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INTRODUCTION TO
LINEAR ALGEBRA
THIRD EDITION

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PREFACE

This preface expresses some personal thoughts. It is my chance to write about how linear algebra can be taught and learned. If we teach pure abstraction, or settle for cookbook formulas, we miss the best part. This course has come a long way, in living up to what it can be.

It may be helpful to mention the web pages connected to this book. So many messages come back with suggestions and encouragement, and I hope that professors and students will make free use of everything. You can directly access web.mit.edu/18.06/www, which is continually updated for the MIT course that is taught every semester. Linear Algebra is also on the OpenCourseWare site ocw.mit.edu, where 18.06 became exceptional by including videos (which you definitely don't have to watch ...). I can briefly indicate part of what is available now:

1. Lecture schedule and current homeworks and exams with solutions
2. The goals of the course and conceptual questions
3. Interactive Java demos for eigenvalues and least squares and more
4. A table of eigenvalue/eigenvector information (see page 362)
5. **Glossary:** A Dictionary for Linear Algebra
6. Linear Algebra Teaching Codes and MATLAB problems
7. Videos of the full course (taught in a real classroom).

These web pages are a resource for professors and students worldwide. My goal is to make this book as useful as possible, with all the course material I can provide.

After this preface, the book will speak for itself. You will see the spirit right away. The goal is to show the beauty of linear algebra, and its value. The emphasis is on understanding—*I try to explain rather than to deduce*. This is a book about real mathematics, not endless drill. I am constantly working with examples (create a matrix, find its nullspace, add another column, see what changes, **ask for help!**). The textbook has to help too, in teaching what students need. The effort is absolutely rewarding, and fortunately this subject is *not too hard*.

The New Edition

A major addition to the book is the large number of Worked Examples, section by section. Their purpose is to connect the text directly to the homework problems. The complete solution to a vector equation $Ax = b$ is $x_{\text{particular}} + x_{\text{nullspace}}$ —and the steps

are explained as clearly as I can. The Worked Example 3.4A converts this explanation into action by taking every step in the solution (starting with the test for solvability). I hope these model examples will bring the content of each section into focus (see 5.1A and 5.2B on determinants). The “Pascal matrices” are a neat link from the amazing properties of Pascal’s triangle to linear algebra.

The book contains new problems of all kinds—more basic practice, applications throughout science and engineering and management, and just fun with matrices. Northwest and southeast matrices wander into Problem 2.4.39. *Google* appears in Chapter 6. Please look at the last exercise in Section 1.1. I hope the problems are a strong point of this book—the newest one is about the six 3 by 3 permutation matrices: What are their determinants and pivots and traces and eigenvalues?

The Glossary is also new, in the book and on the web. I believe students will find it helpful. In addition to defining the important terms of linear algebra, there was also a chance to include many of the key facts for quick reference.

Fortunately, the need for linear algebra is widely recognized. *This subject is absolutely as important as calculus.* I don’t concede anything, when I look at how mathematics is used. There is even a light-hearted essay called “*Too Much Calculus*” on the web page. The century of data has begun! So many applications are discrete rather than continuous, digital rather than analog. The truth is that vectors and matrices have become the language to know.

The Linear Algebra Course

The equation $Ax = b$ uses that language right away. The matrix A times any vector x is a *combination of the columns of A*. The equation is asking for a *combination that produces b*. Our solution comes at three levels and they are all important:

1. ***Direct solution*** by forward elimination and back substitution.
2. ***Matrix solution*** $x = A^{-1}b$ by inverting the matrix.
3. ***Vector space solution*** by looking at the column space and nullspace of A .

And there is another possibility: $Ax = b$ may have no solution. Elimination may lead to $0 = 1$. The matrix approach may fail to find A^{-1} . The vector space approach can look at all combinations Ax of the columns, but b might be outside that column space. Part of mathematics is understanding when $Ax = b$ is solvable, and what to do when it is not (the least squares solution uses $A^T A$ in Chapter 4).

Another part is learning to visualize vectors. A vector v with two components is not hard. Its components v_1 and v_2 tell how far to go across and up—we draw an arrow. A second vector w may be perpendicular to v (and Chapter 1 tells when). If those vectors have six components, we can’t draw them but our imagination keeps trying. In six-dimensional space, we can test quickly for a right angle. It is easy to visualize $2v$ (twice as far) and $-w$ (opposite to w). We can almost see a combination like $2v - w$.

Most important is the effort to imagine *all the combinations* $cv + dw$. They fill a “two-dimensional plane” inside the six-dimensional space. As I write these words, I am not at all sure that I can see this subspace. But linear algebra works easily with vectors and matrices of any size. If we have currents on six edges, or prices for six products, or just position and velocity of an airplane, we are dealing with six dimensions. For image processing or web searches (or the human genome), six might change to a million. It is still linear algebra, and linear combinations still hold the key.

Structure of the Textbook

Already in this preface, you can see the style of the book and its goal. The style is informal but the goal is absolutely serious. Linear algebra is great mathematics, and I certainly hope that each professor who teaches this course will learn something new. The author always does. The student will notice how the applications reinforce the ideas. I hope you will see how this book moves forward, *gradually and steadily*.

I want to note six points about the organization of the book:

1. Chapter 1 provides a brief introduction to vectors and dot products. If the class has met them before, the course can begin with Chapter 2. That chapter solves n by n systems $Ax = b$, and prepares for the whole course.
2. I now use the *reduced row echelon form* more than before. The MATLAB command `rref(A)` produces bases for the row space and column space. Better than that, reducing the combined matrix $[A \ I]$ produces total information about all four of the fundamental subspaces.
3. Those four subspaces are an excellent way to learn about linear independence and bases and dimension. They go to the heart of the matrix, and they are genuinely the key to applications. I hate just making up vector spaces when so many important ones come naturally. If the class sees plenty of examples, independence is virtually understood in advance: A has independent columns when $x = 0$ is the only solution to $Ax = 0$.
4. Section 6.1 introduces eigenvalues for **2 by 2 matrices**. Many courses want to see eigenvalues early. It is absolutely possible to go directly from Chapter 3 to Section 6.1. The determinant is easy for a 2 by 2 matrix, and `eigshow` on the web captures graphically the moment when $Ax = \lambda x$.
5. Every section in Chapters 1 to 7 ends with a highlighted ***Review of the Key Ideas***. The reader can recapture the main points by going carefully through this review.
6. Chapter 8 (*Applications*) has a new section on ***Matrices in Engineering***.

When software is available (and time to use it), I see two possible approaches. One is to carry out instantly the steps of testing linear independence, orthogonalizing by Gram-Schmidt, and solving $Ax = b$ and $Ax = \lambda x$. The Teaching Codes follow the steps described in class—MATLAB and Maple and Mathematica compute a little differently. All can be used (*optionally*) with this book. The other approach is to experiment on bigger problems—like finding the largest determinant of a ± 1 matrix, or

the average size of a pivot. The time to compute $A^{-1}b$ is measured by tic; inv(A)*b; toc. Choose A = rand(1000) and compare with tic; A/b; toc by direct elimination.

A one-semester course that moves steadily will reach eigenvalues. The key idea is to diagonalize A by its eigenvector matrix S. When that succeeds, the eigenvalues appear in $S^{-1}AS$. For symmetric matrices we can choose $S^{-1} = S^T$. When A is rectangular we need U^TAV (U comes from eigenvectors of AA^T and V from A^TA). Chapters 1 to 6 are the heart of a basic course in linear algebra—*theory plus applications*. The beauty of this subject is in the way those come together.

May I end with this thought for professors. You might feel that the direction is right, and wonder if your students are ready. ***Just give them a chance!*** Literally thousands of students have written to me, frequently with suggestions and surprisingly often with thanks. They know when the course has a purpose, because the professor and the book are on their side. Linear algebra is a fantastic subject, enjoy it.

Acknowledgements

This book owes a big debt to readers everywhere. Thousands of students and colleagues have been involved in every step. I have not forgotten the warm welcome for the first sentence written 30 years ago: “I believe that the teaching of linear algebra has become too abstract.” A less formal approach is now widely accepted as the right choice for the basic course. And this course has steadily improved—the homework problems, the lectures, the Worked Examples, even the Web. I really hope you see that linear algebra is not some optional elective, it is *needed*. The first step in all subjects is linear!

I owe a particular debt to friends who offered suggestions and corrections and ideas. David Arnold in California and Mike Kerckhove in Virginia teach this course well. Per-Olof Persson created MATLAB codes for the experiments, as Cleve Moler and Steven Lee did earlier for the Teaching Codes. And the Pascal matrix examples, in the book and on the Web, owe a lot to Alan Edelman (and a little to Pascal). It is just a pleasure to work with friends.

My deepest thanks of all go to Cordula Robinson and Brett Coonley. They created the L^AT_EX pages that you see. Day after day, new words and examples have gone back and forth across the hall. After 2000 problems (and 3000 attempted solutions) this expression of my gratitude to them is almost the last sentence, of work they have beautifully done.

Amy Hendrickson of **texnology.com** produced the book itself, and you will recognize the quality of her ideas. My favorites are the clear boxes that highlight key points. The quilt on the front cover was created by Chris Curtis (it appears in *Great American Quilts: Book 5*, by Oxmoor House). Those houses show nine linear transformations of the plane. (At least they are linear in Figure 7.1, possibly superlinear in the quilt.) Tracy Baldwin has succeeded again to combine art and color and mathematics, in her fourth neat cover for Wellesley-Cambridge Press.

May I dedicate this book to grandchildren who are very precious: Roger, Sophie, Kathryn, Alexander, Scott, Jack, William, Caroline, and Elizabeth. I hope you might take linear algebra one day. Especially I hope you like it. The author is proud of you.

1

INTRODUCTION TO VECTORS

The heart of linear algebra is in two operations—both with vectors. We add vectors to get $\mathbf{v} + \mathbf{w}$. We multiply by numbers c and d to get $c\mathbf{v}$ and $d\mathbf{w}$. Combining those two operations (adding $c\mathbf{v}$ to $d\mathbf{w}$) gives the *linear combination* $c\mathbf{v} + d\mathbf{w}$.

Linear combinations are all-important in this subject! Sometimes we want one particular combination, a specific choice of c and d that produces a desired $c\mathbf{v} + d\mathbf{w}$. Other times we want to visualize *all the combinations* (coming from all c and d). The vectors $c\mathbf{v}$ lie along a line. The combinations $c\mathbf{v} + d\mathbf{w}$ normally fill a two-dimensional plane. (I have to say “two-dimensional” because linear algebra allows higher-dimensional planes.) From four vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}$ in four-dimensional space, their combinations are likely to fill the whole space.

Chapter 1 explains these central ideas, on which everything builds. We start with two-dimensional vectors and three-dimensional vectors, which are reasonable to draw. Then we move into higher dimensions. The really impressive feature of linear algebra is how smoothly it takes that step into n -dimensional space. Your mental picture stays completely correct, even if drawing a ten-dimensional vector is impossible.

This is where the book is going (into n -dimensional space), and the first steps are the operations in Sections 1.1 and 1.2:

1.1 Vector addition $\mathbf{v} + \mathbf{w}$ and linear combinations $c\mathbf{v} + d\mathbf{w}$.

1.2 The dot product $\mathbf{v} \cdot \mathbf{w}$ and the length $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.

VECTORS AND LINEAR COMBINATIONS ■ 1.1

“You can’t add apples and oranges.” In a strange way, this is the reason for vectors! If we keep the number of apples separate from the number of oranges, we have a pair of numbers. That pair is a *two-dimensional vector* \mathbf{v} , with “components” v_1 and v_2 :

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \begin{aligned} v_1 &= \text{number of apples} \\ v_2 &= \text{number of oranges.} \end{aligned}$$

We write \mathbf{v} as a ***column vector***. The main point so far is to have a single letter \mathbf{v} (in ***boldface italic***) for this pair of numbers v_1 and v_2 (in ***lightface italic***).

Even if we don't add v_1 to v_2 , we do *add vectors*. The first components of \mathbf{v} and \mathbf{w} stay separate from the second components:

$$\text{VECTOR ADDITION} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \text{add to} \quad \mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}.$$

You see the reason. We want to add apples to apples. Subtraction of vectors follows the same idea: *The components of $\mathbf{v} - \mathbf{w}$ are $v_1 - w_1$ and _____.*

The other basic operation is *scalar multiplication*. Vectors can be multiplied by 2 or by -1 or by any number c . There are two ways to double a vector. One way is to add $\mathbf{v} + \mathbf{v}$. The other way (the usual way) is to multiply each component by 2:

$$\text{SCALAR MULTIPLICATION} \quad 2\mathbf{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} \quad \text{and} \quad -\mathbf{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}.$$

The components of $c\mathbf{v}$ are cv_1 and cv_2 . The number c is called a “scalar”.

Notice that the sum of $-\mathbf{v}$ and \mathbf{v} is the zero vector. This is **0**, which is not the same as the number zero! The vector **0** has components 0 and 0. Forgive me for hammering away at the difference between a vector and its components. Linear algebra is built on these operations $\mathbf{v} + \mathbf{w}$ and $c\mathbf{v}$ —**adding vectors and multiplying by scalars**.

The order of addition makes no difference: $\mathbf{v} + \mathbf{w}$ equals $\mathbf{w} + \mathbf{v}$. Check that by algebra: The first component is $v_1 + w_1$ which equals $w_1 + v_1$. Check also by an example:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \mathbf{w} + \mathbf{v}.$$

Linear Combinations

By combining these operations, we now form “*linear combinations*” of \mathbf{v} and \mathbf{w} . Multiply \mathbf{v} by c and multiply \mathbf{w} by d ; then add $c\mathbf{v} + d\mathbf{w}$.

DEFINITION *The sum of $c\mathbf{v}$ and $d\mathbf{w}$ is a linear combination of \mathbf{v} and \mathbf{w} .*

Four special linear combinations are: sum, difference, zero, and a scalar multiple $c\mathbf{v}$:

- $1\mathbf{v} + 1\mathbf{w}$ = sum of vectors in Figure 1.1
- $1\mathbf{v} - 1\mathbf{w}$ = difference of vectors in Figure 1.1
- $0\mathbf{v} + 0\mathbf{w}$ = zero vector
- $c\mathbf{v} + 0\mathbf{w}$ = vector $c\mathbf{v}$ in the direction of \mathbf{v}

The zero vector is always a possible combination (when the coefficients are zero). Every time we see a “space” of vectors, that zero vector will be included. It is this big view, taking all the combinations of \mathbf{v} and \mathbf{w} , that makes the subject work.

The figures show how you can visualize vectors. For algebra, we just need the components (like 4 and 2). In the plane, that vector v is represented by an arrow. The arrow goes $v_1 = 4$ units to the right and $v_2 = 2$ units up. It ends at the point whose x, y coordinates are 4, 2. This point is another representation of the vector—so we have three ways to describe v , by an *arrow* or a *point* or a *pair of numbers*.

Using arrows, you can see how to visualize the sum $v + w$:

Vector addition (head to tail) *At the end of v , place the start of w .*

We travel along v and then along w . Or we take the shortcut along $v + w$. We could also go along w and then v . In other words, $w + v$ gives the same answer as $v + w$. These are different ways along the parallelogram (in this example it is a rectangle). The endpoint in Figure 1.1 is the diagonal $v + w$ which is also $w + v$.

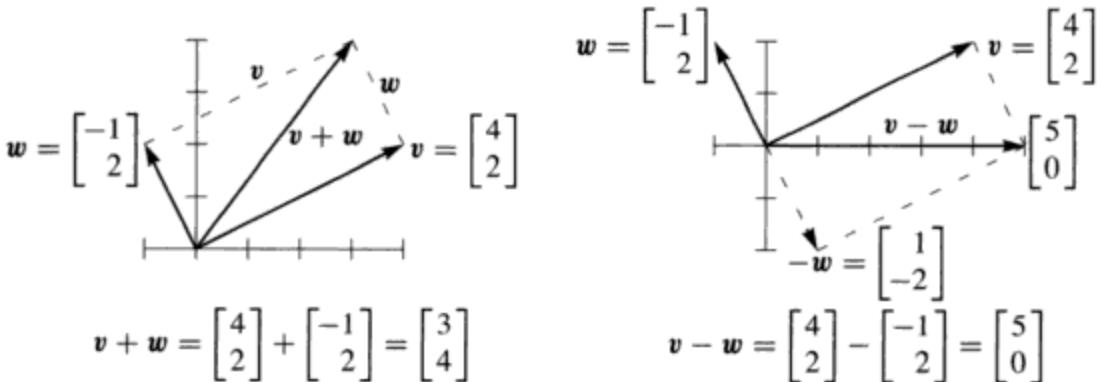


Figure 1.1 Vector addition $v + w$ produces the diagonal of a parallelogram. The linear combination on the right is $v - w$.

The zero vector has $v_1 = 0$ and $v_2 = 0$. It is too short to draw a decent arrow, but you know that $v + \mathbf{0} = v$. For $2v$ we double the length of the arrow. We reverse its direction for $-v$. This reversing gives the subtraction on the right side of Figure 1.1.

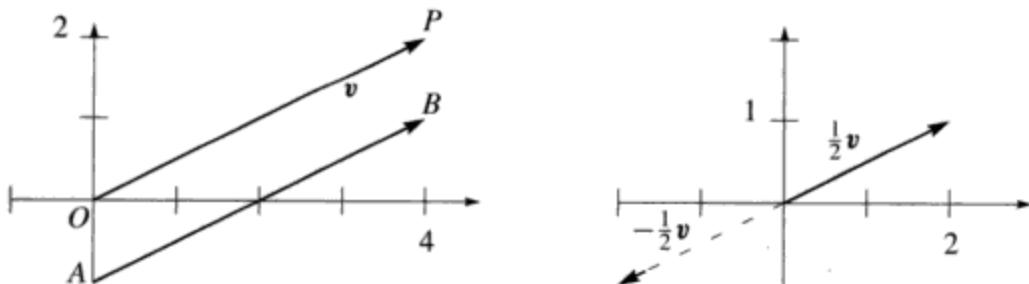


Figure 1.2 The arrow usually starts at the origin $(0, 0)$; $c\mathbf{v}$ is always parallel to \mathbf{v} .

Vectors in Three Dimensions

A vector with two components corresponds to a point in the xy plane. The components of \mathbf{v} are the coordinates of the point: $x = v_1$ and $y = v_2$. The arrow ends at this point (v_1, v_2) , when it starts from $(0, 0)$. Now we allow vectors to have three components (v_1, v_2, v_3) . The xy plane is replaced by three-dimensional space.

Here are typical vectors (still column vectors but with three components):

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v} + \mathbf{w} = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}.$$

The vector \mathbf{v} corresponds to an arrow in 3-space. Usually the arrow starts at the origin, where the xyz axes meet and the coordinates are $(0, 0, 0)$. The arrow ends at the point with coordinates v_1, v_2, v_3 . There is a perfect match between the **column vector** and the **arrow from the origin** and the **point where the arrow ends**.

From now on $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ *is also written as* $\mathbf{v} = (1, 2, 2)$.

The reason for the row form (in parentheses) is to save space. But $\mathbf{v} = (1, 2, 2)$ is not a row vector! It is in actuality a column vector, just temporarily lying down. The row vector $[1 \ 2 \ 2]$ is absolutely different, even though it has the same three components. It is the “transpose” of the column \mathbf{v} .

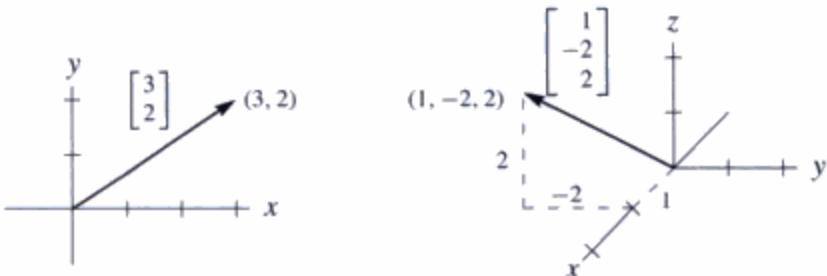


Figure 1.3 Vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ correspond to points (x, y) and (x, y, z) .

In three dimensions, $\mathbf{v} + \mathbf{w}$ is still done a component at a time. The sum has components $v_1 + w_1$ and $v_2 + w_2$ and $v_3 + w_3$. You see how to add vectors in 4 or 5 or n dimensions. When \mathbf{w} starts at the end of \mathbf{v} , the third side is $\mathbf{v} + \mathbf{w}$. The other way around the parallelogram is $\mathbf{w} + \mathbf{v}$. Question: Do the four sides all lie in the same plane? Yes. And the sum $\mathbf{v} + \mathbf{w} - \mathbf{v} - \mathbf{w}$ goes completely around to produce ____.

A typical linear combination of three vectors in three dimensions is $\mathbf{u} + 4\mathbf{v} - 2\mathbf{w}$:

Linear combination $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix}.$

The Important Questions

For one vector \mathbf{u} , the only linear combinations are the multiples $c\mathbf{u}$. For two vectors, the combinations are $c\mathbf{u} + d\mathbf{v}$. For three vectors, the combinations are $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$. Will you take the big step from *one* linear combination to *all* linear combinations? Every c and d and e are allowed. Suppose the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in three-dimensional space:

- 1 What is the picture of all combinations $c\mathbf{u}$?
- 2 What is the picture of all combinations $c\mathbf{u} + d\mathbf{v}$?
- 3 What is the picture of all combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$?

The answers depend on the particular vectors \mathbf{u}, \mathbf{v} , and \mathbf{w} . If they were all zero vectors (a very extreme case), then every combination would be zero. If they are typical nonzero vectors (components chosen at random), here are the three answers. This is the key to our subject:

- 1 The combinations $c\mathbf{u}$ fill a *line*.
- 2 The combinations $c\mathbf{u} + d\mathbf{v}$ fill a *plane*.
- 3 The combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fill *three-dimensional space*.

The line is infinitely long, in the direction of \mathbf{u} (forward and backward, going through the zero vector). It is the plane of all $c\mathbf{u} + d\mathbf{v}$ (combining two lines) that I especially ask you to think about.

Adding all $c\mathbf{u}$ on one line to all $d\mathbf{v}$ on the other line fills in the plane in Figure 1.4.

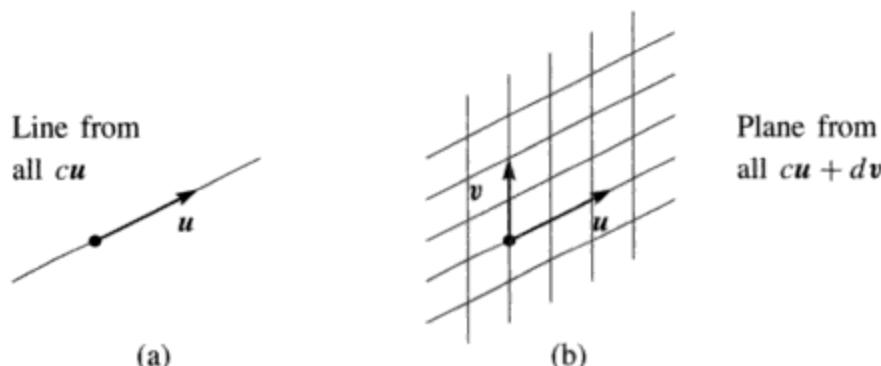


Figure 1.4 (a) The line through \mathbf{u} . (b) The plane containing the lines through \mathbf{u} and \mathbf{v} .

When we include a third vector \mathbf{w} , the multiples $e\mathbf{w}$ give a third line. Suppose that line is not in the plane of \mathbf{u} and \mathbf{v} . Then combining all $e\mathbf{w}$ with all $c\mathbf{u} + d\mathbf{v}$ fills up the whole three-dimensional space.

This is the typical situation! Line, then plane, then space. But other possibilities exist. When w happens to be $cu + dv$, the third vector is in the plane of the first two. The combinations of u, v, w will not go outside that uv plane. We do not get the full three-dimensional space. Please think about the special cases in Problem 1.

■ REVIEW OF THE KEY IDEAS ■

1. A vector v in two-dimensional space has two components v_1 and v_2 .
2. $v + w = (v_1 + w_1, v_2 + w_2)$ and $c v = (cv_1, cv_2)$ are executed a component at a time.
3. A linear combination of u and v and w is $cu + dv + ew$.
4. Take all linear combinations of u , or u and v , or u and v and w . In three dimensions, those combinations typically fill a line, a plane, and the whole space.

■ WORKED EXAMPLES ■

- 1.1 A** Describe all the linear combinations of $v = (1, 1, 0)$ and $w = (0, 1, 1)$. Find a vector that is *not* a combination of v and w .

Solution These are vectors in three-dimensional space \mathbf{R}^3 . Their combinations $c v + d w$ fill a *plane* in \mathbf{R}^3 . The vectors in that plane allow any c and d :

$$c v + d w = c \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ c+d \\ d \end{bmatrix}.$$

Four particular vectors in that plane are $(0, 0, 0)$ and $(2, 3, 1)$ and $(5, 7, 2)$ and $(\sqrt{2}, 0, -\sqrt{2})$. The second component is always the sum of the first and third components. The vector $(1, 1, 1)$ is *not* in the plane.

