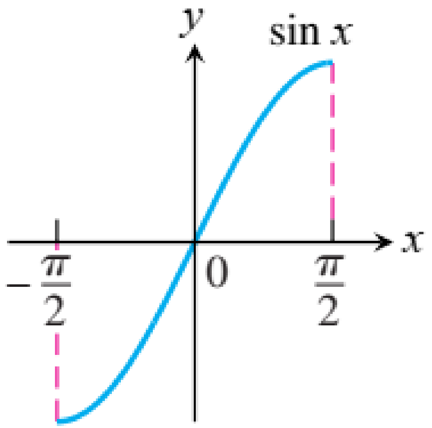
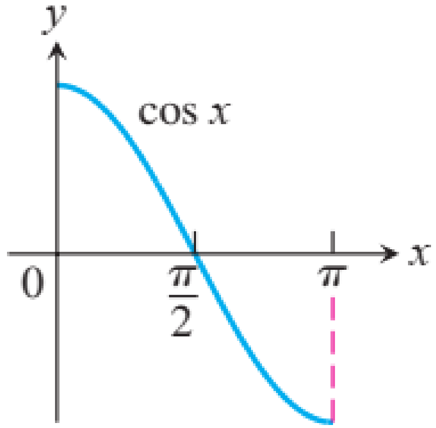
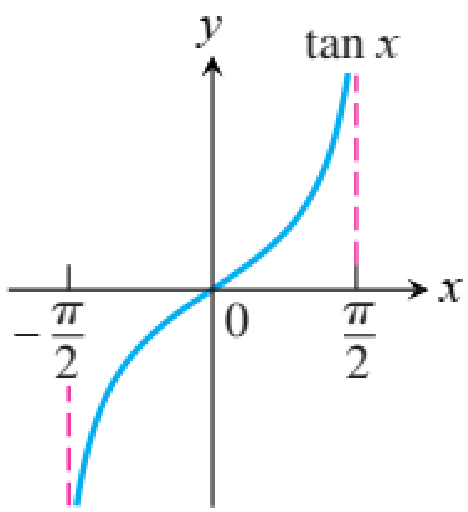
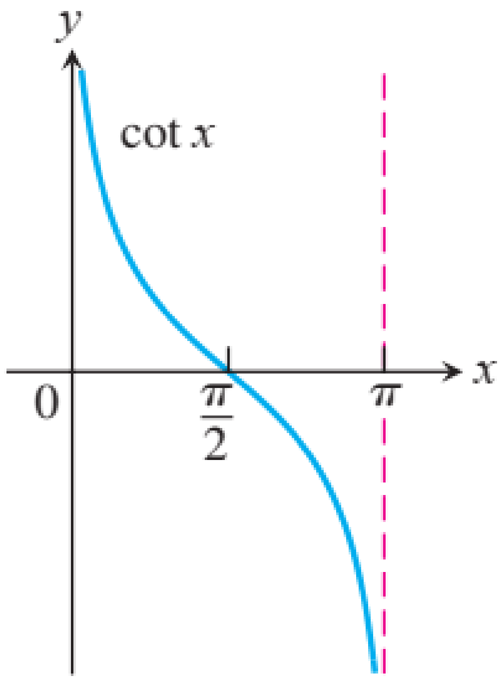
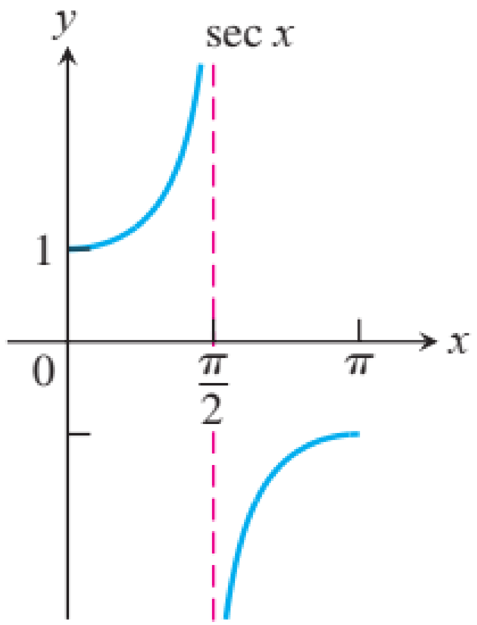
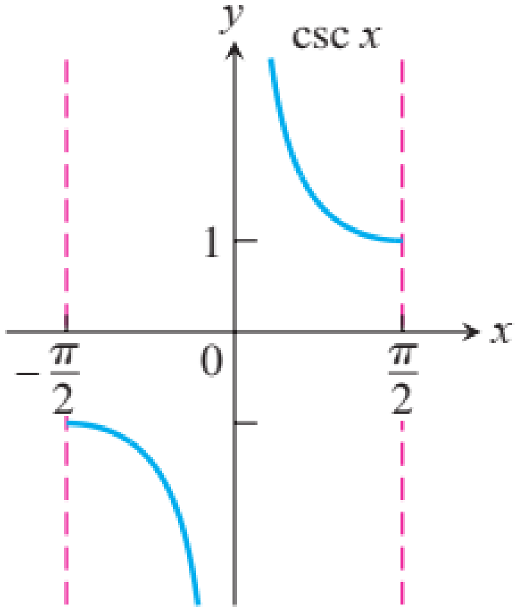
Inverse Trigonometric Functions

