The Precise Definition of a Limit

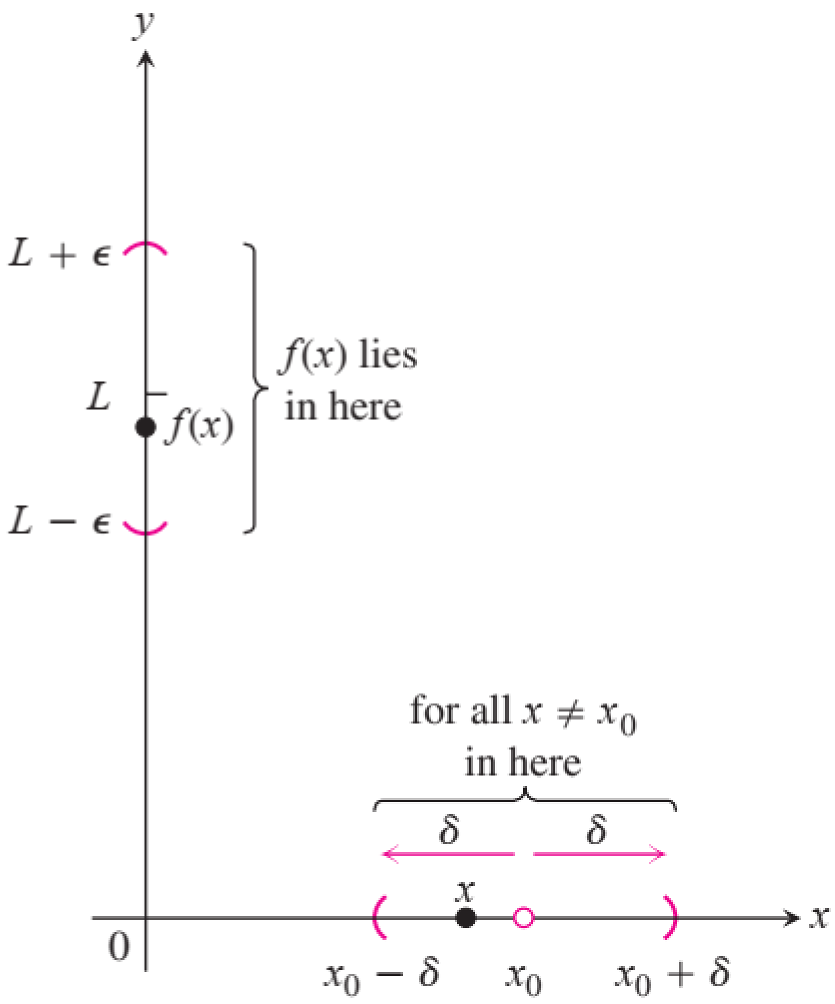
DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$


y
L + ϵ
L
f(x)
L − ϵ
f(x) lies
in here
for all x ≠ x0
in here
δ
δ
x
x
0
x0 − δ
x0
x0 + δ

EXAMPLE 2 Testing the Definition

Show that

$$\lim_{x \to 1} (5x - 3) = 2.$$

Solution Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ϵ of $L = 2$, so

$$|f(x) - 2| < \epsilon.$$

We find δ by working backward from the ϵ-inequality:

$$\begin{aligned} |(5x - 3) - 2| = |5x - 5| &< \epsilon \\ 5|x - 1| &< \epsilon \\ |x - 1| &< \epsilon/5. \end{aligned}$$

Thus, we can take $\delta = \epsilon/5$ (Figure 2.15). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x\to1}(5x - 3) = 2$.

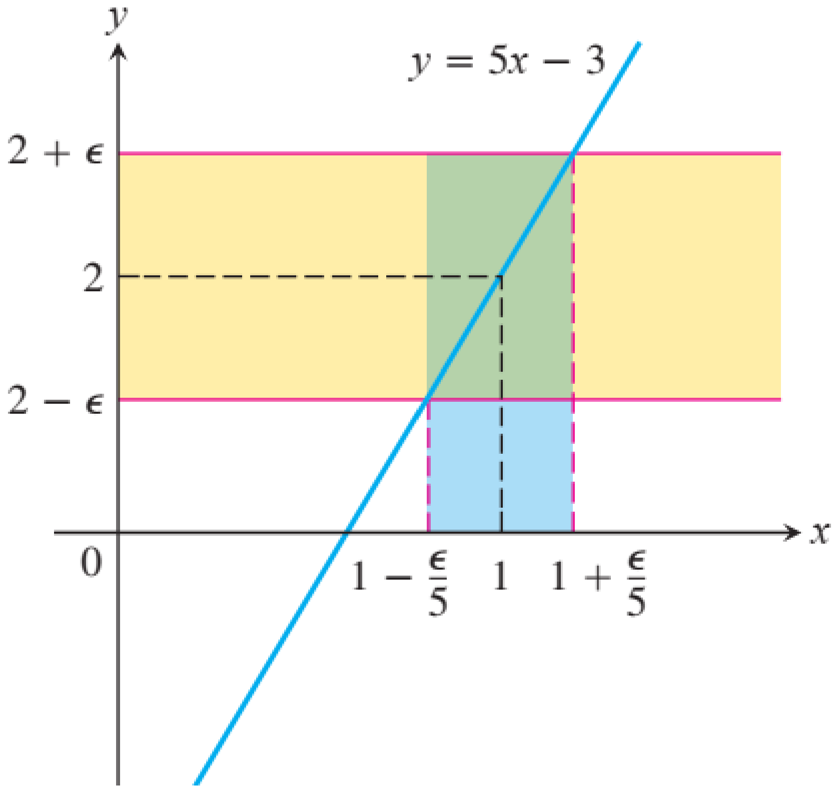


FIGURE 2.15 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

EXAMPLE 3 Limits of the Identity and Constant Functions

Prove:

(a) $\lim_{x \to x_0} x = x_0$ **(b)** $\lim_{x \to x_0} k = k$ (k constant).

