Limits and Continuity

DEFINITION Average Rate of Change over an Interval

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

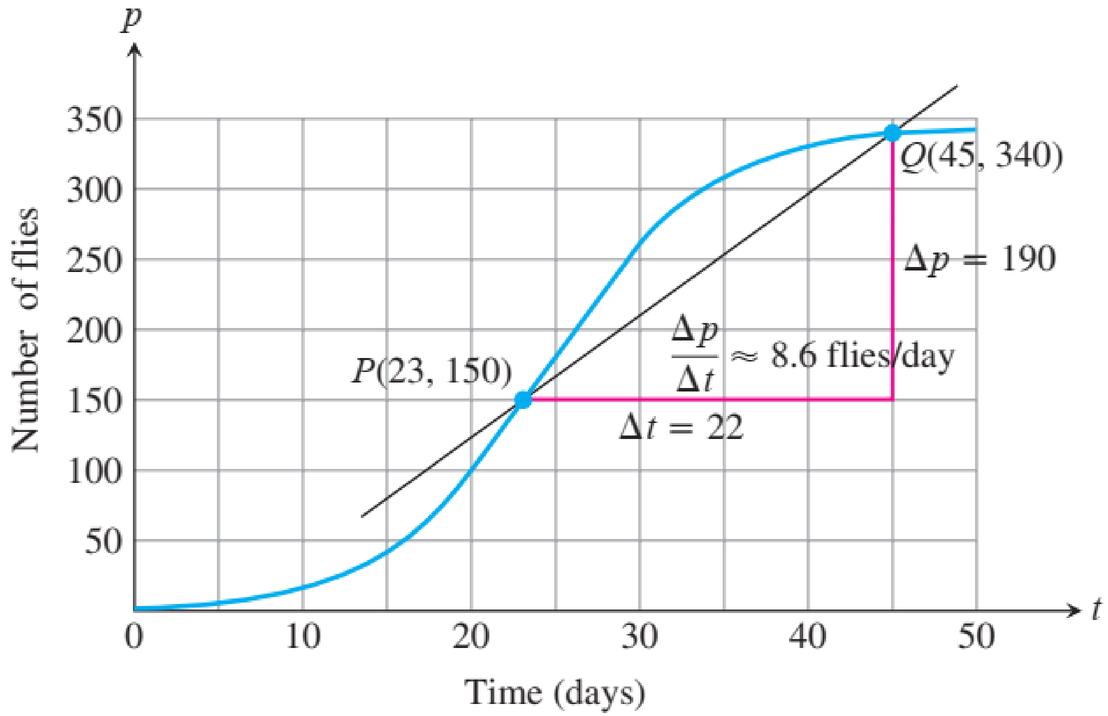


FIGURE 2.2 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line.

EXAMPLE 3 The Average Growth Rate of a Laboratory Population

Figure 2.2 shows how a population of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time, and the points joined by a smooth curve (colored blue in Figure 2.2). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}.$$

This average is the slope of the secant through the points P and Q on the graph in Figure 2.2. ■

Let $f(x)$ be defined on an open interval about x_0, *except possibly at x_0 itself*. If $f(x)$ gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the **limit** L as x approaches x_0, and we write

$$\lim_{x \to x_0} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches x_0 is L". Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to x_0 (on either side of x_0).

EXAMPLE 5 Behavior of a Function Near a Point

How does the function

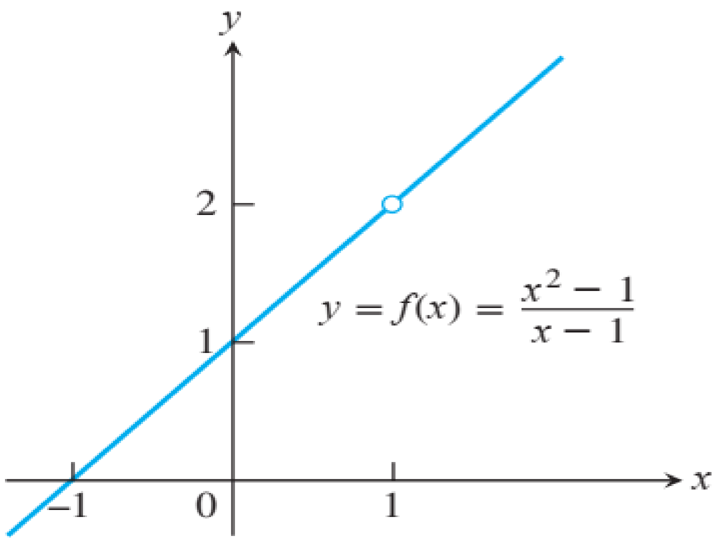
$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

