Integration

Indefinite Integrals and the Substitution Rule

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1,\ n$ rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

If u is any differentiable function, then

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1, n \text{ rational}). \tag{1}$$

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x) \, dx = \int f(u) \, du.$$

EXAMPLE 3 Using Substitution

$$\int \cos(7\theta + 5)\, d\theta = \int \cos u \cdot \frac{1}{7}\, du \qquad \text{Let } u = 7\theta + 5,\ du = 7\, d\theta,\ (1/7)\, du = d\theta.$$

$$= \frac{1}{7}\int \cos u\, du \qquad \text{With the } (1/7) \text{ out front, the integral is now in standard form.}$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate with respect to } u, \text{ Table 4.2.}$$

$$= \frac{1}{7}\sin(7\theta + 5) + C \qquad \text{Replace } u \text{ by } 7\theta + 5.$$

EXAMPLE 4 Using Substitution

$$\int x^2 \sin(x^3)\,dx = \int \sin(x^3) \cdot x^2\,dx$$

$$= \int \sin u \cdot \frac{1}{3}\,du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx.$$

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}(-\cos u) + C \qquad \text{Integrate with respect to } u.$$

$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

■

EXAMPLE 5 Using Identities and Substitution

$$\begin{aligned}
\int \frac{1}{\cos^2 2x}\,dx &= \int \sec^2 2x\,dx && \frac{1}{\cos 2x} = \sec 2x \\
&= \int \sec^2 u \cdot \frac{1}{2}\,du && u = 2x,\ du = 2\,dx,\ dx = (1/2)\,du \\
&= \frac{1}{2}\int \sec^2 u\,du \\
&= \frac{1}{2}\tan u + C && \frac{d}{du}\tan u = \sec^2 u \\
&= \frac{1}{2}\tan 2x + C && u = 2x
\end{aligned}$$

■

EXAMPLE 6 Using Different Substitutions

Evaluate

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}.$$

Substitute $u = z^2 + 1$.

$$\begin{aligned}\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} &= \int \frac{du}{u^{1/3}} && \text{Let } u = z^2+1,\ du = 2z\,dz. \\ &= \int u^{-1/3}\,du && \text{In the form } \int u^n\,du\end{aligned}$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{3}{2} u^{2/3} + C$$

$$= \frac{3}{2} (z^2 + 1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2 + 1.$$

EXAMPLE 7

(a)
$$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx$$
$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$= \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

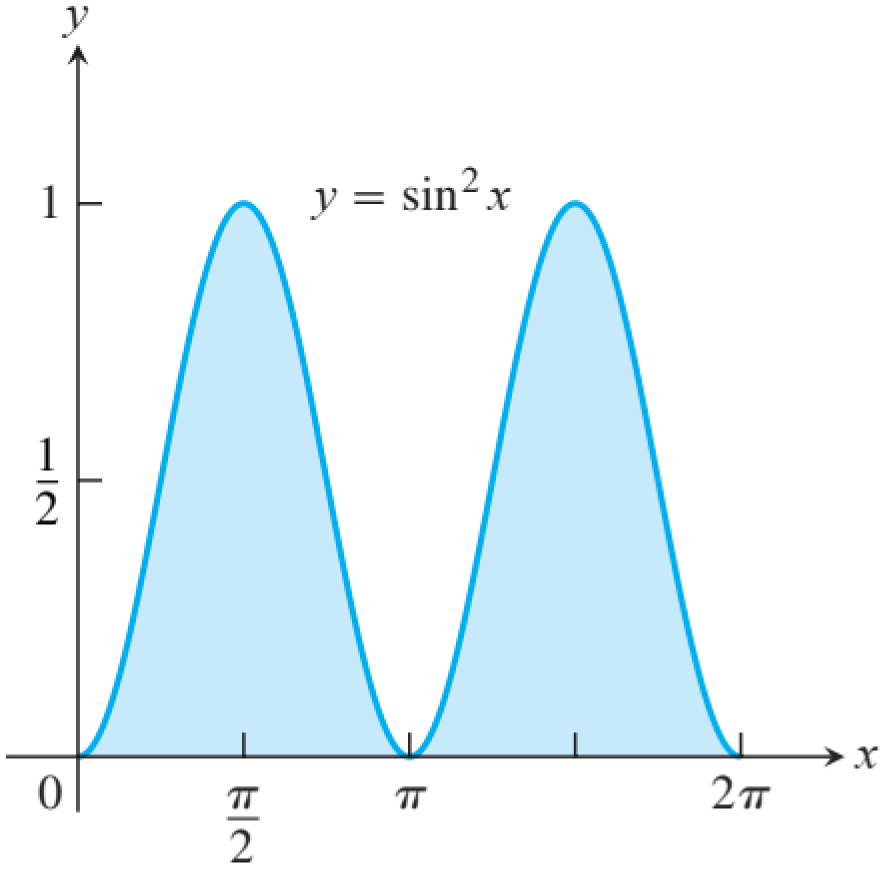
(b)
$$\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2}\, dx$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$
$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$
As in part (a), but with a sign change

EXAMPLE 8 Area Beneath the Curve $y = \sin^2 x$

Figure 5.24 shows the graph of $g(x) = \sin^2 x$ over the interval $[0, 2\pi]$. Find

(a) the definite integral of $g(x)$ over $[0, 2\pi]$.