Another description of this plane through $(0, 0, 0)$ is to know a vector *perpendicular* to the plane. In this case $n = (1, -1, 1)$ is perpendicular, as Section 1.2 will confirm by testing dot products: $v \cdot n = 0$ and $w \cdot n = 0$.

- 1.1 B** For $v = (1, 0)$ and $w = (0, 1)$, describe all the points $c v$ and all the combinations $c v + d w$ with any d and (1) *whole numbers* c (2) *nonnegative* $c \geq 0$.

Solution

- (1) The vectors $c\mathbf{v} = (c, 0)$ with whole numbers c are equally spaced points along the x axis (the direction of \mathbf{v}). They include $(-2, 0), (-1, 0), (0, 0), (1, 0), (2, 0)$. Adding all vectors $d\mathbf{w} = (0, d)$ puts a full line in the y direction through those points. We have infinitely many *parallel lines* from $c\mathbf{v} + d\mathbf{w} = (\text{whole number, any number})$. These are vertical lines in the xy plane, through equally spaced points on the x axis.
- (2) The vectors $c\mathbf{v}$ with $c \geq 0$ fill a “half-line”. It is the *positive* x axis, starting at $(0, 0)$ where $c = 0$. It includes $(\pi, 0)$ but not $(-\pi, 0)$. Adding all vectors $d\mathbf{w}$ puts a full line in the y direction crossing every point on that half-line. Now we have a *half-plane*. It is the right half of the xy plane, where $x \geq 0$.

Problem Set 1.1

Problems 1–9 are about addition of vectors and linear combinations.

- 1 Describe geometrically (as a line, plane, . . .) all linear combinations of

$$(a) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \quad (b) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad (c) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- 2 Draw the vectors $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ in a single xy plane.
- 3 If $\mathbf{v} + \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{v} - \mathbf{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, compute and draw \mathbf{v} and \mathbf{w} .
- 4 From $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, find the components of $3\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - 3\mathbf{w}$ and $c\mathbf{v} + d\mathbf{w}$.
- 5 Compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} + \mathbf{v} + \mathbf{w}$ and $2\mathbf{u} + 2\mathbf{v} + \mathbf{w}$ when

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}.$$

- 6 Every combination of $\mathbf{v} = (1, -2, 1)$ and $\mathbf{w} = (0, 1, -1)$ has components that add to _____. Find c and d so that $c\mathbf{v} + d\mathbf{w} = (4, 2, -6)$.
- 7 In the xy plane mark all nine of these linear combinations:

$$c \begin{bmatrix} 3 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{with } c = 0, 1, 2 \quad \text{and } d = 0, 1, 2.$$

8 Chapter 1 Introduction to Vectors

- 8** The parallelogram in Figure 1.1 has diagonal $\mathbf{v} + \mathbf{w}$. What is its other diagonal? What is the sum of the two diagonals? Draw that vector sum.
- 9** If three corners of a parallelogram are $(1, 1)$, $(4, 2)$, and $(1, 3)$, what are all the possible fourth corners? Draw two of them.

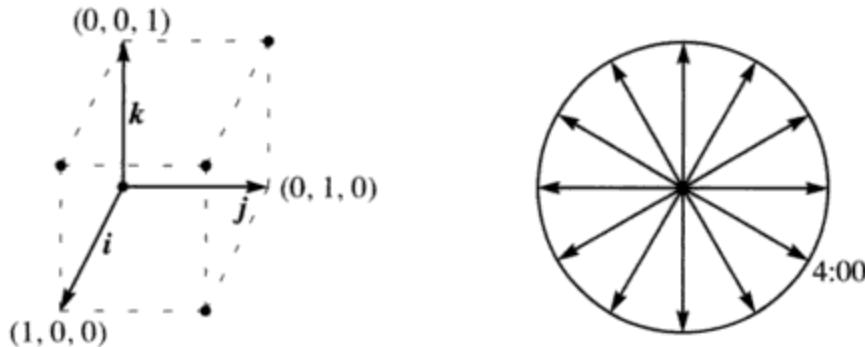


Figure 1.5 Unit cube from i, j, k ; twelve clock vectors.

Problems 10–14 are about special vectors on cubes and clocks.

- 10** Copy the cube and draw the vector sum of $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. The addition $\mathbf{i} + \mathbf{j}$ yields the diagonal of _____.
11 Four corners of the cube are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. What are the other four corners? Find the coordinates of the center point of the cube. The center points of the six faces are _____.
12 How many corners does a cube have in 4 dimensions? How many faces? How many edges? A typical corner is $(0, 0, 1, 0)$.
13 (a) What is the sum \mathbf{V} of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, ..., 12:00?
(b) If the vector to 4:00 is removed, find the sum of the eleven remaining vectors.
(c) What is the unit vector to 1:00?
14 Suppose the twelve vectors start from 6:00 at the bottom instead of $(0, 0)$ at the center. The vector to 12:00 is doubled to $2\mathbf{j} = (0, 2)$. Add the new twelve vectors.

Problems 15–19 go further with linear combinations of \mathbf{v} and \mathbf{w} (Figure 1.6)

- 15** The figure shows $\frac{1}{2}\mathbf{v} + \frac{1}{2}\mathbf{w}$. Mark the points $\frac{3}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$ and $\frac{1}{4}\mathbf{v} + \frac{1}{4}\mathbf{w}$ and $\mathbf{v} + \mathbf{w}$.
16 Mark the point $-\mathbf{v} + 2\mathbf{w}$ and any other combination $c\mathbf{v} + d\mathbf{w}$ with $c + d = 1$. Draw the line of all combinations that have $c + d = 1$.

- 17 Locate $\frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ and $\frac{2}{3}\mathbf{v} + \frac{2}{3}\mathbf{w}$. The combinations $c\mathbf{v} + c\mathbf{w}$ fill out what line? Restricted by $c \geq 0$ those combinations with $c = d$ fill out what half line?

- 18 Restricted by $0 \leq c \leq 1$ and $0 \leq d \leq 1$, shade in all combinations $c\mathbf{v} + d\mathbf{w}$.

- 19 Restricted only by $c \geq 0$ and $d \geq 0$ draw the “cone” of all combinations $c\mathbf{v} + d\mathbf{w}$.

Problems 20–27 deal with $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in three-dimensional space (see Figure 1.6).

- 20 Locate $\frac{1}{3}\mathbf{u} + \frac{1}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ and $\frac{1}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$ in the dashed triangle. Challenge problem: Under what restrictions on c, d, e , will the combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fill in the dashed triangle?

- 21 The three sides of the dashed triangle are $\mathbf{v} - \mathbf{u}$ and $\mathbf{w} - \mathbf{v}$ and $\mathbf{u} - \mathbf{w}$. Their sum is _____. Draw the head-to-tail addition around a plane triangle of $(3, 1)$ plus $(-1, 1)$ plus $(-2, -2)$.

- 22 Shade in the pyramid of combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ with $c \geq 0, d \geq 0, e \geq 0$ and $c + d + e \leq 1$. Mark the vector $\frac{1}{2}(\mathbf{u} + \mathbf{v} + \mathbf{w})$ as inside or outside this pyramid.

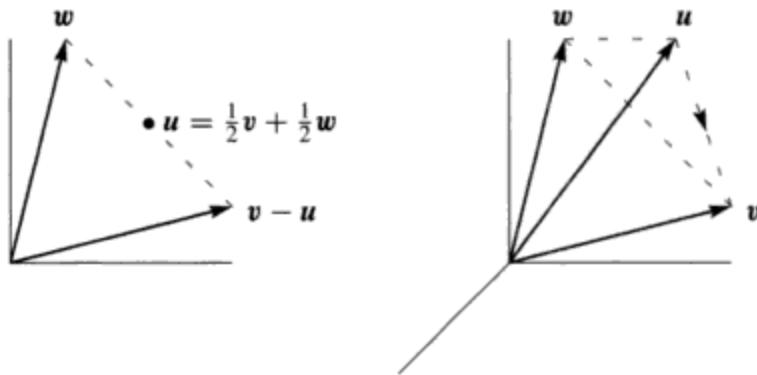


Figure 1.6 Problems 15–19 in a plane Problems 20–27 in 3-dimensional space

- 23 If you look at *all* combinations of those \mathbf{u}, \mathbf{v} , and \mathbf{w} , is there any vector that can't be produced from $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$?
- 24 Which vectors are combinations of \mathbf{u} and \mathbf{v} , and *also* combinations of \mathbf{v} and \mathbf{w} ?
- 25 Draw vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ so that their combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fill only a line. Draw vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ so that their combinations $c\mathbf{u} + d\mathbf{v} + e\mathbf{w}$ fill only a plane.
- 26 What combination of the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ produces $\begin{bmatrix} 14 \\ 8 \end{bmatrix}$? Express this question as two equations for the coefficients c and d in the linear combination.
- 27 *Review Question.* In xyz space, where is the plane of all linear combinations of $\mathbf{i} = (1, 0, 0)$ and $\mathbf{j} = (0, 1, 0)$?

- 28** If (a, b) is a multiple of (c, d) with $abcd \neq 0$, show that (a, c) is a multiple of (b, d) . This is surprisingly important; call it a challenge question. You could use numbers first to see how a, b, c, d are related. The question will lead to:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has dependent rows then it has dependent columns.

And eventually: If $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. That looks so simple...

LENGTHS AND DOT PRODUCTS ■ 1.2

The first section mentioned multiplication of vectors, but it backed off. Now we go forward to define the “dot product” of v and w . This multiplication involves the separate products v_1w_1 and v_2w_2 , but it doesn’t stop there. Those two numbers are added to produce the single number $v \cdot w$.

DEFINITION The *dot product* or *inner product* of $v = (v_1, v_2)$ and $w = (w_1, w_2)$ is the number

$$v \cdot w = v_1w_1 + v_2w_2. \quad (1)$$

Example 1 The vectors $v = (4, 2)$ and $w = (-1, 2)$ have a *zero* dot product:

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -4 + 4 = 0.$$

In mathematics, zero is always a special number. For dot products, it means that *these two vectors are perpendicular*. The angle between them is 90° . When we drew them in Figure 1.1, we saw a rectangle (not just any parallelogram). The clearest example of perpendicular vectors is $i = (1, 0)$ along the x axis and $j = (0, 1)$ up the y axis. Again the dot product is $i \cdot j = 0 + 0 = 0$. Those vectors i and j form a right angle.

The dot product of $v = (1, 2)$ and $w = (2, 1)$ is 4. Please check this. Soon that will reveal the angle between v and w (not 90°).

Example 2 Put a weight of 4 at the point $x = -1$ and a weight of 2 at the point $x = 2$. The x axis will balance on the center point $x = 0$ (like a see-saw). The weights balance because the dot product is $(4)(-1) + (2)(2) = 0$.

This example is typical of engineering and science. The vector of weights is $(w_1, w_2) = (4, 2)$. The vector of distances from the center is $(v_1, v_2) = (-1, 2)$. The weights times the distances, w_1v_1 and w_2v_2 , give the “moments”. The equation for the see-saw to balance is $w_1v_1 + w_2v_2 = 0$.

The dot product $w \cdot v$ equals $v \cdot w$. The order of v and w makes no difference.

Example 3 Dot products enter in economics and business. We have three products to buy and sell. Their prices are (p_1, p_2, p_3) for each unit—this is the “price vector” p .

The quantities we buy or sell are (q_1, q_2, q_3) —positive when we sell, negative when we buy. Selling q_1 units of the first product at the price p_1 brings in $q_1 p_1$. The total income is the dot product $\mathbf{q} \cdot \mathbf{p}$:

$$\text{Income} = (q_1, q_2, q_3) \cdot (p_1, p_2, p_3) = q_1 p_1 + q_2 p_2 + q_3 p_3.$$

A zero dot product means that “the books balance.” Total sales equal total purchases if $\mathbf{q} \cdot \mathbf{p} = 0$. Then \mathbf{p} is perpendicular to \mathbf{q} (in three-dimensional space). With three products, **the vectors are three-dimensional**. A supermarket goes quickly into high dimensions.

Small note: Spreadsheets have become essential in management. They compute linear combinations and dot products. What you see on the screen is a matrix.

Main point To compute the dot product $\mathbf{v} \cdot \mathbf{w}$, multiply each v_i times w_i . Then add.

Lengths and Unit Vectors

An important case is the dot product of a vector *with itself*. In this case $\mathbf{v} = \mathbf{w}$. When the vector is $\mathbf{v} = (1, 2, 3)$, the dot product with itself is $\mathbf{v} \cdot \mathbf{v} = 14$:

$$\mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 + 4 + 9 = 14.$$

The answer is not zero because \mathbf{v} is not perpendicular to itself. Instead of a 90° angle between vectors we have 0° . The dot product $\mathbf{v} \cdot \mathbf{v}$ gives the *length of \mathbf{v} squared*.

DEFINITION The *length* (or *norm*) of a vector \mathbf{v} is the square root of $\mathbf{v} \cdot \mathbf{v}$:

$$\text{length} = \|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

In two dimensions the length is $\sqrt{v_1^2 + v_2^2}$. In three dimensions it is $\sqrt{v_1^2 + v_2^2 + v_3^2}$. By the calculation above, the length of $\mathbf{v} = (1, 2, 3)$ is $\|\mathbf{v}\| = \sqrt{14}$.

We can explain this definition. $\|\mathbf{v}\|$ is just the ordinary length of the arrow that represents the vector. In two dimensions, the arrow is in a plane. If the components are 1 and 2, the arrow is the third side of a right triangle (Figure 1.7). The formula $a^2 + b^2 = c^2$, which connects the three sides, is $1^2 + 2^2 = \|\mathbf{v}\|^2$.

For the length of $\mathbf{v} = (1, 2, 3)$, we used the right triangle formula twice. The vector $(1, 2, 0)$ in the base has length $\sqrt{5}$. This base vector is perpendicular to $(0, 0, 3)$ that goes straight up. So the diagonal of the box has length $\|\mathbf{v}\| = \sqrt{5 + 9} = \sqrt{14}$.

The length of a four-dimensional vector would be $\sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2}$. Thus $(1, 1, 1, 1)$ has length $\sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2$. This is the diagonal through a unit cube in four-dimensional space. The diagonal in n dimensions has length \sqrt{n} .

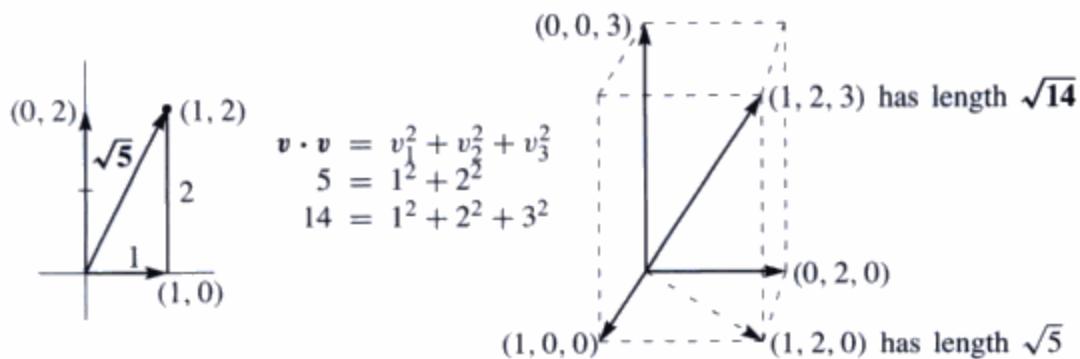


Figure 1.7 The length $\sqrt{v \cdot v}$ of two-dimensional and three-dimensional vectors.

The word “unit” is always indicating that some measurement equals “one.” The unit price is the price for one item. A unit cube has sides of length one. A unit circle is a circle with radius one. Now we define the idea of a “unit vector.”

DEFINITION A *unit vector* u is a vector whose length equals one. Then $u \cdot u = 1$.

An example in four dimensions is $u = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Then $u \cdot u$ is $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. We divided $v = (1, 1, 1, 1)$ by its length $\|v\| = 2$ to get this unit vector.

Example 4 The standard unit vectors along the x and y axes are written i and j . In the xy plane, the unit vector that makes an angle “theta” with the x axis is $(\cos \theta, \sin \theta)$:

$$\text{Unit vectors } i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}.$$

When $\theta = 0$, the horizontal vector u is i . When $\theta = 90^\circ$ (or $\frac{\pi}{2}$ radians), the vertical vector is j . At any angle, the components $\cos \theta$ and $\sin \theta$ produce $u \cdot u = 1$ because $\cos^2 \theta + \sin^2 \theta = 1$. These vectors reach out to the unit circle in Figure 1.8. Thus $\cos \theta$ and $\sin \theta$ are simply the coordinates of that point at angle θ on the unit circle.

In three dimensions, the unit vectors along the axes are i , j , and k . Their components are $(1, 0, 0)$ and $(0, 1, 0)$ and $(0, 0, 1)$. Notice how every three-dimensional vector is a linear combination of i , j , and k . The vector $v = (2, 2, 1)$ is equal to $2i + 2j + k$. Its length is $\sqrt{2^2 + 2^2 + 1^2}$. This is the square root of 9, so $\|v\| = 3$.

Since $(2, 2, 1)$ has length 3, the vector $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ has length 1. Check that $u \cdot u = \frac{4}{9} + \frac{4}{9} + \frac{1}{9} = 1$. To create a unit vector, just divide v by its length $\|v\|$.

1A Unit vectors Divide any nonzero vector v by its length. Then $u = v/\|v\|$ is a unit vector in the same direction as v .

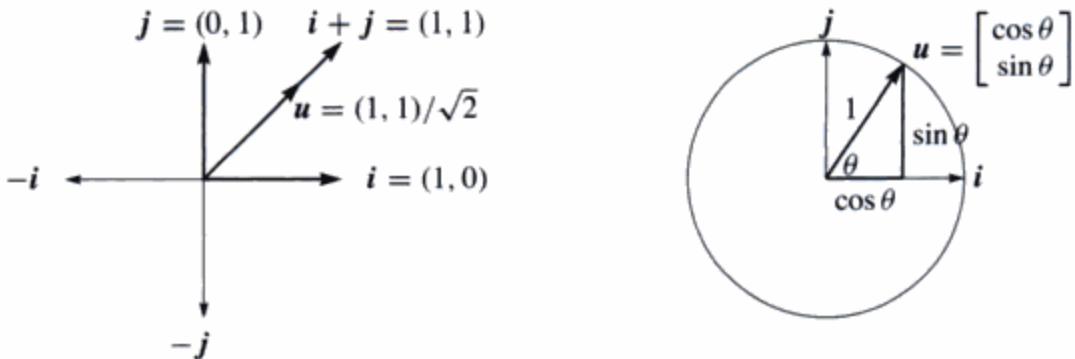


Figure 1.8 The coordinate vectors i and j . The unit vector u at angle 45° (left) and the unit vector $(\cos \theta, \sin \theta)$ at angle θ .

The Angle Between Two Vectors

We stated that perpendicular vectors have $v \cdot w = 0$. The dot product is zero when the angle is 90° . To explain this, we have to connect angles to dot products. Then we show how $v \cdot w$ finds the angle between any two nonzero vectors v and w .

1B Right angles *The dot product is $v \cdot w = 0$ when v is perpendicular to w .*

Proof When v and w are perpendicular, they form two sides of a right triangle. The third side is $v - w$ (the hypotenuse going across in Figure 1.7). The *Pythagoras Law* for the sides of a right triangle is $a^2 + b^2 = c^2$:

$$\text{Perpendicular vectors} \quad \|v\|^2 + \|w\|^2 = \|v - w\|^2 \quad (2)$$

Writing out the formulas for those lengths in two dimensions, this equation is

$$(v_1^2 + v_2^2) + (w_1^2 + w_2^2) = (v_1 - w_1)^2 + (v_2 - w_2)^2. \quad (3)$$

The right side begins with $v_1^2 - 2v_1w_1 + w_1^2$. Then v_1^2 and w_1^2 are on both sides of the equation and they cancel, leaving $-2v_1w_1$. Similarly v_2^2 and w_2^2 cancel, leaving $-2v_2w_2$. (In three dimensions there would also be $-2v_3w_3$.) The last step is to divide by -2 :

$$0 = -2v_1w_1 - 2v_2w_2 \quad \text{which leads to} \quad v_1w_1 + v_2w_2 = 0. \quad (4)$$

Conclusion Right angles produce $v \cdot w = 0$. We have proved **Theorem 1B**. The dot product is zero when the angle is $\theta = 90^\circ$. Then $\cos \theta = 0$. The zero vector $v = \mathbf{0}$ is perpendicular to every vector w because $\mathbf{0} \cdot w$ is always zero.

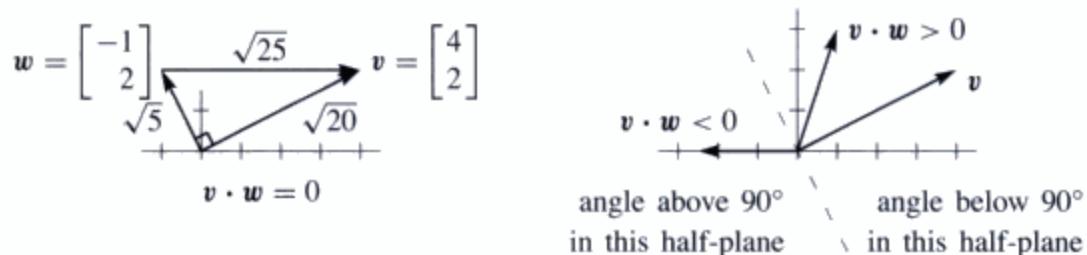


Figure 1.9 Perpendicular vectors have $v \cdot w = 0$. The angle is below 90° when $v \cdot w > 0$.

Now suppose $v \cdot w$ is not zero. It may be positive, it may be negative. The sign of $v \cdot w$ immediately tells whether we are below or above a right angle. The angle is less than 90° when $v \cdot w$ is positive. The angle is above 90° when $v \cdot w$ is negative. Figure 1.9 shows a typical vector $v = (3, 1)$. The angle with $w = (1, 3)$ is less than 90° .

The borderline is where vectors are perpendicular to v . On that dividing line between plus and minus, where we find $w = (1, -3)$, the dot product is zero.

The next page takes one more step, to find the exact angle θ . This is not necessary for linear algebra—you could stop here! Once we have matrices and linear equations, we won't come back to θ . But while we are on the subject of angles, this is the place for the formula.

Start with unit vectors u and U . The sign of $u \cdot U$ tells whether $\theta < 90^\circ$ or $\theta > 90^\circ$. Because the vectors have length 1, we learn more than that. **The dot product $u \cdot U$ is the cosine of θ** . This is true in any number of dimensions.

1C If u and U are unit vectors then $u \cdot U = \cos \theta$. Certainly $|u \cdot U| \leq 1$.

Remember that $\cos \theta$ is never greater than 1. It is never less than -1 . **The dot product of unit vectors is between -1 and 1** .

Figure 1.10 shows this clearly when the vectors are $u = (\cos \theta, \sin \theta)$ and $i = (1, 0)$. The dot product is $u \cdot i = \cos \theta$. That is the cosine of the angle between them.

After rotation through any angle α , these are still unit vectors. Call the vectors $u = (\cos \beta, \sin \beta)$ and $U = (\cos \alpha, \sin \alpha)$. Their dot product is $\cos \alpha \cos \beta + \sin \alpha \sin \beta$. From trigonometry this is the same as $\cos(\beta - \alpha)$. Since $\beta - \alpha$ equals θ (no change in the angle between them) we have reached the formula $u \cdot U = \cos \theta$.

Problem 26 proves $|u \cdot U| \leq 1$ directly, without mentioning angles. The inequality and the cosine formula $u \cdot U = \cos \theta$ are always true for unit vectors.

What if v and w are not unit vectors? Divide by their lengths to get $u = v/\|v\|$ and $U = w/\|w\|$. Then the dot product of those unit vectors u and U gives $\cos \theta$.



Figure 1.10 The dot product of unit vectors is the cosine of the angle θ .

Whatever the angle, this dot product of $v/\|v\|$ with $w/\|w\|$ never exceeds one. That is the “*Schwarz inequality*” for dot products—or more correctly the Cauchy-Schwarz-Buniakowsky inequality. It was found in France and Germany and Russia (and maybe elsewhere—it is the most important inequality in mathematics). With the division by $\|v\| \|w\|$ from rescaling to unit vectors, we have $\cos \theta$:

- 1D (a) COSINE FORMULA** If v and w are nonzero vectors then $\frac{v \cdot w}{\|v\| \|w\|} = \cos \theta$.
(b) SCHWARZ INEQUALITY If v and w are any vectors then $|v \cdot w| \leq \|v\| \|w\|$.

Example 5 Find $\cos \theta$ for $v = [\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}]$ and $w = [\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}]$ in Figure 1.9b.