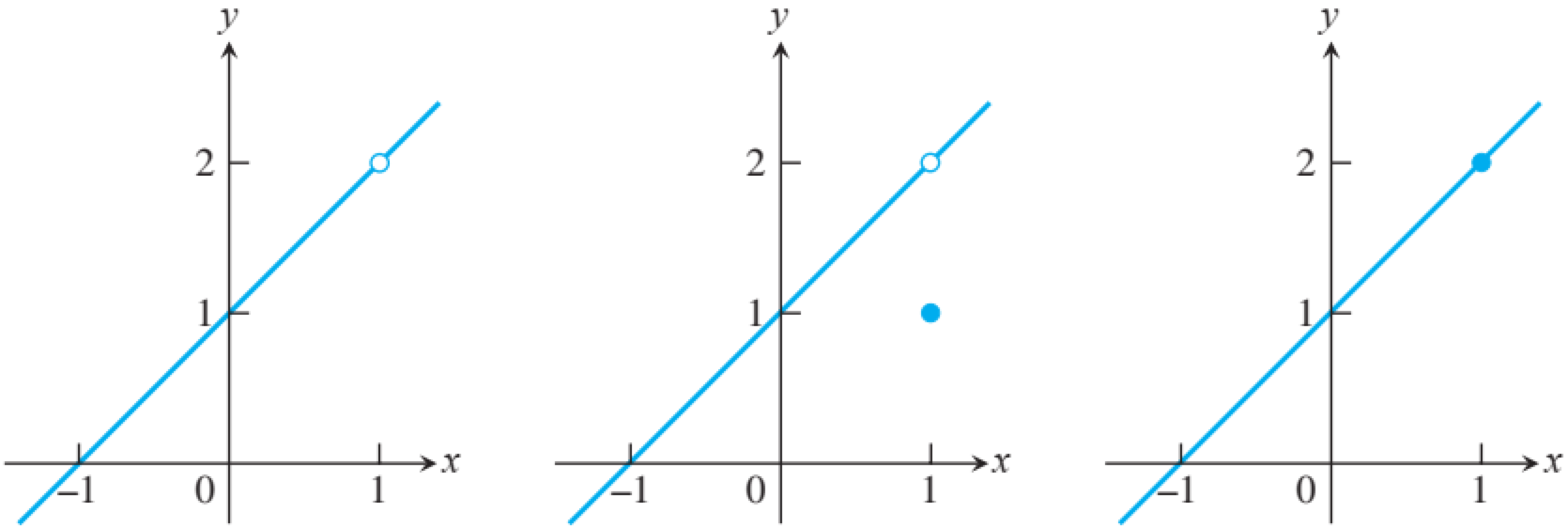
The graph of f is thus the line $y = x + 1$ with the point (1, 2) *removed*. This removed point is shown as a "hole" in Figure 2.4. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ *as close as we want* to 2 by choosing x *close enough* to 1



We say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

EXAMPLE 6 The Limit Value Does Not Depend on How the Function Is Defined at x_0



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

FIGURE 2.5 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).