(b) the area between the graph of the function and the x-axis over $[0, 2\pi]$.



Solution

(a) From Example 7(a), the definite integral is

$$\int_0^{2\pi} \sin^2 x\, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi} = \left[\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right] - \left[\frac{0}{2} - \frac{\sin 0}{4}\right]$$

$$= [\pi - 0] - [0 - 0] = \pi.$$

(b) The function $\sin^2 x$ is nonnegative, so the area is equal to the definite integral, or π. ■

Exercise 5.5 Question # 1-48.

48. $\int 3x^5\sqrt{x^3 + 1}\, dx$

Let $u = x^3 + 1 \Rightarrow du = 3x^2 dx$ and $x^3 = u - 1$. So $\int 3x^5\sqrt{x^3+1}\, dx = \int (u-1)\sqrt{u}\, du = \int \left(u^{3/2} - u^{1/2}\right) du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2} + C$

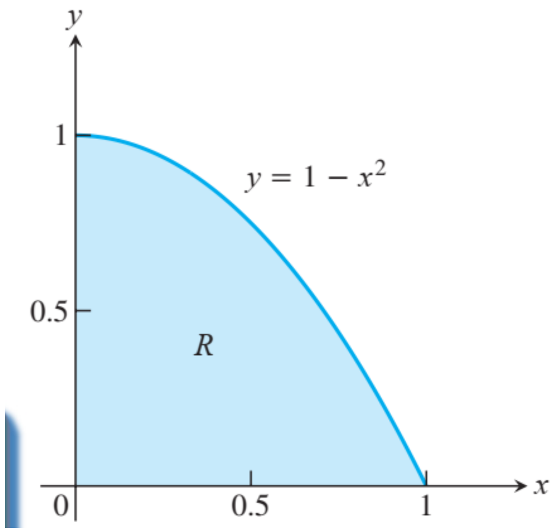
Estimating with Finite Sums

Area

The area of a region with a curved boundary can be approximated by summing the areas of a collection of rectangles. Using more rectangles can increase the accuracy of the approximation.

EXAMPLE 1 Approximating Area

What is the area of the shaded region R that lies above the x-axis, below the graph of $y = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$?



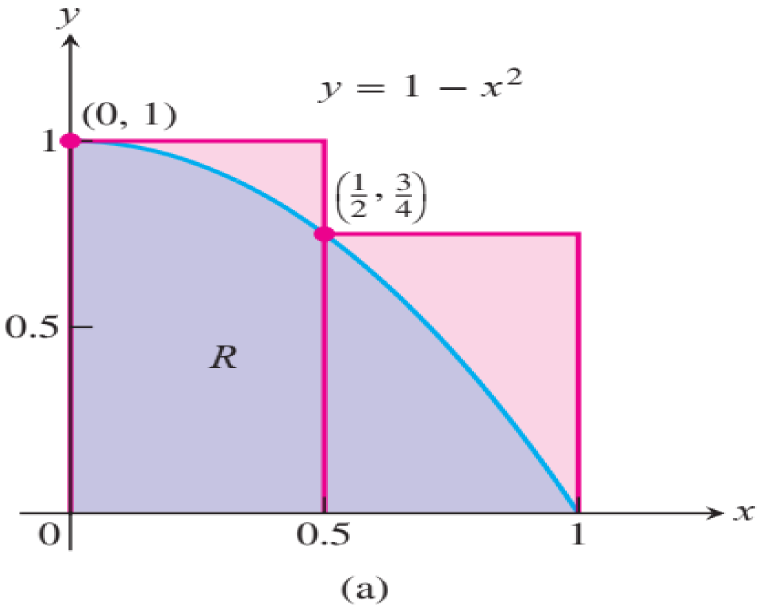


FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area.

The total area of the two rectangles approximates the area A of the region R,

$$A \approx 1 \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} = \frac{7}{8} = 0.875.$$

This estimate is larger than the true area A, since the two rectangles contain R. We say that 0.875 is an **upper sum** because it is obtained by taking the height of each rectangle as the maximum (uppermost) value of $f(x)$ for x a point in the base interval of the rectangle.

. These four rectangles give the approximation

$$A \approx 1 \cdot \frac{1}{4} + \frac{15}{16} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \frac{7}{16} \cdot \frac{1}{4} = \frac{25}{32} = 0.78125,$$

which is still greater than A since the four rectangles contain R.