Solution The dot product is $v \cdot w = 6$. Both v and w have length $\sqrt{10}$. The cosine is

$$\cos \theta = \frac{v \cdot w}{\|v\| \|w\|} = \frac{6}{\sqrt{10}\sqrt{10}} = \frac{3}{5}.$$

The angle is below 90° because $v \cdot w = 6$ is positive. By the Schwarz inequality, $\|v\| \|w\| = 10$ is larger than $v \cdot w = 6$.

Example 6 The dot product of $v = (a, b)$ and $w = (b, a)$ is $2ab$. Both lengths are $\sqrt{a^2 + b^2}$. The Schwarz inequality says that $2ab \leq a^2 + b^2$.

Reason The difference between $a^2 + b^2$ and $2ab$ can never be negative:

$$a^2 + b^2 - 2ab = (a - b)^2 \geq 0.$$

This is more famous if we write $x = a^2$ and $y = b^2$. Then the “geometric mean” \sqrt{xy} is not larger than the “arithmetic mean,” which is the average $\frac{1}{2}(x + y)$:

$$ab \leq \frac{a^2 + b^2}{2} \quad \text{becomes} \quad \sqrt{xy} \leq \frac{x + y}{2}.$$

Notes on Computing

Write the components of \mathbf{v} as $v(1), \dots, v(N)$ and similarly for \mathbf{w} . In FORTRAN, the sum $\mathbf{v} + \mathbf{w}$ requires a loop to add components separately. The dot product also loops to add the separate $v(i)w(i)$:

```
DO 10 I = 1,N           DO 10 I = 1,N
10 VPLUSW(I) = V(I)+W(I) 10 VDOTW = VDOTW + V(I) * W(I)
```

MATLAB works directly with whole vectors, not their components. No loop is needed. When \mathbf{v} and \mathbf{w} have been defined, $\mathbf{v} + \mathbf{w}$ is immediately understood. It is printed unless the line ends in a semicolon. Input \mathbf{v} and \mathbf{w} as rows—the prime ' $'$ at the end transposes them to columns. The combination $2\mathbf{v} + 3\mathbf{w}$ uses $*$ for multiplication.

$$\mathbf{v} = [2 \ 3 \ 4]'; \quad \mathbf{w} = [1 \ 1 \ 1]'; \quad \mathbf{u} = 2 * \mathbf{v} + 3 * \mathbf{w}$$

The dot product $\mathbf{v} \cdot \mathbf{w}$ is usually seen as *a row times a column (with no dot)*:

Instead of $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ we more often see $[1 \ 2] \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ or $\mathbf{v}' * \mathbf{w}$

The length of \mathbf{v} is already known to MATLAB as `norm(v)`. We could define it ourselves as `sqrt(v'*v)`, using the square root function—also known. The cosine we have to define ourselves! Then the angle (in radians) comes from the *arc cosine* (`acos`) function:

$$\text{cosine} = \mathbf{v}' * \mathbf{w} / (\text{norm}(\mathbf{v}) * \text{norm}(\mathbf{w})); \\ \text{angle} = \text{acos}(\text{cosine})$$

An *M*-file would create a new function `cosine(v, w)` for future use. (Quite a few *M*-files have been created especially for this book. They are listed at the end.)

■ REVIEW OF THE KEY IDEAS ■

1. The dot product $\mathbf{v} \cdot \mathbf{w}$ multiplies each component v_i by w_i and adds the $v_i w_i$.
2. The length $\|\mathbf{v}\|$ is the square root of $\mathbf{v} \cdot \mathbf{v}$.
3. The vector $\mathbf{v}/\|\mathbf{v}\|$ is a *unit vector*. Its length is 1.
4. The dot product is $\mathbf{v} \cdot \mathbf{w} = 0$ when \mathbf{v} and \mathbf{w} are perpendicular.
5. The cosine of θ (the angle between any nonzero \mathbf{v} and \mathbf{w}) never exceeds 1:

$$\cos \theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|} \quad \textit{Schwarz inequality} \quad |\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|.$$

■ WORKED EXAMPLES ■

1.2 A For the vectors $\mathbf{v} = (3, 4)$ and $\mathbf{w} = (4, 3)$ test the Schwarz inequality on $\mathbf{v} \cdot \mathbf{w}$ and the triangle inequality on $\|\mathbf{v} + \mathbf{w}\|$. Find $\cos \theta$ for the angle between \mathbf{v} and \mathbf{w} . When will we have *equality* $|\mathbf{v} \cdot \mathbf{w}| = \|\mathbf{v}\| \|\mathbf{w}\|$ and $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$?

Solution The dot product is $\mathbf{v} \cdot \mathbf{w} = (3)(4) + (4)(3) = 24$. The length of \mathbf{v} is $\|\mathbf{v}\| = \sqrt{9+16} = 5$ and also $\|\mathbf{w}\| = 5$. The sum $\mathbf{v} + \mathbf{w} = (7, 7)$ has length $\|\mathbf{v} + \mathbf{w}\| = 7\sqrt{2} \approx 9.9$.

Schwarz inequality	$ \mathbf{v} \cdot \mathbf{w} \leq \ \mathbf{v}\ \ \mathbf{w}\ $ is $24 < 25$.
Triangle inequality	$\ \mathbf{v} + \mathbf{w}\ \leq \ \mathbf{v}\ + \ \mathbf{w}\ $ is $7\sqrt{2} < 10$.
Cosine of angle	$\cos \theta = \frac{24}{25}$ (Thin angle!)

If one vector is a multiple of the other as in $\mathbf{w} = -2\mathbf{v}$, then the angle is 0° or 180° and $|\cos \theta| = 1$ and $|\mathbf{v} \cdot \mathbf{w}|$ equals $\|\mathbf{v}\| \|\mathbf{w}\|$. If the angle is 0° , as in $\mathbf{w} = 2\mathbf{v}$, then $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$. The triangle is flat.

1.2 B Find a unit vector \mathbf{u} in the direction of $\mathbf{v} = (3, 4)$. Find a unit vector \mathbf{U} perpendicular to \mathbf{u} . How many possibilities for \mathbf{U} ?

Solution For a unit vector \mathbf{u} , divide \mathbf{v} by its length $\|\mathbf{v}\| = 5$. For a perpendicular vector \mathbf{V} we can choose $(-4, 3)$ since the dot product $\mathbf{v} \cdot \mathbf{V}$ is $(3)(-4) + (4)(3) = 0$. For a unit vector \mathbf{U} , divide \mathbf{V} by its length $\|\mathbf{V}\|$:

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{(3, 4)}{5} = \left(\frac{3}{5}, \frac{4}{5} \right) \quad \mathbf{U} = \frac{\mathbf{V}}{\|\mathbf{V}\|} = \frac{(-4, 3)}{5} = \left(-\frac{4}{5}, \frac{3}{5} \right)$$

The only other perpendicular unit vector would be $-\mathbf{U} = (\frac{4}{5}, -\frac{3}{5})$.

Problem Set 1.2

- 1** Calculate the dot products $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$ and $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{w} \cdot \mathbf{v}$:

$$\mathbf{u} = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

- 2** Compute the lengths $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$ of those vectors. Check the Schwarz inequalities $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \|\mathbf{v}\|$ and $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$.
- 3** Find unit vectors in the directions of \mathbf{v} and \mathbf{w} in Problem 1, and the cosine of the angle θ . Choose vectors that make 0° , 90° , and 180° angles with \mathbf{w} .
- 4** Find unit vectors \mathbf{u}_1 and \mathbf{u}_2 in the directions of $\mathbf{v} = (3, 1)$ and $\mathbf{w} = (2, 1, 2)$. Find unit vectors \mathbf{U}_1 and \mathbf{U}_2 that are perpendicular to \mathbf{u}_1 and \mathbf{u}_2 .

- 5 For any *unit* vectors \mathbf{v} and \mathbf{w} , find the dot products (actual numbers) of
 (a) \mathbf{v} and $-\mathbf{v}$ (b) $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ (c) $\mathbf{v} - 2\mathbf{w}$ and $\mathbf{v} + 2\mathbf{w}$
- 6 Find the angle θ (from its cosine) between
- (a) $\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$
 (c) $\mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$ (d) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$.
- 7 (a) Describe every vector $\mathbf{w} = (w_1, w_2)$ that is perpendicular to $\mathbf{v} = (2, -1)$.
 (b) The vectors that are perpendicular to $\mathbf{V} = (1, 1, 1)$ lie on a _____.
 (c) The vectors that are perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a _____.
- 8 True or false (give a reason if true or a counterexample if false):
- (a) If \mathbf{u} is perpendicular (in three dimensions) to \mathbf{v} and \mathbf{w} , then \mathbf{v} and \mathbf{w} are parallel.
 (b) If \mathbf{u} is perpendicular to \mathbf{v} and \mathbf{w} , then \mathbf{u} is perpendicular to $\mathbf{v} + 2\mathbf{w}$.
 (c) If \mathbf{u} and \mathbf{v} are perpendicular unit vectors then $\|\mathbf{u} - \mathbf{v}\| = \sqrt{2}$.
- 9 The slopes of the arrows from $(0, 0)$ to (v_1, v_2) and (w_1, w_2) are v_2/v_1 and w_2/w_1 . If the product v_2w_2/v_1w_1 of those slopes is -1 , show that $\mathbf{v} \cdot \mathbf{w} = 0$ and the vectors are perpendicular.
- 10 Draw arrows from $(0, 0)$ to the points $\mathbf{v} = (1, 2)$ and $\mathbf{w} = (-2, 1)$. Multiply their slopes. That answer is a signal that $\mathbf{v} \cdot \mathbf{w} = 0$ and the arrows are _____.
- 11 If $\mathbf{v} \cdot \mathbf{w}$ is negative, what does this say about the angle between \mathbf{v} and \mathbf{w} ? Draw a 2-dimensional vector \mathbf{v} (an arrow), and show where to find all \mathbf{w} 's with $\mathbf{v} \cdot \mathbf{w} < 0$.
- 12 With $\mathbf{v} = (1, 1)$ and $\mathbf{w} = (1, 5)$ choose a number c so that $\mathbf{w} - c\mathbf{v}$ is perpendicular to \mathbf{v} . Then find the formula that gives this number c for any nonzero \mathbf{v} and \mathbf{w} .
- 13 Find two vectors \mathbf{v} and \mathbf{w} that are perpendicular to $(1, 0, 1)$ and to each other.
- 14 Find three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ that are perpendicular to $(1, 1, 1, 1)$ and to each other.
- 15 The geometric mean of $x = 2$ and $y = 8$ is $\sqrt{xy} = 4$. The arithmetic mean is larger: $\frac{1}{2}(x+y) = \text{_____}$. This came in Example 6 from the Schwarz inequality for $\mathbf{v} = (\sqrt{2}, \sqrt{8})$ and $\mathbf{w} = (\sqrt{8}, \sqrt{2})$. Find $\cos \theta$ for this \mathbf{v} and \mathbf{w} .
- 16 How long is the vector $\mathbf{v} = (1, 1, \dots, 1)$ in 9 dimensions? Find a unit vector \mathbf{u} in the same direction as \mathbf{v} and a vector \mathbf{w} that is perpendicular to \mathbf{v} .
- 17 What are the cosines of the angles α, β, θ between the vector $(1, 0, -1)$ and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ along the axes? Check the formula $\cos^2 \alpha + \cos^2 \beta + \cos^2 \theta = 1$.

Problems 18–31 lead to the main facts about lengths and angles in triangles.

- 18 The parallelogram with sides $\mathbf{v} = (4, 2)$ and $\mathbf{w} = (-1, 2)$ is a rectangle. Check the Pythagoras formula $a^2 + b^2 = c^2$ which is for *right triangles only*:

$$(\text{length of } \mathbf{v})^2 + (\text{length of } \mathbf{w})^2 = (\text{length of } \mathbf{v} + \mathbf{w})^2.$$

- 19 In this 90° case, $a^2 + b^2 = c^2$ also works for $\mathbf{v} - \mathbf{w}$:

$$(\text{length of } \mathbf{v})^2 + (\text{length of } \mathbf{w})^2 = (\text{length of } \mathbf{v} - \mathbf{w})^2.$$

Give an example of \mathbf{v} and \mathbf{w} (not at right angles) for which this equation fails.

- 20 (Rules for dot products) These equations are simple but useful:

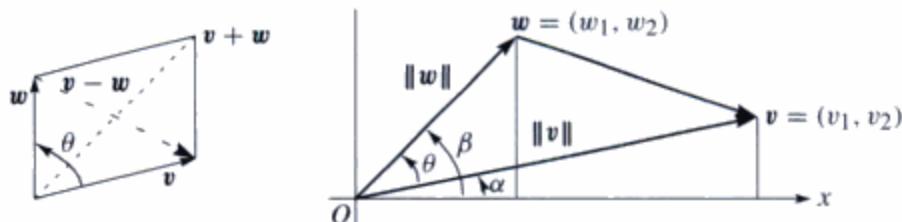
(1) $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$ (2) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ (3) $(c\mathbf{v}) \cdot \mathbf{w} = c(\mathbf{v} \cdot \mathbf{w})$

Use (1) and (2) with $\mathbf{u} = \mathbf{v} + \mathbf{w}$ to prove $\|\mathbf{v} + \mathbf{w}\|^2 = \mathbf{v} \cdot \mathbf{v} + 2\mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{w}$.

- 21 The *triangle inequality* says: $(\text{length of } \mathbf{v} + \mathbf{w}) \leq (\text{length of } \mathbf{v}) + (\text{length of } \mathbf{w})$. Problem 20 found $\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + 2\mathbf{v} \cdot \mathbf{w} + \|\mathbf{w}\|^2$. Use the Schwarz inequality $\mathbf{v} \cdot \mathbf{w} \leq \|\mathbf{v}\| \|\mathbf{w}\|$ to turn this into the triangle inequality:

$$\|\mathbf{v} + \mathbf{w}\|^2 \leq (\|\mathbf{v}\| + \|\mathbf{w}\|)^2 \quad \text{or} \quad \|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|.$$

- 22 A right triangle in three dimensions still obeys $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 = \|\mathbf{v} + \mathbf{w}\|^2$. Show how this leads in Problem 20 to $v_1 w_1 + v_2 w_2 + v_3 w_3 = 0$.



- 23 The figure shows that $\cos \alpha = v_1 / \|\mathbf{v}\|$ and $\sin \alpha = v_2 / \|\mathbf{v}\|$. Similarly $\cos \beta$ is _____ and $\sin \beta$ is _____. The angle θ is $\beta - \alpha$. Substitute into the formula $\cos \beta \cos \alpha + \sin \beta \sin \alpha$ for $\cos(\beta - \alpha)$ to find $\cos \theta = \mathbf{v} \cdot \mathbf{w} / \|\mathbf{v}\| \|\mathbf{w}\|$.

- 24 With \mathbf{v} and \mathbf{w} at angle θ , the “Law of Cosines” comes from $(\mathbf{v} - \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w})$:

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 - 2\|\mathbf{v}\| \|\mathbf{w}\| \cos \theta + \|\mathbf{w}\|^2.$$

If $\theta < 90^\circ$ show that $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$ is larger than $\|\mathbf{v} - \mathbf{w}\|^2$ (the third side).

- 25 The Schwarz inequality $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$ by algebra instead of trigonometry:

- (a) Multiply out both sides of $(v_1 w_1 + v_2 w_2)^2 \leq (v_1^2 + v_2^2)(w_1^2 + w_2^2)$.
- (b) Show that the difference between those sides equals $(v_1 w_2 - v_2 w_1)^2$. This cannot be negative since it is a square—so the inequality is true.

- 26 One-line proof of the Schwarz inequality $|\mathbf{u} \cdot \mathbf{U}| \leq 1$ for unit vectors:

$$|\mathbf{u} \cdot \mathbf{U}| \leq |\mathbf{u}_1| |\mathbf{U}_1| + |\mathbf{u}_2| |\mathbf{U}_2| \leq \frac{\mathbf{u}_1^2 + \mathbf{U}_1^2}{2} + \frac{\mathbf{u}_2^2 + \mathbf{U}_2^2}{2} = \frac{1+1}{2} = 1.$$

Put $(\mathbf{u}_1, \mathbf{u}_2) = (.6, .8)$ and $(\mathbf{U}_1, \mathbf{U}_2) = (.8, .6)$ in that whole line and find $\cos \theta$.

- 27 Why is $|\cos \theta|$ never greater than 1 in the first place?
- 28 Pick any numbers that add to $x + y + z = 0$. Find the angle between your vector $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$. Challenge question: Explain why $\mathbf{v} \cdot \mathbf{w} / \|\mathbf{v}\| \|\mathbf{w}\|$ is always $-\frac{1}{2}$.
- 29 (*Recommended*) If $\|\mathbf{v}\| = 5$ and $\|\mathbf{w}\| = 3$, what are the smallest and largest values of $\|\mathbf{v} - \mathbf{w}\|$? What are the smallest and largest values of $\mathbf{v} \cdot \mathbf{w}$?
- 30 If $\mathbf{v} = (1, 2)$ draw all vectors $\mathbf{w} = (x, y)$ in the xy plane with $\mathbf{v} \cdot \mathbf{w} = 5$. Which is the shortest \mathbf{w} ?
- 31 Can three vectors in the xy plane have $\mathbf{u} \cdot \mathbf{v} < 0$ and $\mathbf{v} \cdot \mathbf{w} < 0$ and $\mathbf{u} \cdot \mathbf{w} < 0$? I don't know how many vectors in xyz space can have all negative dot products. (Four of those vectors in the plane would be impossible...).

2

SOLVING LINEAR EQUATIONS

VECTORS AND LINEAR EQUATIONS ■ 2.1

The central problem of linear algebra is to solve a system of equations. Those equations are linear, which means that the unknowns are only multiplied by numbers—we never see x times y . Our first example of a linear system is certainly not big. It has two equations in two unknowns. But you will see how far it leads:

$$\begin{array}{rcl} x & - & 2y = 1 \\ 3x & + & 2y = 11 \end{array} \quad (1)$$

We begin *a row at a time*. The first equation $x - 2y = 1$ produces a straight line in the xy plane. The point $x = 1, y = 0$ is on the line because it solves that equation. The point $x = 3, y = 1$ is also on the line because $3 - 2 = 1$. If we choose $x = 101$ we find $y = 50$. The slope of this particular line is $\frac{1}{2}$ (y increases by 50 when x changes by 100). But slopes are important in calculus and this is linear algebra!

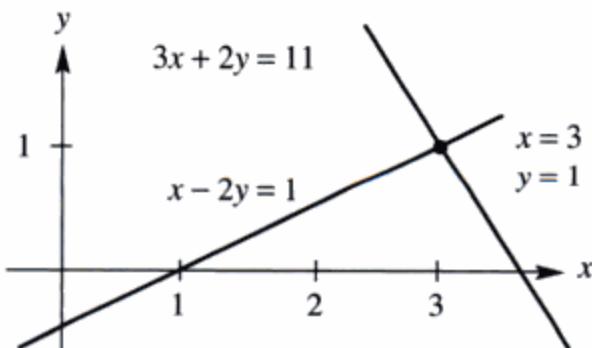


Figure 2.1 *Row picture:* The point $(3, 1)$ where the lines meet is the solution.

Figure 2.1 shows that line $x - 2y = 1$. The second line in this “row picture” comes from the second equation $3x + 2y = 11$. You can’t miss the intersection point

where the two lines meet. *The point $x = 3, y = 1$ lies on both lines.* That point solves both equations at once. This is the solution to our system of linear equations.

R *The row picture shows two lines meeting at a single point.*

Turn now to the column picture. I want to recognize the linear system as a “vector equation”. Instead of numbers we need to see *vectors*. If you separate the original system into its columns instead of its rows, you get

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix} = b. \quad (2)$$

This has two column vectors on the left side. The problem is *to find the combination of those vectors that equals the vector on the right*. We are multiplying the first column by x and the second column by y , and adding. With the right choices $x = 3$ and $y = 1$, this produces $3(\text{column 1}) + 1(\text{column 2}) = b$.

C *The column picture combines the column vectors on the left side to produce the vector b on the right side.*

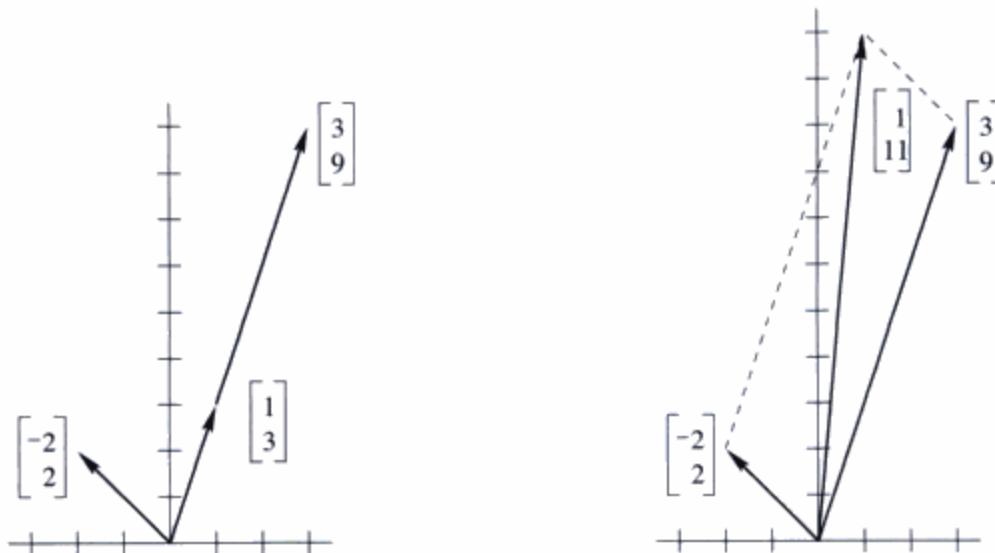


Figure 2.2 *Column picture:* A combination of columns produces the right side $(1,11)$.

Figure 2.2 is the “column picture” of two equations in two unknowns. The first part shows the two separate columns, and that first column multiplied by 3. This multiplication by a *scalar* (a number) is one of the two basic operations in linear algebra:

Scalar multiplication $3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$

If the components of a vector v are v_1 and v_2 , then $c v$ has components $c v_1$ and $c v_2$.

The other basic operation is *vector addition*. We add the first components and the second components separately. The vector sum is $(1, 11)$ as desired:

$$\text{Vector addition} \quad \begin{bmatrix} 3 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

The graph in Figure 2.2 shows a parallelogram. The sum $(1, 11)$ is along the diagonal:

$$\text{The sides are } \begin{bmatrix} 3 \\ 9 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 2 \end{bmatrix}. \text{ The diagonal sum is } \begin{bmatrix} 3 - 2 \\ 9 + 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

We have multiplied the original columns by $x = 3$ and $y = 1$. That combination produces the vector $b = (1, 11)$ on the right side of the linear equations.

To repeat: The left side of the vector equation is a *linear combination* of the columns. The problem is to find the right coefficients $x = 3$ and $y = 1$. We are combining scalar multiplication and vector addition into one step. That step is crucially important, because it contains both of the basic operations:

$$\text{Linear combination} \quad 3 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Of course the solution $x = 3, y = 1$ is the same as in the row picture. I don't know which picture you prefer! I suspect that the two intersecting lines are more familiar at first. You may like the row picture better, but only for one day. My own preference is to combine column vectors. It is a lot easier to see a combination of four vectors in four-dimensional space, than to visualize how four hyperplanes might possibly meet at a point. (*Even one hyperplane is hard enough. . .*)

The *coefficient matrix* on the left side of the equations is the 2 by 2 matrix A :

$$\text{Coefficient matrix} \quad A = \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}.$$

This is very typical of linear algebra, to look at a matrix by rows and by columns. Its rows give the row picture and its columns give the column picture. Same numbers, different pictures, same equations. We write those equations as a matrix problem $Ax = b$:

$$\text{Matrix equation} \quad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

The row picture deals with the two rows of A . The column picture combines the columns. The numbers $x = 3$ and $y = 1$ go into the solution vector x . Then

$$Ax = b \quad \text{is} \quad \begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}.$$

Three Equations in Three Unknowns

The three unknowns are x, y, z . The linear equations $Ax = b$ are

$$\begin{array}{rcl} x + 2y + 3z & = & 6 \\ 2x + 5y + 2z & = & 4 \\ 6x - 3y + z & = & 2 \end{array} \quad (3)$$

We look for numbers x, y, z that solve all three equations at once. Those desired numbers might or might not exist. For this system, they do exist. When the number of unknowns matches the number of equations, there is *usually* one solution. Before solving the problem, we visualize it both ways:

R *The row picture shows three planes meeting at a single point.*

C *The column picture combines three columns to produce the vector $(6, 4, 2)$.*

In the row picture, each equation is a *plane* in three-dimensional space. The first plane comes from the first equation $x + 2y + 3z = 6$. That plane crosses the x and y and z axes at the points $(6, 0, 0)$ and $(0, 3, 0)$ and $(0, 0, 2)$. Those three points solve the equation and they determine the whole plane.

The vector $(x, y, z) = (0, 0, 0)$ does not solve $x + 2y + 3z = 6$. Therefore the plane in Figure 2.3 does not contain the origin.