Solution

(a) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |x - x_0| < \epsilon .$$

The implication will hold if δ equals ϵ or any smaller positive number
This proves that $\lim_{x \to x_0} x = x_0$.

(b) Let $\epsilon > 0$ be given. We must find $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \text{implies} \quad |k - k| < \epsilon .$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold

This proves that $\lim_{x \to x_0} k = k$.

EXAMPLE 4 Finding Delta Algebraically

For the limit $\lim_{x \to 5} \sqrt{x-1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x-1} - 2| < 1.$$

Solution We organize the search into two steps, as discussed below.

1. *Solve the inequality* $|\sqrt{x-1} - 2| < 1$ *to find an interval containing* $x_0 = 5$ *on which the inequality holds for all* $x \neq x_0$.

$$\begin{aligned} |\sqrt{x-1} - 2| &< 1 \\ -1 < \sqrt{x-1} - 2 &< 1 \\ 1 < \sqrt{x-1} &< 3 \\ 1 < x - 1 &< 9 \\ 2 < x &< 10 \end{aligned}$$

The inequality holds for all x in the open interval $(2, 10)$, so it holds for all $x \neq 5$ in this interval as well

2. *Find a value of* $\delta > 0$ *to place the centered interval* $5 - \delta < x < 5 + \delta$ (centered at $x_0 = 5$) *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Figure 2.18). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x - 1} - 2| < 1$ (Figure 2.19)

$$0 < |x - 5| < 3 \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.

2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

EXAMPLE 5 Finding Delta Algebraically

Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\epsilon > 0$ there exists a $\delta > 0$ such that for all x

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \epsilon.$$

1. *Solve the inequality $|f(x) - 4| < \epsilon$ to find an open interval containing $x_0 = 2$ on which the inequality holds for all $x \neq x_0$.*

For $x \neq x_0 = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \epsilon$:

$$|x^2 - 4| < \epsilon$$
$$-\epsilon < x^2 - 4 < \epsilon$$

$$\begin{aligned}
4 - \epsilon &< x^2 < 4 + \epsilon \\
\sqrt{4 - \epsilon} &< |x| < \sqrt{4 + \epsilon} \qquad \text{Assumes } \epsilon < 4\text{; see below.} \\
\sqrt{4 - \epsilon} &< x < \sqrt{4 + \epsilon}. \qquad \text{An open interval about } x_0 = 2 \text{ that solves the inequality}
\end{aligned}$$

The inequality $|f(x) - 4| < \epsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$ (Figure 2.20).

2. *Find a value of* $\delta > 0$ *that places the centered interval* $(2 - \delta, 2 + \delta)$ *inside the interval* $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $\left(\sqrt{4 - \epsilon}, \sqrt{4 + \epsilon}\right)$. In other words, take $\delta = \min\left\{2 - \sqrt{4 - \epsilon}, \sqrt{4 + \epsilon} - 2\right\}$, the *minimum* (the smaller) of the two numbers $2 - \sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon} - 2$. If δ has this or any smaller positive value, the inequality $0 < |x - 2| < \delta$ will automatically place x between $\sqrt{4 - \epsilon}$ and $\sqrt{4 + \epsilon}$ to make $|f(x) - 4| < \epsilon$. For all x,

$$0 < |x - 2| < \delta \qquad \Rightarrow \qquad |f(x) - 4| < \epsilon.$$

This completes the proof.

Exercise 2.3 Questions 31 to 45

Each of Exercises 31–36 gives a function $f(x)$, a point x_0, and a positive number ϵ. Find $L = \lim_{x \to x_0} f(x)$. Then find a number $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

31. $\lim_{x \to 3} (3 - 2x) = 3 - 2(3) = -3$

Step 1: $|(3 - 2x) - (-3)| < 0.02 \Rightarrow -0.02 < 6 - 2x < 0.02 \Rightarrow -6.02 < -2x < -5.98 \Rightarrow 3.01 > x > 2.99$ or $2.99 < x < 3.01.$

Step 2: $0 < |x - 3| < \delta \Rightarrow -\delta < x - 3 < \delta \Rightarrow -\delta + 3 < x < \delta + 3.$

Then $-\delta + 3 = 2.99 \Rightarrow \delta = 0.01$, or $\delta + 3 = 3.01 \Rightarrow \delta = 0.01$; thus $\delta = 0.01.$

33. $\lim\limits_{x \to 2} \frac{x^2-4}{x-2} = \lim\limits_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim\limits_{x \to 2} (x+2) = 2+2 = 4,\ x \neq 2$