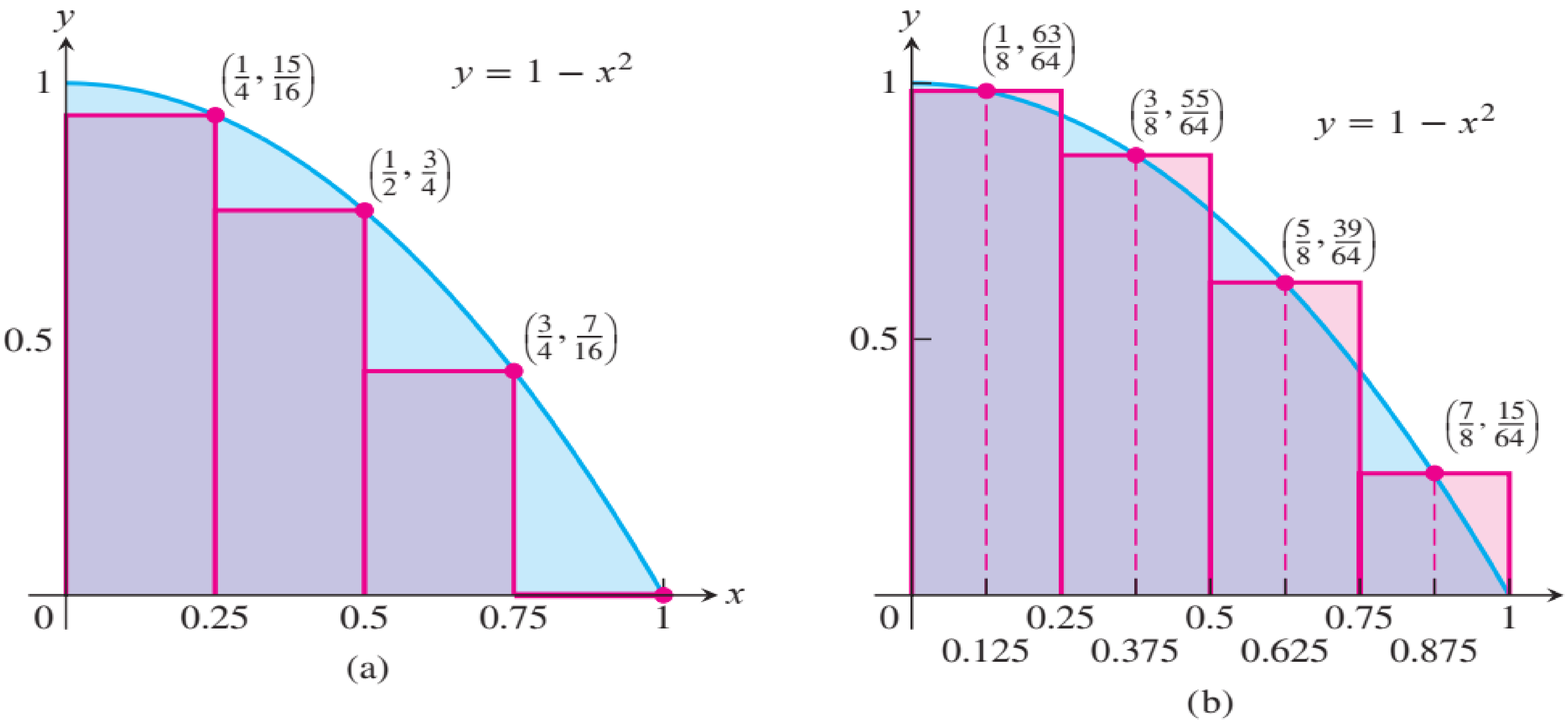


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases.

Summing these rectangles with heights equal to the mini- mum value of $f(x)$ for x a point in each base subinterval, gives a **lower sum** approximation to the area,

$$A \approx \frac{15}{16}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{1}{4}+\frac{7}{16}\cdot\frac{1}{4}+0\cdot\frac{1}{4}=\frac{17}{32}=0.53125.$$

This estimate is smaller than the area A since the rectangles all lie inside of the region R. The true value of A lies somewhere between these lower and upper sums:

$$0.53125 < A < 0.78125.$$

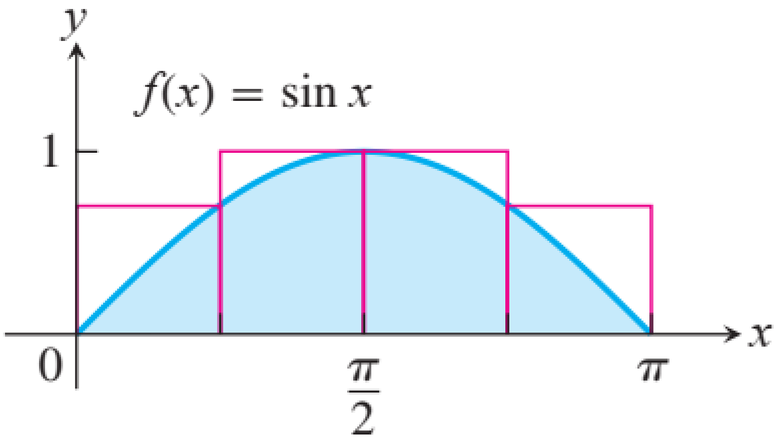
Summary

The area under the graph of a positive function, the distance traveled by a moving object that doesn't change direction, and the average value of a nonnegative function over an interval can all be approximated by finite sums. First we subdivide the interval into subintervals, treating the appropriate function f as if it were constant over each particular subinterval. Then we multiply the width of each subinterval by the value of f at some point within it, and add these products together. If the interval $[a, b]$ is subdivided into n subintervals of equal widths $\Delta x = (b - a)/n$, and if $f(c_k)$ is the value of f at the chosen point c_k in the kth subinterval, this process gives a finite sum of the form