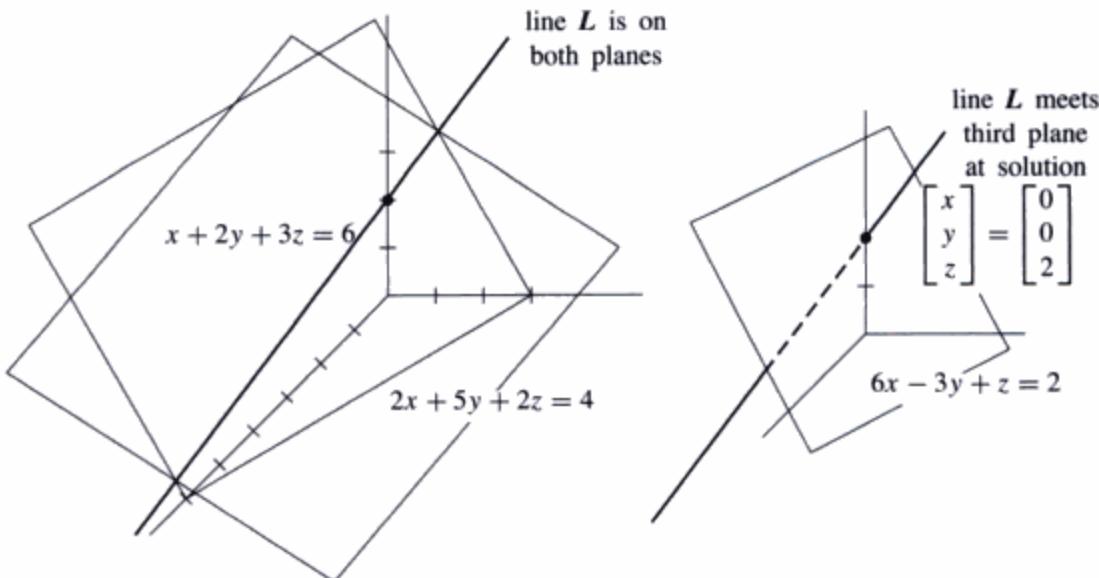


Figure 2.3 Row picture of three equations: Three planes meet at a point.

The plane $x + 2y + 3z = 0$ does pass through the origin, and it is parallel to $x + 2y + 3z = 6$. When the right side increases to 6, the plane moves away from the origin.

The second plane is given by the second equation $2x + 5y + 2z = 4$. It intersects the first plane in a line L . The usual result of two equations in three unknowns is a line L of solutions.

The third equation gives a third plane. It cuts the line L at a single point. That point lies on all three planes and it solves all three equations. It is harder to draw this triple intersection point than to imagine it. The three planes meet at the solution (which we haven't found yet). The column form shows immediately why $z = 2$!

The column picture starts with the vector form of the equations:

$$x \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + z \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}. \quad (4)$$

The unknown numbers x, y, z are the coefficients in this linear combination. We want to multiply the three column vectors by the correct numbers x, y, z to produce $\mathbf{b} = (6, 4, 2)$.

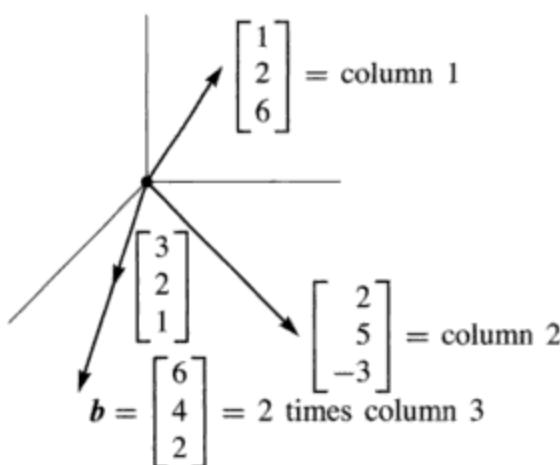


Figure 2.4 Column picture: $(x, y, z) = (0, 0, 2)$ because $2(3, 2, 1) = (6, 4, 2) = \mathbf{b}$.

Figure 2.4 shows this column picture. Linear combinations of those columns can produce any vector \mathbf{b} ! The combination that produces $\mathbf{b} = (6, 4, 2)$ is just 2 times the third column. The coefficients we need are $x = 0$, $y = 0$, and $z = 2$. This is also the intersection point of the three planes in the row picture. It solves the system:

Correct combination $0 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$

The Matrix Form of the Equations

We have three rows in the row picture and three columns in the column picture (plus the right side). The three rows and three columns contain nine numbers. *These nine numbers fill a 3 by 3 matrix.* The “coefficient matrix” has the rows and columns that have so far been kept separate:

$$\text{The coefficient matrix is } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix}.$$

The capital letter A stands for all nine coefficients (in this square array). The letter b denotes the column vector with components 6, 4, 2. The unknown x is also a column vector, with components x, y, z . (We use boldface because it is a vector, x because it is unknown.) By rows the equations were (3), by columns they were (4), and now by matrices they are (5). The shorthand is $Ax = b$:

$$\text{Matrix equation} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}. \quad (5)$$

We multiply the matrix A times the unknown vector x to get the right side b .

Basic question: What does it mean to “multiply A times x ”? We can multiply by rows or by columns. Either way, $Ax = b$ must be a correct representation of the three equations. You do the same nine multiplications either way.

Multiplication by rows Ax comes from *dot products*, each row times the column x :

$$Ax = \begin{bmatrix} (\text{row 1}) \cdot x \\ (\text{row 2}) \cdot x \\ (\text{row 3}) \cdot x \end{bmatrix}. \quad (6)$$

Multiplication by columns Ax is a *combination of column vectors*:

$$Ax = x \text{ (column 1)} + y \text{ (column 2)} + z \text{ (column 3)}. \quad (7)$$

When we substitute the solution $x = (0, 0, 2)$, the multiplication Ax produces b :

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 6 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2 \text{ times column 3} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}.$$

The first dot product in row multiplication is $(1, 2, 3) \cdot (0, 0, 2) = 6$. The other dot products are 4 and 2. Multiplication by columns is simply 2 times column 3.

This book sees Ax as a combination of the columns of A .

Example 1 Here are 3 by 3 matrices A and I , with three ones and six zeros:

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \quad Ix = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

If you are a row person, the product of every row $(1, 0, 0)$ with $(4, 5, 6)$ is 4. If you are a column person, the linear combination is 4 times the first column $(1, 1, 1)$. In that matrix A , the second and third columns are zero vectors.

The example with Ix deserves a careful look, because the matrix I is special. It has ones on the “main diagonal”. Off that diagonal, all the entries are zeros. *Whatever vector this matrix multiplies, that vector is not changed.* This is like multiplication by 1, but for matrices and vectors. The exceptional matrix in this example is the 3 by 3 **identity matrix**:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{always yields the multiplication } Ix = x.$$

Matrix Notation

The first row of a 2 by 2 matrix contains a_{11} and a_{12} . The second row contains a_{21} and a_{22} . The first index gives the row number, so that a_{ij} is an entry in row i . The second index j gives the column number. But those subscripts are not convenient on a keyboard! Instead of a_{ij} it is easier to type $A(i, j)$. **The entry $a_{57} = A(5, 7)$ would be in row 5, column 7.**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} A(1, 1) & A(1, 2) \\ A(2, 1) & A(2, 2) \end{bmatrix}.$$

For an m by n matrix, the row index i goes from 1 to m . The column index j stops at n . There are mn entries in the matrix. A square matrix (order n) has n^2 entries.

Multiplication in MATLAB

I want to express A and x and their product Ax using MATLAB commands. This is a first step in learning that language. I begin by defining the matrix A and the vector x . This vector is a 3 by 1 matrix, with three rows and one column. Enter matrices a row at a time, and use a semicolon to signal the end of a row:

$$A = [1 \ 2 \ 3; \ 2 \ 5 \ 2; \ 6 \ -3 \ 1] \\ x = [0; 0; 2]$$

Here are three ways to multiply Ax in MATLAB. In reality, $A * x$ is the way to do it. MATLAB is a high level language, and it works with matrices:

Matrix multiplication $b = A * x$

We can also pick out the first row of A (as a smaller matrix!). The notation for that 1 by 3 submatrix is $A(1, :)$. **Here the colon symbol keeps all columns of row 1:**

$$\text{Row at a time } b = [A(1, :) * x; A(2, :) * x; A(3, :) * x]$$

Those are dot products, row times column, 1 by 3 matrix times 3 by 1 matrix.

The other way to multiply uses the columns of A . The first column is the 3 by 1 submatrix $A(:, 1)$. Now the colon symbol $:$ is keeping all rows of column 1. This column multiplies $x(1)$ and the other columns multiply $x(2)$ and $x(3)$:

$$\text{Column at a time } b = A(:, 1) * x(1) + A(:, 2) * x(2) + A(:, 3) * x(3)$$

I think that matrices are stored by columns. Then multiplying a column at a time will be a little faster. So $A * x$ is actually executed by columns.

You can see the same choice in a FORTRAN-type structure, which operates on single entries of A and x . This lower level language needs an outer and inner “DO loop”. When the outer loop uses the row number I , multiplication is a row at a time. The inner loop $J = 1, 3$ goes along each row I .

When the outer loop uses J , multiplication is a column at a time. I will do that in MATLAB , which needs two more lines “end” “end” to close “for I ” and “for J ”:

FORTRAN by rows

```
DO 10  I = 1, 3
DO 10  J = 1, 3
10    B(I) = B(I) + A(I, J) * X(J)
```

MATLAB by columns

```
for J = 1 : 3
for I = 1 : 3
b(I) = b(I) + A(I, J) * x(J)
```

Notice that MATLAB is sensitive to upper case versus lower case (capital letters and small letters). If the matrix is A then its entries are $A(I, J)$ not $a(I, J)$.

I think you will prefer the higher level $A * x$. FORTRAN won't appear again in this book. *Maple* and *Mathematica* and graphing calculators also operate at the higher level. Multiplication is $A \cdot x$ in *Mathematica*. It is **multiply**(A, x); or **evalm**($A \&* x$); in *Maple*. Those languages allow symbolic entries a, b, x, \dots and not only real numbers. Like MATLAB's Symbolic Toolbox, they give the symbolic answer.

■ REVIEW OF THE KEY IDEAS ■

1. The basic operations on vectors are multiplication $c\mathbf{v}$ and vector addition $\mathbf{v} + \mathbf{w}$.
2. Together those operations give linear combinations $c\mathbf{v} + d\mathbf{w}$.

3. Matrix-vector multiplication Ax can be executed by rows (dot products). But it should be understood as a combination of the columns of A !
4. Column picture: $Ax = b$ asks for a combination of columns to produce b .
5. Row picture: Each equation in $Ax = b$ gives a line ($n = 2$) or a plane ($n = 3$) or a “hyperplane” ($n > 3$). They intersect at the solution or solutions.

■ WORKED EXAMPLES ■

2.1 A Describe the column picture of these three equations. Solve by careful inspection of the columns (instead of elimination):

$$\begin{aligned} x + 3y + 2z &= -3 \\ 2x + 2y + 2z &= -2 \quad \text{which is } Ax = b : \\ 3x + 5y + 4z &= -5 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & 3 & 2 & -3 \\ 2 & 2 & 2 & -2 \\ 3 & 5 & 4 & -5 \end{array} \right]$$

Solution The column picture asks for a linear combination that produces b from the three columns of A . In this example b is *minus the second column*. So the solution is $x = 0, y = -1, z = 0$. To show that $(0, -1, 0)$ is the *only* solution we have to know that “ A is invertible” and “the columns are independent” and “the determinant isn’t zero”. Those words are not yet defined but the test comes from elimination: We need (and we find!) a full set of three nonzero pivots.

If the right side changes to $b = (4, 4, 8) =$ sum of the first two columns, then the right combination has $x = 1, y = 1, z = 0$. The solution becomes $x = (1, 1, 0)$.

2.1 B This system has *no solution*, because the three planes in the row picture don’t pass through a point. No combination of the three columns produces b :

$$\begin{aligned} x + 3y + 5z &= 4 \\ x + 2y - 3z &= 5 \\ 2x + 5y + 2z &= 8 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 1 & 2 & -3 & 5 \\ 2 & 5 & 2 & 8 \end{array} \right] = \left[\begin{array}{c} 4 \\ 5 \\ 8 \end{array} \right] = b$$

- (1) Multiply the equations by $1, 1, -1$ and add to show that these planes don’t meet at a point. Are any two of the planes parallel? What are the equations of planes parallel to $x + 3y + 5z = 4$?
- (2) Take the dot product of each column (and also b) with $y = (1, 1, -1)$. How do those dot products show that the system has no solution?
- (3) Find three right side vectors b^* and b^{**} and b^{***} that *do* allow solutions.

Solution

- (1) Multiplying the equations by 1, 1, -1 and adding gives

$$\begin{array}{r} x + 3y + 5z = 4 \\ x + 2y - 3z = 5 \\ \hline -[2x + 5y + 2z = 8] \\ \hline 0x + 0y + 0z = 1 \quad \text{No Solution} \end{array}$$

The planes don't meet at any point, but no two planes are parallel. For a plane parallel to $x + 3y + 5z = 4$, just change the "4". The parallel plane $x + 3y + 5z = 0$ goes through the origin $(0, 0, 0)$. And the equation multiplied by any nonzero constant still gives the same plane, as in $2x + 6y + 10z = 8$.

- (2) The dot product of each column with $y = (1, 1, -1)$ is *zero*. On the right side, $y \cdot b = (1, 1, -1) \cdot (4, 5, 8) = 1$ is *not zero*. So a solution is impossible. (If a combination of columns could produce b , take dot products with y . Then a combination of zeros would produce 1.)
- (3) There is a solution when b is a combination of the columns. These three examples b^*, b^{**}, b^{***} have solutions $x^* = (1, 0, 0)$ and $x^{**} = (1, 1, 1)$ and $x^{***} = (0, 0, 0)$:

$$b^* = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \text{first column} \quad b^{**} = \begin{bmatrix} 9 \\ 0 \\ 9 \end{bmatrix} = \text{sum of columns} \quad b^{***} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem Set 2.1

Problems 1–9 are about the row and column pictures of $Ax = b$.

- 1 With $A = I$ (the identity matrix) draw the planes in the row picture. Three sides of a box meet at the solution $x = (x, y, z) = (2, 3, 4)$:

$$\begin{array}{l} 1x + 0y + 0z = 2 \\ 0x + 1y + 0z = 3 \\ 0x + 0y + 1z = 4 \end{array} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

- 2 Draw the vectors in the column picture of Problem 1. Two times column 1 plus three times column 2 plus four times column 3 equals the right side b .
- 3 If the equations in Problem 1 are multiplied by 2, 3, 4 they become $\widehat{A}\widehat{x} = \widehat{b}$:

$$\begin{array}{l} 2x + 0y + 0z = 4 \\ 0x + 3y + 0z = 9 \\ 0x + 0y + 4z = 16 \end{array} \quad \text{or} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 16 \end{bmatrix}$$

Why is the row picture the same? Is the solution \widehat{x} the same as x ? What is changed in the column picture—the columns or the right combination to give \widehat{b} ?

- 4 If equation 1 is added to equation 2, which of these are changed: the planes in the row picture, the column picture, the coefficient matrix, the solution? The new equations in Problem 1 would be $x = 2$, $x + y = 5$, $z = 4$.
- 5 Find a point with $z = 2$ on the intersection line of the planes $x + y + 3z = 6$ and $x - y + z = 4$. Find the point with $z = 0$ and a third point halfway between.
- 6 The first of these equations plus the second equals the third:

$$\begin{aligned}x + y + z &= 2 \\x + 2y + z &= 3 \\2x + 3y + 2z &= 5.\end{aligned}$$

The first two planes meet along a line. The third plane contains that line, because if x, y, z satisfy the first two equations then they also _____. The equations have infinitely many solutions (the whole line \mathbf{L}). Find three solutions on \mathbf{L} .

- 7 Move the third plane in Problem 6 to a parallel plane $2x + 3y + 2z = 9$. Now the three equations have no solution—why not? The first two planes meet along the line \mathbf{L} , but the third plane doesn't _____ that line.
- 8 In Problem 6 the columns are $(1, 1, 2)$ and $(1, 2, 3)$ and $(1, 1, 2)$. This is a “singular case” because the third column is _____. Find two combinations of the columns that give $\mathbf{b} = (2, 3, 5)$. This is only possible for $\mathbf{b} = (4, 6, c)$ if $c = \underline{\hspace{2cm}}$.
- 9 Normally 4 “planes” in 4-dimensional space meet at a _____. Normally 4 column vectors in 4-dimensional space can combine to produce \mathbf{b} . What combination of $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$ produces $\mathbf{b} = (3, 3, 3, 2)$? What 4 equations for x, y, z, t are you solving?

Problems 10–15 are about multiplying matrices and vectors.

- 10 Compute each $A\mathbf{x}$ by dot products of the rows with the column vector:

$$(a) \quad \begin{bmatrix} 1 & 2 & 4 \\ -2 & 3 & 1 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \quad (b) \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

- 11 Compute each $A\mathbf{x}$ in Problem 10 as a combination of the columns:

$$10(a) \text{ becomes } A\mathbf{x} = 2 \begin{bmatrix} 1 \\ -2 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}.$$

How many separate multiplications for $A\mathbf{x}$, when the matrix is “3 by 3”?

32 Chapter 2 Solving Linear Equations

- 12** Find the two components of Ax by rows or by columns:

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

- 13** Multiply A times x to find three components of Ax :

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- 14** (a) A matrix with m rows and n columns multiplies a vector with _____ components to produce a vector with _____ components.
 (b) The planes from the m equations $Ax = b$ are in _____-dimensional space. The combination of the columns of A is in _____-dimensional space.
- 15** Write $2x + 3y + z + 5t = 8$ as a matrix A (how many rows?) multiplying the column vector $x = (x, y, z, t)$ to produce b . The solutions x fill a plane or “hyperplane” in 4-dimensional space. *The plane is 3-dimensional with no 4D volume.*

Problems 16–23 ask for matrices that act in special ways on vectors.

- 16** (a) What is the 2 by 2 identity matrix? I times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} x \\ y \end{bmatrix}$.
 (b) What is the 2 by 2 exchange matrix? P times $\begin{bmatrix} x \\ y \end{bmatrix}$ equals $\begin{bmatrix} y \\ x \end{bmatrix}$.
- 17** (a) What 2 by 2 matrix R rotates every vector by 90° ? R times $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} y \\ -x \end{bmatrix}$.
 (b) What 2 by 2 matrix rotates every vector by 180° ?
- 18** Find the matrix P that multiplies (x, y, z) to give (y, z, x) . Find the matrix Q that multiplies (y, z, x) to bring back (x, y, z) .
- 19** What 2 by 2 matrix E subtracts the first component from the second component? What 3 by 3 matrix does the same?

$$E \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \text{and} \quad E \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}.$$

- 20** What 3 by 3 matrix E multiplies (x, y, z) to give $(x, y, z+x)$? What matrix E^{-1} multiplies (x, y, z) to give $(x, y, z-x)$? If you multiply $(3, 4, 5)$ by E and then multiply by E^{-1} , the two results are (_____) and (______).
- 21** What 2 by 2 matrix P_1 projects the vector (x, y) onto the x axis to produce $(x, 0)$? What matrix P_2 projects onto the y axis to produce $(0, y)$? If you multiply $(5, 7)$ by P_1 and then multiply by P_2 , you get (_____) and (______).

- 22 What 2 by 2 matrix R rotates every vector through 45° ? The vector $(1, 0)$ goes to $(\sqrt{2}/2, \sqrt{2}/2)$. The vector $(0, 1)$ goes to $(-\sqrt{2}/2, \sqrt{2}/2)$. Those determine the matrix. Draw these particular vectors in the xy plane and find R .
- 23 Write the dot product of $(1, 4, 5)$ and (x, y, z) as a matrix multiplication Ax . The matrix A has one row. The solutions to $Ax = \mathbf{0}$ lie on a _____ perpendicular to the vector _____. The columns of A are only in _____-dimensional space.
- 24 In MATLAB notation, write the commands that define this matrix A and the column vectors \mathbf{x} and \mathbf{b} . What command would test whether or not $Ax = b$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad x = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

- 25 The MATLAB commands $A = \text{eye}(3)$ and $v = [3:5]'$ produce the 3 by 3 identity matrix and the column vector $(3, 4, 5)$. What are the outputs from $A*v$ and $v'*v$? (Computer not needed!) If you ask for $v*A$, what happens?
- 26 If you multiply the 4 by 4 all-ones matrix $A = \text{ones}(4,4)$ and the column $v = \text{ones}(4,1)$, what is $A*v$? (Computer not needed.) If you multiply $B = \text{eye}(4) + \text{ones}(4,4)$ times $w = \text{zeros}(4,1) + 2 * \text{ones}(4,1)$, what is $B*w$?

Questions 27–29 are a review of the row and column pictures.

- 27 Draw the two pictures in two planes for the equations $x - 2y = 0$, $x + y = 6$.
- 28 For two linear equations in three unknowns x, y, z , the row picture will show (2 or 3) (lines or planes) in (2 or 3)-dimensional space. The column picture is in (2 or 3)-dimensional space. The solutions normally lie on a _____.
- 29 For four linear equations in two unknowns x and y , the row picture shows four _____. The column picture is in _____-dimensional space. The equations have no solution unless the vector on the right side is a combination of _____.
- 30 Start with the vector $\mathbf{u}_0 = (1, 0)$. Multiply again and again by the same “Markov matrix” A below. The next three vectors are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$:

$$\mathbf{u}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \mathbf{u}_2 = A\mathbf{u}_1 = \underline{\hspace{2cm}} \quad \mathbf{u}_3 = A\mathbf{u}_2 = \underline{\hspace{2cm}}.$$

What property do you notice for all four vectors $\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$?

- 31 With a computer, continue from $\mathbf{u}_0 = (1, 0)$ to \mathbf{u}_7 , and from $\mathbf{v}_0 = (0, 1)$ to \mathbf{v}_7 . What do you notice about \mathbf{u}_7 and \mathbf{v}_7 ? Here are two MATLAB codes, one with while and one with for. They plot \mathbf{u}_0 to \mathbf{u}_7 —you can use other languages:

```

u = [1 ; 0]; A = [.8 .3 ; .2 .7];
x = u; k = [0 : 7];
while size(x,2) <= 7
    u = A * u; x = [x u];
end
plot(k, x)

u = [1 ; 0]; A = [.8 .3 ; .2 .7];
x = u; k = [0 : 7];
for j=1 : 7
    u = A * u; x = [x u];
end
plot(k, x)

```

- 32 The \mathbf{u} 's and \mathbf{v} 's in Problem 31 are approaching a steady state vector \mathbf{s} . Guess that vector and check that $A\mathbf{s} = \mathbf{s}$. If you start with \mathbf{s} , you stay with \mathbf{s} .
- 33 This MATLAB code allows you to input \mathbf{x}_0 with a mouse click, by ginput. With $t = 1$, A rotates vectors by *theta*. The plot will show $A\mathbf{x}_0, A^2\mathbf{x}_0, \dots$ going around a circle ($t > 1$ will spiral out and $t < 1$ will spiral in). You can change *theta* and the stop at $j=10$. We plan to put this code on web.mit.edu/18.06/www:

```

theta = 15*pi/180; t = 1.0;
A = t* [cos(theta) -sin(theta) ; sin(theta) cos(theta)];
disp('Click to select starting point')
[x1 , x2] = ginput(1); x = [x1 ; x2];
for j=1:10
    x = [x A*x( : , end)];
end
plot(x(1,:), x(2,:), 'o')
hold off

```

- 34 Invent a 3 by 3 **magic matrix** M_3 with entries 1, 2, ..., 9. All rows and columns and diagonals add to 15. The first row could be 8, 3, 4. What is M_3 times (1, 1, 1)? What is M_4 times (1, 1, 1, 1) if this magic matrix has entries 1, ..., 16?

THE IDEA OF ELIMINATION ■ 2.2

This chapter explains a systematic way to solve linear equations. The method is called “**elimination**”, and you can see it immediately in our 2 by 2 example. Before elimination, x and y appear in both equations. After elimination, the first unknown x has disappeared from the second equation:

Before	$x - 2y = 1$	After	$x - 2y = 1$	<i>(multiply by 3 and subtract)</i>
	$3x + 2y = 11$		$8y = 8$	<i>(x has been eliminated)</i>

The last equation $8y = 8$ instantly gives $y = 1$. Substituting for y in the first equation leaves $x - 2 = 1$. Therefore $x = 3$ and the solution $(x, y) = (3, 1)$ is complete.

Elimination produces an **upper triangular system**—this is the goal. The nonzero coefficients 1, -2 , 8 form a triangle. The last equation $8y = 8$ reveals $y = 1$, and we go up the triangle to x . This quick process is called **back substitution**. It is used for upper triangular systems of any size, after forward elimination is complete.

Important point: The original equations have the same solution $x = 3$ and $y = 1$. Figure 2.5 repeats this original system as a pair of lines, intersecting at the solution point $(3, 1)$. After elimination, the lines still meet at the same point! One line is horizontal because its equation $8y = 8$ does not contain x .