EXAMPLE 7 Finding Limits by Calculating $f(x_0)$

(a) $\lim_{x \to 2} (4) = 4$

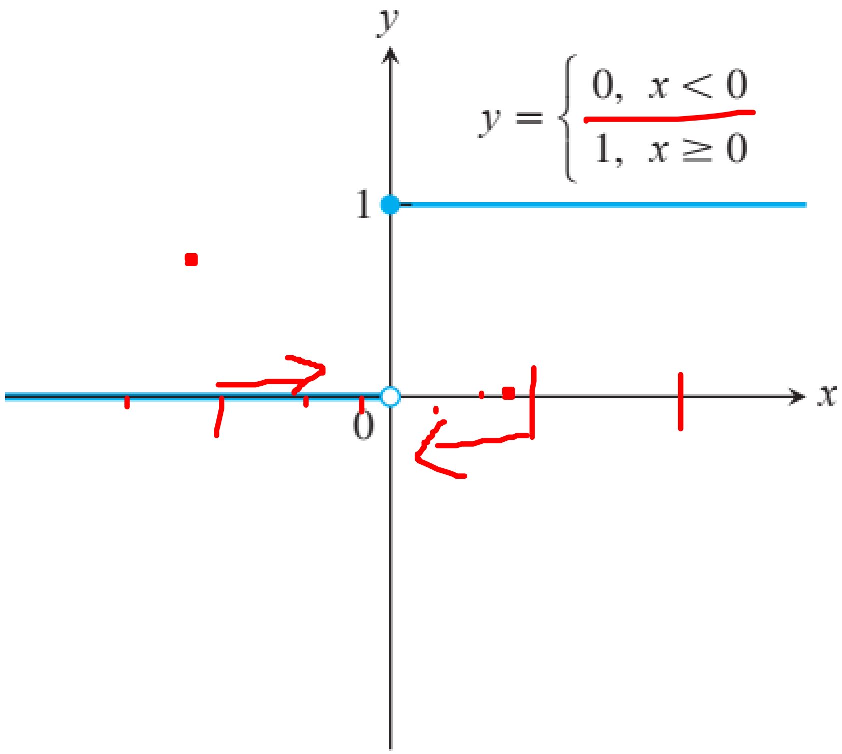
(b) $\lim_{x \to -13} (4) = 4$

(c) $\lim_{x \to 3} x = 3$

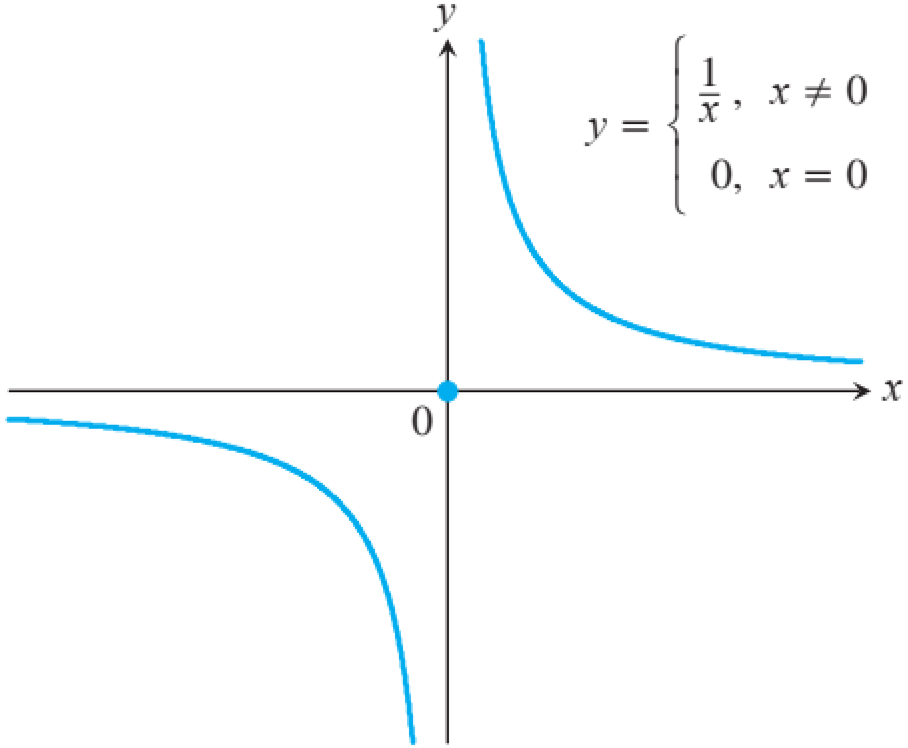
(d) $\lim_{x \to 2} (5x - 3) = 10 - 3 = 7$

(e) $\lim_{x \to -2} \frac{3x + 4}{x + 5} = \frac{-6 + 4}{-2 + 5} = -\frac{2}{3}$