Step 1: $\left|\left(\frac{x^2-4}{x-2}\right) - 4\right| < 0.05 \Rightarrow -0.05 < \frac{(x+2)(x-2)}{(x-2)} - 4 < 0.05 \Rightarrow 3.95 < x+2 < 4.05,\ x \neq 2$

$\Rightarrow 1.95 < x < 2.05,\ x \neq 2.$

Step 2: $|x-2| < \delta \Rightarrow -\delta < x-2 < \delta \Rightarrow -\delta + 2 < x < \delta + 2.$

Then $-\delta + 2 = 1.95 \Rightarrow \delta = 0.05$, or $\delta + 2 = 2.05 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

34. $\lim\limits_{x \to -5} \frac{x^2+6x+5}{x+5} = \lim\limits_{x \to -5} \frac{(x+5)(x+1)}{(x+5)} = \lim\limits_{x \to -5} (x+1) = -4,\ x \neq -5.$

Step 1: $\left|\left(\frac{x^2+6x+5}{x+5}\right) - (-4)\right| < 0.05 \Rightarrow -0.05 < \frac{(x+5)(x+1)}{(x+5)} + 4 < 0.05 \Rightarrow -4.05 < x+1 < -3.95,\ x \neq -5$

$\Rightarrow -5.05 < x < -4.95,\ x \neq -5.$

Step 2: $|x-(-5)| < \delta \Rightarrow -\delta < x+5 < \delta \Rightarrow -\delta - 5 < x < \delta - 5.$

Then $-\delta - 5 = -5.05 \Rightarrow \delta = 0.05$, or $\delta - 5 = -4.95 \Rightarrow \delta = 0.05$; thus $\delta = 0.05$.

39. Step 1: $\left|\sqrt{x-5}-2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{x-5}-2 < \epsilon \Rightarrow 2-\epsilon < \sqrt{x-5} < 2+\epsilon \Rightarrow (2-\epsilon)^2 < x-5 < (2+\epsilon)^2$

$\Rightarrow (2-\epsilon)^2+5 < x < (2+\epsilon)^2+5.$

Step 2: $|x-9| < \delta \Rightarrow -\delta < x-9 < \delta \Rightarrow -\delta+9 < x < \delta+9.$

Then $-\delta+9 = \epsilon^2-4\epsilon+9 \Rightarrow \delta = 4\epsilon-\epsilon^2$, or $\delta+9 = \epsilon^2+4\epsilon+9 \Rightarrow \delta = 4\epsilon+\epsilon^2$. Thus choose the smaller distance, $\delta = 4\epsilon-\epsilon^2$.

40. Step 1: $\left|\sqrt{4-x}-2\right| < \epsilon \Rightarrow -\epsilon < \sqrt{4-x}-2 < \epsilon \Rightarrow 2-\epsilon < \sqrt{4-x} < 2+\epsilon \Rightarrow (2-\epsilon)^2 < 4-x < (2+\epsilon)^2$

$\Rightarrow -(2+\epsilon)^2 < x-4 < -(2-\epsilon)^2 \Rightarrow -(2+\epsilon)^2+4 < x < -(2-\epsilon)^2+4.$

Step 2: $|x-0| < \delta \Rightarrow -\delta < x < \delta.$

Then $-\delta = -(2+\epsilon)^2+4 = -\epsilon^2-4\epsilon \Rightarrow \delta = 4\epsilon+\epsilon^2$, or $\delta = -(2-\epsilon)^2+4 = 4\epsilon-\epsilon^2$. Thus choose the smaller distance, $\delta = 4\epsilon-\epsilon^2$.

44. Step 1: $\left|\frac{1}{x^2}-\frac{1}{3}\right|<\epsilon \Rightarrow -\epsilon<\frac{1}{x^2}-\frac{1}{3}<\epsilon \Rightarrow \frac{1}{3}-\epsilon<\frac{1}{x^2}<\frac{1}{3}+\epsilon \Rightarrow \frac{1-3\epsilon}{3}<\frac{1}{x^2}<\frac{1+3\epsilon}{3} \Rightarrow \frac{3}{1-3\epsilon}>x^2>\frac{3}{1+3\epsilon}$

$\Rightarrow \sqrt{\frac{3}{1+3\epsilon}}<|x|<\sqrt{\frac{3}{1-3\epsilon}}$, or $\sqrt{\frac{3}{1+3\epsilon}}<x<\sqrt{\frac{3}{1-3\epsilon}}$ for x near $\sqrt{3}$.