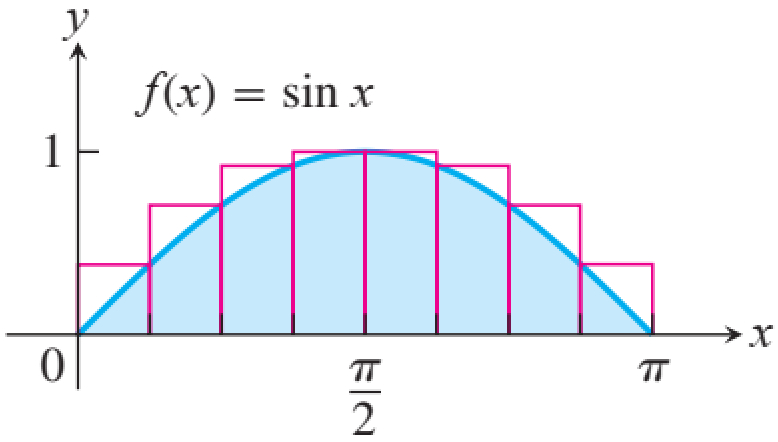
$$f(c_1)\,\Delta x + f(c_2)\,\Delta x + f(c_3)\,\Delta x + \cdots + f(c_n)\,\Delta x.$$

The choices for the c_k could maximize or minimize the value of f in the kth subinterval, or give some value in between. The true value lies somewhere between the approximations given by upper sums and lower sums. The finite sum approximations we looked at improved as we took more subintervals of thinner width.

Estimate the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.



(a)



(b)

$$A \approx \left(\sin\frac{\pi}{4}\right)\cdot\frac{\pi}{4} + \left(\sin\frac{\pi}{2}\right)\cdot\frac{\pi}{4} + \left(\sin\frac{\pi}{2}\right)\cdot\frac{\pi}{4} + \left(\sin\frac{3\pi}{4}\right)\cdot\frac{\pi}{4}$$

$$= \left(\frac{1}{\sqrt{2}} + 1 + 1 + \frac{1}{\sqrt{2}}\right)\cdot\frac{\pi}{4} \approx (3.42)\cdot\frac{\pi}{4} \approx 2.69.$$

$$A \approx \left(\sin\frac{\pi}{8} + \sin\frac{\pi}{4} + \sin\frac{3\pi}{8} + \sin\frac{\pi}{2} + \sin\frac{\pi}{2} + \sin\frac{5\pi}{8} + \sin\frac{3\pi}{4} + \sin\frac{7\pi}{8}\right)\cdot\frac{\pi}{8}$$

$$\approx (.38 + .71 + .92 + 1 + 1 + .92 + .71 + .38)\cdot\frac{\pi}{8} = (6.02)\cdot\frac{\pi}{8} \approx 2.365.$$

The Definite Integral

The function is the integrand.

Upper limit of integration

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

THEOREM 1 The Existence of Definite Integrals

A continuous function is integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

TABLE 5.3 Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

DEFINITION Area Under a Curve as a Definite Integral

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve** $\boldsymbol{y = f(x)}$ **over** $[\boldsymbol{a, b}]$ is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

THEOREM 4 The Fundamental Theorem of Calculus Part 1

If f is continuous on $[a, b]$ then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^2}\,dt = \frac{1}{1+x^2}$$

$$\begin{aligned}\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t\sin t\,dt &= \frac{d}{dx}\left(-\int_5^x 3t\sin t\,dt\right) \qquad \text{Rule 1}\\ &= -\frac{d}{dx}\int_5^x 3t\sin t\,dt\\ &= -3x\sin x\end{aligned}$$

THEOREM 4 (Continued) The Fundamental Theorem of Calculus Part 2

If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$$\int_{-\pi/4}^{0} \sec x \tan x \, dx = \sec x \Big]_{-\pi/4}^{0} = \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$$

$$\begin{aligned}
\int_{1}^{4} \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx &= \left[x^{3/2} + \frac{4}{x}\right]_1^4 \\
&= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right] \\
&= [8 + 1] - [5] = 4.
\end{aligned}$$

EXAMPLE

Find a function $y = f(x)$ on the domain $(-\pi/2, \pi/2)$ with derivative

$$\frac{dy}{dx} = \tan x$$

that satisfies the condition $f(3) = 5$.

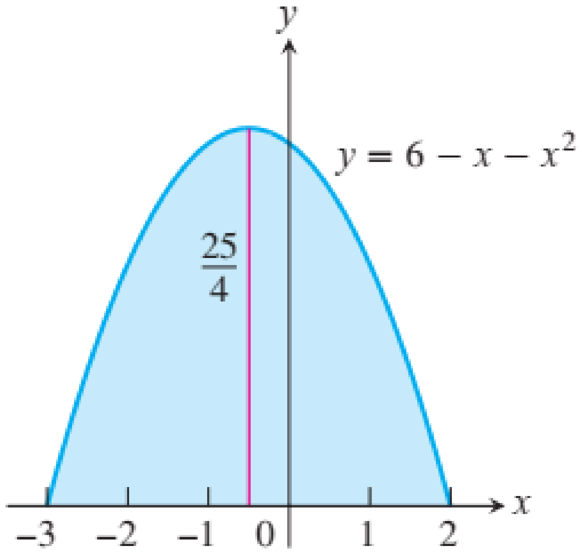
Solution The Fundamental Theorem makes it easy to construct a function with derivative $\tan x$ that equals 0 at $x = 3$:

$$y = \int_3^x \tan t \, dt.$$

Since $y(3) = \int_3^3 \tan t \, dt = 0$, we have only to add 5 to this function to construct one with derivative $\tan x$ whose value at $x = 3$ is 5:

$$f(x) = \int_3^x \tan t \, dt + 5.$$ ■

EXAMPLE Calculate the area bounded by the x-axis and the parabola $y = 6 - x - x^2$.



