right) = 2\pi \left(\frac{12-4-3}{12} \right) = \frac{10\pi}{12} = \frac{5\pi}{6} \end{aligned}$$



Lengths of Plane Curves

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DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

EXAMPLE

Find the length of the circle of radius r defined parametrically by

$$x = r \cos t \quad \text{and} \quad y = r \sin t, \quad 0 \leq t \leq 2\pi.$$

Solution As t varies from 0 to 2π , the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

We find

$$\frac{dx}{dt} = -r \sin t, \quad \frac{dy}{dt} = r \cos t$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

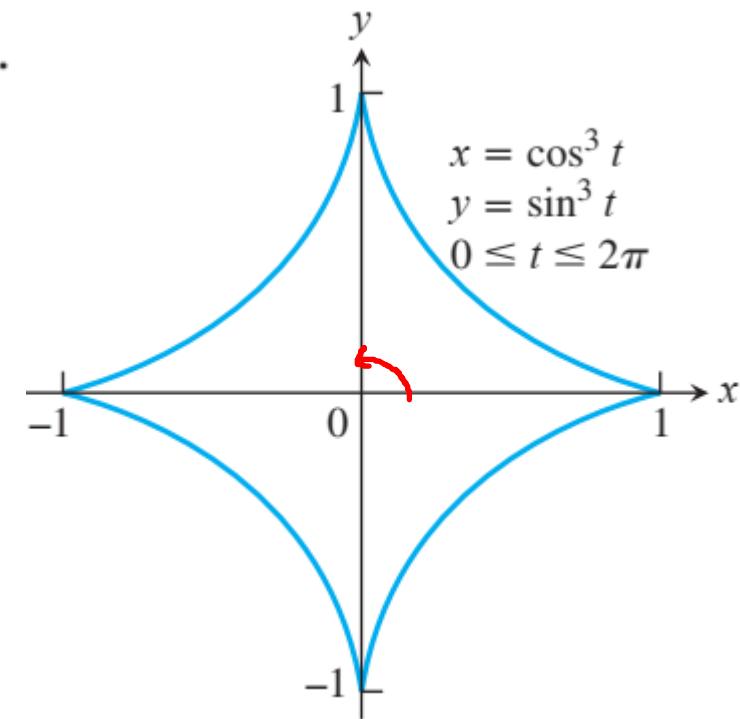
So

$$L = \int_0^{2\pi} \sqrt{r^2} dt = r [t]_0^{2\pi} = 2\pi r.$$

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EXAMPLE Find the length of

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi.$$



Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have

$$x = \cos^3 t, \quad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3 \cos^2 t(-\sin t)]^2 = 9 \cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3 \sin^2 t(\cos t)]^2 = 9 \sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9 \cos^2 t \sin^2 t (\underbrace{\cos^2 t + \sin^2 t}_1)}$$

$$= \sqrt{9 \cos^2 t \sin^2 t}$$

$$= 3 |\cos t \sin t|$$

$$= 3 \cos t \sin t.$$

$\cos t \sin t \geq 0$ for
 $0 \leq t \leq \pi/2$

Therefore,

$$\begin{aligned}\text{Length of first-quadrant portion} &= \int_0^{\pi/2} 3 \cos t \sin t \, dt \\ &= \frac{3}{2} \int_0^{\pi/2} \sin 2t \, dt \\ &= -\frac{3}{4} \cos 2t \Big|_0^{\pi/2} = \frac{3}{2}.\end{aligned}$$

$$\cos t \sin t = (1/2) \sin 2t$$

The length of the astroid is four times this: $4(3/2) = 6$.



Formula for the Length of $y = f(x)$, $a \leq x \leq b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{1 + [f'(x)]^2} dx. \quad (2)$$

EXAMPLE

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \leq x \leq 1.$$

Solution We use Equation (2) with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = \underline{2\sqrt{2}x^{1/2}}$$

$$\left(\frac{dy}{dx}\right)^2 = (2\sqrt{2}x^{1/2})^2 = 8x.$$

The length of the curve from $x = 0$ to $x = 1$ is

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^1 \sqrt{1 + 8x} dx \\ &= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big|_0^1 = \frac{13}{6}. \end{aligned}$$

Eq. (2) with
 $a = 0, b = 1$
Let $u = 1 + 8x$,
integrate, and
replace u by
 $1 + 8x$. ■

At a point on a curve where dy/dx fails to exist, dx/dy may exist and we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation (2):

Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d \sqrt{1 + [g'(y)]^2} dy. \quad (3)$$

EXAMPLE

Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3} \left(\frac{x}{2}\right)^{-1/3} \left(\frac{1}{2}\right) = \frac{1}{3} \left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (2).

We therefore rewrite the equation to express x in terms of y :

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2}$$

Raise both sides
to the power 3/2.

$$x = 2y^{3/2}.$$

Solve for x .

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Figure 6.27).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = \underline{\underline{3y^{1/2}}}$$

is continuous on $[0, 1]$. We may therefore use Equation (3) to find the curve's length:

$$\begin{aligned} L &= \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 \sqrt{1 + 9y} dy && \text{Eq. (3) with } c = 0, d = 1. \\ &= \frac{1}{9} \cdot \frac{2}{3} (1 + 9y)^{3/2} \Big|_0^1 && \text{Let } u = 1 + 9y, \\ &= \frac{2}{27} (10\sqrt{10} - 1) \approx 2.27. && \text{du}/9 = dy, \\ &&& \text{integrate, and} \\ &&& \text{substitute back.} \end{aligned}$$

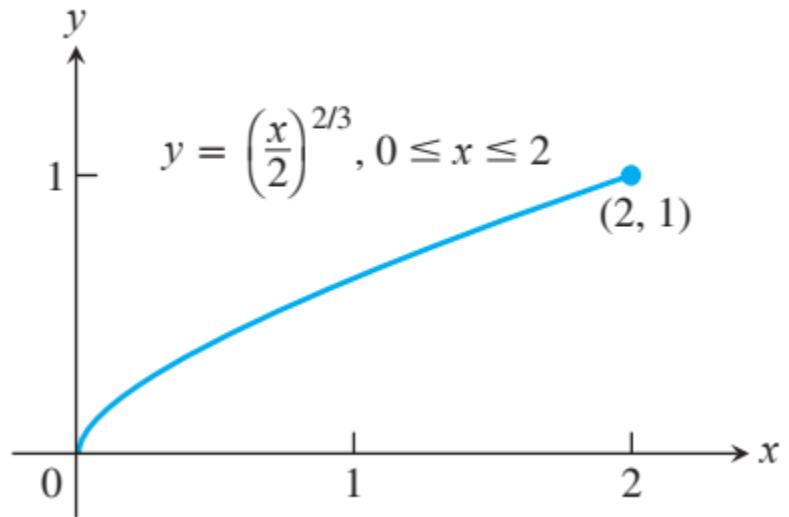


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).