How did we get from the first pair of lines to the second pair? We subtracted 3 times the first equation from the second equation. The step that eliminates x from equation 2 is the fundamental operation in this chapter. We use it so often that we look at it closely:

To eliminate x : Subtract a multiple of equation 1 from equation 2.

Three times $x - 2y = 1$ gives $3x - 6y = 3$. When this is subtracted from $3x + 2y = 11$, the right side becomes 8. The main point is that $3x$ cancels $3x$. What remains on the left side is $2y - (-6y)$ or $8y$, and x is eliminated.

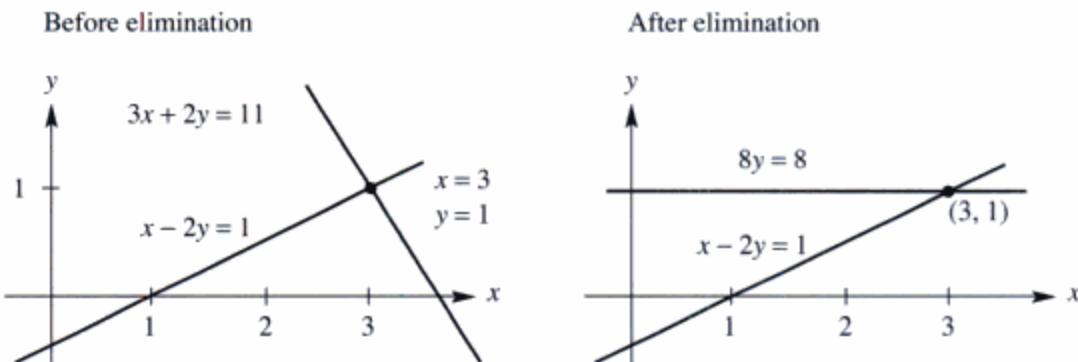


Figure 2.5 Two lines meet at the solution. So does the new line $8y = 8$.

Ask yourself how that multiplier $\ell = 3$ was found. The first equation contains x . **The first pivot is 1** (the coefficient of x). The second equation contains $3x$, so the first equation was multiplied by 3. Then subtraction $3x - 3x$ produced the zero.

You will see the multiplier rule if we change the first equation to $4x - 8y = 4$. (Same straight line but the first pivot becomes 4.) The correct multiplier is now $\ell = \frac{3}{4}$. *To find the multiplier, divide the coefficient “3” to be eliminated by the pivot “4”:*

$$\begin{array}{l} 4x - 8y = 4 \\ 3x + 2y = 11 \end{array} \quad \begin{array}{l} \text{Multiply equation 1 by } \frac{3}{4} \\ \text{Subtract from equation 2} \end{array} \quad \begin{array}{l} 4x - 8y = 4 \\ 8y = 8. \end{array}$$

The final system is triangular and the last equation still gives $y = 1$. Back substitution produces $4x - 8 = 4$ and $4x = 12$ and $x = 3$. We changed the numbers but not the lines or the solution. *Divide by the pivot to find that multiplier $\ell = \frac{3}{4}$:*

$$\begin{array}{ll} \text{Pivot} & = \text{first nonzero in the row that does the elimination} \\ \text{Multiplier} & = (\text{entry to eliminate}) \text{ divided by (pivot)} = \frac{3}{4}. \end{array}$$

The new second equation starts with the second pivot, which is 8. We would use it to eliminate y from the third equation if there were one. *To solve n equations we want n pivots. The pivots are on the diagonal of the triangle after elimination.*

You could have solved those equations for x and y without reading this book. It is an extremely humble problem, but we stay with it a little longer. Even for a 2 by 2 system, elimination might break down and we have to see how. By understanding the possible breakdown (when we can't find a full set of pivots), you will understand the whole process of elimination.

Breakdown of Elimination

Normally, elimination produces the pivots that take us to the solution. But failure is possible. At some point, the method might ask us to *divide by zero*. We can't do it. The process has to stop. There might be a way to adjust and continue—or failure may be unavoidable. Example 1 fails with no solution. Example 2 fails with too many solutions. Example 3 succeeds by exchanging the equations.

Example 1 Permanent failure with no solution. Elimination makes this clear:

$$\begin{array}{l} x - 2y = 1 \\ 3x - 6y = 11 \end{array} \quad \begin{array}{l} \text{Subtract 3 times eqn. 1 from eqn. 2} \\ \text{0y = 8.} \end{array} \quad \begin{array}{l} x - 2y = 1 \\ 0y = 8. \end{array}$$

The last equation is $0y = 8$. There is *no* solution. Normally we divide the right side 8 by the second pivot, but *this system has no second pivot. (Zero is never allowed as a pivot!)* The row and column pictures of this 2 by 2 system show that failure was unavoidable. If there is no solution, elimination must certainly have trouble.

The row picture in Figure 2.6 shows parallel lines—which never meet. A solution must lie on both lines. With no meeting point, the equations have no solution.

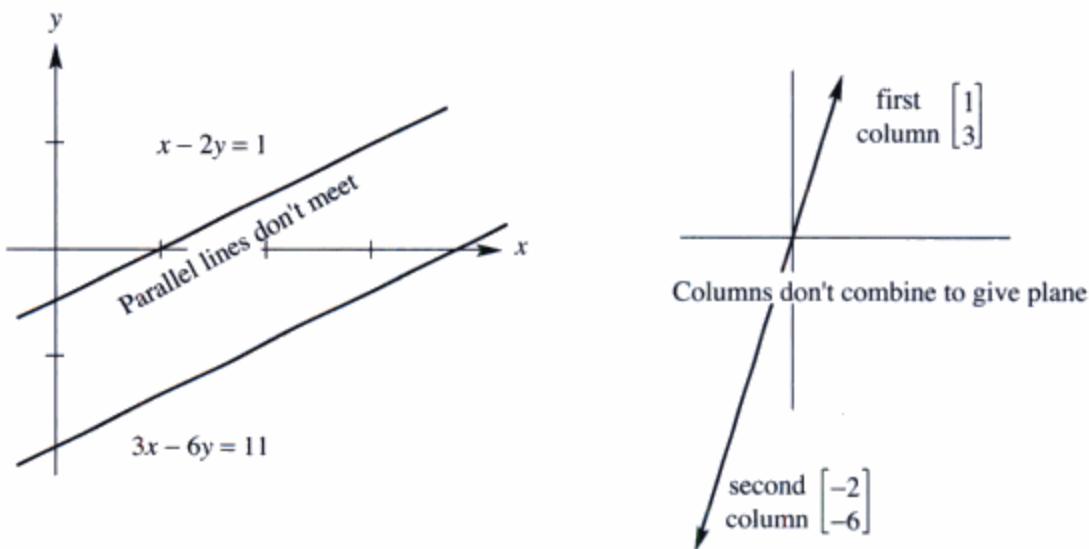


Figure 2.6 Row picture and column picture for Example 1: *no solution*.

The column picture shows the two columns $(1, 3)$ and $(-2, -6)$ in the same direction. *All combinations of the columns lie along a line.* But the column from the right side is in a different direction $(1, 11)$. No combination of the columns can produce this right side—therefore no solution.

When we change the right side to $(1, 3)$, failure shows as a whole line of solutions. Instead of no solution there are infinitely many:

Example 2 Permanent failure with infinitely many solutions:

$$\begin{array}{l} x - 2y = 1 \quad \text{Subtract 3 times} \\ 3x - 6y = 3 \quad \text{eqn. 1 from eqn. 2} \end{array} \qquad \begin{array}{l} x - 2y = 1 \\ 0y = 0. \end{array}$$

Every y satisfies $0y = 0$. There is really only one equation $x - 2y = 1$. The unknown y is “*free*”. After y is freely chosen, x is determined as $x = 1 + 2y$.

In the row picture, the parallel lines have become the same line. Every point on that line satisfies both equations. We have a whole line of solutions.

In the column picture, the right side $(1, 3)$ is now the same as the first column. So we can choose $x = 1$ and $y = 0$. We can also choose $x = 0$ and $y = -\frac{1}{2}$; the second column times $-\frac{1}{2}$ equals the right side. There are infinitely many other solutions. Every (x, y) that solves the row problem also solves the column problem.

Elimination can go wrong in a third way—but this time it can be fixed. *Suppose the first pivot position contains zero.* We refuse to allow zero as a pivot. When the first equation has no term involving x , we can exchange it with an equation below:

Example 3 Temporary failure but a row exchange produces two pivots:

$$\begin{array}{ll} 0x + 2y = 4 & \text{Exchange the} \\ 3x - 2y = 5 & \text{two equations} \end{array} \qquad \begin{array}{l} 3x - 2y = 5 \\ 2y = 4. \end{array}$$

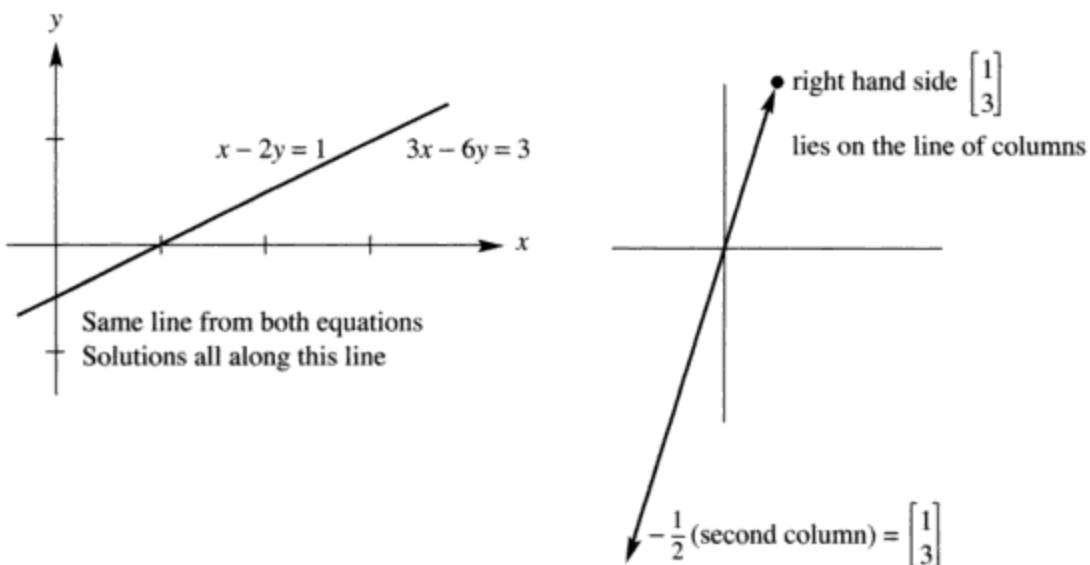


Figure 2.7 Row and column pictures for Example 2: *infinitely many solutions*.

The new system is already triangular. This small example is ready for back substitution. The last equation gives $y = 2$, and then the first equation gives $x = 3$. The row picture is normal (two intersecting lines). The column picture is also normal (column vectors not in the same direction). The pivots 3 and 2 are normal—but an exchange was required to put the rows in a good order.

Examples 1 and 2 are **singular**—there is no second pivot. Example 3 is **nonsingular**—there is a full set of pivots and exactly one solution. Singular equations have no solution or infinitely many solutions. Pivots must be nonzero because we have to divide by them.

Three Equations in Three Unknowns

To understand Gaussian elimination, you have to go beyond 2 by 2 systems. Three by three is enough to see the pattern. For now the matrices are square—an equal number of rows and columns. Here is a 3 by 3 system, specially constructed so that all steps lead to whole numbers and not fractions:

$$\begin{aligned} \mathbf{2}x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \tag{1}$$

What are the steps? The first pivot is the boldface 2 (upper left). Below that pivot we want to create zeros. The first multiplier is the ratio $4/2 = 2$. Multiply the pivot equation by $\ell_{21} = 2$ and subtract. Subtraction removes the $4x$ from the second equation:

Step 1 Subtract 2 times equation 1 from equation 2.

We also eliminate $-2x$ from equation 3—still using the first pivot. The quick way is to add equation 1 to equation 3. Then $2x$ cancels $-2x$. We do exactly that, but the rule in this book is to *subtract rather than add*. The systematic pattern has multiplier $\ell_{31} = -2/2 = -1$. Subtracting -1 times an equation is the same as adding:

Step 2 Subtract -1 times equation 1 from equation 3.

The two new equations involve only y and z . The second pivot (boldface) is 1:

$$\begin{aligned} 1y + 1z &= 4 \\ 1y + 5z &= 12 \end{aligned}$$

We have reached a 2 by 2 system. The final step eliminates y to make it 1 by 1:

Step 3 Subtract equation 2_{new} from 3_{new}. The multiplier is 1. Then $4z = 8$.

The original system $Ax = b$ has been converted into a triangular system $Ux = c$:

$2x + 4y - 2z = 2$	has become	$2x + 4y - 2z = 2$
$4x + 9y - 3z = 8$		$1y + 1z = 4$
$-2x - 3y + 7z = 10$		$4z = 8$.

(2)

The goal is achieved—forward elimination is complete. *Notice the pivots 2,1,4 along the diagonal.* Those pivots 1 and 4 were hidden in the original system! Elimination brought them out. This triangle is ready for back substitution, which is quick:

$$(4z = 8 \text{ gives } z = 2) \quad (y + z = 4 \text{ gives } y = 2) \quad (\text{equation 1 gives } x = -1)$$

The solution is $(x, y, z) = (-1, 2, 2)$. The row picture has three planes from three equations. All the planes go through this solution. The original planes are sloping, but the last plane $4z = 8$ after elimination is horizontal.

The column picture shows a combination of column vectors producing the right side b . The coefficients in that combination Ax are $-1, 2, 2$ (the solution):

$$(-1) \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} \text{ equals } \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}. \quad (3)$$

The numbers x, y, z multiply columns 1, 2, 3 in the original system $Ax = b$ and also in the triangular system $Ux = c$.

For a 4 by 4 problem, or an n by n problem, elimination proceeds the same way. Here is the whole idea of forward elimination, column by column:

Column 1. Use the first equation to create zeros below the first pivot.

Column 2. Use the new equation 2 to create zeros below the second pivot.

Columns 3 to n . Keep going to find the other pivots and the triangular U .

After column 2 we have $\begin{bmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ 0 & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ 0 & 0 & \mathbf{x} & \mathbf{x} \\ 0 & 0 & \mathbf{x} & \mathbf{x} \end{bmatrix}$. We want $\begin{bmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \end{bmatrix}$. (4)

The result of forward elimination is an upper triangular system. It is nonsingular if there is a full set of n pivots (never zero!). *Question:* Which x could be changed to boldface \mathbf{x} because the pivot is known? Here is a final example to show the original $Ax = b$, the triangular system $Ux = c$, and the solution from back substitution:

$$\begin{array}{l} x + y + z = 6 \\ x + 2y + 2z = 9 \\ x + 2y + 3z = 10 \end{array} \quad \begin{array}{l} x + y + z = 6 \\ y + z = 3 \\ z = 1 \end{array} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

All multipliers are 1. All pivots are 1. All planes meet at the solution $(3, 2, 1)$. The columns combine with coefficients 3, 2, 1 to give $b = (6, 9, 10)$ and $c = (6, 3, 1)$.

■ REVIEW OF THE KEY IDEAS ■

1. A linear system becomes upper triangular after elimination.
2. The upper triangular system is solved by back substitution (starting at the bottom).
3. Elimination subtracts ℓ_{ij} times equation j from equation i , to make the (i, j) entry zero.
4. The multiplier is $\ell_{ij} = \frac{\text{entry to eliminate in row } i}{\text{pivot in row } j}$. Pivots can not be zero!
5. A zero in the pivot position can be repaired if there is a nonzero below it.
6. When breakdown is permanent, the system has no solution or infinitely many.

■ WORKED EXAMPLES ■

2.2 A When elimination is applied to this matrix A , what are the first and second pivots? What is the multiplier ℓ_{21} in the first step (ℓ_{21} times row 1 is subtracted from row 2)? What entry in the 2, 2 position (instead of 9) would force an exchange of rows 2 and 3? Why is the multiplier $\ell_{31} = 0$, subtracting 0 times row 1 from row 3?

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 6 & 9 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

Solution The first pivot is 3. The multiplier ℓ_{21} is $\frac{6}{3} = 2$. When 2 times row 1 is subtracted from row 2, the second pivot is revealed as 7. If we reduce the entry “9” to “2”, that drop of 7 in the (2, 2) position would force a row exchange. (The second row would start with 6, 2 which is an exact multiple of 3, 1 in the first row. Zero will appear in the second pivot position.) The multiplier ℓ_{31} is zero because $a_{31} = 0$. A zero at the start of a row needs no elimination.

2.2 B Use elimination to reach upper triangular matrices U . Solve by back substitution or explain why this is impossible. What are the pivots (never zero)? Exchange equations when necessary. The only difference is the $-x$ in equation (3).

$$\begin{array}{ll} x + y + z = 7 & x + y + z = 7 \\ x + y - z = 5 & x + y - z = 5 \\ x - y + z = 3 & -x - y + z = 3 \end{array}$$

Solution For the first system, subtract equation 1 from equations 2 and 3 (the multipliers are $\ell_{21} = 1$ and $\ell_{31} = 1$). The 2, 2 entry becomes zero, so exchange equations:

$$\begin{array}{ll} x + y + z = 7 & x + y + z = 7 \\ 0y - 2z = -2 & \text{exchanges into} \\ -2y + 0z = -4 & -2y + 0z = -4 \\ & -2z = -2 \end{array}$$

Then back substitution gives $z = 1$ and $y = 2$ and $x = 4$. The pivots are 1, -2 , -2 .

For the second system, subtract equation 1 from equation 2 as before. Add equation 1 to equation 3. This leaves zero in the 2, 2 entry *and below*:

$$\begin{array}{ll} x + y + z = 7 & \text{There is } \textit{no pivot} \text{ in column 2.} \\ 0y - 2z = -2 & \text{A further elimination step gives } 0z = 8 \\ 0y + 2z = 10 & \text{The three planes don't meet!} \end{array}$$

Plane 1 meets plane 2 in a line. Plane 1 meets plane 3 in a parallel line. *No solution*.

If we change the “3” in the original third equation to “−5” then elimination would leave $2z = 2$ instead of $2z = 10$. Now $z = 1$ would be consistent—we have moved the third plane. Substituting $z = 1$ in the first equation leaves $x + y = 6$. There are infinitely many solutions! *The three planes now meet along a whole line*.

Problem Set 2.2

Problems 1–10 are about elimination on 2 by 2 systems.

- 1 What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11. \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. The numbers 1 and 11 have no influence on those pivots.

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- 2 Solve the triangular system of Problem 1 by back substitution, y before x . Verify that x times $(2, 10)$ plus y times $(3, 9)$ equals $(1, 11)$. If the right side changes to $(4, 44)$, what is the new solution?
- 3 What multiple of equation 1 should be *subtracted* from equation 2?

$$\begin{aligned} 2x - 4y &= 6 \\ -x + 5y &= 0. \end{aligned}$$

After this elimination step, solve the triangular system. If the right side changes to $(-6, 0)$, what is the new solution?

- 4 What multiple ℓ of equation 1 should be subtracted from equation 2?

$$\begin{aligned} ax + by &= f \\ cx + dy &= g. \end{aligned}$$

The first pivot is a (assumed nonzero). Elimination produces what formula for the second pivot? What is y ? The second pivot is missing when $ad = bc$.

- 5 Choose a right side which gives no solution and another right side which gives infinitely many solutions. What are two of those solutions?

$$\begin{aligned} 3x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

- 6 Choose a coefficient b that makes this system singular. Then choose a right side g that makes it solvable. Find two solutions in that singular case.

$$\begin{aligned} 2x + by &= 16 \\ 4x + 8y &= g. \end{aligned}$$

- 7 For which numbers a does elimination break down (1) permanently (2) temporarily?

$$\begin{aligned} ax + 3y &= -3 \\ 4x + 6y &= -6. \end{aligned}$$

Solve for x and y after fixing the second breakdown by a row exchange.

- 8 For which three numbers k does elimination break down? Which is fixed by a row exchange? In each case, is the number of solutions 0 or 1 or ∞ ?

$$\begin{aligned} kx + 3y &= 6 \\ 3x + ky &= -6. \end{aligned}$$

- 9** What test on b_1 and b_2 decides whether these two equations allow a solution? How many solutions will they have? Draw the column picture.

$$3x - 2y = b_1$$

$$6x - 4y = b_2.$$

- 10** In the xy plane, draw the lines $x + y = 5$ and $x + 2y = 6$ and the equation $y = \underline{\hspace{2cm}}$ that comes from elimination. The line $5x - 4y = c$ will go through the solution of these equations if $c = \underline{\hspace{2cm}}$.

Problems 11–20 study elimination on 3 by 3 systems (and possible failure).

- 11** Reduce this system to upper triangular form by two row operations:

$$2x + 3y + z = 8$$

$$4x + 7y + 5z = 20$$

$$-2y + 2z = 0.$$

Circle the pivots. Solve by back substitution for z, y, x .

- 12** Apply elimination (circle the pivots) and back substitution to solve

$$2x - 3y = 3$$

$$4x - 5y + z = 7$$

$$2x - y - 3z = 5.$$

List the three row operations: Subtract $\underline{\hspace{2cm}}$ times row $\underline{\hspace{2cm}}$ from row $\underline{\hspace{2cm}}$.

- 13** Which number d forces a row exchange, and what is the triangular system (not singular) for that d ? Which d makes this system singular (no third pivot)?

$$2x + 5y + z = 0$$

$$4x + dy + z = 2$$

$$y - z = 3.$$

- 14** Which number b leads later to a row exchange? Which b leads to a missing pivot? In that singular case find a nonzero solution x, y, z .

$$x + by = 0$$

$$x - 2y - z = 0$$

$$y + z = 0.$$

- 15** (a) Construct a 3 by 3 system that needs two row exchanges to reach a triangular form and a solution.
 (b) Construct a 3 by 3 system that needs a row exchange to keep going, but breaks down later.

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- 16** If rows 1 and 2 are the same, how far can you get with elimination (allowing row exchange)? If columns 1 and 2 are the same, which pivot is missing?

$$\begin{array}{ll} 2x - y + z = 0 & 2x + 2y + z = 0 \\ 2x - y + z = 0 & 4x + 4y + z = 0 \\ 4x + y + z = 2 & 6x + 6y + z = 2. \end{array}$$

- 17** Construct a 3 by 3 example that has 9 different coefficients on the left side, but rows 2 and 3 become zero in elimination. How many solutions to your system with $\mathbf{b} = (1, 10, 100)$ and how many with $\mathbf{b} = (0, 0, 0)$?
- 18** Which number q makes this system singular and which right side t gives it infinitely many solutions? Find the solution that has $z = 1$.

$$\begin{array}{l} x + 4y - 2z = 1 \\ x + 7y - 6z = 6 \\ 3y + qz = t. \end{array}$$

- 19** (Recommended) It is impossible for a system of linear equations to have exactly two solutions. *Explain why.*
- (a) If (x, y, z) and (X, Y, Z) are two solutions, what is another one?
 (b) If 25 planes meet at two points, where else do they meet?

- 20** Three planes can fail to have an intersection point, when no two planes are parallel. The system is singular if row 3 of A is a _____ of the first two rows. Find a third equation that can't be solved if $x + y + z = 0$ and $x - 2y - z = 1$.

Problems 21–23 move up to 4 by 4 and n by n .

- 21** Find the pivots and the solution for these four equations:

$$\begin{array}{ll} 2x + y & = 0 \\ x + 2y + z & = 0 \\ y + 2z + t = 0 & \\ z + 2t = 5. & \end{array}$$

- 22** This system has the same pivots and right side as Problem 21. How is the solution different (if it is)?

$$\begin{array}{ll} 2x - y & = 0 \\ -x + 2y - z & = 0 \\ -y + 2z - t = 0 & \\ -z + 2t = 5. & \end{array}$$

- 23 If you extend Problems 21–22 following the 1, 2, 1 pattern or the $-1, 2, -1$ pattern, what is the fifth pivot? What is the n th pivot?
- 24 If elimination leads to these equations, find three possible original matrices A :

$$\begin{aligned}x + y + z &= 0 \\y + z &= 0 \\3z &= 0.\end{aligned}$$

- 25 For which two numbers a will elimination fail on $A = \begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$?
- 26 For which three numbers a will elimination fail to give three pivots?

$$A = \begin{bmatrix} a & 2 & 3 \\ a & a & 4 \\ a & a & a \end{bmatrix}.$$

- 27 Look for a matrix that has row sums 4 and 8, and column sums 2 and s :

$$\text{Matrix } = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{array}{ll} a+b=4 & a+c=2 \\ c+d=8 & b+d=s \end{array}$$

The four equations are solvable only if $s = \underline{\hspace{2cm}}$. Then find two different matrices that have the correct row and column sums. *Extra credit:* Write down the 4 by 4 system $Ax = b$ with $x = (a, b, c, d)$ and make A triangular by elimination.