Domain restrictions that make the trigonometric functions one-to-one

Function	**Domain**	**Range**	
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	
$\cos x$	$[0, \pi]$	$[-1, 1]$	

$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	

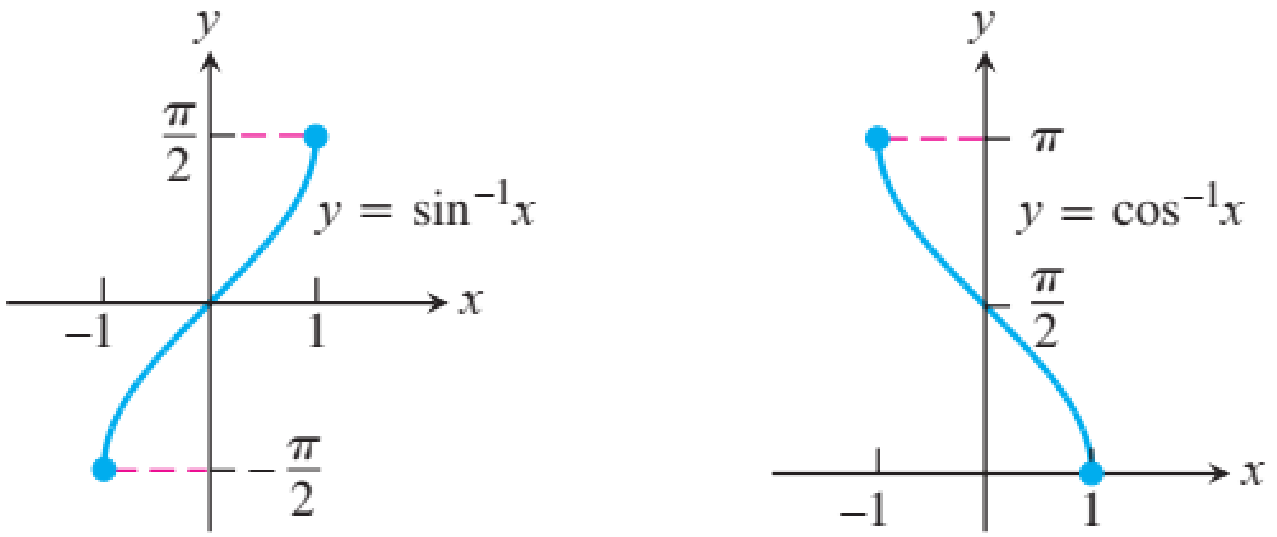
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$
\begin{aligned}
y &= \sin^{-1} x &\quad \text{or} \quad && y &= \arcsin x \\
y &= \cos^{-1} x &\quad \text{or} \quad && y &= \arccos x \\
y &= \tan^{-1} x &\quad \text{or} \quad && y &= \arctan x \\
y &= \cot^{-1} x &\quad \text{or} \quad && y &= \operatorname{arccot} x \\
y &= \sec^{-1} x &\quad \text{or} \quad && y &= \operatorname{arcsec} x \\
y &= \csc^{-1} x &\quad \text{or} \quad && y &= \operatorname{arccsc} x
\end{aligned}
$$

