Applications of Definite Integrals

Volumes by Slicing and Rotation About an Axis

In this section we define volumes of solids whose cross-sections are plane regions. A **cross-section** of a solid S is the plane region formed by intersecting S with a plane (Figure 6.1).

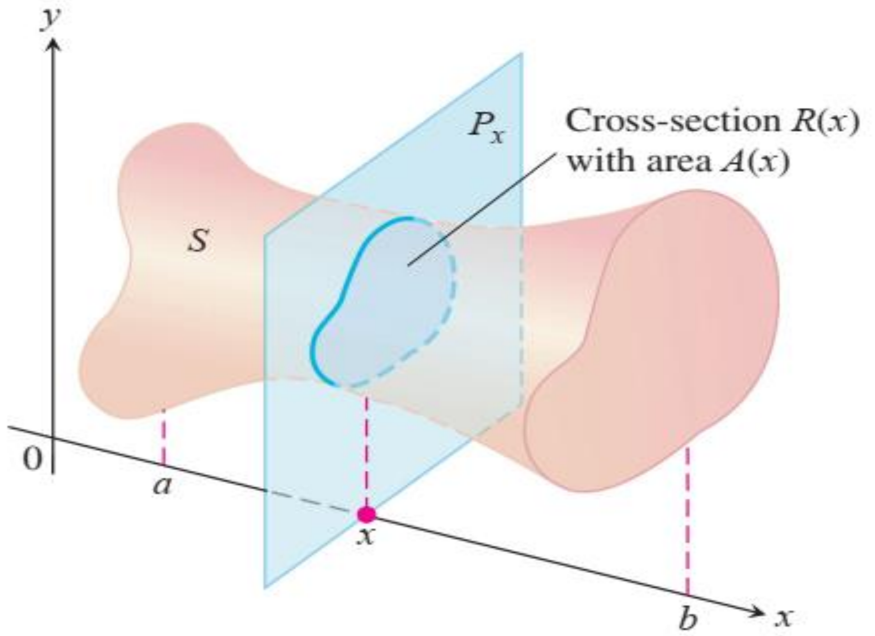


FIGURE 6.1 A cross-section of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

DEFINITION Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$

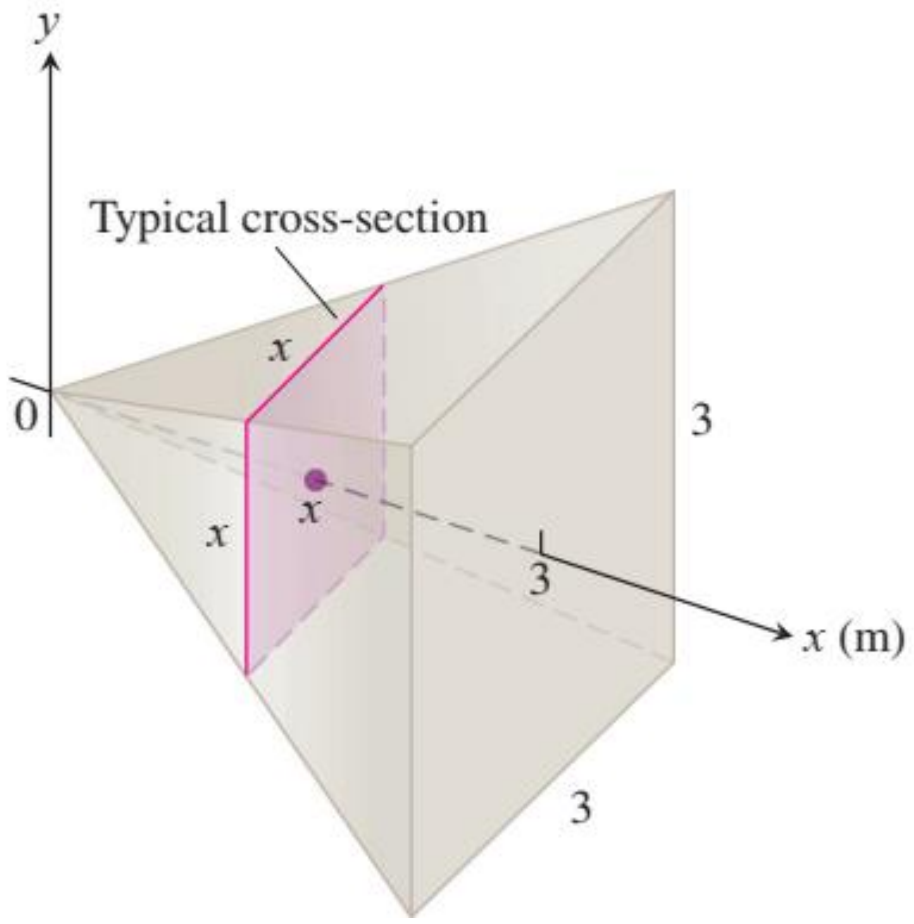
To apply the formula in the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