- 28 Elimination in the usual order gives what pivot matrix and what solution to this “lower triangular” system? We are really solving by *forward substitution*:

$$\begin{aligned}3x &= 3 \\6x + 2y &= 8 \\9x - 2y + z &= 9.\end{aligned}$$

- 29 Create a MATLAB command $A(2, :) = \dots$ for the new row 2, to subtract 3 times row 1 from the existing row 2 if the matrix A is already known.
- 30 Find experimentally the average first and second and third pivot sizes (use the absolute value) in MATLAB’s $A = \text{rand}(3, 3)$. The average of $\text{abs}(A(1, 1))$ should be 0.5 but I don’t know the others.

ELIMINATION USING MATRICES ■ 2.3

We now combine two ideas—elimination and matrices. The goal is to express all the steps of elimination (and the final result) in the clearest possible way. In a 3 by 3 example, elimination could be described in words. For larger systems, a long list of steps would be hopeless. You will see how to subtract a multiple of one row from another row—*using matrices*.

The matrix form of a linear system is $Ax = b$. Here are b , x , and A :

- 1 The vector of right sides is b .
- 2 The vector of unknowns is x . (The unknowns change to x_1 , x_2 , x_3 , . . . because we run out of letters before we run out of numbers.)
- 3 The coefficient matrix is A . In this chapter A is square.

The example in the previous section has the beautifully short form $Ax = b$:

$$\begin{array}{l} 2x_1 + 4x_2 - 2x_3 = 2 \\ 4x_1 + 9x_2 - 3x_3 = 8 \\ -2x_1 - 3x_2 + 7x_3 = 10 \end{array} \quad \text{is the same as} \quad \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}. \quad (1)$$

The nine numbers on the left go into the matrix A . That matrix not only sits beside x , it *multiplies* x . The rule for “ A times x ” is exactly chosen to yield the three equations.

Review of A times x . A matrix times a vector gives a vector. The matrix is square when the number of equations (three) matches the number of unknowns (three). Our matrix is 3 by 3. A general square matrix is n by n . Then the vector x is in n -dimensional space. This example is in 3-dimensional space:

$$\text{The unknown is } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and the solution is } x = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

Key point: $Ax = b$ represents the row form and also the column form of the equations. We can multiply by taking a column of A at a time:

$$Ax = (-1) \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}. \quad (2)$$

This rule is used so often that we express it once more for emphasis.

2A *The product Ax is a combination of the columns of A .* Components of x multiply columns: $Ax = x_1 \text{ times (column 1)} + \cdots + x_n \text{ times (column } n\text{)}$.

One point to repeat about matrix notation: The entry in row 1, column 1 (the top left corner) is called a_{11} . The entry in row 1, column 3 is a_{13} . The entry in row 3, column 1 is a_{31} . (Row number comes before column number.) The word “entry” for a matrix corresponds to the word “component” for a vector. General rule: **The entry in row i , column j of the matrix A is a_{ij} .**

Example 1 This matrix has $a_{ij} = 2i + j$. Then $a_{11} = 3$. Also $a_{12} = 4$ and $a_{21} = 5$. Here is Ax with numbers and letters:

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 4 \cdot 1 \\ 5 \cdot 2 + 6 \cdot 1 \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix}.$$

The first component of Ax is $6 + 4 = 10$. That is the product of the row $[3 \ 4]$ with the column $(2, 1)$. **A row times a column gives a dot product!**

The i th component of Ax involves row i , which is $[a_{i1} \ a_{i2} \ \dots \ a_{in}]$. The short formula for its dot product with x uses “sigma notation”:

2B The i th component of Ax is $a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n$. This is

$$\sum_{j=1}^n a_{ij}x_j.$$

The sigma symbol \sum is an instruction to add. Start with $j = 1$ and stop with $j = n$. Start the sum with $a_{i1}x_1$ and stop with $a_{in}x_n$.¹

The Matrix Form of One Elimination Step

$Ax = b$ is a convenient form for the original equation. What about the elimination steps? The first step in this example subtracts 2 times the first equation from the second equation. On the right side, 2 times the first component of b is subtracted from the second component:

$$b = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \text{ changes to } b_{\text{new}} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}.$$

We want to do that subtraction with a matrix! The same result $b_{\text{new}} = Eb$ is achieved when we multiply an “elimination matrix” E times b . It subtracts $2b_1$ from b_2 :

$$\text{The elimination matrix is } E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplication by E subtracts 2 times row 1 from row 2. Rows 1 and 3 stay the same:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

Notice how $b_1 = 2$ and $b_3 = 10$ stay the same. The first and third rows of E are the first and third rows of the identity matrix I . The new second component is the number 4 that appeared after the elimination step. This is $b_2 - 2b_1$.

¹Einstein shortened this even more by omitting the \sum . The repeated j in $a_{ij}x_j$ automatically meant addition. He also wrote the sum as $a_i^j x_j$. Not being Einstein, we include the \sum .

It is easy to describe the “elementary matrices” or “elimination matrices” like E . Start with the identity matrix I . *Change one of its zeros to the multiplier $-\ell$:*

2C The *identity matrix* has 1’s on the diagonal and otherwise 0’s. Then $Ib = b$.
The *elementary matrix or elimination matrix* E_{ij} that subtracts a multiple ℓ of row j from row i has the extra nonzero entry $-\ell$ in the i, j position.

Example 2

$$\text{Identity } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Elimination } E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

When you multiply I times b , you get b . But E_{31} subtracts ℓ times the first component from the third component. With $\ell = 4$ we get $9 - 4 = 5$:

$$Ib = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} \quad \text{and} \quad Eb = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

What about the left side of $Ax = b$? The multiplier $\ell = 4$ was chosen to produce a zero, by subtracting 4 times the pivot. E_{31} **creates a zero in the (3, 1) position**.

The notation fits this purpose. Start with A . Apply E ’s to produce zeros below the pivots (the first E is E_{21}). End with a triangular U . We now look in detail at those steps.

First a small point. The vector x stays the same. The solution is not changed by elimination. (That may be more than a small point.) It is the coefficient matrix that is changed! When we start with $Ax = b$ and multiply by E , the result is $EAx = Eb$. The new matrix EA is the result of multiplying E times A .

Matrix Multiplication

The big question is: **How do we multiply two matrices?** When the first matrix is E (an elimination matrix), there is already an important clue. We know A , and we know what it becomes after the elimination step. To keep everything right, we hope and expect that EA is

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix} \quad (\text{with the zero}).$$

This step does not change rows 1 and 3 of A . Those rows are unchanged in EA —only row 2 is different. *Twice the first row has been subtracted from the second row.* Matrix multiplication agrees with elimination—and the new system of equations is $EAx = Eb$.

EAx is simple but it involves a subtle idea. Multiplying both sides of the original equation gives $E(Ax) = Eb$. With our proposed multiplication of matrices, this is also

$(EA)x = Eb$. The first was E times Ax , the second is EA times x . They are the same! The parentheses are not needed. We just write $EAx = Eb$.

When multiplying ABC , you can do BC first or you can do AB first. This is the point of an “associative law” like $3 \times (4 \times 5) = (3 \times 4) \times 5$. We multiply 3 times 20, or we multiply 12 times 5. Both answers are 60. That law seems so obvious that it is hard to imagine it could be false. But the “commutative law” $3 \times 4 = 4 \times 3$ looks even more obvious. For matrices, EA is different from AE .

2D ASSOCIATIVE LAW

$$A(BC) = (AB)C$$

NOT COMMUTATIVE LAW

$$\text{Often } AB \neq BA.$$

There is another requirement on matrix multiplication. Suppose B has only one column (this column is b). The matrix-matrix law for EB should be consistent with the old matrix-vector law for Eb . Even more, we should be able to *multiply matrices a column at a time*:

If B has several columns b_1, b_2, b_3 , then EB has columns Eb_1, Eb_2, Eb_3 .

This holds true for the matrix multiplication above (where the matrix is A instead of B). If you multiply column 1 of A by E , you get column 1 of EA :

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and } E(\text{column } j \text{ of } A) = \text{column } j \text{ of } EA.$$

This requirement deals with columns, while elimination deals with rows. The next section describes each individual entry of the product. The beauty of matrix multiplication is that all three approaches (rows, columns, whole matrices) come out right.

The Matrix P_{ij} for a Row Exchange

To subtract row j from row i we use E_{ij} . To exchange or “permute” those rows we use another matrix P_{ij} . Row exchanges are needed when zero is in the pivot position. Lower down that pivot column may be a nonzero. By exchanging the two rows, we have a pivot (never zero!) and elimination goes forward.

What matrix P_{23} exchanges row 2 with row 3? We can find it by exchanging rows of the identity matrix I :

Permutation matrix $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

This is a **row exchange matrix**. Multiplying by P_{23} exchanges components 2 and 3 of any column vector. Therefore it also exchanges rows 2 and 3 of any matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}.$$

On the right, P_{23} is doing what it was created for. With zero in the second pivot position and “6” below it, the exchange puts 6 into the pivot.

Matrices *act*. They don’t just sit there. We will soon meet other permutation matrices, which can change the order of several rows. Rows 1, 2, 3 can be moved to 3, 1, 2. Our P_{23} is one particular permutation matrix—it exchanges rows 2 and 3.

2E Row Exchange Matrix P_{ij} is the identity matrix with rows i and j reversed. When P_{ij} multiplies a matrix A , it exchanges rows i and j of A .

To exchange equations 1 and 3 multiply by $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Usually row exchanges are not required. The odds are good that elimination uses only the E_{ij} . But the P_{ij} are ready if needed, to move a pivot up to the diagonal.

The Augmented Matrix

This book eventually goes far beyond elimination. Matrices have all kinds of practical applications, in which they are multiplied. Our best starting point was a square E times a square A , because we met this in elimination—and we know what answer to expect for EA . The next step is to allow a *rectangular matrix*. It still comes from our original equations, but now it includes the right side b .

Key idea: Elimination does the same row operations to A and to b . **We can include b as an extra column and follow it through elimination.** The matrix A is enlarged or “augmented” by the extra column b :

$$\text{Augmented matrix } [A \ b] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}.$$

Elimination acts on whole rows of this matrix. The left side and right side are both multiplied by E , to subtract 2 times equation 1 from equation 2. With $[A \ b]$ those steps happen together:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}.$$

The new second row contains 0, 1, 1, 4. The new second equation is $x_2 + x_3 = 4$. Matrix multiplication works by rows and at the same time by columns:

- R** (by rows): Each row of E acts on $[A \ b]$ to give a row of $[EA \ Eb]$.
- C** (by columns): E acts on each column of $[A \ b]$ to give a column of $[EA \ Eb]$.

Notice again that word “acts.” This is essential. Matrices do something! The matrix A acts on x to produce b . The matrix E operates on A to give EA . The whole process of elimination is a sequence of row operations, alias matrix multiplications. A goes to $E_{21}A$ which goes to $E_{31}E_{21}A$. Finally $E_{32}E_{31}E_{21}A$ is a triangular matrix.

The right side is included in the augmented matrix. The end result is a triangular system of equations. We stop for exercises on multiplication by E , before writing down the rules for all matrix multiplications (including block multiplication).

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = x_1$ times column 1 + \cdots + x_n times column n . And $(Ax)_i = \sum_{j=1}^n a_{ij}x_j$.
2. Identity matrix = I , elimination matrix = E_{ij} , exchange matrix = P_{ij} .
3. Multiplying $Ax = b$ by E_{21} subtracts a multiple ℓ_{21} of equation 1 from equation 2. The number $-\ell_{21}$ is the (2, 1) entry of the elimination matrix E_{21} .
4. For the augmented matrix $[A \ b]$, that elimination step gives $[E_{21}A \ E_{21}b]$.
5. When A multiplies any matrix B , it multiplies each column of B separately.

■ WORKED EXAMPLES ■

2.3 A What 3 by 3 matrix E_{21} subtracts 4 times row 1 from row 2? What matrix P_{32} exchanges row 2 and row 3? If you multiply A on the *right* instead of the left, describe the results AE_{21} and AP_{32} .

Solution By doing those operations on the identity matrix I , we find

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Multiplying by E_{21} on the right side will subtract 4 times **column 2** from **column 1**. Multiplying by P_{32} on the right will exchange **columns 2** and **3**.

2.3 B Write down the augmented matrix $[A \ b]$ with an extra column:

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

Apply E_{21} and then P_{32} to reach a triangular system. Solve by back substitution. What combined matrix $P_{32}E_{21}$ will do both steps at once?

Solution The augmented matrix and the result of using E_{21} are

$$[A \ b] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad E_{21}[A \ b] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

P_{32} exchanges equation 2 and 3. Back substitution produces (x, y, z) :

$$P_{32} E_{21}[A \ b] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

For the matrix $P_{32} E_{21}$ that does both steps at once, apply P_{32} to E_{21} !

$$P_{32} E_{21} = \text{exchange the rows of } E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 1 & 0 \end{bmatrix}.$$

2.3 C Multiply these matrices in two ways: first, rows of A times columns of B to find each entry of AB , and second, columns of A times rows of B to produce two matrices that add to AB . How many separate ordinary multiplications are needed?

$$AB = \begin{bmatrix} 3 & 4 \\ 1 & 5 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} = (3 \text{ by } 2)(2 \text{ by } 2)$$

Solution Rows of A times columns of B are dot products of vectors:

$$(\text{row 1}) \cdot (\text{column 1}) = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 10 \quad \text{is the (1, 1) entry of } AB$$

$$(\text{row 2}) \cdot (\text{column 1}) = \begin{bmatrix} 1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 7 \quad \text{is the (2, 1) entry of } AB$$

The first columns of AB are $(10, 7, 4)$ and $(16, 9, 8)$. We need 6 dot products, 2 multiplications each, 12 in all $(3 \cdot 2 \cdot 2)$. The same AB comes from *columns of A times rows of B*:

$$AB = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 5 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 16 \\ 7 & 9 \\ 4 & 8 \end{bmatrix}.$$

Problem Set 2.3

Problems 1–15 are about elimination matrices.

- 1 Write down the 3 by 3 matrices that produce these elimination steps:
 - E_{21} subtracts 5 times row 1 from row 2.
 - E_{32} subtracts -7 times row 2 from row 3.
 - P exchanges rows 1 and 2, then rows 2 and 3.
- 2 In Problem 1, applying E_{21} and then E_{32} to the column $\mathbf{b} = (1, 0, 0)$ gives $E_{32}E_{21}\mathbf{b} = \underline{\hspace{2cm}}$. Applying E_{32} before E_{21} gives $E_{21}E_{32}\mathbf{b} = \underline{\hspace{2cm}}$. When E_{32} comes first, row $\underline{\hspace{2cm}}$ feels no effect from row $\underline{\hspace{2cm}}$.
- 3 Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U ?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E 's to get one matrix M that does elimination: $MA = U$.

- 4 Include $\mathbf{b} = (1, 0, 0)$ as a fourth column in Problem 3 to produce $[A \ \mathbf{b}]$. Carry out the elimination steps on this augmented matrix to solve $A\mathbf{x} = \mathbf{b}$.
- 5 Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is $\underline{\hspace{2cm}}$. If you change a_{33} to $\underline{\hspace{2cm}}$, there is no third pivot.
- 6 If every column of A is a multiple of $(1, 1, 1)$, then $A\mathbf{x}$ is always a multiple of $(1, 1, 1)$. Do a 3 by 3 example. How many pivots are produced by elimination?
- 7 Suppose E_{31} subtracts 7 times row 1 from row 3. To reverse that step you should $\underline{\hspace{2cm}}$ 7 times row $\underline{\hspace{2cm}}$ to row $\underline{\hspace{2cm}}$. This “inverse matrix” is $R_{31} = \underline{\hspace{2cm}}$.
- 8 Suppose E_{31} subtracts 7 times row 1 from row 3. What matrix R_{31} is changed into I ? Then $E_{31}R_{31} = I$ where Problem 7 has $R_{31}E_{31} = I$. Both are true!
- 9
 - E_{21} subtracts row 1 from row 2 and then P_{23} exchanges rows 2 and 3. What matrix $M = P_{23}E_{21}$ does both steps at once?
 - P_{23} exchanges rows 2 and 3 and then E_{31} subtracts row 1 from row 3. What matrix $M = E_{31}P_{23}$ does both steps at once? Explain why the M 's are the same but the E 's are different.
- 10
 - What 3 by 3 matrix E_{13} will add row 3 to row 1?
 - What matrix adds row 1 to row 3 and *at the same time* row 3 to row 1?
 - What matrix adds row 1 to row 3 and *then* adds row 3 to row 1?

- 11 Create a matrix that has $a_{11} = a_{22} = a_{33} = 1$ but elimination produces two negative pivots without row exchanges. (The first pivot is 1.)

- 12 Multiply these matrices:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

- 13 Explain these facts. If the third column of B is all zero, the third column of EB is all zero (for any E). If the third row of B is all zero, the third row of EB might *not* be zero.
- 14 This 4 by 4 matrix will need elimination matrices E_{21} and E_{32} and E_{43} . What are those matrices?

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

- 15 Write down the 3 by 3 matrix that has $a_{ij} = 2i - 3j$. This matrix has $a_{32} = 0$, but elimination still needs E_{32} to produce a zero in the 3, 2 position. Which previous step destroys the original zero and what is E_{32} ?

Problems 16–23 are about creating and multiplying matrices.

- 16 Write these ancient problems in a 2 by 2 matrix form $Ax = b$ and solve them:
- X is twice as old as Y and their ages add to 33.
 - $(x, y) = (2, 5)$ and $(3, 7)$ lie on the line $y = mx + c$. Find m and c .
- 17 The parabola $y = a + bx + cx^2$ goes through the points $(x, y) = (1, 4)$ and $(2, 8)$ and $(3, 14)$. Find and solve a matrix equation for the unknowns (a, b, c) .
- 18 Multiply these matrices in the orders EF and FE and E^2 :

$$E = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}.$$

Also compute $E^2 = EE$ and $F^3 = FFF$.

- 19 Multiply these row exchange matrices in the orders PQ and QP and P^2 :

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Find four matrices whose squares are $M^2 = I$.

- 20** (a) Suppose all columns of B are the same. Then all columns of EB are the same, because each one is E times ____.
- (b) Suppose all rows of B are $[1 \ 2 \ 4]$. Show by example that all rows of EB are *not* $[1 \ 2 \ 4]$. It is true that those rows are ____.
- 21** If E adds row 1 to row 2 and F adds row 2 to row 1, does EF equal FE ?
- 22** The entries of A and \mathbf{x} are a_{ij} and x_j . So the first component of $A\mathbf{x}$ is $\sum a_{1j}x_j = a_{11}x_1 + \dots + a_{1n}x_n$. If E_{21} subtracts row 1 from row 2, write a formula for
- the third component of $A\mathbf{x}$
 - the $(2, 1)$ entry of $E_{21}A$
 - the $(2, 1)$ entry of $E_{21}(E_{21}A)$
 - the first component of $EA\mathbf{x}$.
- 23** The elimination matrix $E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ subtracts 2 times row 1 of A from row 2 of A . The result is EA . What is the effect of $E(EA)$? In the opposite order AE , we are subtracting 2 times ____ of A from _____. (Do examples.)

Problems 24–29 include the column b in the augmented matrix $[A \ b]$.

- 24** Apply elimination to the 2 by 3 augmented matrix $[A \ b]$. What is the triangular system $U\mathbf{x} = \mathbf{c}$? What is the solution \mathbf{x} ?

$$A\mathbf{x} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 17 \end{bmatrix}.$$

- 25** Apply elimination to the 3 by 4 augmented matrix $[A \ b]$. How do you know this system has no solution? Change the last number 6 so there *is* a solution.

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}.$$

- 26** The equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x}^* = \mathbf{b}^*$ have the same matrix A . What double augmented matrix should you use in elimination to solve both equations at once? Solve both of these equations by working on a 2 by 4 matrix:

$$\begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- 27** Choose the numbers a, b, c, d in this augmented matrix so that there is (a) no solution (b) infinitely many solutions.

$$[A \ b] = \begin{bmatrix} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{bmatrix}$$

Which of the numbers a, b, c , or d have no effect on the solvability?

- 28 If $AB = I$ and $BC = I$ use the associative law to prove $A = C$.
- 29 Choose two matrices $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det M = ad - bc = 1$ and with a, b, c, d positive integers. Prove that every such matrix M either has

EITHER row 1 \leq row 2 OR row 2 \leq row 1.

Subtraction makes $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}M$ or $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}M$ nonnegative but smaller than M . If you continue and reach I , write your M 's as products of the inverses $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

- 30 Find the triangular matrix E that reduces “*Pascal's matrix*” to a smaller Pascal:

$$E \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Challenge question: Which M (from several E 's) reduces Pascal all the way to I ?

RULES FOR MATRIX OPERATIONS ■ 2.4

I will start with basic facts. A matrix is a rectangular array of numbers or “entries.” When A has m rows and n columns, it is an “ m by n ” matrix. Matrices can be added if their shapes are the same. They can be multiplied by any constant c . Here are examples of $A + B$ and $2A$, for 3 by 2 matrices:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 4 & 4 \\ 9 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 7 & 8 \\ 9 & 9 \end{bmatrix} \quad \text{and} \quad 2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \\ 0 & 0 \end{bmatrix}.$$

Matrices are added exactly as vectors are—one entry at a time. We could even regard a column vector as a matrix with only one column (so $n = 1$). The matrix $-A$ comes from multiplication by $c = -1$ (reversing all the signs). Adding A to $-A$ leaves the *zero matrix*, with all entries zero.

The 3 by 2 zero matrix is different from the 2 by 3 zero matrix. Even zero has a shape (several shapes) for matrices. All this is only common sense.

The entry in row i and column j is called a_{ij} or $A(i, j)$. The n entries along the first row are $a_{11}, a_{12}, \dots, a_{1n}$. The lower left entry in the matrix is a_{m1} and the lower right is a_{mn} . The row number i goes from 1 to m . The column number j goes from 1 to n .

Matrix addition is easy. The serious question is **matrix multiplication**. When can we multiply A times B , and what is the product AB ? We cannot multiply when A and B are 3 by 2. They don't pass the following test:

To multiply AB : *If A has n columns, B must have n rows.*

If A has two columns, B must have two rows. When A is 3 by 2, the matrix B can be 2 by 1 (a vector) or 2 by 2 (square) or 2 by 20. Every column of B is ready to be multiplied by A . Then AB is 3 by 1 (a vector) or 3 by 2 or 3 by 20.

Suppose A is m by n and B is n by p . We can multiply. The product AB is m by p .

$$\begin{bmatrix} \text{m rows} \\ n \text{ columns} \end{bmatrix} \begin{bmatrix} n \text{ rows} \\ \text{p columns} \end{bmatrix} = \begin{bmatrix} \text{m rows} \\ \text{p columns} \end{bmatrix}.$$

A row times a column is an extreme case. Then 1 by n multiplies n by 1. The result is 1 by 1. That single number is the “dot product.”

In every case AB is filled with dot products. For the top corner, the $(1, 1)$ entry of AB is (row 1 of A) \cdot (column 1 of B). To multiply matrices, take all these dot products: *(each row of A) \cdot (each column of B)*.

2F The entry in row i and column j of AB is $(\text{row } i \text{ of } A) \cdot (\text{column } j \text{ of } B)$.

Figure 2.8 picks out the second row ($i = 2$) of a 4 by 5 matrix A . It picks out the third column ($j = 3$) of a 5 by 6 matrix B . Their dot product goes into row 2 and column 3 of AB . The matrix AB has *as many rows as A* (4 rows), and *as many columns as B* .

$$\begin{bmatrix} * & & & & a_{i1} & a_{i2} & \cdots & a_{i5} \\ * & & & & b_{1j} & * & * & * \\ * & & & & b_{2j} & & & \\ * & & & & b_{3j} & & & \\ * & & & & b_{4j} & & & \end{bmatrix} = \begin{bmatrix} * & * & (AB)_{ij} & * & * & * \\ * & * & * & & & \\ * & * & * & & & \end{bmatrix}$$

A is 4 by 5 B is 5 by 6 AB is 4 by 6

Figure 2.8 Here $i = 2$ and $j = 3$. Then $(AB)_{23}$ is (row 2) \cdot (column 3) $= \sum a_{2k}b_{k3}$.

Example 1 Square matrices can be multiplied if and only if they have the same size:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}.$$

The first dot product is $1 \cdot 2 + 1 \cdot 3 = 5$. Three more dot products give 6, 1, and 0. Each dot product requires two multiplications—thus eight in all.

If A and B are n by n , so is AB . It contains n^2 dot products, row of A times column of B . Each dot product needs n multiplications, so *the computation of AB uses n^3 separate multiplications*. For $n = 100$ we multiply a million times. For $n = 2$ we have $n^3 = 8$.

Mathematicians thought until recently that AB absolutely needed $2^3 = 8$ multiplications. Then somebody found a way to do it with 7 (and extra additions). By breaking n by n matrices into 2 by 2 blocks, this idea also reduced the count for large matrices. Instead of n^3 it went below $n^{2.8}$, and the exponent keeps falling.¹ The best

¹Maybe the exponent won't stop falling before 2. No number in between looks special.

at this moment is n^2 .³⁷⁶ But the algorithm is so awkward that scientific computing is done the regular way: n^2 dot products in AB , and n multiplications for each one.