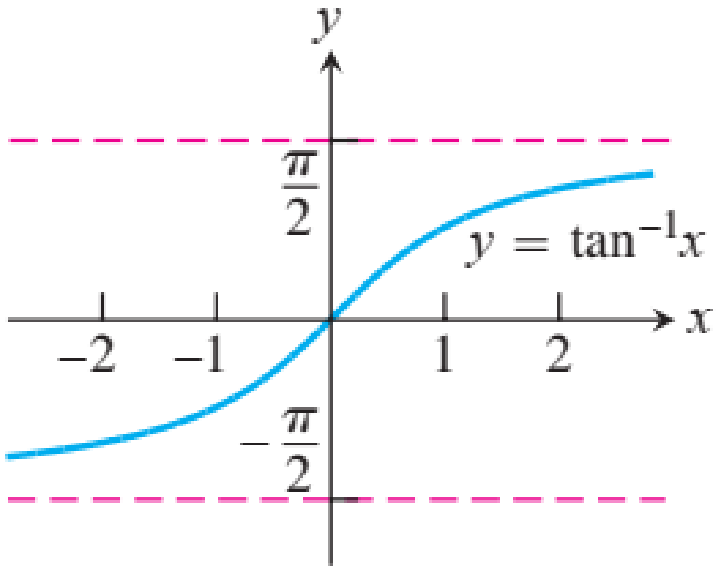
Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



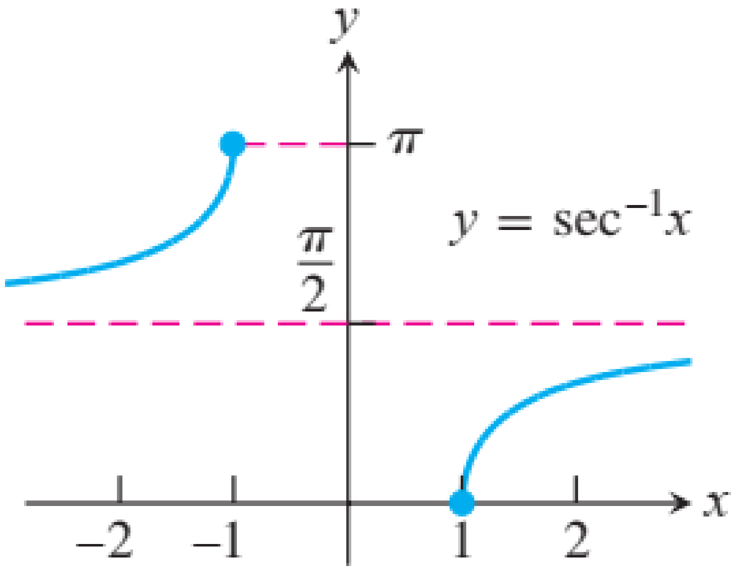
(a) (b)

Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



(c)

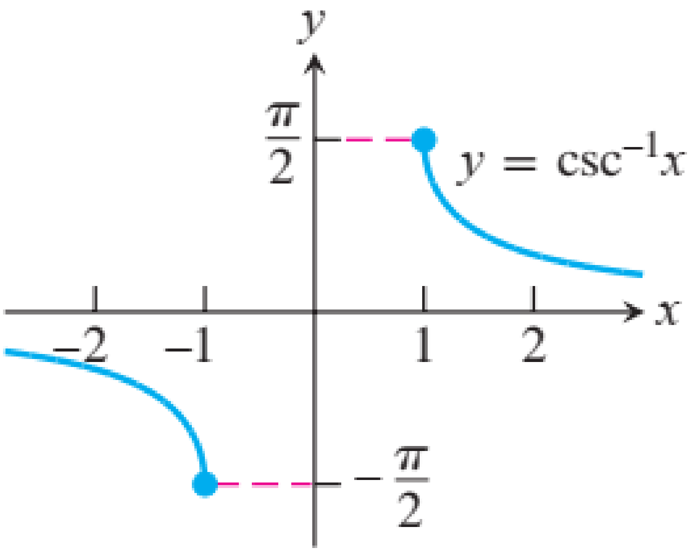
Domain: $x \leq -1$ or $x \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$



(d)