EXAMPLE 1 Volume of a Pyramid

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



Solution

1. *A sketch*. We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section (Figure 6.5).
2. *A formula for $A(x)$*. The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

3. *The limits of integration*. The squares lie on the planes from $x = 0$ to $x = 3$.
4. *Integrate to find the volume*.

$$V = \int_0^3 A(x)\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\bigg]_0^3 = 9 \text{ m}^3$$

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EXAMPLE

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

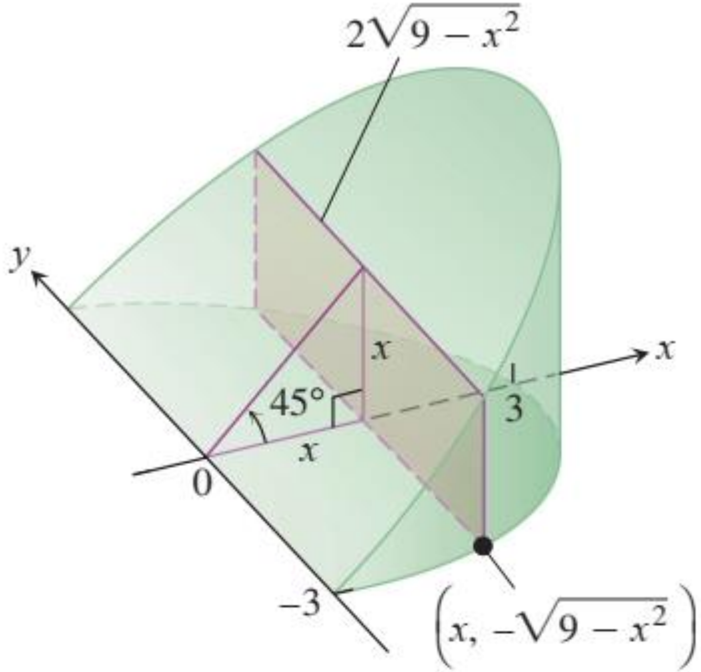

FIGURE 6.7 The wedge of Example 3, sliced perpendicular to the x-axis. The cross-sections are rectangles.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x-axis (Figure 6.7). The cross-section at x is a rectangle of area

$$A(x) = (\text{height})(\text{width}) = (x)\left(2\sqrt{9 - x^2}\right)$$
$$= 2x\sqrt{9 - x^2}\,.$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$V = \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9 - x^2}\,dx$$
$$= -\frac{2}{3}(9 - x^2)^{3/2}\Big]_0^3$$
$$= 0 + \frac{2}{3}(9)^{3/2}$$
$$= 18.$$

Let $u = 9 - x^2$, $du = -2x\,dx$, integrate, and substitute back.

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Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume gives