Solution We find where the curve crosses the x-axis by setting

$$y = 0 = 6 - x - x^2 = (3 + x)(2 - x),$$

which gives

$$x = -3 \qquad \text{or} \qquad x = 2.$$

The curve is sketched in Figure 5.21, and is nonnegative on $[-3, 2]$.

The area is

$$\int_{-3}^{2} (6 - x - x^2)\, dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3} \right]_{-3}^{2}$$

$$= \left(12 - 2 - \frac{8}{3} \right) - \left(-18 - \frac{9}{2} + \frac{27}{3} \right) = 20\tfrac{5}{6}.$$

The curve in Figure 5.21 is an arch of a parabola, and it is interesting to note that the area under such an arch is exactly equal to two-thirds the base times the altitude:

$$\frac{2}{3}(5)\left(\frac{25}{4}\right) = \frac{125}{6} = 20\tfrac{5}{6}.$$

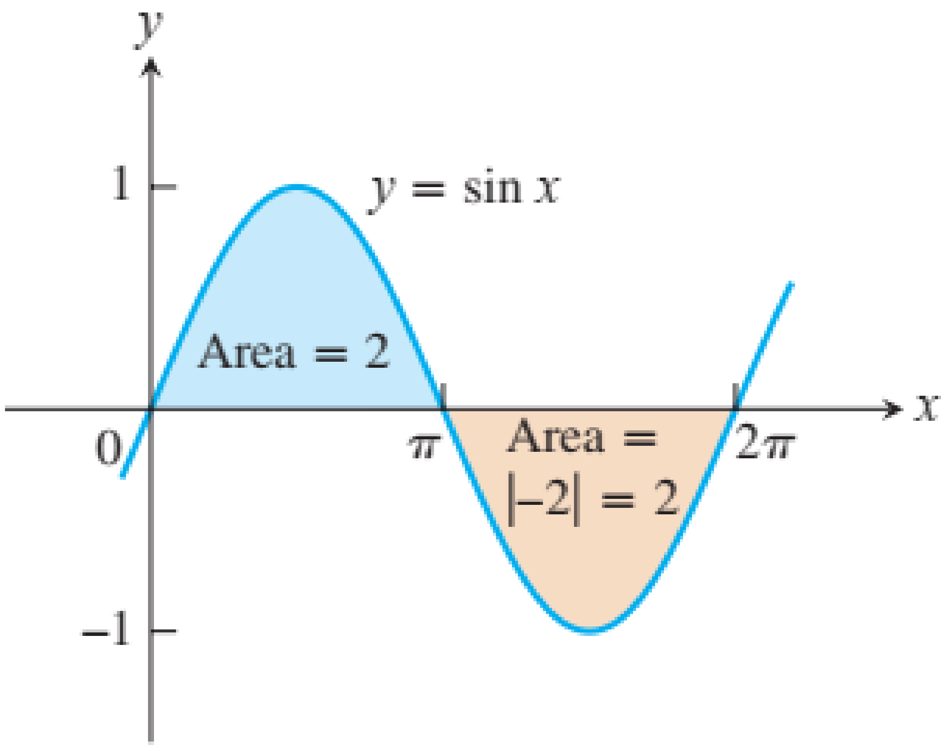
■

EXAMPLE

Figure 5.22 shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.



Solution The definite integral for $f(x) = \sin x$ is given by

$$\int_0^{2\pi} \sin x\, dx = -\cos x \Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.

The area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin x$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin x\, dx = -\cos x \Big]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1 - 1] = 2.$$

$$\int_{\pi}^{2\pi} \sin x\, dx = -\cos x \Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1 - (-1)] = -2.$$

The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$\text{Area} = |2| + |-2| = 4.$$

■

Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

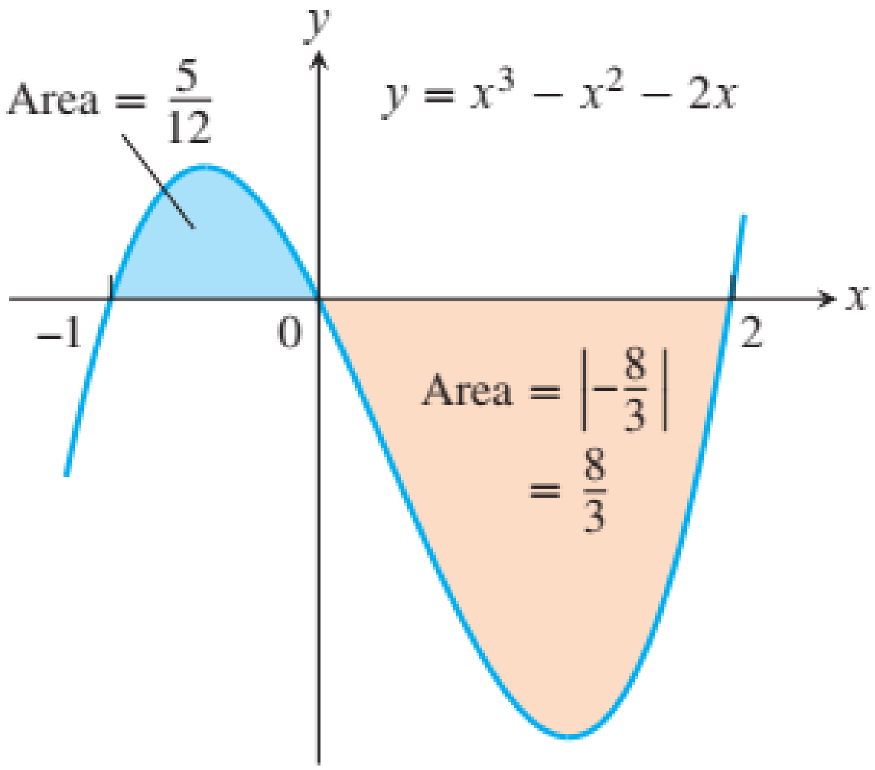
1. Subdivide $[a, b]$ at the zeros of f.

