Transcendental Functions

Inverse Functions and Their Derivatives

DEFINITION One-to-One Function

A function $f(x)$ is **one-to-one** on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.

x1 is not equal to x2 implies fx1 is not equal to fx2
or fx1 = fx2 implies x1=x2

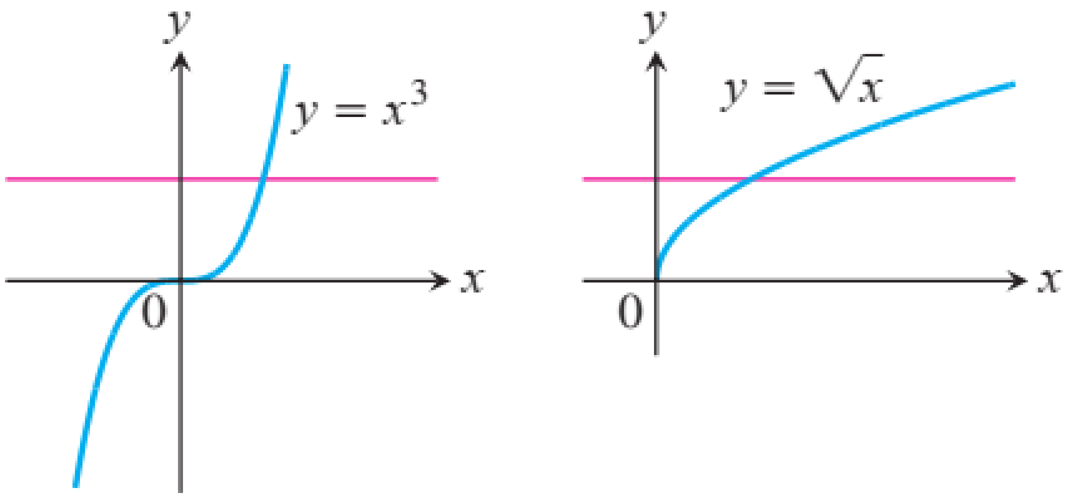
EXAMPLE 1 Domains of One-to-One Functions

(a) $f(x) = \sqrt{x}$ is one-to-one on any domain of nonnegative numbers because $\sqrt{x_1} \neq \sqrt{x_2}$ whenever $x_1 \neq x_2$.

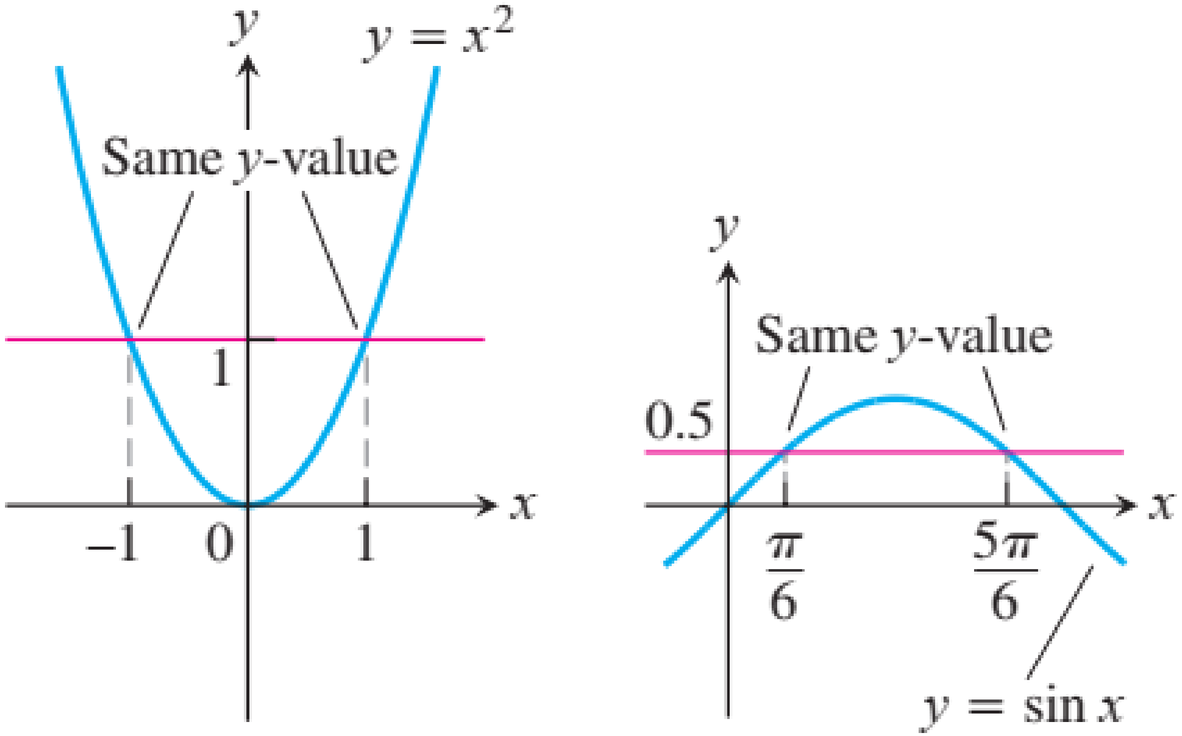
(b) $g(x) = \sin x$ is *not* one-to-one on the interval $[0, \pi]$ because $\sin(\pi/6) = \sin(5\pi/6)$. The sine *is* one-to-one on $[0, \pi/2]$, however, because it is a strictly increasing function on $[0, \pi/2]$. ■