$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.}$$

EXAMPLE

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

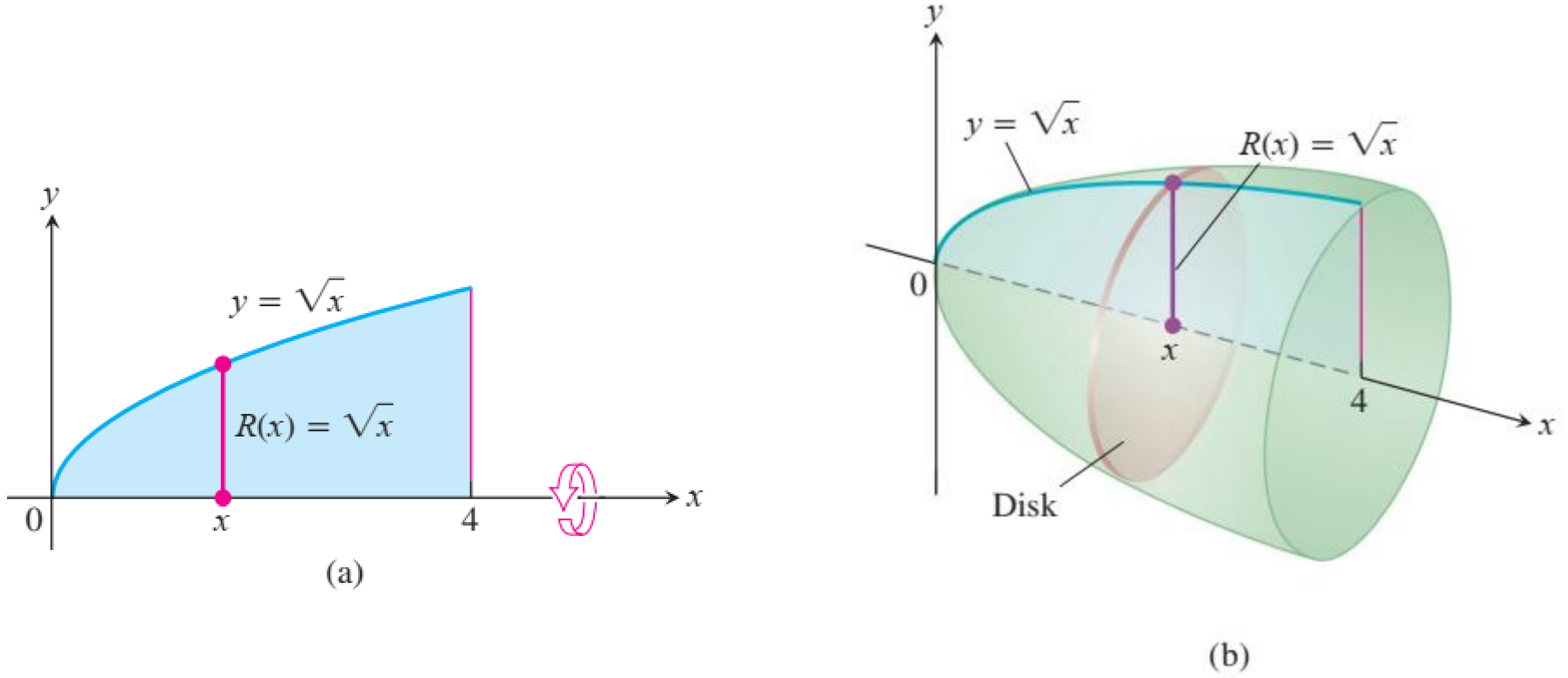

FIGURE 6.8

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned}
V &= \int_a^b \pi[R(x)]^2\,dx \\
&= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad\qquad R(x) = \sqrt{x} \\
&= \pi\int_0^4 x\,dx = \pi\,\frac{x^2}{2}\bigg]_0^4 = \pi\,\frac{(4)^2}{2} = 8\pi.
\end{aligned}$$