2. Integrate f over each subinterval.

3. Add the absolute values of the integrals.

EXAMPLE

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.



Solution First find the zeros of f. Since

$$f(x) = x^3 - x^2 - 2x = x(x^2 - x - 2) = x(x + 1)(x - 2),$$

the zeros are $x = 0, -1$, and 2 (Figure 5.23). The zeros subdivide $[-1, 2]$ into two subintervals: $[-1, 0]$, on which $f \geq 0$, and $[0, 2]$, on which $f \leq 0$. We integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

The total enclosed area is obtained by adding the absolute values of the calculated integrals,

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}.$$

■

In Exercises 37–42, find the total area between the region and the x-axis.

39. $y = x^3 - 3x^2 + 2x, \quad 0 \le x \le 2$

$x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0$

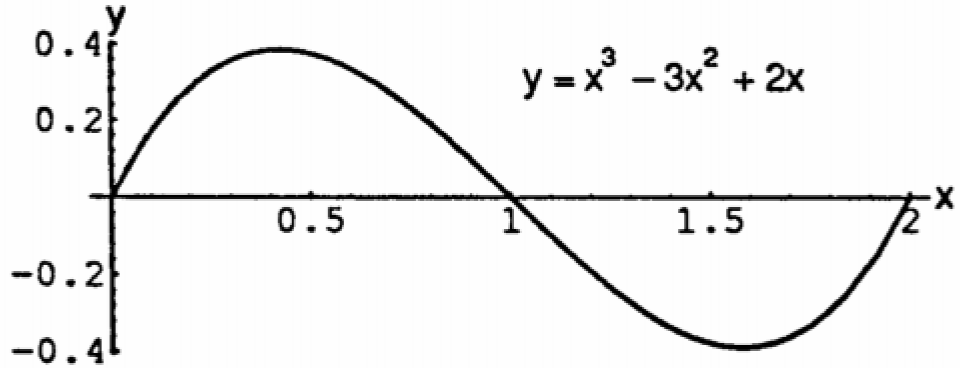
$\Rightarrow x(x-2)(x-1) = 0 \Rightarrow x = 0, 1, \text{ or } 2;$

$\text{Area} = \int_0^1 (x^3 - 3x^2 + 2x)dx - \int_1^2 (x^3 - 3x^2 + 2x)dx$

$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 - \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2$

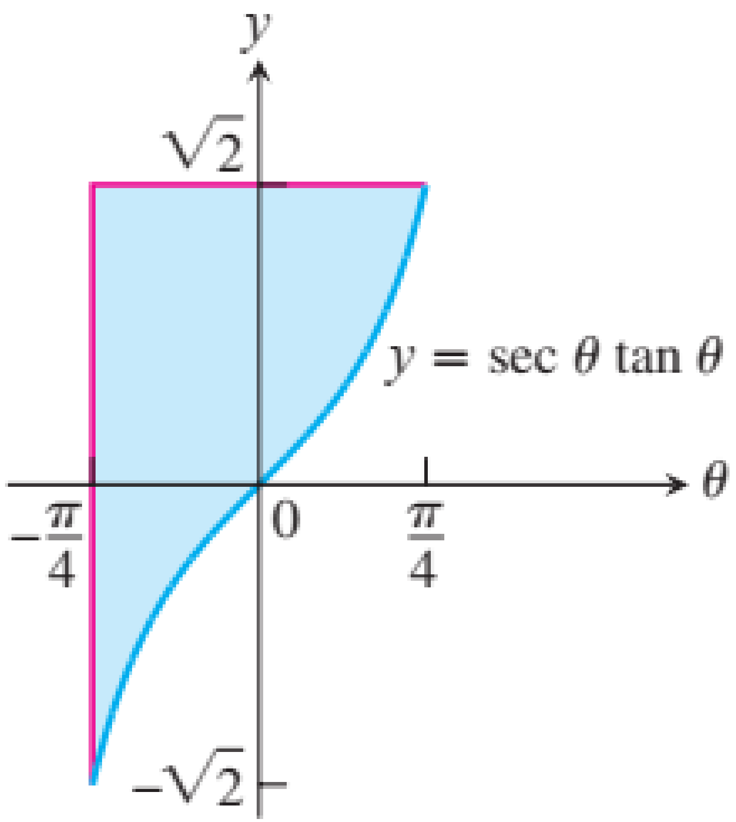
$= \left(\frac{1^4}{4} - 1^3 + 1^2\right) - \left(\frac{0^4}{4} - 0^3 + 0^2\right)$

$- \left[\left(\frac{2^4}{4} - 2^3 + 2^2\right) - \left(\frac{1^4}{4} - 1^3 + 1^2\right)\right] = \frac{1}{2}$



Find the areas of the shaded regions in Exercises 43–46.