Domain: $x \leq -1$ or $x \geq 1$

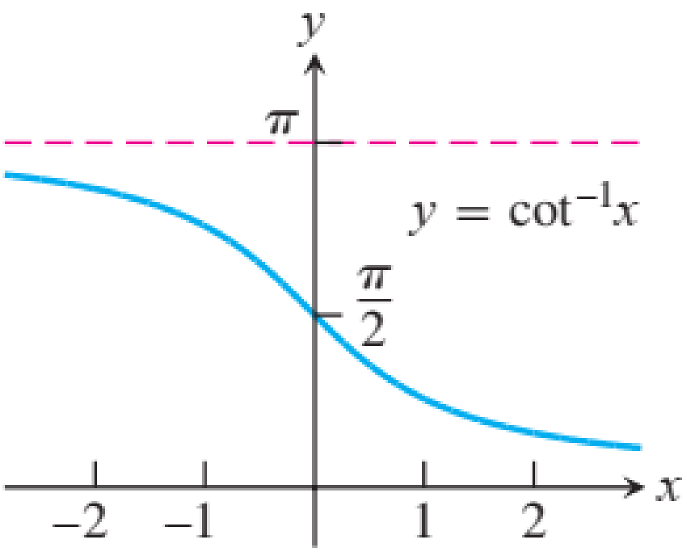
Range: $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}, y \neq 0$



(e)