The Horizontal Line Test for One-to-One Functions

A function $y = f(x)$ is one-to-one if and only if its graph intersects each horizontal line at most once.



One-to-one: Graph meets each horizontal line at most once.



Not one-to-one: Graph meets one or more horizontal lines more than once.

DEFINITION Inverse Function

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

$$f^{-1}(a) = b \text{ if } f(b) = a.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

$$(f^{-1} \circ f)(x) = x, \qquad \text{for all } x \text{ in the domain of } f$$

$$(f \circ f^{-1})(y) = y, \qquad \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f\text{)}$$

EXAMPLE 2 Finding an Inverse Function

Find the inverse of $y = \frac{1}{2}x + 1$, expressed as a function of x.

Solution

1. *Solve for x in terms of y:*

$$\begin{aligned} y &= \frac{1}{2}x + 1 \\ 2y &= x + 2 \\ x &= 2y - 2. \end{aligned}$$

2. *Interchange x and y:* $y = 2x - 2.$

The inverse of the function $f(x) = (1/2)x + 1$ is the function $f^{-1}(x) = 2x - 2$. To check, we verify that both composites give the identity function:

$$f^{-1}(f(x)) = 2\left(\frac{1}{2}x + 1\right) - 2 = x + 2 - 2 = x$$

$$f(f^{-1}(x)) = \frac{1}{2}(2x - 2) + 1 = x - 1 + 1 = x.$$

EXAMPLE 3 Finding an Inverse Function

Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

Solution We first solve for x in terms of y:

$$y = x^2$$

$$\sqrt{y} = \sqrt{x^2} = |x| = x \qquad |x| = x \text{ because } x \geq 0$$

We then interchange x and y, obtaining

$$y = \sqrt{x}.$$

The inverse of the function $y = x^2, x \geq 0$, is the function $y = \sqrt{x}$

Derivatives of Inverses of Differentiable Functions

If we calculate the derivatives of $f(x) = (1/2)x + 1$ and its inverse $f^{-1}(x) = 2x - 2$ from Example 2, we see that

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{2}x + 1\right) = \frac{1}{2}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{d}{dx}(2x - 2) = 2.$$

THEOREM 1 The Derivative Rule for Inverses

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain. The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}} \tag{1}$$

EXAMPLE 4 Applying Theorem 1

The function $f(x) = x^2, x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$ have derivatives $f'(x) = 2x$ and $(f^{-1})'(x) = 1/(2\sqrt{x})$.