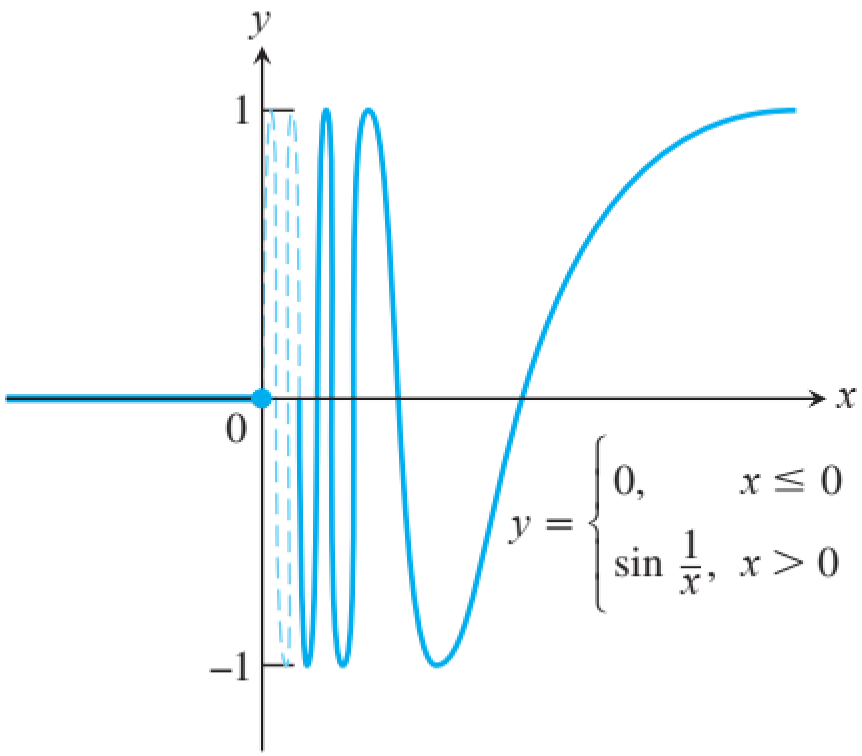
Some ways that limits can fail to exist are illustrated in Figure 2.7 and described in the next example.



(a) Unit step function $U(x)$



(b) $g(x)$



(c) $f(x)$

EXAMPLE 9 A Function May Fail to Have a Limit at a Point in Its Domain

Discuss the behavior of the following functions as $x \longrightarrow 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin \dfrac{1}{x}, & \mathrm{x} > 0 \end{cases}$

(a) It *jumps:* The **unit step function** $U(x)$ has no limit as $x \longrightarrow 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \longrightarrow 0$ (Figure 2.7a).

(b) It *grows too large to have a limit*: $g(x)$ has no limit as $x \rightarrow 0$ because the values of g grow arbitrarily large in absolute value as $x \rightarrow 0$ and do not stay close to *any* real number (Figure 2.7b).

(c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \rightarrow 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any one number as $x \rightarrow 0$ (Figure 2.7c). ■

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim\limits_{x \rightarrow 0} \dfrac{x}{|x|}$

6. $\lim\limits_{x \rightarrow 1} \dfrac{1}{x-1}$

21. $\lim_{x \to 2} 2x$

22. $\lim_{x \to 0} 2x$

23. $\lim_{x \to 1/3} (3x - 1)$

24. $\lim_{x \to 1} \dfrac{-1}{(3x - 1)}$

25. $\lim_{x \to -1} 3x(2x - 1)$

26. $\lim_{x \to -1} \dfrac{3x^2}{2x - 1}$

27. $\lim_{x \to \pi/2} x \sin x$

28. $\lim_{x \to \pi} \dfrac{\cos x}{1 - \pi}$

Calculating Limits Using the Limit Laws

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x \to c}(f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x \to c}(f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $$\lim_{x \to c} (k \cdot f(x)) = k \cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 1 Using the Limit Laws

Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 8 in Section 2.1) and the properties of limits to find the following limits.