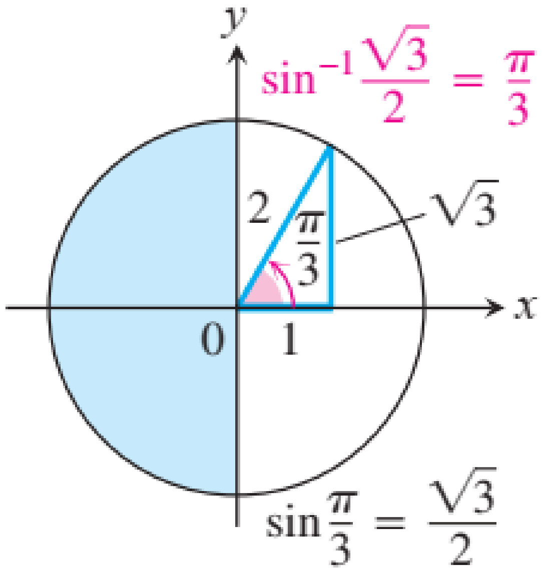
Domain: $-\infty < x < \infty$

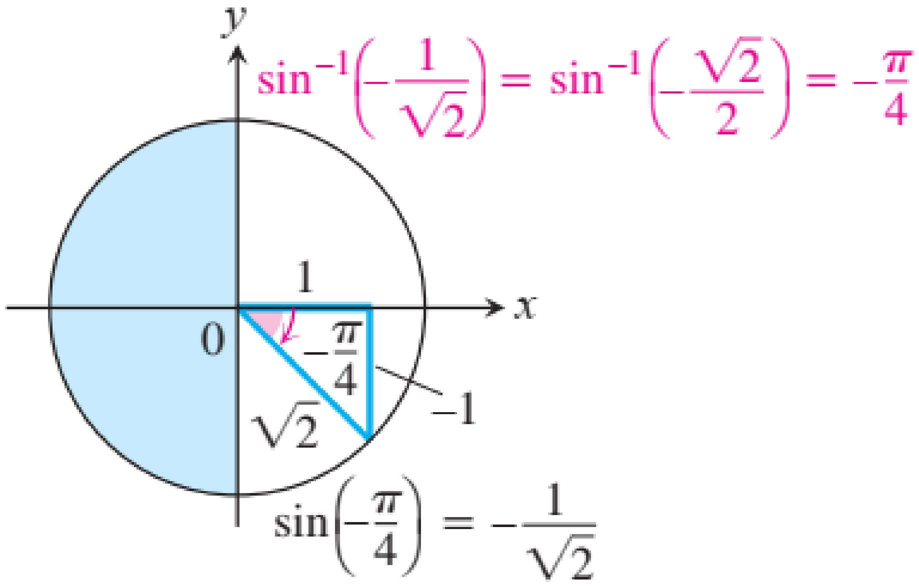
Range: $0 < y < \pi$



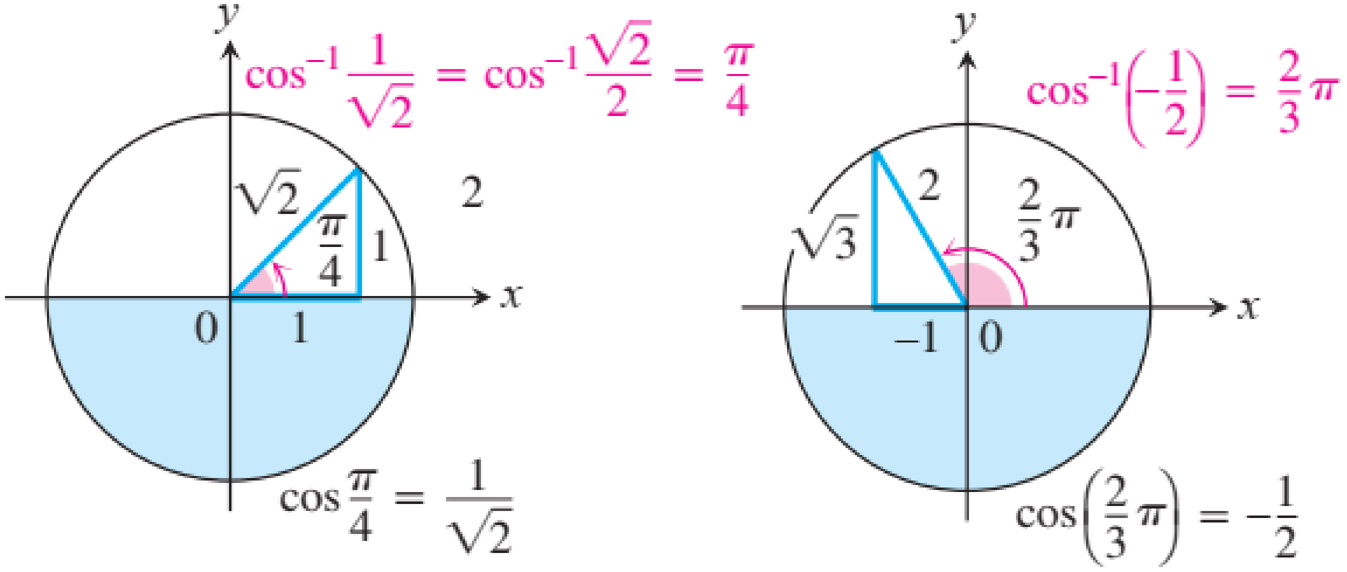
(f)