Theorem 1 predicts that the derivative of $f^{-1}(x)$ is

$$\begin{aligned}(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\ &= \frac{1}{2(f^{-1}(x))} \\ &= \frac{1}{2(\sqrt{x})}.\end{aligned}$$

EXAMPLE 5 Finding a Value of the Inverse Derivative

Let $f(x) = x^3 - 2$. Find the value of df^{-1}/dx at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Solution

$$\left.\frac{df}{dx}\right|_{x=2} = \left.3x^2\right|_{x=2} = 12$$

$$\left.\frac{df^{-1}}{dx}\right|_{x=f(2)} = \frac{1}{\left.\dfrac{df}{dx}\right|_{x=2}} = \frac{1}{12} \qquad \text{Eq. (1)}$$

Natural Logarithms

DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

DEFINITION The Number *e*

The number e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = 1$$

The Derivative of $y = \ln x$

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0 \tag{1}$$

EXAMPLE 1 Derivatives of Natural Logarithms

(a) $\dfrac{d}{dx}\ln 2x = \dfrac{1}{2x}\dfrac{d}{dx}(2x) = \dfrac{1}{2x}(2) = \dfrac{1}{x}$

(b) Equation (1) with $u = x^2 + 3$ gives

$$\frac{d}{dx}\ln(x^2 + 3) = \frac{1}{x^2 + 3}\cdot\frac{d}{dx}(x^2 + 3) = \frac{1}{x^2 + 3}\cdot 2x = \frac{2x}{x^2 + 3}.$$

■

THEOREM 2 Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ r rational

EXAMPLE 2 Interpreting the Properties of Logarithms

(a) $\ln 6 = \ln(2 \cdot 3) = \ln 2 + \ln 3$ Product

(b) $\ln 4 - \ln 5 = \ln \frac{4}{5} = \ln 0.8$ Quotient

(c) $\ln \frac{1}{8} = -\ln 8$ Reciprocal

$= -\ln 2^3 = -3 \ln 2$ Power ■

EXAMPLE 3 Applying the Properties to Function Formulas

(a) $\ln 4 + \ln \sin x = \ln(4 \sin x)$ Product

(b) $\ln \frac{x+1}{2x-3} = \ln(x+1) - \ln(2x-3)$ Quotient

(c) $\ln \sec x = \ln \dfrac{1}{\cos x} = -\ln \cos x$ Reciprocal

(d) $\ln \sqrt[3]{x+1} = \ln (x+1)^{1/3} = \dfrac{1}{3} \ln (x+1)$ Power ■

EXAMPLE 6 Using Logarithmic Differentiation

Find dy/dx if

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2+1)(x+3)^{1/2}}{x-1} \\
&= \ln\left((x^2+1)(x+3)^{1/2}\right) - \ln(x-1) && \text{Rule 2} \\
&= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1) && \text{Rule 1} \\
&= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1). && \text{Rule 3}
\end{aligned}$$

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1}{2}\cdot\frac{1}{x+3} - \frac{1}{x-1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right). \quad \blacksquare$$

A direct computation in Example 6, using the Quotient and Product Rules, would be much longer.

$$y = (\tan\theta)\sqrt{2\theta+1} = (\tan\theta)(2\theta+1)^{1/2} \Rightarrow \ln y = \ln(\tan\theta) + \frac{1}{2}\ln(2\theta+1) \Rightarrow \frac{1}{y}\frac{dy}{d\theta} = \frac{\sec^2\theta}{\tan\theta} + \left(\frac{1}{2}\right)\left(\frac{2}{2\theta+1}\right)$$

$$\Rightarrow \frac{dy}{d\theta} = (\tan\theta)\sqrt{2\theta+1}\left(\frac{\sec^2\theta}{\tan\theta} + \frac{1}{2\theta+1}\right) = (\sec^2\theta)\sqrt{2\theta+1} + \frac{\tan\theta}{\sqrt{2\theta+1}}$$

The Exponential Function

DEFINITION The Natural Exponential Function

For every real number x, $e^x = \ln^{-1} x = \exp x$.