45.



On $\left[-\frac{\pi}{4}, 0\right]$: The area of the rectangle bounded by the lines $y = \sqrt{2}$, $y = 0$, $\theta = 0$, and $\theta = -\frac{\pi}{4}$ is $\sqrt{2}\left(\frac{\pi}{4}\right)$ $= \frac{\pi\sqrt{2}}{4}$. The area between the curve $y = \sec\theta\tan\theta$ and $y = 0$ is $-\int_{-\pi/4}^{0} \sec\theta\tan\theta\, d\theta = [-\sec\theta]_{-\pi/4}^{0}$ $= (-\sec 0) - \left(-\sec\left(-\frac{\pi}{4}\right)\right) = \sqrt{2} - 1$. Therefore the area of the shaded region on $\left[-\frac{\pi}{4}, 0\right]$ is $\frac{\pi\sqrt{2}}{4} + \left(\sqrt{2} - 1\right)$.

On $\left[0, \frac{\pi}{4}\right]$: The area of the rectangle bounded by $\theta = \frac{\pi}{4}$, $\theta = 0$, $y = \sqrt{2}$, and $y = 0$ is $\sqrt{2}\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{4}$. The area under the curve $y = \sec\theta\tan\theta$ is $\int_{0}^{\pi/4} \sec\theta\tan\theta\, d\theta = [\sec\theta]_{0}^{\pi/4} = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$. Therefore the area of the shaded region on $\left[0, \frac{\pi}{4}\right]$ is $\frac{\pi\sqrt{2}}{4} - \left(\sqrt{2} - 1\right)$. Thus, the area of the total shaded region is $\left(\frac{\pi\sqrt{2}}{4} + \sqrt{2} - 1\right) + \left(\frac{\pi\sqrt{2}}{4} - \sqrt{2} + 1\right) = \frac{\pi\sqrt{2}}{2}$.

Substitution and Area Between Curves

THEOREM 6 Substitution in Definite Integrals

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

EXAMPLE 1 Substitution by Two Methods

Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3 + 1}\, dx$.

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 6.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$= \int_{0}^{2} \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3+1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3} \qquad \blacksquare$$

Theorem 7

Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

DEFINITION Area Between Curves

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_{a}^{b} [f(x) - g(x)]\,dx.$$

EXAMPLE 4 Area Between Intersecting Curves

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.30). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

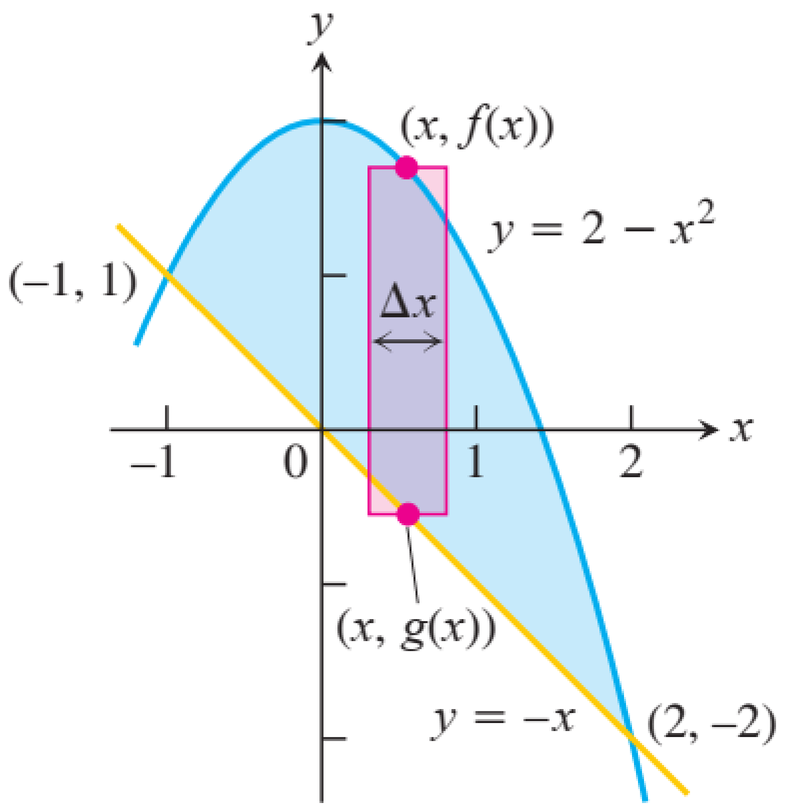
$$2 - x^2 = -x \qquad \text{Equate } f(x) \text{ and } g(x).$$

$$x^2 - x - 2 = 0 \qquad \text{Rewrite.}$$

$$(x + 1)(x - 2) = 0 \qquad \text{Factor.}$$

$$x = -1, \qquad x = 2. \qquad \text{Solve.}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.



y
$(x, f(x))$
$y = 2 - x^2$
$(-1, 1)$
Δx
x
-1
0
1
2
$(x, g(x))$
$y = -x$
$(2, -2)$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1, b = 2$.
The area between the curves is