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EXAMPLE

The circle

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

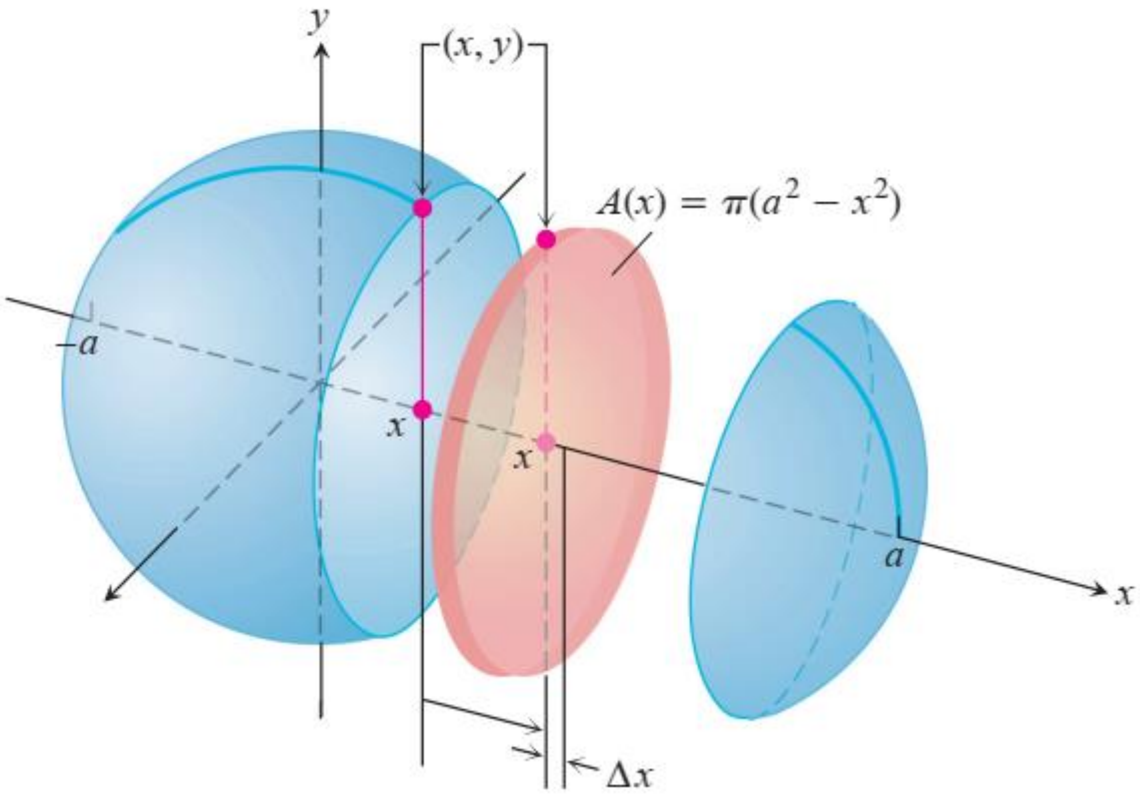


FIGURE 6.9

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x-axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

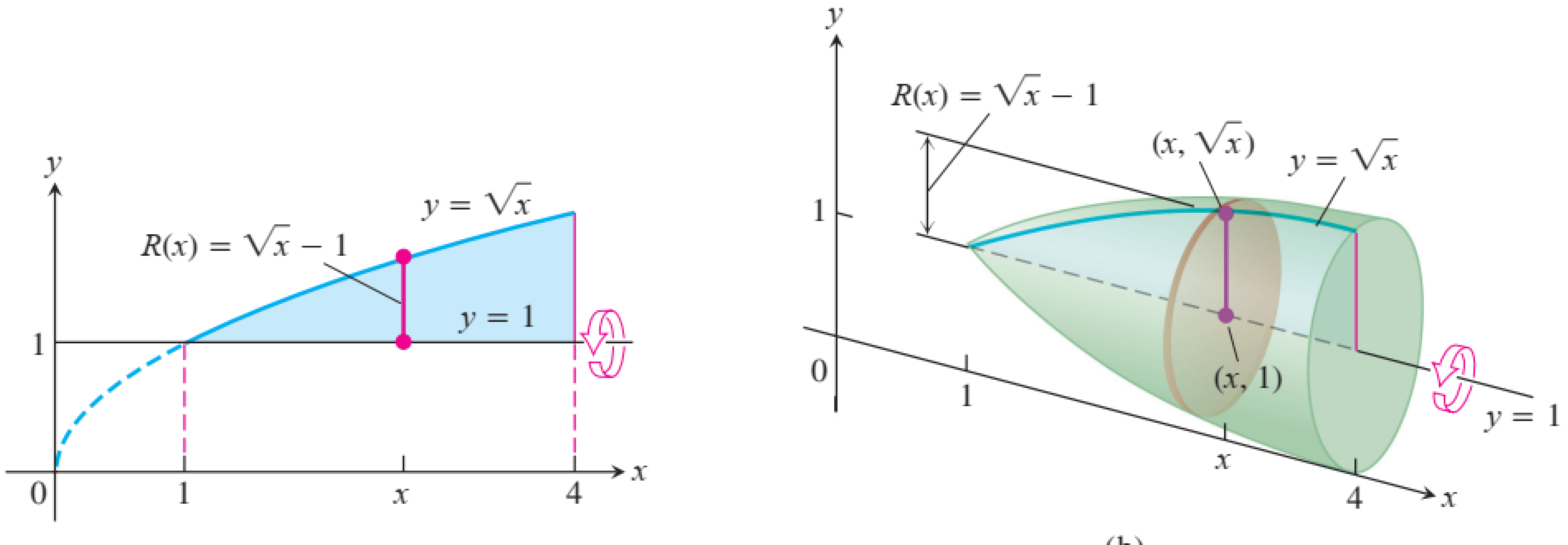
$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