Example 2 Suppose A is a row vector (1 by 3) and B is a column vector (3 by 1). Then AB is 1 by 1 (only one entry, the dot product). On the other hand B times A (*a column times a row*) is a full 3 by 3 matrix. This multiplication is allowed!

$$\text{Column times row: } \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}.$$

A row times a column is an “inner” product—that is another name for dot product. A column times a row is an “outer” product. These are extreme cases of matrix multiplication, with very thin matrices. They follow the rule for shapes in multiplication: (n by 1) times (1 by n). The product of column times row is n by n .

Example 3 will show how to multiply AB using columns times rows.

Rows and Columns of AB

In the big picture, A multiplies each column of B . The result is a column of AB . In that column, we are combining the columns of A . **Each column of AB is a combination of the columns of A .** That is the column picture of matrix multiplication:

Column of AB is (matrix A) times (column of B).

The row picture is reversed. Each row of A multiplies the whole matrix B . The result is a row of AB . It is a combination of the rows of B :

$$[\text{row } i \text{ of } A] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = [\text{row } i \text{ of } AB].$$

We see row operations in elimination (E times A). We see columns in A times x . The “row-column picture” has the dot products of rows with columns. Believe it or not, **there is also a “column-row picture.”** Not everybody knows that columns 1, ..., n of A multiply rows 1, ..., n of B and add up to the same answer AB .

The Laws for Matrix Operations

May I put on record six laws that matrices do obey, while emphasizing an equation they don’t obey? The matrices can be square or rectangular, and the laws involving $A + B$ are all simple and all obeyed. Here are three addition laws:

$$\begin{aligned} A + B &= B + A && \text{(commutative law)} \\ c(A + B) &= cA + cB && \text{(distributive law)} \\ A + (B + C) &= (A + B) + C && \text{(associative law).} \end{aligned}$$

Three more laws hold for multiplication, but $AB = BA$ is not one of them:

- $AB \neq BA$ (the commutative “law” is *usually broken*)
- $C(A + B) = CA + CB$ (distributive law from the left)
- $(A + B)C = AC + BC$ (distributive law from the right)
- $A(BC) = (AB)C$ (associative law for ABC) (*parentheses not needed*).

When A and B are not square, AB is a different size from BA . These matrices can’t be equal—even if both multiplications are allowed. For square matrices, almost any example shows that AB is different from BA :

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is true that $AI = IA$. All square matrices commute with I and also with cI . Only these matrices cI commute with all other matrices.

The law $A(B + C) = AB + AC$ is proved a column at a time. Start with $A(b + c) = Ab + Ac$ for the first column. That is the key to everything—*linearity*. Say no more.

The law $A(BC) = (AB)C$ means that you can multiply BC first or AB first. The direct proof is sort of awkward (Problem 16) but this law is extremely useful. We highlighted it above; it is the key to the way we multiply matrices.

Look at the special case when $A = B = C =$ square matrix. Then $(A \text{ times } A^2) = (A^2 \text{ times } A)$. The product in either order is A^3 . The matrix powers A^p follow the same rules as numbers:

$$A^p = AAA \cdots A \text{ (} p \text{ factors)} \quad (A^p)(A^q) = A^{p+q} \quad (A^p)^q = A^{pq},$$

Those are the ordinary laws for exponents. A^3 times A^4 is A^7 (seven factors). A^3 to the fourth power is A^{12} (twelve A ’s). When p and q are zero or negative these rules still hold, provided A has a “ -1 power”—which is the *inverse matrix* A^{-1} . Then $A^0 = I$ is the identity matrix (no factors).

For a number, a^{-1} is $1/a$. For a matrix, the inverse is written A^{-1} . (It is *never* I/A , except this is allowed in MATLAB.) Every number has an inverse except $a = 0$. To decide when A has an inverse is a central problem in linear algebra. Section 2.5 will start on the answer. This section is a Bill of Rights for matrices, to say when A and B can be multiplied and how.

Block Matrices and Block Multiplication

We have to say one more thing about matrices. They can be cut into *blocks* (which are smaller matrices). This often happens naturally. Here is a 4 by 6 matrix broken into blocks of size 2 by 2—and each block is just I :

$$A = \left[\begin{array}{cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array} \right] = \begin{bmatrix} I & I & I \\ I & I & I \end{bmatrix}.$$

If B is also 4 by 6 and its block sizes match the block sizes in A , you can add $A + B$ a block at a time.

We have seen block matrices before. The right side vector b was placed next to A in the “augmented matrix.” Then $[A \ b]$ has two blocks of different sizes. Multiplying by an elimination matrix gave $[EA \ Eb]$. No problem to multiply blocks times blocks, when their shapes permit:

2G Block multiplication If the cuts between columns of A match the cuts between rows of B , then block multiplication of AB is allowed:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & \cdots \\ B_{21} & \cdots \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & \cdots \\ A_{21}B_{11} + A_{22}B_{21} & \cdots \end{bmatrix}. \quad (1)$$

This equation is the same as if the blocks were numbers (which are 1 by 1 blocks). We are careful to keep A 's in front of B 's, because BA can be different. The cuts between rows of A give cuts between rows of AB . Any column cuts in B are also column cuts in AB .

Main point When matrices split into blocks, it is often simpler to see how they act. The block matrix of I 's above is much clearer than the original 4 by 6 matrix A .

Example 3 (Important special case) Let the blocks of A be its n columns. Let the blocks of B be its n rows. Then block multiplication AB adds up *columns times rows*:

$$AB = \begin{bmatrix} | & & | \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & \mathbf{b}_1 & - \\ \vdots & & \\ - & \mathbf{b}_n & - \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 + \cdots + \mathbf{a}_n\mathbf{b}_n \end{bmatrix}. \quad (2)$$

This is another way to multiply matrices! Compare it with the usual rows times columns. Row 1 of A times column 1 of B gave the (1, 1) entry in AB . Now *column 1 of A*

times *row* 1 of B gives a full matrix—not just a single number. Look at this example:

$$\begin{aligned} \begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} [3 \ 2] + \begin{bmatrix} 4 \\ 5 \end{bmatrix} [1 \ 0] \\ &= \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}. \end{aligned} \quad (3)$$

We stop there so you can see columns multiplying rows. If a 2 by 1 matrix (a column) multiplies a 1 by 2 matrix (a row), the result is 2 by 2. That is what we found. Dot products are “inner products,” these are “outer products.”

When you add the two matrices at the end of equation (3), you get the correct answer AB . In the top left corner the answer is $3 + 4 = 7$. This agrees with the row-column dot product of $(1, 4)$ with $(3, 1)$.

Summary The usual way, rows times columns, gives four dot products (8 multiplications). The new way, columns times rows, gives two full matrices (8 multiplications). The eight multiplications, and also the four additions, are all the same. You just execute them in a different order.

Example 4 (Elimination by blocks) Suppose the first column of A contains 1, 3, 4. To change 3 and 4 to 0 and 0, multiply the pivot row by 3 and 4 and subtract. Those row operations are really multiplications by elimination matrices E_{21} and E_{31} :

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

The “block idea” is to do both eliminations with one matrix E . That matrix clears out the whole first column of A below the pivot $a = 2$:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad \text{multiplies} \quad \begin{bmatrix} 1 & x & x \\ 3 & x & x \\ 4 & x & x \end{bmatrix} \quad \text{to give} \quad EA = \begin{bmatrix} 1 & x & x \\ \mathbf{0} & x & x \\ \mathbf{0} & x & x \end{bmatrix}.$$

Block multiplication gives a formula for EA . The matrix A has four blocks a, b, c, D : the pivot, the rest of row 1, the rest of column 1, and the rest of the matrix. Watch how E multiplies A by blocks:

$$EA = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline -c/a & I \end{array} \right] \left[\begin{array}{c|c} a & b \\ \hline c & D \end{array} \right] = \left[\begin{array}{c|c} a & b \\ \hline \mathbf{0} & D - cb/a \end{array} \right]. \quad (4)$$

Elimination multiplies the first row $[a \ b]$ by c/a . It subtracts from c to get zeros in the first column. It subtracts from D to get $D - cb/a$. This is ordinary elimination, a column at a time—written in blocks.

■ REVIEW OF THE KEY IDEAS ■

1. The (i, j) entry of AB is (row i of A) \cdot (column j of B).
2. An m by n matrix times an n by p matrix uses mnp separate multiplications.
3. A times BC equals AB times C (surprisingly important).
4. AB is also the sum of these matrices: (column j of A) times (row j of B).
5. Block multiplication is allowed when the block shapes match correctly.

■ WORKED EXAMPLES ■

2.4 A Put yourself in the position of the author! I want to show you matrix multiplications that are *special*, but mostly I am stuck with small matrices. There is one terrific family of **Pascal matrices**, and they come in all sizes, and above all they have real meaning. I think 4 by 4 is a good size to show some of their amazing patterns.

Here is the lower triangular Pascal matrix L . Its entries come from “*Pascal’s triangle*”. I will multiply L times the **ones** vector, and the **powers** vector:

$$\begin{array}{l} \text{Pascal} \\ \text{matrix} \end{array} \left[\begin{array}{cccc} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right] = \left[\begin{array}{c} 1 \\ 2 \\ 4 \\ 8 \end{array} \right] \quad \left[\begin{array}{cccc} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{array} \right] \left[\begin{array}{c} 1 \\ x \\ x^2 \\ x^3 \end{array} \right] = \left[\begin{array}{c} 1 \\ 1+x \\ (1+x)^2 \\ (1+x)^3 \end{array} \right].$$

Each row of L leads to the next row: *Add an entry to the one on its left to get the entry below*. In symbols $\ell_{ij} + \ell_{i,j-1} = \ell_{i+1,j}$. The numbers after 1, 3, 3, 1 would be 1, 4, 6, 4, 1. Pascal lived in the 1600’s, long before matrices, but his triangle fits perfectly into L .

Multiplying by **ones** is the same as adding up each row, to get powers of 2. In fact **powers** = **ones** when $x = 1$. By writing out the last rows of L times **powers**, you see the entries of L as the “binomial coefficients” that are so essential to gamblers:

$$1 + 2x + 1x^2 = (1+x)^2 \qquad 1 + 3x + 3x^2 + 1x^3 = (1+x)^3$$

The number “3” counts the ways to get Heads once and Tails twice in three coin flips: HTT and THT and TTH. The other “3” counts the ways to get Heads twice: HHT and HTH and THH. Those are examples of “ i choose j ” = the number of ways to get j heads in i coin flips. That number is exactly ℓ_{ij} , if we start counting rows and columns of L at $i = 0$ and $j = 0$ (and remember $0! = 1$):

$$\ell_{ij} = \binom{i}{j} = i \text{ choose } j = \frac{i!}{j!(i-j)!} \qquad \binom{4}{2} = \frac{4!}{2!2!} = 6$$

There are six ways to choose two aces out of four aces. We will see Pascal’s triangle and these matrices again. Here are the questions I want to ask now:

1. What is $H = L^2$? This is the “hypercube matrix”.
2. Multiply H times **ones** and **powers**.
3. The last row of H is 8, 12, 6, 1. A cube has 8 corners, 12 edges, 6 faces, 1 box.
What would the next row of H tell about a hypercube in 4D?

Solution Multiply L times L to get the hypercube matrix $H = L^2$:

$$\begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix} = H.$$

Now multiply H times the vectors of **ones** and **powers**:

$$\begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 9 \\ 27 \end{bmatrix} \quad \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 4 & 4 & 1 & \\ 8 & 12 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2+x \\ (2+x)^2 \\ (2+x)^3 \end{bmatrix}$$

If $x = 1$ we get the powers of 3. If $x = 0$ we get powers of 2 (where do 1, 2, 4, 8 appear in H ?). Where L changed x to $1+x$, applying L again changes $1+x$ to $2+x$.

How do the rows of H count corners and edges and faces of a cube? A square in **2D** has 4 corners, 4 edges, 1 face. Add one dimension at a time:

*Connect two squares to get a **3D** cube. Connect two cubes to get a **4D** hypercube.*

The cube has 8 corners and 12 edges: 4 edges in each square and 4 between the squares. The cube has 6 faces: 1 in each square and 4 faces between the squares. This row 8, 12, 6, 1 of H will lead to the next row (one more dimension) by $2h_{ij} + h_{i,j-1} = h_{i+1,j}$.

Can you see this in four dimensions? The hypercube has 16 corners, no problem. It has 12 edges from one cube, 12 from the other cube, 8 that connect corners between those cubes: total $2 \times 12 + 8 = 32$ edges. It has 6 faces from each separate cube and 12 more from connecting pairs of edges: total $2 \times 6 + 12 = 24$ faces. It has one box from each cube and 6 more from connecting pairs of faces: total $2 \times 1 + 6 = 8$ boxes. And sure enough, the next row of H is 16, 32, 24, 8, 1.

2.4 B For these matrices, when does $AB = BA$? When does $BC = CB$? When does A times BC equal AB times C ? Give the conditions on their entries p, q, r, z :

$$A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$

If $p, q, r, 1, z$ are 4 by 4 blocks instead of numbers, do the answers change?

Solution First of all, A times BC always equals AB times C . We don't need parentheses in $A(BC) = (AB)C = ABC$. But we do need to keep the matrices in this order A, B, C . Compare AB with BA :

$$AB = \begin{bmatrix} p & p \\ q & q+r \end{bmatrix} \quad BA = \begin{bmatrix} p+q & r \\ q & r \end{bmatrix}.$$

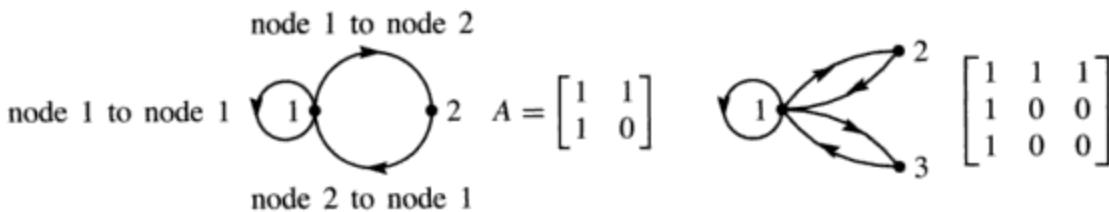
We only have $AB = BA$ if $q = 0$ and $p = r$. Now compare BC with CB :

$$BC = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \quad CB = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}.$$

B and C happen to commute. One explanation is that the diagonal part of B is I , which commutes with all 2 by 2 matrices. The off-diagonal part of B looks exactly like C (except for a scalar factor z) and every matrix commutes with itself.

When p, q, r, z are 4 by 4 blocks and 1 changes to the 4 by 4 identity matrix, all these products remain correct. So the answers are the same. (If the I 's in B were changed to blocks t, t, t , then BC would have the block tz and CB would have the block zt . Those would normally be different—the order is important in block multiplication.)

2.4 C A **directed graph** starts with n nodes. There are n^2 possible edges—each edge leaves one of the n nodes and enters one of the n nodes (possibly itself). The n by n **adjacency matrix** has $a_{ij} = 1$ when an edge leaves node i and enters node j ; if no edge then $a_{ij} = 0$. Here are two directed graphs and their adjacency matrices:



The i, j entry of A^2 is $a_{i1}a_{1j} + \dots + a_{in}a_{nj}$. Why does that sum count the *two-step paths* from i to any node to j ? The i, j entry of A^k counts k -step paths:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{counts the paths with two edges} \quad \begin{bmatrix} 1 \text{ to } 2 \text{ to } 1, 1 \text{ to } 1 \text{ to } 1 & 1 \text{ to } 1 \text{ to } 2 \\ 2 \text{ to } 1 \text{ to } 1 & 2 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

List all of the 3-step paths between each pair of nodes and compare with A^3 . When A^k has **no zeros**, that number k is the **diameter** of the graph—the number of edges needed to connect the most distant pair of nodes. What is the diameter of the second graph?

Solution The number $a_{ik}a_{kj}$ will be “1” if there is an edge from node i to k and an edge from k to j . This is a 2-step path. The number $a_{ik}a_{kj}$ will be “0” if either of

those edges (i to k , k to j) is missing. So the sum of $a_{ik}a_{kj}$ is the number of 2-step paths leaving i and entering j . Matrix multiplication is just right for this count.

The 3-step paths are counted by A^3 ; we look at paths to node 2:

$$A^3 = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{array}{l} \text{counts the paths} \\ \text{with three steps} \end{array} \quad \begin{bmatrix} \dots & 1 \text{ to } 1 \text{ to } 1 \text{ to } 2, 1 \text{ to } 2 \text{ to } 1 \text{ to } 2 \\ \dots & 2 \text{ to } 1 \text{ to } 1 \text{ to } 2 \end{bmatrix}$$

These A^k contain the Fibonacci numbers 0, 1, 1, 2, 3, 5, 8, 13, ... coming in Section 6.2. Fibonacci's rule $F_{k+2} = F_{k+1} + F_k$ (as in $13 = 8 + 5$) shows up in $(A)(A^k) = A^{k+1}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} = \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix} = A^{k+1}.$$

There are 13 six-step paths from node 1 to node 1, but I can't find them all.

A^k also counts words. A path like 1 to 1 to 2 to 1 corresponds to the number 1121 or the word **aaba**. The number 2 (the letter **b**) is not allowed to repeat because the graph has no edge from node 2 to node 2. The i, j entry of A^k counts the allowed numbers (or words) of length $k + 1$ that start with the i th letter and end with the j th.

The second graph also has diameter 2; A^2 has no zeros.

Problem Set 2.4

Problems 1–17 are about the laws of matrix multiplication.

- 1 A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. All entries are 1. Which of these matrix operations are allowed, and what are the results?

$$BA \qquad AB \qquad ABD \qquad DBA \qquad A(B + C).$$

- 2 What rows or columns or matrices do you multiply to find

- (a) the third column of AB ?
- (b) the first row of AB ?
- (c) the entry in row 3, column 4 of AB ?
- (d) the entry in row 1, column 1 of CDE ?

- 3 Add AB to AC and compare with $A(B + C)$:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}.$$

- 4 In Problem 3, multiply A times BC . Then multiply AB times C .

- 5 Compute A^2 and A^3 . Make a prediction for A^5 and A^n :

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}.$$

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- 6** Show that $(A + B)^2$ is different from $A^2 + 2AB + B^2$, when

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Write down the correct rule for $(A + B)(A + B) = A^2 + \underline{\hspace{2cm}} + B^2$.

- 7** True or false. Give a specific example when false:

- (a) If columns 1 and 3 of B are the same, so are columns 1 and 3 of AB .
- (b) If rows 1 and 3 of B are the same, so are rows 1 and 3 of AB .
- (c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of ABC .
- (d) $(AB)^2 = A^2B^2$.

- 8** How is each row of DA and EA related to the rows of A , when

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}?$$

How is each column of AD and AE related to the columns of A ?

- 9** Row 1 of A is added to row 2. This gives EA below. Then column 1 of EA is added to column 2 to produce $(EA)F$:

$$EA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

$$\text{and } (EA)F = (EA) \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ a+c & a+c+b+d \end{bmatrix}.$$

- (a) Do those steps in the opposite order. First add column 1 of A to column 2 by AF , then add row 1 of AF to row 2 by $E(AF)$.
 - (b) Compare with $(EA)F$. What law is obeyed by matrix multiplication?
- 10** Row 1 of A is again added to row 2 to produce EA . Then F adds row 2 of EA to row 1. The result is $F(EA)$:

$$F(EA) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix} = \begin{bmatrix} 2a+c & 2b+d \\ a+c & b+d \end{bmatrix}.$$

- (a) Do those steps in the opposite order: first add row 2 to row 1 by FA , then add row 1 of FA to row 2.
 - (b) What law is or is not obeyed by matrix multiplication?
- 11** (3 by 3 matrices) Choose the only B so that for every matrix A
- (a) $BA = 4A$
 - (b) $BA = 4B$

- (c) BA has rows 1 and 3 of A reversed and row 2 unchanged
 (d) All rows of BA are the same as row 1 of A .
- 12** Suppose $AB = BA$ and $AC = CA$ for these two particular matrices B and C :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ commutes with } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Prove that $a = d$ and $b = c = 0$. Then A is a multiple of I . The only matrices that commute with B and C and all other 2 by 2 matrices are $A = \text{multiple of } I$.

- 13** Which of the following matrices are guaranteed to equal $(A - B)^2$: $A^2 - B^2$, $(B - A)^2$, $A^2 - 2AB + B^2$, $A(A - B) - B(A - B)$, $A^2 - AB - BA + B^2$?

- 14** True or false:

- (a) If A^2 is defined then A is necessarily square.
 (b) If AB and BA are defined then A and B are square.
 (c) If AB and BA are defined then AB and BA are square.
 (d) If $AB = B$ then $A = I$.

- 15** If A is m by n , how many separate multiplications are involved when

- (a) A multiplies a vector \mathbf{x} with n components?
 (b) A multiplies an n by p matrix B ?
 (c) A multiplies itself to produce A^2 ? Here $m = n$.

- 16** To prove that $(AB)C = A(BC)$, use the column vectors $\mathbf{b}_1, \dots, \mathbf{b}_n$ of B . First suppose that C has only one column \mathbf{c} with entries c_1, \dots, c_n :

AB has columns $A\mathbf{b}_1, \dots, A\mathbf{b}_n$ and $B\mathbf{c}$ has one column $c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n$.

Then $(AB)\mathbf{c} = c_1A\mathbf{b}_1 + \dots + c_nA\mathbf{b}_n$ equals $A(c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n) = A(B\mathbf{c})$.

Linearity gives equality of those two sums, and $(AB)\mathbf{c} = A(B\mathbf{c})$. The same is true for all other _____ of C . Therefore $(AB)C = A(BC)$.

- 17** For $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 0 & 6 \end{bmatrix}$, compute these answers *and nothing more*:

- (a) column 2 of AB
 (b) row 2 of AB
 (c) row 2 of $AA = A^2$
 (d) row 2 of $AAA = A^3$.

Problems 18–20 use a_{ij} for the entry in row i , column j of A .

- 18** Write down the 3 by 3 matrix A whose entries are

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- (a) a_{ij} = minimum of i and j
 (b) $a_{ij} = (-1)^{i+j}$
 (c) $a_{ij} = i/j.$
- 19** What words would you use to describe each of these classes of matrices? Give a 3 by 3 example in each class. Which matrix belongs to all four classes?
- (a) $a_{ij} = 0$ if $i \neq j$
 (b) $a_{ij} = 0$ if $i < j$
 (c) $a_{ij} = a_{ji}$
 (d) $a_{ij} = a_{1j}.$
- 20** The entries of A are a_{ij} . Assuming that zeros don't appear, what is
- (a) the first pivot?
 (b) the multiplier ℓ_{31} of row 1 to be subtracted from row 3?
 (c) the new entry that replaces a_{32} after that subtraction?
 (d) the second pivot?

Problems 21–25 involve powers of A .

- 21** Compute A^2, A^3, A^4 and also $A\mathbf{v}, A^2\mathbf{v}, A^3\mathbf{v}, A^4\mathbf{v}$ for

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix}.$$

- 22** Find all the powers A^2, A^3, \dots and $AB, (AB)^2, \dots$ for

$$A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

- 23** By trial and error find real nonzero 2 by 2 matrices such that

$$A^2 = -I \quad BC = 0 \quad DE = -ED \quad (\text{not allowing } DE = 0).$$

- 24** (a) Find a nonzero matrix A for which $A^2 = 0$.
 (b) Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$.
- 25** By experiment with $n = 2$ and $n = 3$ predict A^n for

$$A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

Problems 26–34 use column-row multiplication and block multiplication.

- 26 Multiply AB using columns times rows:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} [3 \ 3 \ 0] + \underline{\quad} = \underline{\quad}.$$

- 27 The product of upper triangular matrices is always upper triangular:

$$AB = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row times column is dot product (Row 2 of A) · (column 1 of B) = 0. Which other dot products give zeros?

Column times row is full matrix Draw x 's and 0's in (column 2 of A) times (row 2 of B) and in (column 3 of A) times (row 3 of B).

- 28 Draw the cuts in A (2 by 3) and B (3 by 4) and AB to show how each of the four multiplication rules is really a block multiplication:

- (1) Matrix A times columns of B .
 - (2) Rows of A times matrix B .
 - (3) Rows of A times columns of B .
 - (4) Columns of A times rows of B .
- 29 Draw cuts in A and x to multiply Ax a column at a time: $x_1(\text{column 1}) + \dots$.
- 30 Which matrices E_{21} and E_{31} produce zeros in the (2, 1) and (3, 1) positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}.$$

Find the single matrix $E = E_{31}E_{21}$ that produces both zeros at once. Multiply EA .

- 31 Block multiplication says in the text that column 1 is eliminated by

$$EA = \begin{bmatrix} 1 & \mathbf{0} \\ -c/a & I \end{bmatrix} \begin{bmatrix} a & b \\ c & D \end{bmatrix} = \begin{bmatrix} a & b \\ \mathbf{0} & D - cb/a \end{bmatrix}.$$

In Problem 30, what are c and D and what is $D - cb/a$?