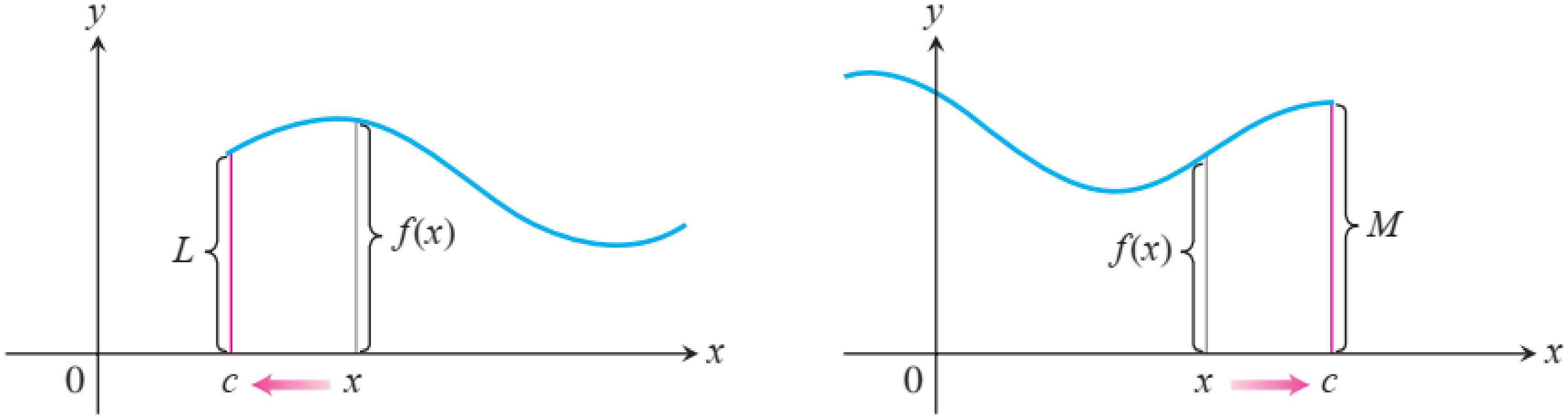
Step 2: $\left|x-\sqrt{3}\right|<\delta \Rightarrow -\delta<x-\sqrt{3}<\delta \Rightarrow \sqrt{3}-\delta<x<\sqrt{3}+\delta$.

Then $\sqrt{3}-\delta=\sqrt{\frac{3}{1+3\epsilon}} \Rightarrow \delta=\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}$, or $\sqrt{3}+\delta=\sqrt{\frac{3}{1-3\epsilon}} \Rightarrow \delta=\sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}$.

Choose $\delta=\min\left\{\sqrt{3}-\sqrt{\frac{3}{1+3\epsilon}}, \sqrt{\frac{3}{1-3\epsilon}}-\sqrt{3}\right\}$.

One-Sided Limits and Limits at Infinity

One-Sided Limits



(a) $\lim_{x \to c^+} f(x) = L$

(b) $\lim_{x \to c^-} f(x) = M$

FIGURE 2.22 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

EXAMPLE 1 One-Sided Limits for a Semicircle

The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.23. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \quad \text{and} \quad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have ordinary two-sided limits at either -2 or 2. ■

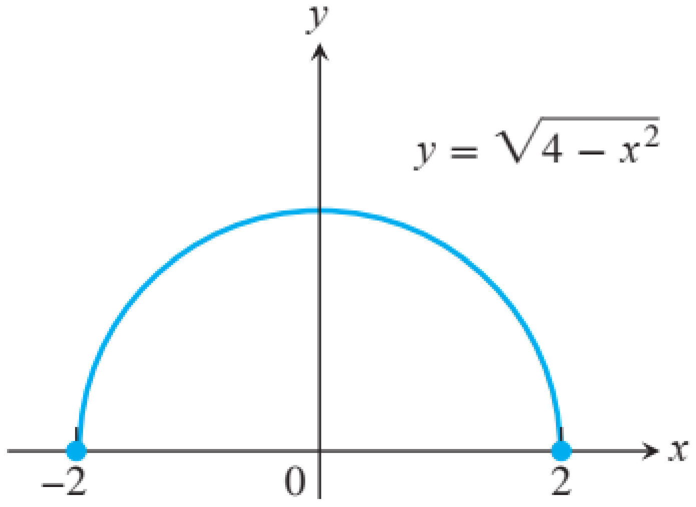
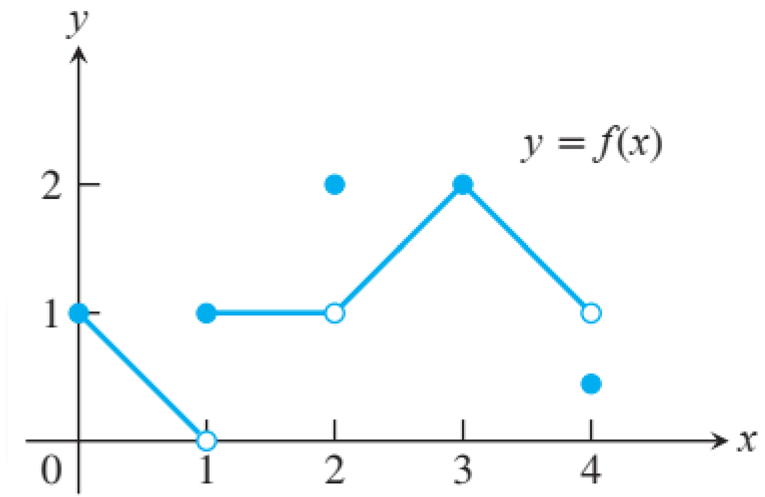


FIGURE 2.23 $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$ and $\lim_{x \to -2^+} \sqrt{4 - x^2} = 0$ (Example 1).