(a) $\lim\limits_{x\to c}(x^3 + 4x^2 - 3)$ **(b)** $\lim\limits_{x\to c}\dfrac{x^4 + x^2 - 1}{x^2 + 5}$ **(c)** $\lim\limits_{x\to -2}\sqrt{4x^2 - 3}$

Solution

(a)
$$\begin{aligned}\lim_{x\to c}(x^3 + 4x^2 - 3) &= \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3 && \text{Sum and Difference Rules}\\ &= c^3 + 4c^2 - 3 && \text{Product and Multiple Rules}\end{aligned}$$

$$\textbf{(b)}\quad \lim_{x\to c}\frac{x^4+x^2-1}{x^2+5}=\frac{\lim_{x\to c}(x^4+x^2-1)}{\lim_{x\to c}(x^2+5)}$$

Quotient Rule

$$=\frac{\lim_{x\to c}x^4+\lim_{x\to c}x^2-\lim_{x\to c}1}{\lim_{x\to c}x^2+\lim_{x\to c}5}$$

Sum and Difference Rules

$$=\frac{c^4+c^2-1}{c^2+5}$$

Power or Product Rule

$$\textbf{(c)}\quad \lim_{x\to -2}\sqrt{4x^2-3}=\sqrt{\lim_{x\to -2}(4x^2-3)}$$

Power Rule with $r/s = {}^1\!/_2$

$$=\sqrt{\lim_{x\to -2}4x^2-\lim_{x\to -2}3}$$

Difference Rule

$$=\sqrt{4(-2)^2-3}$$

Product and Multiple Rules

$$=\sqrt{16-3}$$

$$=\sqrt{13}$$

■

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 2 Limit of a Rational Function

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

This result is similar to the second limit in Example 1 with $c = -1$, now done in one step. ■

EXAMPLE 3 Canceling a Common Factor

Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling the $(x - 1)$'s gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by substitution:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.8. ■

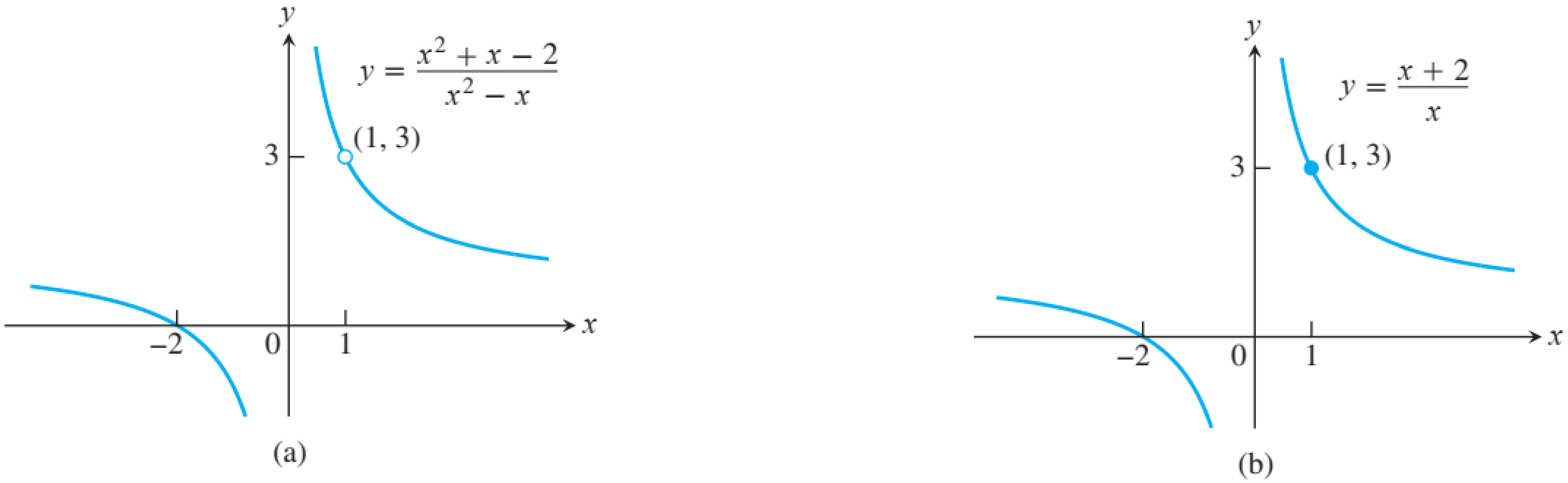


FIGURE 2.8 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 3).