EXAMPLE 1 Common Values of $\sin^{-1} x$



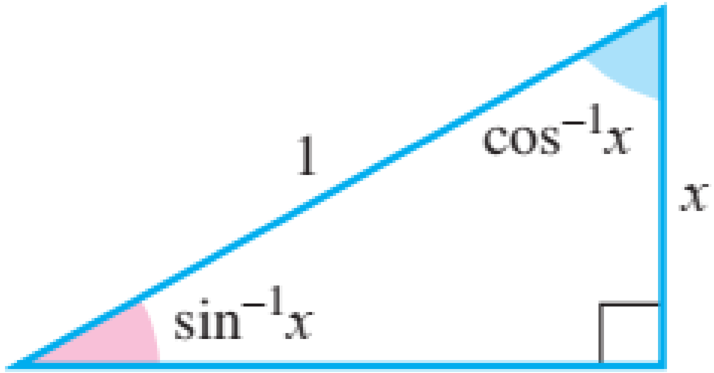


EXAMPLE 2 Common Values of $\cos^{-1} x$

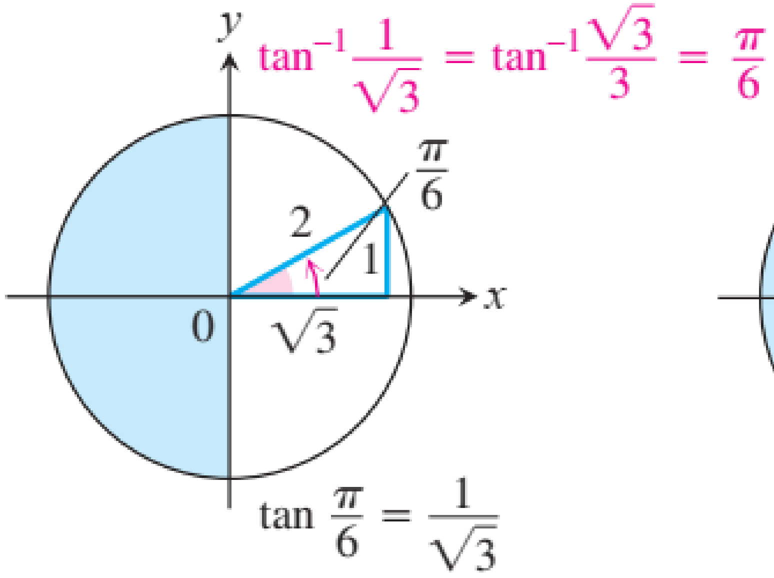


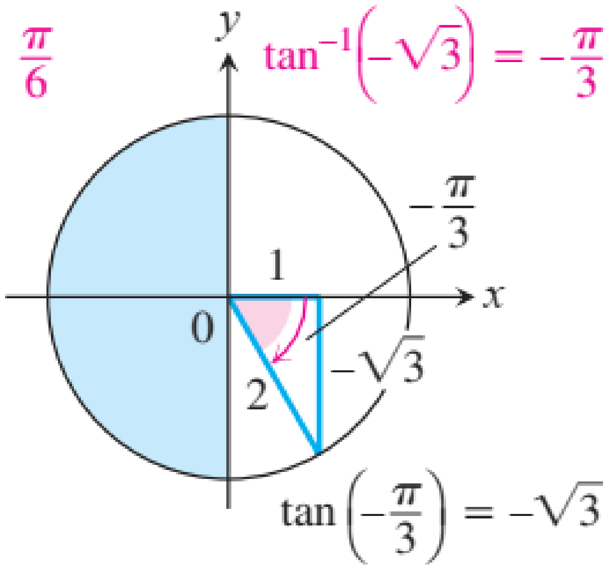
Also, we can see from the triangle in Figure 7.21 that for $x > 0$,

$$\sin^{-1} x + \cos^{-1} x = \pi/2.$$



EXAMPLE 3 Common Values of $\tan^{-1} x$





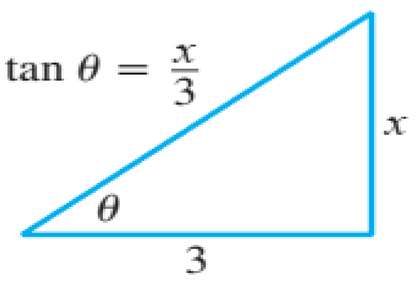
EXAMPLE 5 Find $\sec\left(\tan^{-1}\frac{x}{3}\right)$.

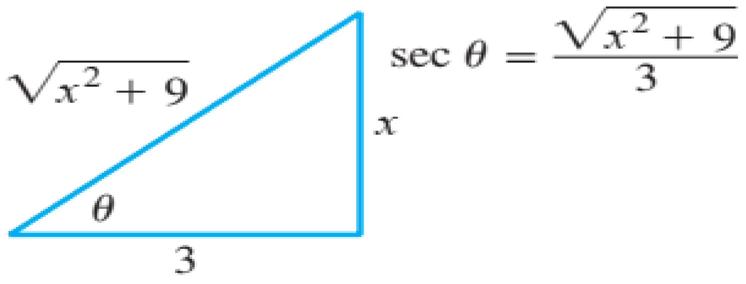
Solution We let $\theta = \tan^{-1}(x/3)$ (to give the angle a name) and picture θ in a right triangle with

$$\tan\theta = \text{opposite/adjacent} = x/3.$$

The length of the triangle's hypotenuse is

$$\sqrt{x^2 + 3^2} = \sqrt{x^2 + 9}.$$





Thus,

$$\sec\left(\tan^{-1}\frac{x}{3}\right) = \sec\theta$$

$$= \frac{\sqrt{x^2+9}}{3}. \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

■

The Derivative of $y = \sin^{-1} u$

We find the derivative of $y = \sin^{-1} x$ by applying Theorem 1 with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$.

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}} && \sin(\sin^{-1} x) = x
\end{aligned}$$

Alternate Derivation: Instead of applying Theorem 1 directly, we can find the derivative of $y = \sin^{-1} x$ using implicit differentiation as follows:

$$\sin y = x \qquad y = \sin^{-1} x \Leftrightarrow \sin y = x$$

$$\frac{d}{dx}(\sin y) = 1 \qquad \text{Derivative of both sides with respect to } x$$

$$\cos y \, \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \qquad \text{We can divide because } \cos y > 0 \text{ for } -\pi/2 < y < \pi/2.$$

$$= \frac{1}{\sqrt{1 - x^2}} \qquad \cos y = \sqrt{1 - \sin^2 y}$$

If u is a differentiable function of x with $|u| < 1$, we apply the Chain Rule to get

$$\boxed{\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \qquad |u| < 1.}$$

EXAMPLE 7 Applying the Derivative Formula

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}$$

■

The Derivative of $y = \tan^{-1} u$

We find the derivative of $y = \tan^{-1} x$ by applying Theorem 1 with $f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$. Theorem 1 can be applied because the derivative of $\tan x$ is positive for $-\pi/2 < x < \pi/2$.