THEOREM 6

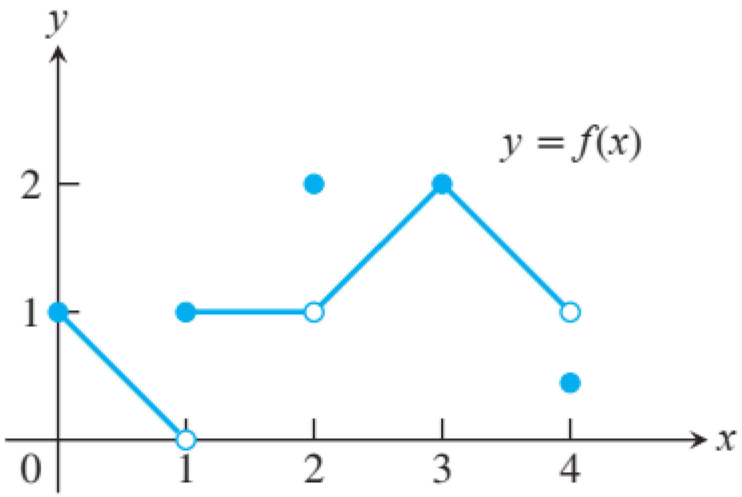
A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \qquad \Leftrightarrow \qquad \lim_{x \to c^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to c^+} f(x) = L.$$



At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,

$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.



At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

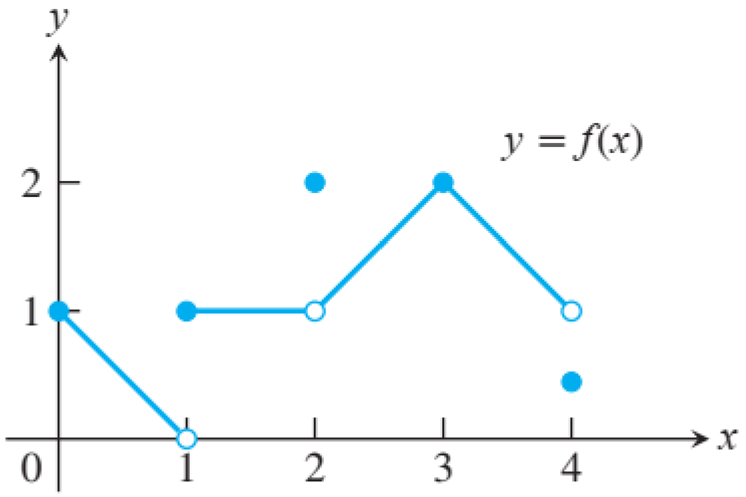
$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,

$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.



At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

Precise Definitions of One-Sided Limits

DEFINITIONS **Right-Hand, Left-Hand Limits**

We say that $f(x)$ has **right-hand limit L at x_0**, and write

$$\lim_{x \to x_0^+} f(x) = L \qquad \text{(See Figure 2.25)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 < x < x_0 + \delta \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

We say that f has **left-hand limit L at x_0**, and write

$$\lim_{x \to x_0^-} f(x) = L \qquad \text{(See Figure 2.26)}$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x

$$x_0 - \delta < x < x_0 \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

EXAMPLE 3 Applying the Definition to Find Delta

Prove that

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

Solution Let $\epsilon > 0$ be given. Here $x_0 = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that for all x