EXAMPLE

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



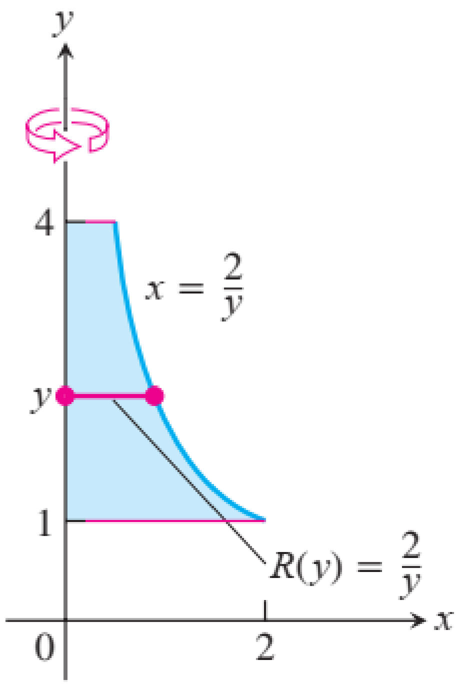
(a) (b)

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

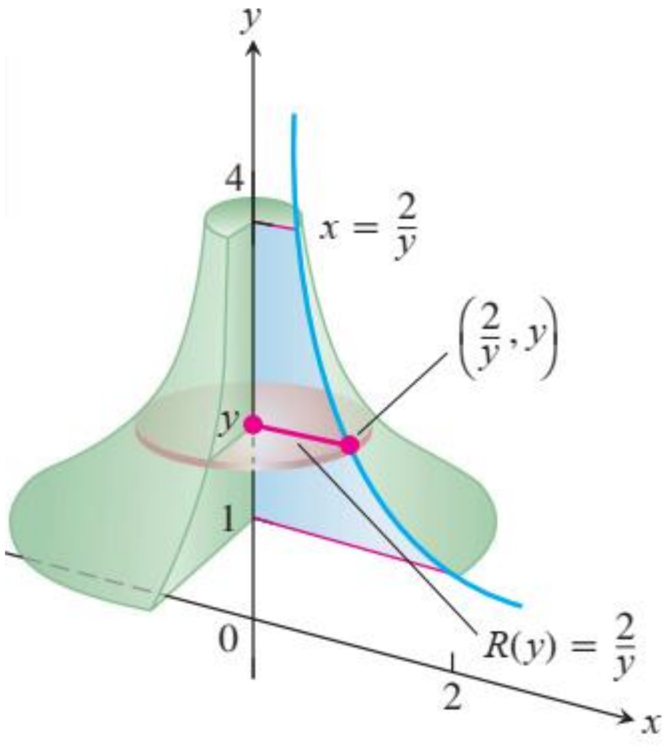
$$
\begin{aligned}
V &= \int_1^4 \pi[R(x)]^2\,dx \\
&= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx \\
&= \pi \int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \\
&= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}.
\end{aligned}
$$

EXAMPLE

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.



(a)



Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(y)]^2 \, dy \\ &= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \\ &= \pi \int_1^4 \frac{4}{y^2}\, dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] \\ &= 3\pi. \end{aligned}$$