EXAMPLE 4 Creating and Canceling a Common Factor

Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

The preliminary algebra rationalizes the numerator:

$$\begin{aligned}
\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\
&= \frac{x^2 + 100 - 100}{x^2\left(\sqrt{x^2 + 100} + 10\right)} \\
&= \frac{x^2}{x^2\left(\sqrt{x^2 + 100} + 10\right)} && \text{Common factor } x^2 \\
&= \frac{1}{\sqrt{x^2 + 100} + 10}. && \text{Cancel } x^2 \text{ for } x \neq 0
\end{aligned}$$

Therefore,

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 100} + 10}$$

$$= \frac{1}{\sqrt{0^2 + 100} + 10}$$

Denominator not 0 at $x = 0$; substitute

$$= \frac{1}{20} = 0.05.$$

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

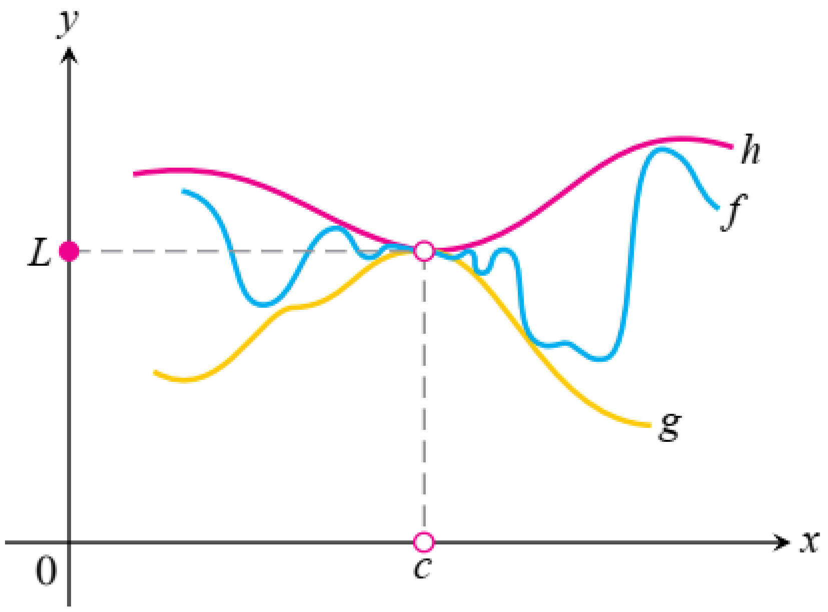


FIGURE 2.9 The graph of f is sandwiched between the graphs of g and h.

EXAMPLE 5 Applying the Sandwich Theorem

Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \neq 0,$$

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x\to 0}(1 - (x^2/4)) = 1$$

and

$$\lim_{x\to 0}(1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$

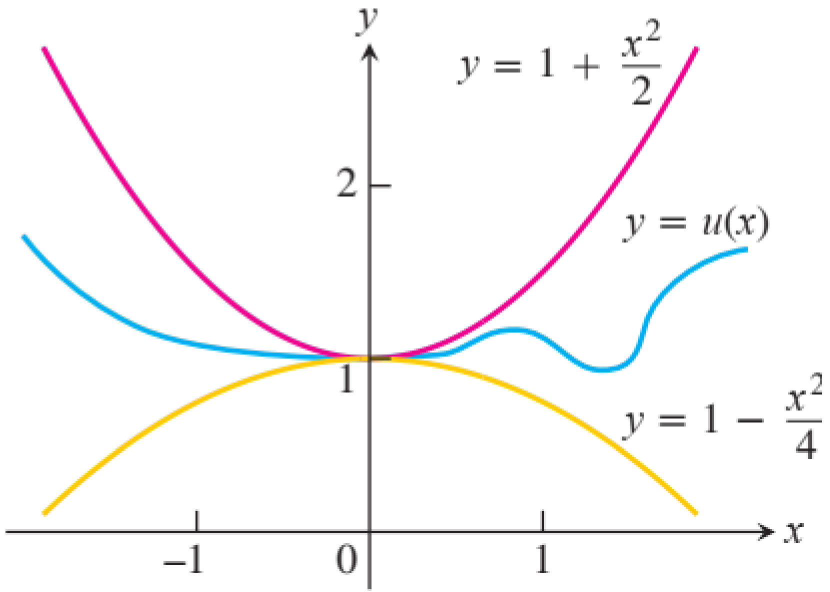


FIGURE 2.10 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 5).