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 1} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
&= \frac{1}{1 + x^2} && \tan(\tan^{-1} x) = x
\end{aligned}
$$

The derivative is defined for all real numbers. If u is a differentiable function of x, we get the Chain Rule form:

$$\boxed{\frac{d}{dx}\left(\tan^{-1} u\right) = \frac{1}{1 + u^2}\frac{du}{dx}.}$$

The Derivative of $y = \sec^{-1} u$

Since the derivative of $\sec x$ is positive for $0 < x < \pi/2$ and $\pi/2 < x < \pi$, Theorem 1 says that the inverse function $y = \sec^{-1} x$ is differentiable. Instead of applying the formula in Theorem 1 directly, we find the derivative of $y = \sec^{-1} x$, $|x| > 1$, using implicit differentiation and the Chain Rule as follows:

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx} x \qquad \text{Differentiate both sides.}$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

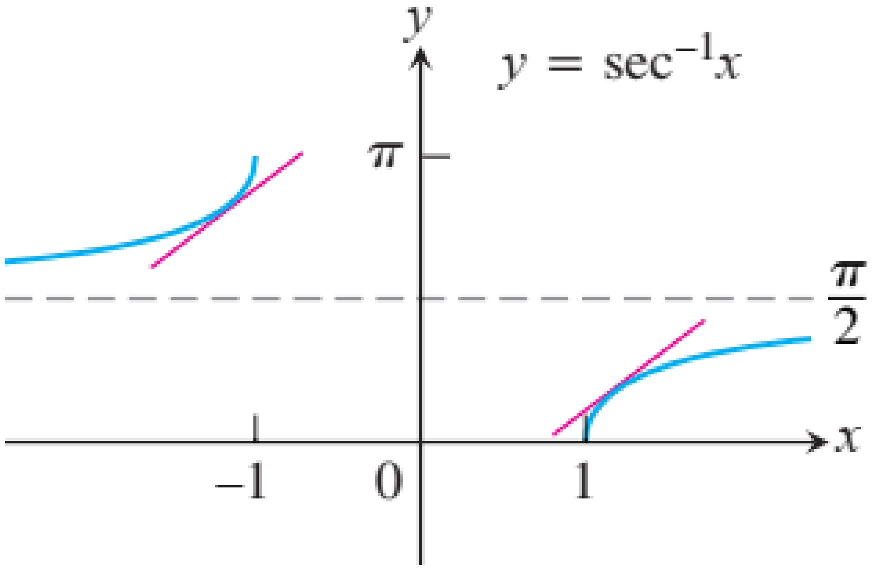
Since $|x| > 1$, y lies in $(0, \pi/2) \cup (\pi/2, \pi)$ and $\sec y \tan y \neq 0$.

To express the result in terms of x, we use the relationships

$$\sec y = x \qquad \text{and} \qquad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

to get

$$\frac{dy}{dx} = \pm \frac{1}{x\sqrt{x^2 - 1}} .$$



Thus,

$$\frac{d}{dx}\sec^{-1}x = \begin{cases} +\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

With the absolute value symbol, we can write a single expression that eliminates the "±" ambiguity:

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}.$$

If u is a differentiable function of x with $|u| > 1$, we have the formula

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

EXAMPLE 9 Using the Formula

$$\begin{aligned}
\frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2 - 1}}\frac{d}{dx}(5x^4) \\
&= \frac{1}{5x^4\sqrt{25x^8 - 1}}(20x^3) \qquad 5x^4 > 0 \\
&= \frac{4}{x\sqrt{25x^8 - 1}}
\end{aligned}$$

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1} x\right) \qquad \text{Identity}$$

$$= -\frac{d}{dx}(\sin^{-1} x)$$

$$= -\frac{1}{\sqrt{1 - x^2}} \qquad \text{Derivative of arcsine}$$

TABLE 7.3 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1 - u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1 + u^2}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1 + u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2 - 1}}, \quad |u| > 1$$