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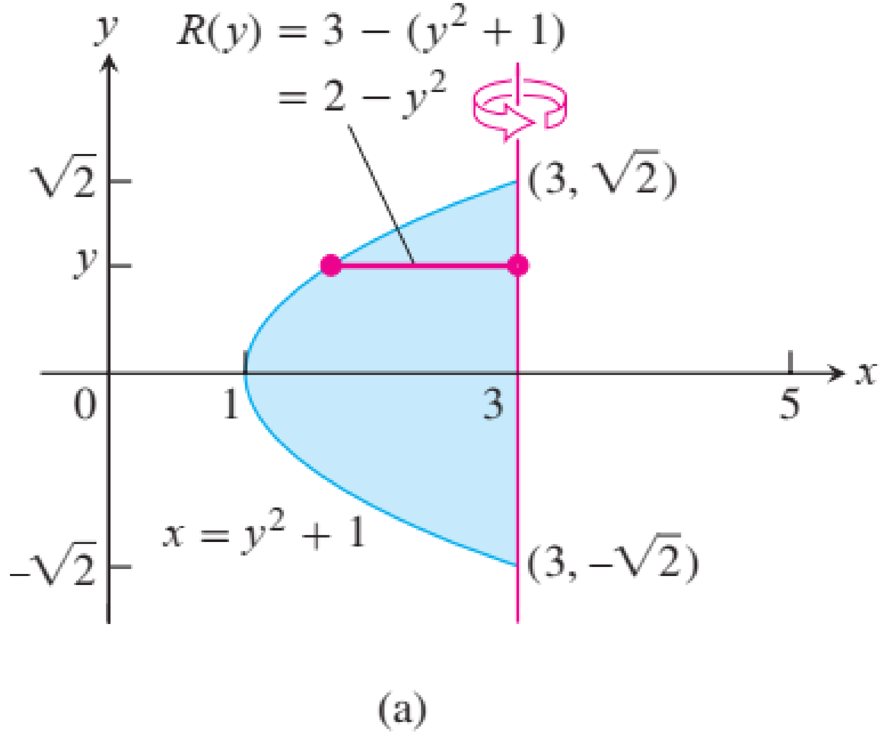
EXAMPLE

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$. The volume is

$$
\begin{aligned}
V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 \, dy \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 \, dy \qquad R(y) = 3 - (y^2 + 1) = 2 - y^2 \\
&= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] \, dy \\
&= \pi \left[4y - \frac{4}{3} y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \frac{64\pi\sqrt{2}}{15}.
\end{aligned}
$$



$R(y) = 3 - (y^2 + 1)$
$= 2 - y^2$
y
x
$\sqrt{2}$
y
0
1
3
5
$-\sqrt{2}$
$(3, \sqrt{2})$
$(3, -\sqrt{2})$
$x = y^2 + 1$
(a)
$R(y) = 2 - y^2$
$x = 3$
y
x
$\sqrt{2}$
y
0
1
3
5
$-\sqrt{2}$
$x = y^2 + 1$
(b)

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

$$\text{Outer radius:} \quad R(x)$$

$$\text{Inner radius:} \quad r(x)$$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi\left([R(x)]^2 - [r(x)]^2\right).$$

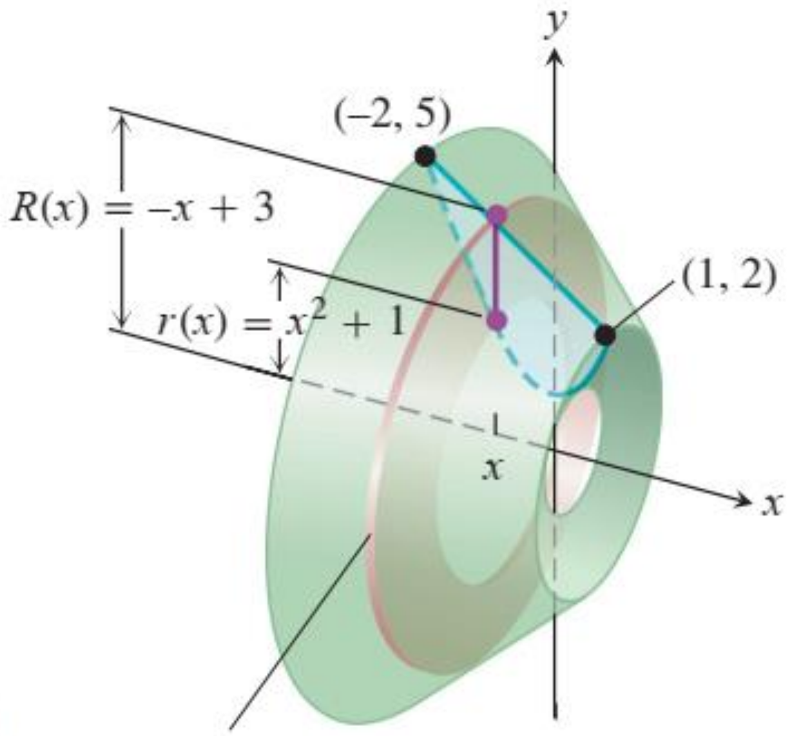
Consequently, the definition of volume gives

$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.}$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a slab is a circular washer of outer radius $R(x)$ and inner radius $r(x)$.