$$0 < x < \delta \qquad \Rightarrow \qquad |\sqrt{x} - 0| < \epsilon,$$

or

$$0 < x < \delta \qquad \Rightarrow \qquad \sqrt{x} < \epsilon.$$

Squaring both sides of this last inequality gives

$$x < \epsilon^2 \qquad \text{if} \qquad 0 < x < \delta.$$

If we choose $\delta = \epsilon^2$ we have

$$0 < x < \delta = \epsilon^2 \qquad \Rightarrow \qquad \sqrt{x} < \epsilon,$$

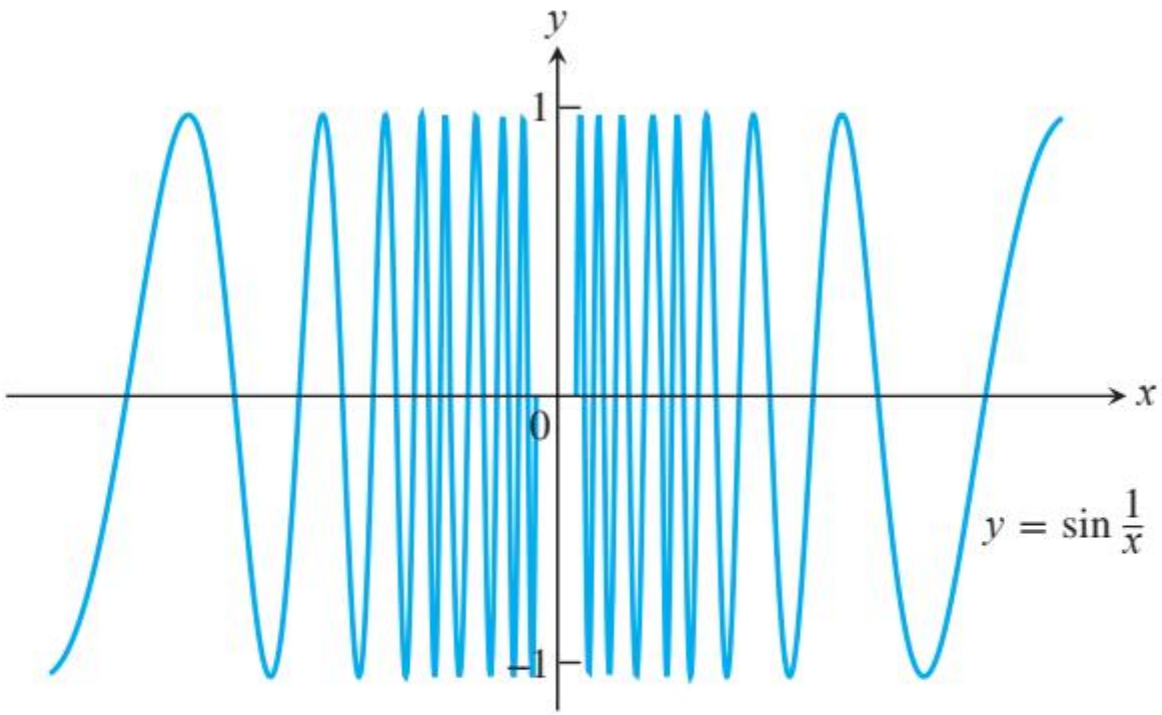
or

$$0 < x < \epsilon^2 \quad \Rightarrow \quad |\sqrt{x} - 0| < \epsilon .$$

According to the definition, this shows that $\lim_{x \to 0^+} \sqrt{x} = 0$ (Figure 2.27). ■

EXAMPLE 4 A Function Oscillating Too Much

Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.28).



THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \tag{1}$$

EXAMPLE 5 Using $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Show that **(a)** $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and **(b)** $\lim_{x \to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos h = 1 - 2\sin^2(h/2)$, we calculate

$$\begin{aligned}
\lim_{h \to 0} \frac{\cos h - 1}{h} &= \lim_{h \to 0} -\frac{2\sin^2(h/2)}{h} \\
&= -\lim_{\theta \to 0} \frac{\sin \theta}{\theta} \sin \theta && \text{Let } \theta = h/2. \\
&= -(1)(0) = 0.
\end{aligned}$$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{5x} &= \lim_{x\to 0}\frac{(2/5)\cdot\ \sin 2x}{(2/5)\cdot 5x}\\ &= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x}\\ &= \frac{2}{5}(1) = \frac{2}{5}\end{aligned}$$

Now, Eq. (1) applies with $\theta = 2x$.

■

Limits at Infinity

DEFINITIONS **Limit as x approaches ∞ or $-\infty$**

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \qquad \Rightarrow \qquad |f(x) - L| < \epsilon.$$

THEOREM 8 **Limit Laws as $x \to \pm\infty$**

If L, M, and k, are real numbers and

$$\lim_{x \to \pm\infty} f(x) = L \qquad \text{and} \qquad \lim_{x \to \pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\displaystyle\lim_{x \to \pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{x \to \pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\displaystyle\lim_{x \to \pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\displaystyle\lim_{x \to \pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\displaystyle\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x \to \pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

EXAMPLE 8 Numerator and Denominator of Same Degree

$$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$ Divide numerator and denominator by x^2.

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$ See Fig. 2.33. ■

EXAMPLE 9 Degree of Numerator Less Than Degree of Denominator

$$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$ Divide numerator and denominator by x^3.

$$= \frac{0 + 0}{2 - 0} = 0$$ See Fig. 2.34. ■

DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

The curve

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x \to \infty} f(x) = \frac{5}{3} \qquad \text{and} \qquad \lim_{x \to -\infty} f(x) = \frac{5}{3}.$$

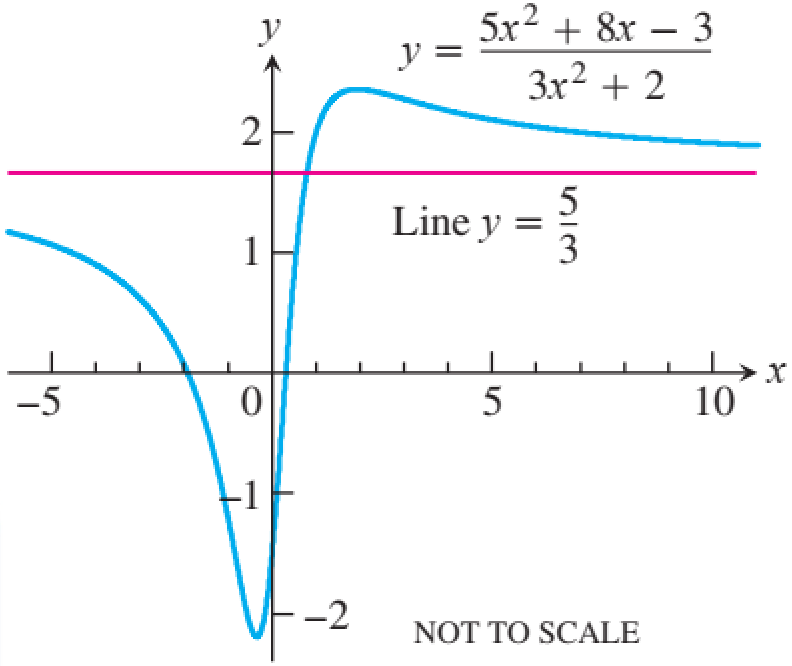


FIGURE 2.33 The graph of the function in Example 8. The graph approaches the line $y = 5/3$ as $|x|$ increases.

