

# Areas of Surfaces of Revolution

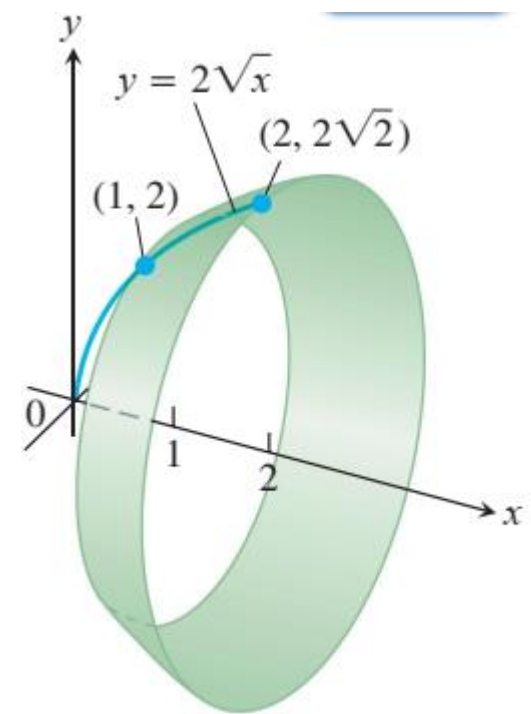
## DEFINITION Surface Area for Revolution About the $x$ -Axis

If the function  $f(x) \geq 0$  is continuously differentiable on  $[a, b]$ , the **area** of the surface generated by revolving the curve  $y = f(x)$  about the  $x$ -axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx. \quad (3)$$

## EXAMPLE

Find the area of the surface generated by revolving the curve  $y = 2\sqrt{x}$ ,  $1 \leq x \leq 2$ , about the  $x$ -axis (Figure 6.48).



**Solution** We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{Eq. (3)}$$

with

$$a = 1, \quad b = 2, \quad y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\begin{aligned} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} \\ &= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}. \end{aligned}$$

With these substitutions,

$$\begin{aligned} S &= \int_1^2 2\pi \cdot 2\sqrt{x} \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx \\ &= 4\pi \cdot \frac{2}{3} (x+1)^{3/2} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}). \end{aligned}$$



## Revolution About the $y$ -Axis

For revolution about the  $y$ -axis, we interchange  $x$  and  $y$  in Equation (3).

### Surface Area for Revolution About the $y$ -Axis

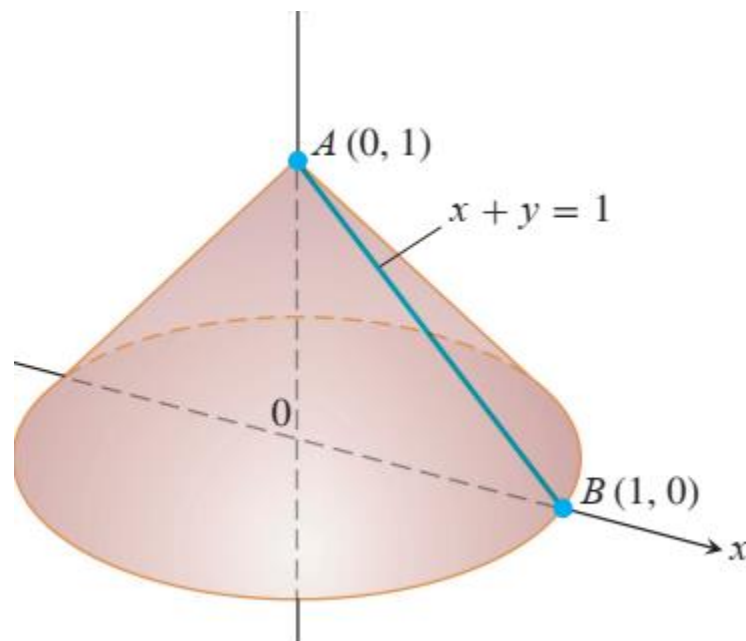
If  $x = g(y) \geq 0$  is continuously differentiable on  $[c, d]$ , the area of the surface generated by revolving the curve  $x = g(y)$  about the  $y$ -axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2} dy. \quad (4)$$

## EXAMPLE

The line segment  $x = 1 - y$ ,  $0 \leq y \leq 1$ , is revolved about the  $y$ -axis to generate the cone in Figure 6.49. Find its lateral surface area (which excludes the base area).