EXAMPLE

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.



Washer cross section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

$$\text{Outer radius:} \quad R(x) = -x + 3$$

$$\text{Inner radius:} \quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2, \quad x &= 1 \end{aligned}$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \\ &= \int_{-2}^{1} \pi((-x+3)^2 - (x^2+1)^2)\,dx \\ &= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx \\ &= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \end{aligned}$$

Values from Steps 2 and 3

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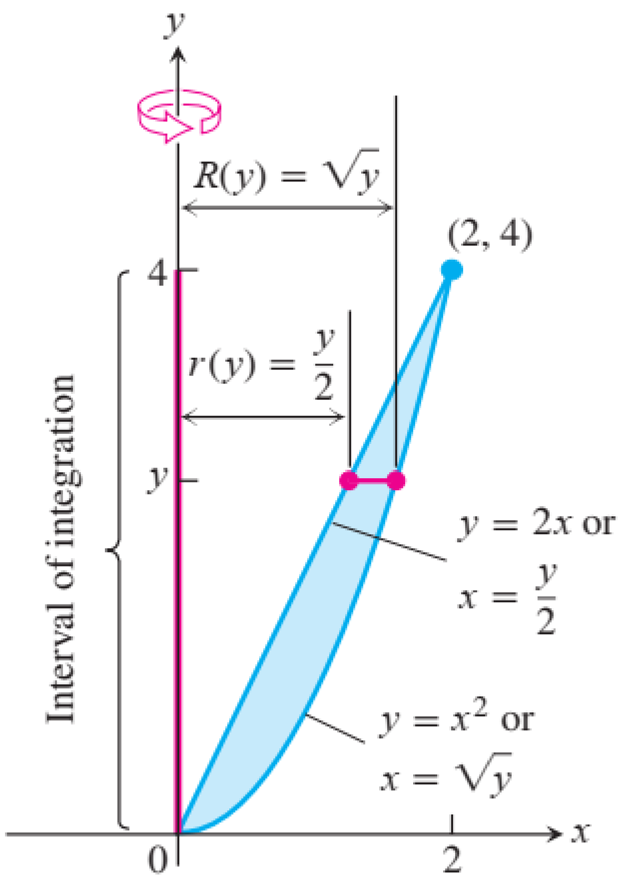
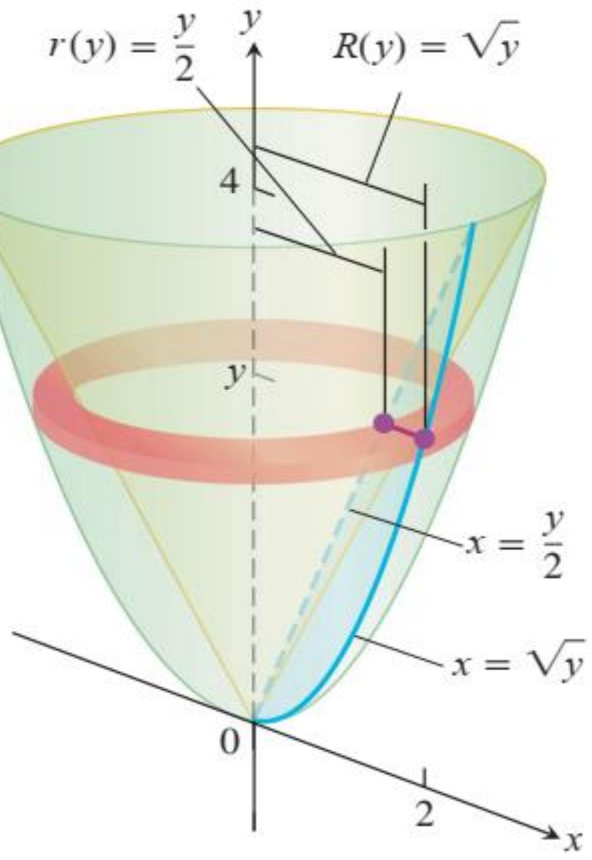
EXAMPLE

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.



(a)



(b)

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy \\ &= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy \\ &= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi. \end{aligned}$$

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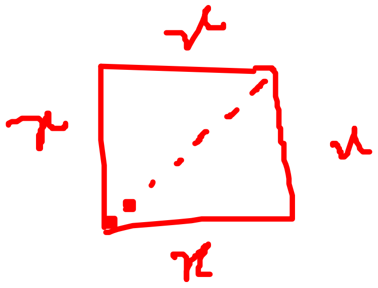
Find the volumes of the solids in Exercises 3–10.