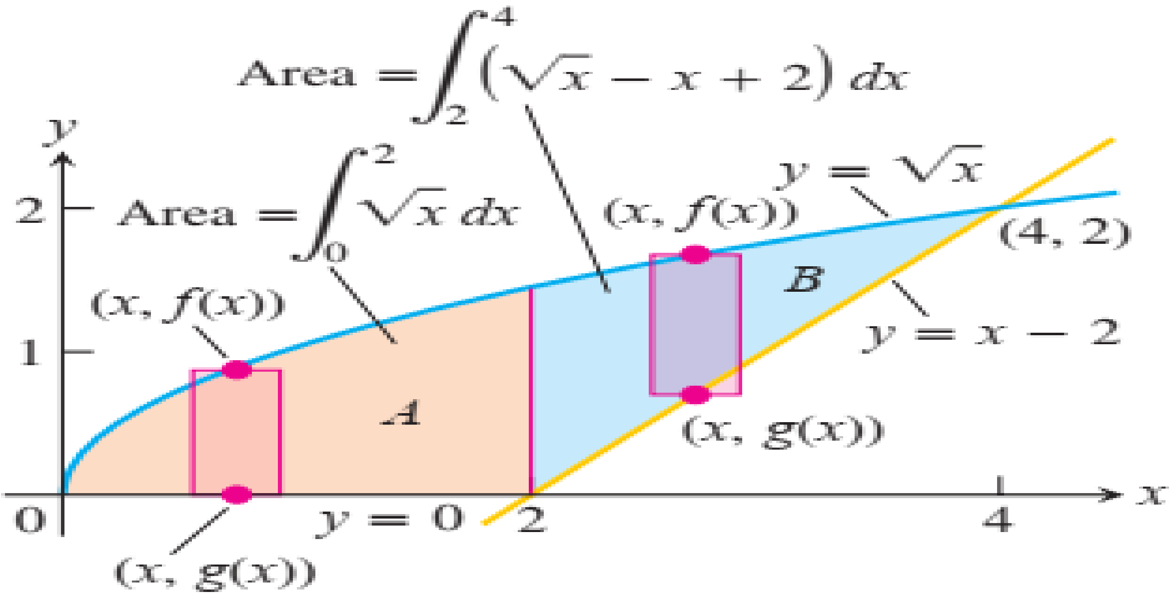
$$A = \int_a^b [f(x) - g(x)]\, dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\, dx$$
$$= \int_{-1}^{2} (2 + x - x^2)\, dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

Exercise 5.6, Question # 1-24.

EXAMPLE

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 5.31) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (there is agreement at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.31.



The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\begin{aligned}
\sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x).\\
x &= (x-2)^2 = x^2 - 4x + 4 && \text{Square both sides.}\\
x^2 - 5x + 4 &= 0 && \text{Rewrite.}\\
(x-1)(x-4) &= 0 && \text{Factor.}\\
x = 1, \qquad x &= 4. && \text{Solve.}
\end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\text{For } 0 \le x \le 2: \qquad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$$

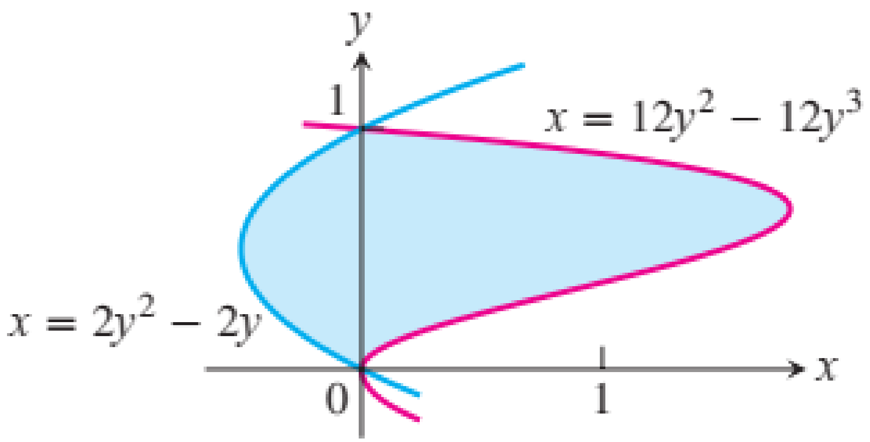
$$\text{For } 2 \le x \le 4: \qquad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$$

We add the area of subregions A and B to find the total area:

$$\begin{aligned}
\text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B} \\
&= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 \\
&= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right) \\
&= \frac{2}{3}(8) - 2 = \frac{10}{3}.
\end{aligned}$$

■

33.



For the sketch given, $c = 0$, $d = 1$; $f(y) - g(y) = (12y^2 - 12y^3) - (2y^2 - 2y) = 10y^2 - 12y^3 + 2y$;

$$A = \int_0^1 (10y^2 - 12y^3 + 2y)\,dy = \int_0^1 10y^2\,dy - \int_0^1 12y^3\,dy + \int_0^1 2y\,dy = \left[\tfrac{10}{3}\,y^3\right]_0^1 - \left[\tfrac{12}{4}\,y^4\right]_0^1 + \left[\tfrac{2}{2}\,y^2\right]_0^1$$

$$= \left(\tfrac{10}{3} - 0\right) - (3 - 0) + (1 - 0) = \tfrac{4}{3}$$