**FIGURE 6.49** Revolving line segment  $AB$  about the  $y$ -axis generates a cone whose



## Solution

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$\begin{aligned} S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^1 2\pi(1 - y)\sqrt{2} dy \\ &= 2\pi\sqrt{2} \left[ y - \frac{y^2}{2} \right]_0^1 = 2\pi\sqrt{2} \left( 1 - \frac{1}{2} \right) \\ &= \pi\sqrt{2}. \end{aligned}$$

## Surface Area of Revolution for Parametrized Curves

If a smooth curve  $x = f(t)$ ,  $y = g(t)$ ,  $a \leq t \leq b$ , is traversed exactly once as  $t$  increases from  $a$  to  $b$ , then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the  $x$ -axis ( $y \geq 0$ ):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (5)$$

2. Revolution about the  $y$ -axis ( $x \geq 0$ ):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (6)$$

## EXAMPLE

The standard parametrization of the circle of radius 1 centered at the point  $(0, 1)$  in the  $xy$ -plane is

$$x = \cos t, \quad y = 1 + \sin t, \quad 0 \leq t \leq 2\pi.$$

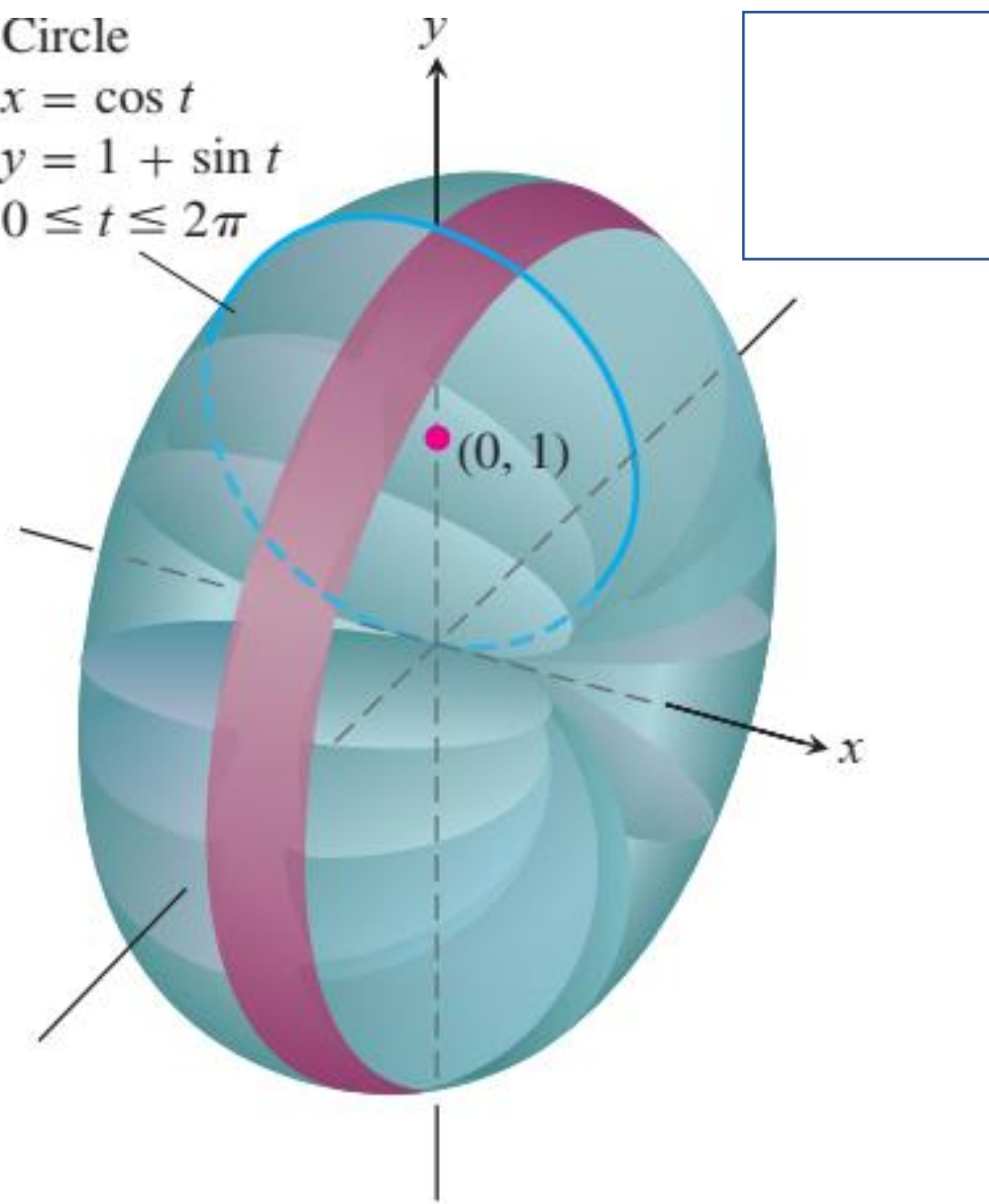
Use this parametrization to find the area of the surface swept out by revolving the circle about the  $x$ -axis (Figure 6.50).

Circle

$$x = \cos t$$

$$y = 1 + \sin t$$

$$0 \leq t \leq 2\pi$$



**Solution** We evaluate the formula

$$\begin{aligned} S &= \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \int_0^{2\pi} 2\pi(1 + \sin t) \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_1} dt \\ &= 2\pi \int_0^{2\pi} (1 + \sin t) dt \\ &= 2\pi [t - \cos t]_0^{2\pi} = 4\pi^2. \end{aligned}$$

Eq. (5) for revolution  
about the  $x$ -axis;  
 $y = 1 + \sin t > 0$