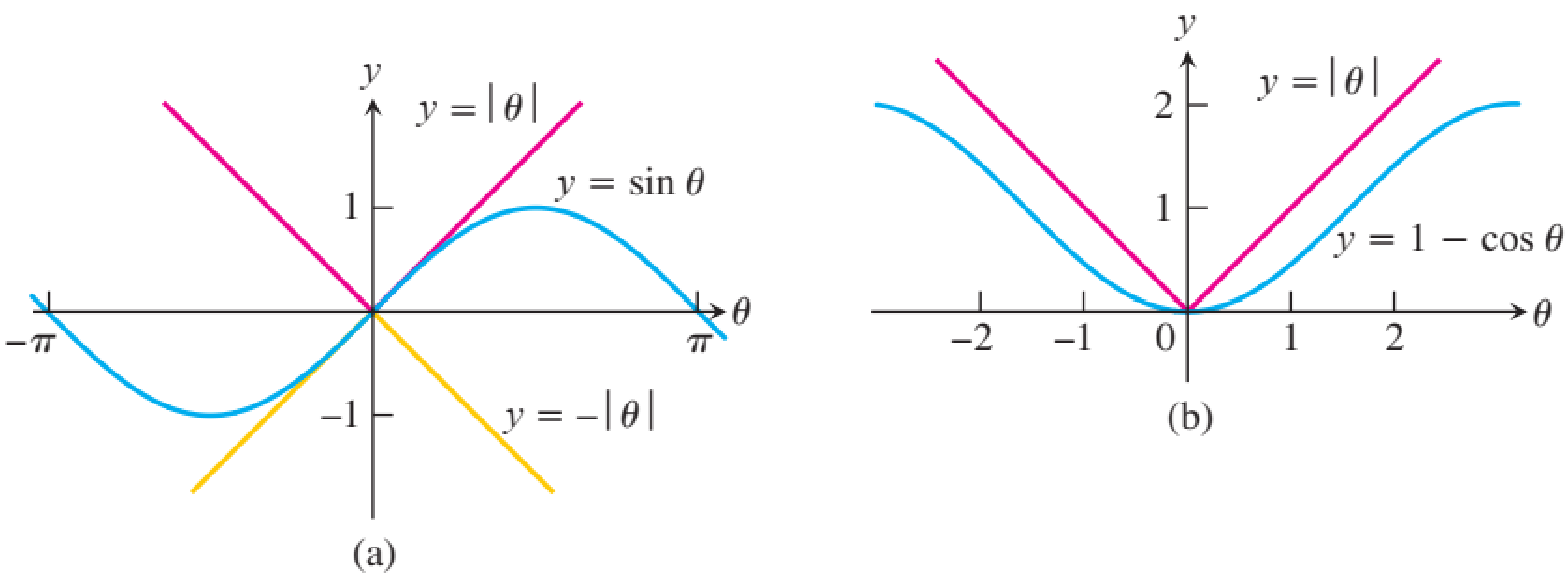
EXAMPLE 6 More Applications of the Sandwich Theorem

(a) (Figure 2.11a). It follows from the definition of $\sin\theta$ that $-|\theta| \leq \sin\theta \leq |\theta|$ for all θ, and since $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0}|\theta| = 0$, we have

$$\lim_{\theta\to 0} \sin\theta = 0.$$

(b) (Figure 2.11b). From the definition of $\cos\theta$, $0 \leq 1 - \cos\theta \leq |\theta|$ for all θ, and we have $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ or

$$\lim_{\theta\to 0} \cos\theta = 1.$$



THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \le \lim_{x \to c} g(x).$$

Exercise 2.2 Question 1 to 34, 49 to 52