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0) \tag{2}$$

$$\ln(e^x) = x \qquad (\text{all } x) \tag{3}$$

The domain of $\ln x$ is $(0, \infty)$ and its range is $(-\infty, \infty)$. So the domain of e^x is $(-\infty, \infty)$ and its range is $(0, \infty)$.

EXAMPLE 1 Using the Inverse Equations

$$e^{\ln(x^2+1)} = x^2 + 1$$
$$e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8$$

EXAMPLE 2 Solving for an Exponent

Find k if $e^{2k} = 10$.

Solution Take the natural logarithm of both sides:

$$\begin{aligned} e^{2k} &= 10 \\ \ln e^{2k} &= \ln 10 \\ 2k &= \ln 10 && \text{Eq. (3)} \\ k &= \frac{1}{2}\ln 10. \end{aligned}$$

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

EXAMPLE 3 Evaluating Exponential Functions

(a) $2^{\sqrt{3}} = e^{\sqrt{3} \ln 2} \approx e^{1.20} \approx 3.32$

(b) $2^{\pi} = e^{\pi \ln 2} \approx e^{2.18} \approx 8.8$

THEOREM 3 Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$

2. $e^{-x} = \dfrac{1}{e^x}$

3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1 x_2} = (e^{x_2})^{x_1}$

The Derivative of e^x

$$\frac{d}{dx}(e^x) = e^x.$$

EXAMPLE 5 Differentiating an Exponential

$$\begin{aligned}\frac{d}{dx}(5e^x) &= 5\frac{d}{dx}e^x \\ &= 5e^x\end{aligned}$$

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}. \tag{6}$$

EXAMPLE 6 Applying the Chain Rule with Exponentials

(a) $\dfrac{d}{dx}e^{-x} = e^{-x}\dfrac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (6) with $u = -x$

(b) $\dfrac{d}{dx}e^{\sin x} = e^{\sin x}\dfrac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ Eq. (6) with $u = \sin x$ ■

THEOREM 4 The Number *e* as a Limit

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

$$y = e^{(4\sqrt{x}+x^2)} \;\Rightarrow\; y' = e^{(4\sqrt{x}+x^2)} \tfrac{d}{dx}\left(4\sqrt{x}+x^2\right) \;\Rightarrow\; y' = \left(\tfrac{2}{\sqrt{x}}+2x\right)e^{(4\sqrt{x}+x^2)}$$

$$y = xe^x - e^x \;\Rightarrow\; y' = (e^x + xe^x) - e^x = xe^x$$

a^x and $\log_a x$

The Derivative of a^u

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx}. \tag{1}$$

EXAMPLE 1 Differentiating General Exponential Functions

(a) $$\frac{d}{dx} 3^x = 3^x \ln 3$$

(b) $$\frac{d}{dx} 3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x} \ln 3$$

(c) $$\frac{d}{dx} 3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$$

EXAMPLE 2 Differentiating a General Power Function

Find dy/dx if $y = x^x, \quad x > 0.$

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad a^x \text{ with } a = x.$$

Then differentiate as usual:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} e^{x \ln x} \\ &= e^{x \ln x} \frac{d}{dx}(x \ln x) \\ &= x^x \left(x \cdot \frac{1}{x} + \ln x \right) \\ &= x^x (1 + \ln x). \end{aligned}$$

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$$\log_a x \text{ is the inverse function of } a^x.$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0) \tag{3}$$

$$\log_a (a^x) = x \qquad (\text{all } x) \tag{4}$$

Evaluation of $\log_a x$

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a} \tag{5}$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}\log_{10}(3x + 1) = \frac{1}{\ln 10} \cdot \frac{1}{3x + 1}\frac{d}{dx}(3x + 1) = \frac{3}{(\ln 10)(3x + 1)}$$