The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.



$$A(x) = \frac{(\text{diagonal})^2}{2} = \frac{\left(\sqrt{x} - (-\sqrt{x})\right)^2}{2} = 2x$$

$a = 0,\ b = 4;$

$$V = \int_a^b A(x)\,dx = \int_0^4 2x\,dx = [x^2]_0^4 = 16$$

$A = x \cdot x$

$x^2 + x^2 = d^2$ | $a^2 + b^2 = c^2$

$2x^2 = d^2$

$\Rightarrow x^2 = d^2/2$

so $A = d^2/2$

10. The base of the solid is the disk $x^2 + y^2 \leq 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

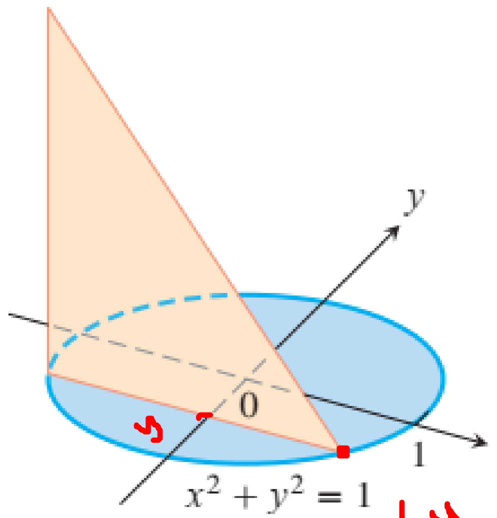
$$A(y) = \tfrac{1}{2}(\text{leg})(\text{leg}) = \tfrac{1}{2}\left[\sqrt{1-y^2} - \left(-\sqrt{1-y^2}\right)\right]^2$$

$$= \tfrac{1}{2}\left(2\sqrt{1-y^2}\right)^2 = 2\left(1-y^2\right);$$

$c = -1,\ d = 1;$

$$V = \int_c^d A(y)\,dy = \int_{-1}^{1} 2(1-y^2)\,dy$$

$$= 2\left[y - \tfrac{y^3}{3}\right]_{-1}^{1} = 4\left(1 - \tfrac{1}{3}\right) = \tfrac{8}{3}$$

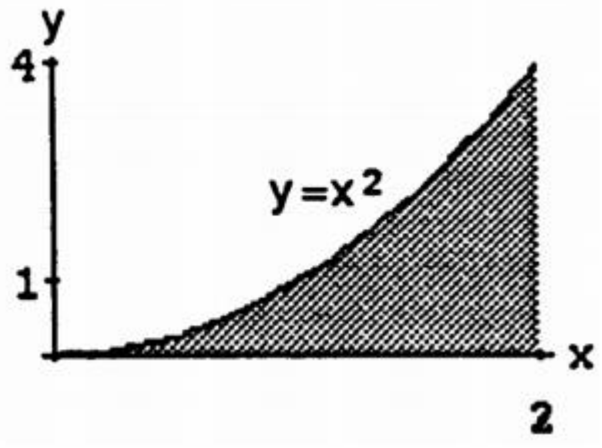


$(y, \sqrt{1-y^2})$

$\therefore\ x = \pm\sqrt{1-y^2}$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 17–22 about the x-axis.

17. $y = x^2, \quad y = 0, \quad x = 2$



. $R(x) = x^2 \Rightarrow V = \int_0^2 \pi[R(x)]^2\,dx = \pi \int_0^2 (x^2)^2\,dx$

$= \pi \int_0^2 x^4\,dx = \pi \left[\frac{x^5}{5}\right]_0^2 = \frac{32\pi}{5}$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 33–38 about the x-axis.

37. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

$r(x) = \sec x$ and $R(x) = \sqrt{2}$

$\Rightarrow V = \int_{-\pi/4}^{\pi/4} \pi \left([R(x)]^2 - [r(x)]^2\right) dx$

$= \pi \int_{-\pi/4}^{\pi/4} (2 - \sec^2 x)\, dx = \pi[2x - \tan x]_{-\pi/4}^{\pi/4}$

$= \pi\left[\left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)\right] = \pi(\pi - 2)$