Exercise 2.4

Questions 1-4, 11, 13, 15, 21,23,25, 27, 47, 49, 51.

Infinite Limits and Vertical Asymptotes

DEFINITIONS Infinity, Negative Infinity as Limits

1. We say that ***f*(*x*) approaches infinity as *x* approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = \infty,$$

if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) > B.$$

2. We say that ***f*(*x*) approaches negative infinity as *x* approaches x_0**, and write

$$\lim_{x \to x_0} f(x) = -\infty,$$

if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \qquad \Rightarrow \qquad f(x) < -B.$$

DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

EXAMPLE 5 Looking for Asymptotes

Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x + 3}{x + 2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to -2$, where the denominator is zero.

The asymptotes are quickly revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$.

$$\begin{array}{r@{}l} & \phantom{x + {}}1 \\ x + 2 \,\big) & \overline{x + 3} \\ & \underline{x + 2} \\ & \phantom{x + {}}1 \end{array}$$

This result enables us to rewrite y:

$$y = 1 + \frac{1}{x + 2}.$$

We now see that the curve in question is the graph of $y = 1/x$ shifted 1 unit up and 2 units left (Figure 2.43). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$. ■

EXERCISES 2.5

3. $\lim\limits_{x \to 2^-} \dfrac{3}{x-2}$

$$\lim_{x \to 2^-} \frac{3}{x-2} = -\infty$$

18. (a) $\lim\limits_{x \to 1^+} \frac{x}{x^2-1} = \lim\limits_{x \to 1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{positive}}\right)$

(b) $\lim\limits_{x \to 1^-} \frac{x}{x^2-1} = \lim\limits_{x \to 1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{positive}}{\text{positive}\cdot\text{negative}}\right)$

(c) $\lim\limits_{x \to -1^+} \frac{x}{x^2-1} = \lim\limits_{x \to -1^+} \frac{x}{(x+1)(x-1)} = \infty$ $\left(\frac{\text{negative}}{\text{positive}\cdot\text{negative}}\right)$

(d) $\lim\limits_{x \to -1^-} \frac{x}{x^2-1} = \lim\limits_{x \to -1^-} \frac{x}{(x+1)(x-1)} = -\infty$ $\left(\frac{\text{negative}}{\text{negative}\cdot\text{negative}}\right)$

19. (a) $\lim\limits_{x \to 0^+} \frac{x^2}{2} - \frac{1}{x} = 0 + \lim\limits_{x \to 0^+} \frac{1}{-x} = -\infty$ $\left(\frac{1}{\text{negative}}\right)$

(b) $\lim\limits_{x \to 0^-} \frac{x^2}{2} - \frac{1}{x} = 0 + \lim\limits_{x \to 0^-} \frac{1}{-x} = \infty$ $\left(\frac{1}{\text{positive}}\right)$

(c) $\lim\limits_{x \to \sqrt[3]{2}} \frac{x^2}{2} - \frac{1}{x} = \frac{2^{2/3}}{2} - \frac{1}{2^{1/3}} = 2^{-1/3} - 2^{-1/3} = 0$

(d) $\lim\limits_{x \to -1} \frac{x^2}{2} - \frac{1}{x} = \frac{1}{2} - \left(\frac{1}{-1}\right) = \frac{3}{2}$

20. (a) $\lim\limits_{x \to -2^+} \frac{x^2-1}{2x+4} = \infty$ $\left(\frac{\text{positive}}{\text{positive}}\right)$

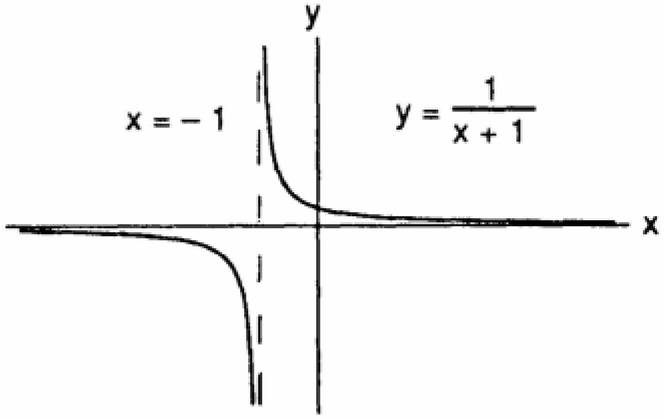
(b) $\lim\limits_{x \to -2^-} \frac{x^2-1}{2x+4} = -\infty$

(c) $\lim\limits_{x \to 1^+} \frac{x^2-1}{2x+4} = \lim\limits_{x \to 1^+} \frac{(x+1)(x-1)}{2x+4} = \frac{2 \cdot 0}{2+4} = 0$

(d) $\lim\limits_{x \to 0^-} \frac{x^2-1}{2x+4} = \frac{-1}{4}$

Find Asymptotes

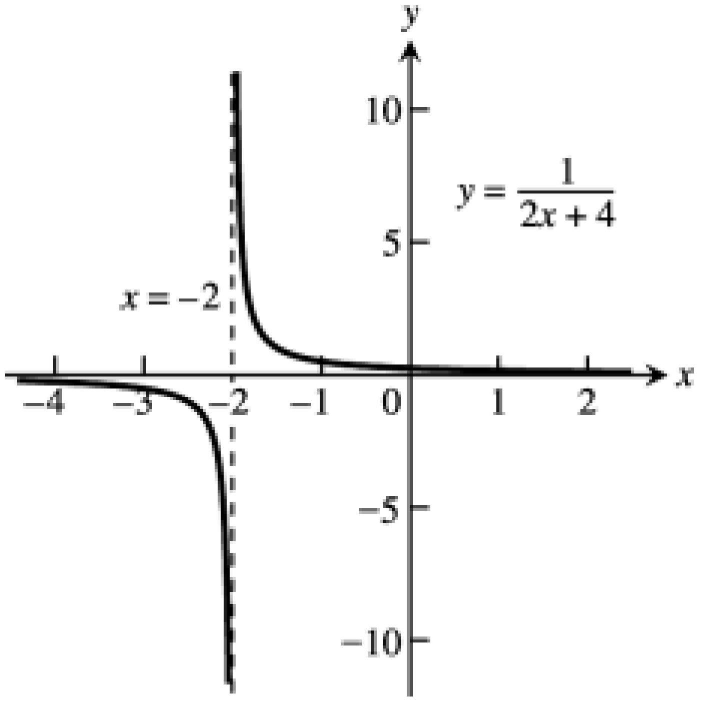
$y = \frac{1}{x+1}$



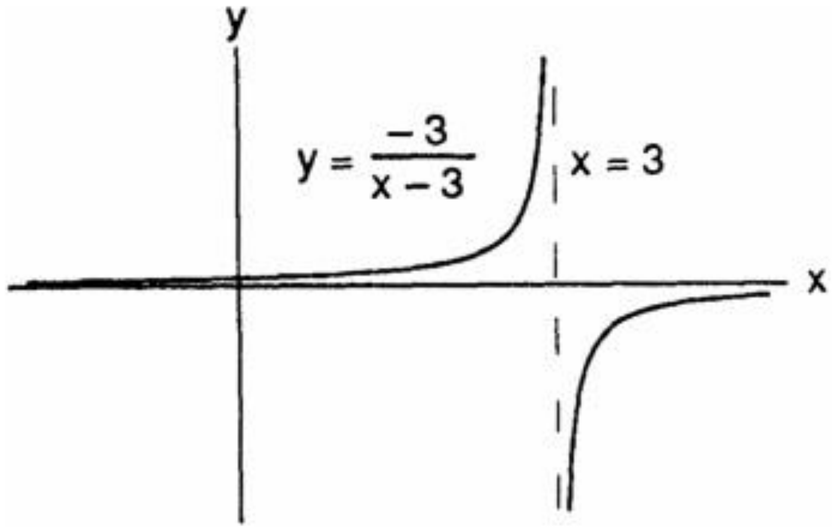
x=-1 is vertical asy. since y appro. infty as x appro. -1 right

Here you need to get y appro to infty or -infty

29. $y = \frac{1}{2x+4}$



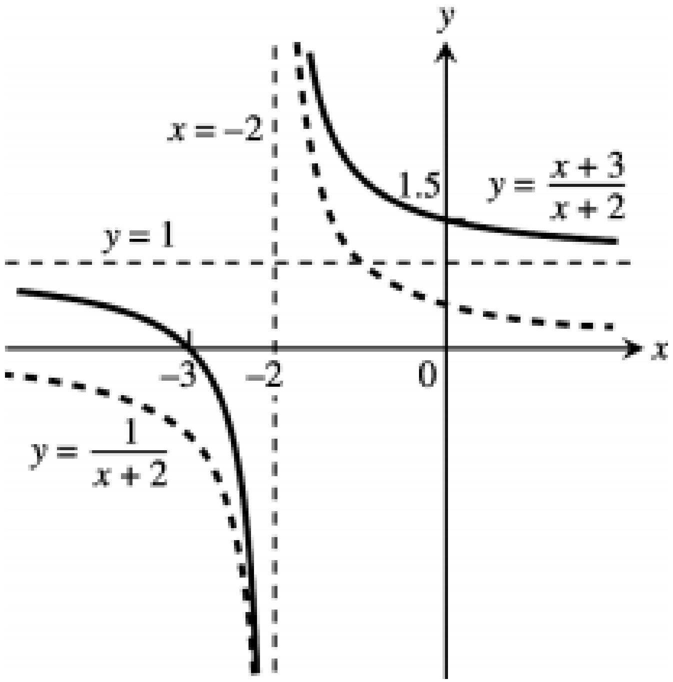
30. $y = \frac{-3}{x-3}$



Vertical Asy: x=3, since
y appro. infty as x appro. 3 from left

Vertical Asy: x=-2, since
y appr. infty as x appro. -2 from right

31. $y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$



Vertical Asy: x=-2, since
y appro. to infty as x appro. to -2 from right

Horizontal Asy: y=1, since
y appro. to 1 as x appro. to
infty