- 32 With $i^2 = -1$, the product of $(A+iB)$ and $(x+iy)$ is $Ax+iBx+iAy-By$. Use blocks to separate the real part without i from the imaginary part that multiplies i :

$$\begin{bmatrix} A & -B \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax - By \\ ? \end{bmatrix} \begin{array}{l} \text{real part} \\ \text{imaginary part} \end{array}$$

- 33 Suppose you solve $Ax = b$ for three special right sides b :

$$Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the three solutions x_1, x_2, x_3 are the columns of a matrix X , what is A times X ?

- 34 If the three solutions in Question 33 are $x_1 = (1, 1, 1)$ and $x_2 = (0, 1, 1)$ and $x_3 = (0, 0, 1)$, solve $Ax = b$ when $b = (3, 5, 8)$. Challenge problem: What is A ?
- 35 *Elimination for a 2 by 2 block matrix:* When you multiply the first block row by CA^{-1} and subtract from the second row, what is the “Schur complement” S that appears?

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}.$$

- 36 Find all matrices $A = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ that satisfy $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}A$.
- 37 Suppose a “circle graph” has 5 nodes connected (in both directions) by edges around a circle. What is its adjacency matrix from Worked Example 2.4 C? What are A^2 and A^3 and the diameter of this graph?
- 38 If 5 edges in Question 37 go in one direction only, from nodes 1, 2, 3, 4, 5 to 2, 3, 4, 5, 1, what are A and A^2 and the diameter of this one-way circle?
- 39 If you multiply a *northwest matrix* A and a *southeast matrix* B , what type of matrices are AB and BA ? “Northwest” and “southeast” mean zeros below and above the antidiagonal going from $(1, n)$ to $(n, 1)$.

INVERSE MATRICES ■ 2.5

Suppose A is a square matrix. We look for an “*inverse matrix*” A^{-1} of the same size, such that A^{-1} times A equals I . Whatever A does, A^{-1} undoes. Their product is the identity matrix—which does nothing. But A^{-1} might not exist.

What a matrix mostly does is to multiply a vector \mathbf{x} . Multiplying $A\mathbf{x} = \mathbf{b}$ by A^{-1} gives $A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$. The left side is just \mathbf{x} ! The product $A^{-1}A$ is like multiplying by a number and then dividing by that number. An ordinary number has an inverse if it is not zero—matrices are more complicated and more interesting. The matrix A^{-1} is called “ A inverse.”

DEFINITION The matrix A is *invertible* if there exists a matrix A^{-1} such that

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I. \quad (1)$$

Not all matrices have inverses. This is the first question we ask about a square matrix: Is A invertible? We don’t mean that we immediately calculate A^{-1} . In most problems we never compute it! Here are six “notes” about A^{-1} .

Note 1 *The inverse exists if and only if elimination produces n pivots* (row exchanges allowed). Elimination solves $A\mathbf{x} = \mathbf{b}$ without explicitly using A^{-1} .

Note 2 The matrix A cannot have two different inverses. Suppose $BA = I$ and also $AC = I$. Then $B = C$, according to this “proof by parentheses”:

$$B(AC) = (BA)C \quad \text{gives} \quad BI = IC \quad \text{or} \quad B = C. \quad (2)$$

This shows that a *left-inverse* B (multiplying from the left) and a *right-inverse* C (multiplying A from the right to give $AC = I$) must be the *same matrix*.

Note 3 If A is invertible, the one and only solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$:

Multiply $A\mathbf{x} = \mathbf{b}$ by A^{-1} . Then $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}$.

Note 4 (Important) *Suppose there is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \mathbf{0}$. Then A cannot have an inverse.* No matrix can bring $\mathbf{0}$ back to \mathbf{x} .

If A is invertible, then $A\mathbf{x} = \mathbf{0}$ can only have the zero solution $\mathbf{x} = \mathbf{0}$.

Note 5 A 2 by 2 matrix is invertible if and only if $ad - bc$ is not zero:

$$\text{2 by 2 Inverse: } \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (3)$$

This number $ad - bc$ is the *determinant* of A . A matrix is invertible if its determinant is not zero (Chapter 5). The test for n pivots is usually decided before the determinant appears.

Note 6 A diagonal matrix has an inverse provided no diagonal entries are zero:

$$\text{If } A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \text{ then } A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}.$$

Example 1 The 2 by 2 matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ is not invertible. It fails the test in Note 5, because $ad - bc$ equals $2 - 2 = 0$. It fails the test in Note 3, because $Ax = \mathbf{0}$ when $x = (2, -1)$. It fails to have two pivots as required by Note 1. Elimination turns the second row of A into a zero row.

The Inverse of a Product AB

For two nonzero numbers a and b , the sum $a + b$ might or might not be invertible. The numbers $a = 3$ and $b = -3$ have inverses $\frac{1}{3}$ and $-\frac{1}{3}$. Their sum $a + b = 0$ has no inverse. But the product $ab = -9$ does have an inverse, which is $\frac{1}{3}$ times $-\frac{1}{3}$.

For two matrices A and B , the situation is similar. It is hard to say much about the invertibility of $A + B$. But the *product* AB has an inverse, whenever the factors A and B are separately invertible (and the same size). The important point is that A^{-1} and B^{-1} come in *reverse order*:

2H If A and B are invertible then so is AB . The inverse of a product AB is

$$(AB)^{-1} = B^{-1}A^{-1}. \quad (4)$$

To see why the order is reversed, multiply AB times $B^{-1}A^{-1}$. The inside step is $BB^{-1} = I$:

$$(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I.$$

We moved parentheses to multiply BB^{-1} first. Similarly $B^{-1}A^{-1}$ times AB equals I . This illustrates a basic rule of mathematics: Inverses come in reverse order. It is also common sense: If you put on socks and then shoes, the first to be taken off are the _____. The same idea applies to three or more matrices:

$$\text{Reverse order } (ABC)^{-1} = C^{-1}B^{-1}A^{-1}. \quad (5)$$

Example 2 Inverse of an Elimination Matrix. If E subtracts 5 times row 1 from row 2, then E^{-1} adds 5 times row 1 to row 2:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiply EE^{-1} to get the identity matrix I . Also multiply $E^{-1}E$ to get I . We are adding and subtracting the same 5 times row 1. Whether we add and then subtract (this is EE^{-1}) or subtract and then add (this is $E^{-1}E$), we are back at the start.

For square matrices, an inverse on one side is automatically an inverse on the other side. If $AB = I$ then automatically $BA = I$. In that case B is A^{-1} . This is very useful to know but we are not ready to prove it.

Example 3 Suppose F subtracts 4 times row 2 from row 3, and F^{-1} adds it back:

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \quad \text{and} \quad F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}.$$

Now multiply F by the matrix E in Example 2 to find FE . Also multiply E^{-1} times F^{-1} to find $(FE)^{-1}$. Notice the orders FE and $E^{-1}F^{-1}$!

$$FE = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 20 & -4 & 1 \end{bmatrix} \quad \text{is inverted by} \quad E^{-1}F^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}. \quad (6)$$

The result is strange but correct. The product FE contains “20” but its inverse doesn’t. E subtracts 5 times row 1 from row 2. Then F subtracts 4 times the new row 2 (changed by row 1) from row 3. **In this order FE , row 3 feels an effect from row 1.**

In the order $E^{-1}F^{-1}$, that effect does not happen. First F^{-1} adds 4 times row 2 to row 3. After that, E^{-1} adds 5 times row 1 to row 2. There is no 20, because row 3 doesn’t change again. **In this order, row 3 feels no effect from row 1.**

For elimination with normal order FE , the product of inverses $E^{-1}F^{-1}$ is quick. The multipliers fall into place below the diagonal of 1’s.

This special property of $E^{-1}F^{-1}$ and $E^{-1}F^{-1}G^{-1}$ will be useful in the next section. We will explain it again, more completely. In this section our job is A^{-1} , and we expect some serious work to compute it. Here is a way to organize that computation.

Calculating A^{-1} by Gauss-Jordan Elimination

I hinted that A^{-1} might not be explicitly needed. The equation $Ax = b$ is solved by $x = A^{-1}b$. But it is not necessary or efficient to compute A^{-1} and multiply it times b . **Elimination goes directly to x .** Elimination is also the way to calculate A^{-1} , as we now show. The Gauss-Jordan idea is to solve $AA^{-1} = I$, finding each column of A^{-1} .

A multiplies the first column of A^{-1} (call that x_1) to give the first column of I (call that e_1). This is our equation $Ax_1 = e_1 = (1, 0, 0)$. Each of the columns x_1, x_2, x_3 of A^{-1} is multiplied by A to produce a column of I :

$$AA^{-1} = A[x_1 \ x_2 \ x_3] = [e_1 \ e_2 \ e_3] = I. \quad (7)$$

To invert a 3 by 3 matrix A , we have to solve three systems of equations: $Ax_1 = e_1$ and $Ax_2 = e_2 = (0, 1, 0)$ and $Ax_3 = e_3 = (0, 0, 1)$. This already shows why

computing A^{-1} is expensive. We must solve n equations for its n columns. To solve $Ax = b$ without A^{-1} , we deal only with *one* column.

In defense of A^{-1} , we want to say that its cost is not n times the cost of one system $Ax = b$. Surprisingly, the cost for n columns is only multiplied by 3. This saving is because the n equations $Ax_i = e_i$ all involve the same matrix A . Working with the right sides is relatively cheap, because elimination only has to be done once on A . The complete A^{-1} needs n^3 elimination steps, where a single x needs $n^3/3$. The next section calculates these costs.

The **Gauss-Jordan method** computes A^{-1} by solving all n equations together. Usually the “augmented matrix” has one extra column b , from the right side of the equations. Now we have three right sides e_1, e_2, e_3 (when A is 3 by 3). They are the columns of I , so the augmented matrix is really the block matrix $[A \ I]$. Here is a worked-out example when A has 2's on the main diagonal and -1's next to the 2's:

$$\begin{aligned} [A \ e_1 \ e_2 \ e_3] &= \left[\begin{array}{cccccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \quad \text{Start Gauss-Jordan} \\ &\rightarrow \left[\begin{array}{cccccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right] \quad (\frac{1}{2} \text{ row 1 + row 2}) \\ &\rightarrow \left[\begin{array}{cccccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array} \right] \quad (\frac{2}{3} \text{ row 2 + row 3}) \end{aligned}$$

We are now halfway. The matrix in the first three columns is U (upper triangular). The pivots $2, \frac{3}{2}, \frac{4}{3}$ are on its diagonal. Gauss would finish by back substitution. The contribution of Jordan is *to continue with elimination!* He goes all the way to the “**reduced echelon form**”. Rows are added to rows above them, to produce *zeros above the pivots*:

$$\begin{aligned} &\rightarrow \left[\begin{array}{cccccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array} \right] \quad (\frac{3}{4} \text{ row 3 + row 2}) \\ &\rightarrow \left[\begin{array}{cccccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array} \right] \quad (\frac{2}{3} \text{ row 2 + row 1}) \end{aligned}$$

The last Gauss-Jordan step is to divide each row by its pivot. The new pivots are 1. We have reached I in the first half of the matrix, because A is invertible. ***The three columns of A^{-1} are in the second half of $[I A^{-1}]$:***

$$\begin{array}{l} \text{(divide by 2)} \\ \text{(divide by } \frac{3}{2} \text{)} \\ \text{(divide by } \frac{4}{3} \text{)} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array} \right] = [I \ x_1 \ x_2 \ x_3].$$

Starting from the 3 by 6 matrix $[A \ I]$, we ended with $[I \ A^{-1}]$. Here is the whole Gauss-Jordan process on one line:

Multiply $[A \ I]$ by A^{-1} to get $[I \ A^{-1}]$.

The elimination steps gradually create the inverse matrix. For large matrices, we probably don't want A^{-1} at all. But for small matrices, it can be very worthwhile to know the inverse. We add three observations about this particular A^{-1} because it is an important example. We introduce the words *symmetric*, *tridiagonal*, and *determinant* (Chapter 5):

1. A is *symmetric* across its main diagonal. So is A^{-1} .
2. A is *tridiagonal* (only three nonzero diagonals). But A^{-1} is a full matrix with no zeros. That is another reason we don't often compute A^{-1} .
3. The product of pivots is $2(\frac{3}{2})(\frac{4}{3}) = 4$. This number 4 is the *determinant* of A .

$$A^{-1} \text{ involves division by the determinant} \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}. \quad (8)$$

Example 4 Find A^{-1} by Gauss-Jordan elimination starting from $A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$. There are two row operations and then a division to put 1's in the pivots:

$$\begin{aligned} [A \ I] &= \begin{bmatrix} 2 & 3 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 2 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \frac{7}{2} & -\frac{3}{2} \\ 0 & 1 & -2 & 1 \end{bmatrix} = [I \ A^{-1}]. \end{aligned}$$

The reduced echelon form of $[A \ I]$ is $[I \ A^{-1}]$. This A^{-1} involves division by the determinant $2 \cdot 7 - 3 \cdot 4 = 2$. The code for $X = \text{inverse}(A)$ has three important lines!

```
I = eye(n, n); % Define the identity matrix
R = rref([A I]); % Eliminate on the augmented matrix
X = R(:, n+1:n+n) % Pick A^-1 from the last n columns of R
```

A must be invertible, or elimination will not reduce it (in the left half of R) to I .

Singular versus Invertible

We come back to the central question. Which matrices have inverses? The start of this section proposed the pivot test: *A^{-1} exists exactly when A has a full set of n pivots.* (Row exchanges allowed.) Now we can prove that by Gauss-Jordan elimination:

1. With n pivots, elimination solves all the equations $Ax_i = e_i$. The columns x_i go into A^{-1} . Then $AA^{-1} = I$ and A^{-1} is at least a **right-inverse**.
2. Elimination is really a sequence of multiplications by E 's and P 's and D^{-1} :

$$(D^{-1} \cdots E \cdots P \cdots E)A = I. \quad (9)$$

D^{-1} divides by the pivots. The matrices E produce zeros below and above the pivots. P will exchange rows if needed (see Section 2.7). The product matrix in equation (9) is evidently a **left-inverse**. With n pivots we reach $A^{-1}A = I$.

The right-inverse equals the left-inverse. That was Note 2 in this section. So a square matrix with a full set of pivots will always have a two-sided inverse.

Reasoning in reverse will now show that A must have n pivots if $AC = I$. Then we deduce that C is also a left-inverse. Here is one route to those conclusions:

1. If A doesn't have n pivots, elimination will lead to a *zero row*.
2. Those elimination steps are taken by an invertible M . So a row of MA is zero.
3. If $AC = I$ then $MAC = M$. The zero row of MA , times C , gives a zero row of M .
4. The invertible matrix M can't have a zero row! A *must* have n pivots if $AC = I$.
5. Then equation (9) displays the left inverse in $BA = I$, and Note 2 proves $B = C$.

That argument took five steps, but the outcome is short and important.

21 A complete test for invertibility of a square matrix A comes from elimination. *A^{-1} exists (and Gauss-Jordan finds it) exactly when A has n pivots.* The full argument shows more:

If $AC = I$ then $CA = I$ and $C = A^{-1}$!

Example 5 If L is lower triangular with 1's on the diagonal, so is L^{-1} .

Use the Gauss-Jordan method to construct L^{-1} . Start by subtracting multiples of pivot rows from rows *below*. Normally this gets us halfway to the inverse, but for L it gets us all the way. L^{-1} appears on the right when I appears on the left:

$$\begin{aligned} [L \ I] &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 5 & 1 & -4 & 0 & 1 \end{bmatrix} \quad (3 \text{ times row 1 from row 2}) \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 11 & -5 & 1 \end{bmatrix} \quad (4 \text{ times row 1 from row 3}) \\ &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 11 & -5 & 1 \end{bmatrix} = [I \ L^{-1}]. \end{aligned}$$

When L goes to I by elimination, I goes to L^{-1} . In other words, the product of elimination matrices $E_{32}E_{31}E_{21}$ is L^{-1} . All pivots are 1's (a full set). L^{-1} is lower triangular. The strange entry “11” in L^{-1} does not appear in $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = L$.

■ REVIEW OF THE KEY IDEAS ■

1. The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.
2. A is invertible if and only if it has n pivots (row exchanges allowed).
3. If $Ax = \mathbf{0}$ for a nonzero vector x , then A has no inverse.
4. The inverse of AB is the reverse product $B^{-1}A^{-1}$.
5. The Gauss-Jordan method solves $AA^{-1} = I$ to find the n columns of A^{-1} . The augmented matrix $[A \ I]$ is row-reduced to $[I \ A^{-1}]$.

■ WORKED EXAMPLES ■

2.5 A Three of these matrices are invertible, and three are singular. Find the inverse when it exists. Give reasons for noninvertibility (zero determinant, too few pivots, nonzero solution to $Ax = \mathbf{0}$) for the other three, in that order. The matrices A, B, C, D, E, F are

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \begin{bmatrix} 4 & 3 \\ 8 & 7 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 0 \end{bmatrix} \quad \begin{bmatrix} 6 & 6 \\ 6 & 6 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution

$$B^{-1} = \frac{1}{4} \begin{bmatrix} 7 & -3 \\ -8 & 4 \end{bmatrix} \quad C^{-1} = \frac{1}{36} \begin{bmatrix} 0 & 6 \\ 6 & -6 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

A is not invertible because its determinant is $4 \cdot 6 - 3 \cdot 8 = 24 - 24 = 0$. D is not invertible because there is only one pivot; the second row becomes zero when the first row is subtracted. F is not invertible because a combination of the columns (the second column minus the first column) is zero—in other words $F\mathbf{x} = \mathbf{0}$ has the solution $\mathbf{x} = (-1, 1, 0)$.

Of course all three reasons for noninvertibility would apply to each of A, D, F .

2.5 B Apply the Gauss-Jordan method to find the inverse of this triangular “Pascal matrix” $A = \text{abs}(\text{pascal}(4,1))$. You see **Pascal’s triangle**—adding each entry to the entry on its left gives the entry below. The entries are “binomial coefficients”:

Triangular Pascal matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$.

Solution Gauss-Jordan starts with $[A \ I]$ and produces zeros by subtracting row 1:

$$[A \ I] = \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 3 & 1 & -1 & 0 & 0 & 1 \end{array} \right].$$

The next stage creates zeros below the second pivot, using multipliers 2 and 3. Then the last stage subtracts 3 times the new row 3 from the new row 4:

$$\rightarrow \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 1 & 2 & -3 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|ccccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array} \right] = [I \ A^{-1}].$$

All the pivots were 1! So we didn’t need to divide rows by pivots to get I . The inverse matrix A^{-1} looks like A itself, except odd-numbered diagonals are multiplied by -1 .

Please notice that 4 by 4 matrix A^{-1} , we will see Pascal matrices again. The same pattern continues to n by n Pascal matrices—the inverse has “alternating diagonals”.

Problem Set 2.5

- 1 Find the inverses (directly or from the 2 by 2 formula) of A, B, C :

$$A = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}.$$

- 2 For these “permutation matrices” find P^{-1} by trial and error (with 1’s and 0’s):

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 3 Solve for the columns of $A^{-1} = [\mathbf{x} \ \mathbf{t}]$:

$$\begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 10 & 20 \\ 20 & 50 \end{bmatrix} \begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- 4 Show that $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has no inverse by trying to solve for the column (x, y) :

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x & t \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{must include} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- 5 Find an upper triangular U (not diagonal) with $U^2 = I$ and $U = U^{-1}$.

- 6 (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

- (b) If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, find two matrices $B \neq C$ such that $AB = AC$.

- 7 (Important) If A has row 1 + row 2 = row 3, show that A is not invertible:

- (a) Explain why $A\mathbf{x} = (1, 0, 0)$ cannot have a solution.

- (b) Which right sides (b_1, b_2, b_3) might allow a solution to $A\mathbf{x} = \mathbf{b}$?

- (c) What happens to row 3 in elimination?

- 8 If A has column 1 + column 2 = column 3, show that A is not invertible:

- (a) Find a nonzero solution \mathbf{x} to $A\mathbf{x} = \mathbf{0}$. The matrix is 3 by 3.

- (b) Elimination keeps column 1 + column 2 = column 3. Explain why there is no third pivot.

- 9 Suppose A is invertible and you exchange its first two rows to reach B . Is the new matrix B invertible and how would you find B^{-1} from A^{-1} ?

- 10 Find the inverses (in any legal way) of

$$A = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 2 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 7 & 6 \end{bmatrix}.$$

- 11 (a) Find invertible matrices A and B such that $A + B$ is not invertible.

- (b) Find singular matrices A and B such that $A + B$ is invertible.

- 12 If the product $C = AB$ is invertible (A and B are square), then A itself is invertible. Find a formula for A^{-1} that involves C^{-1} and B .
- 13 If the product $M = ABC$ of three square matrices is invertible, then B is invertible. (So are A and C .) Find a formula for B^{-1} that involves M^{-1} and A and C .
- 14 If you add row 1 of A to row 2 to get B , how do you find B^{-1} from A^{-1} ?

Notice the order. The inverse of $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix}$ is ____.

- 15 Prove that a matrix with a column of zeros cannot have an inverse.
- 16 Multiply $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ times $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. What is the inverse of each matrix if $ad \neq bc$?
- 17 (a) What matrix E has the same effect as these three steps? Subtract row 1 from row 2, subtract row 1 from row 3, then subtract row 2 from row 3.
 (b) What single matrix L has the same effect as these three reverse steps? Add row 2 to row 3, add row 1 to row 3, then add row 1 to row 2.
- 18 If B is the inverse of A^2 , show that AB is the inverse of A .
- 19 Find the numbers a and b that give the inverse of $5 * \text{eye}(4) - \text{ones}(4,4)$:

$$\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}.$$

What are a and b in the inverse of $6 * \text{eye}(5) - \text{ones}(5,5)$?

- 20 Show that $A = 4 * \text{eye}(4) - \text{ones}(4,4)$ is *not* invertible: Multiply $A * \text{ones}(4,1)$.
- 21 There are sixteen 2 by 2 matrices whose entries are 1's and 0's. How many of them are invertible?

Questions 22–28 are about the Gauss-Jordan method for calculating A^{-1} .

- 22 Change I into A^{-1} as you reduce A to I (by row operations):

$$[A \ I] = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [A \ I] = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix}$$

- 23 Follow the 3 by 3 text example but with plus signs in A . Eliminate above and below the pivots to reduce $[A \ I]$ to $[I \ A^{-1}]$:

$$[A \ I] = \begin{bmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

- 24 Use Gauss-Jordan elimination on $[A \ I]$ to solve $AA^{-1} = I$:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- 25 Find A^{-1} and B^{-1} (if they exist) by elimination on $[A \ I]$ and $[B \ I]$:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

- 26 What three matrices E_{21} and E_{12} and D^{-1} reduce $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ to the identity matrix? Multiply $D^{-1}E_{12}E_{21}$ to find A^{-1} .

- 27 Invert these matrices A by the Gauss-Jordan method starting with $[A \ I]$:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

- 28 Exchange rows and continue with Gauss-Jordan to find A^{-1} :

$$[A \ I] = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}.$$

- 29 True or false (with a counterexample if false and a reason if true):

- (a) A 4 by 4 matrix with a row of zeros is not invertible.
- (b) A matrix with 1's down the main diagonal is invertible.
- (c) If A is invertible then A^{-1} is invertible.
- (d) If A is invertible then A^2 is invertible.

- 30 For which three numbers c is this matrix not invertible, and why not?

$$A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}.$$

- 31 Prove that A is invertible if $a \neq 0$ and $a \neq b$ (find the pivots or A^{-1}):

$$A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}.$$

- 32 This matrix has a remarkable inverse. Find A^{-1} by elimination on $[A \ I]$. Extend to a 5 by 5 “alternating matrix” and guess its inverse; then multiply to confirm.

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

- 33 Use the 4 by 4 inverse in Question 32 to solve $Ax = (1, 1, 1, 1)$.
- 34 Suppose P and Q have the same rows as I but in any order. Show that $P - Q$ is singular by solving $(P - Q)x = \mathbf{0}$.
- 35 Find and check the inverses (assuming they exist) of these block matrices:

$$\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \quad \begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \quad \begin{bmatrix} 0 & I \\ I & D \end{bmatrix}.$$

- 36 If an invertible matrix A commutes with C (this means $AC = CA$) show that A^{-1} commutes with C . If also B commutes with C , show that AB commutes with C . Translation: If $AC = CA$ and $BC = CB$ then $(AB)C = C(AB)$.
- 37 Could a 4 by 4 matrix A be invertible if every row contains the numbers 0, 1, 2, 3 in some order? What if every row of B contains 0, 1, 2, -3 in some order?
- 38 In the worked example 2.5 B, the triangular Pascal matrix A has an inverse with “alternating diagonals”. Check that this A^{-1} is DAD , where the diagonal matrix D has alternating entries 1, -1, 1, -1. Then $ADAD = I$, so what is the inverse of $AD = \text{pascal}(4,1)$?
- 39 The Hilbert matrices have $H_{ij} = 1/(i + j - 1)$. Ask MATLAB for the exact 6 by 6 inverse $\text{invhilb}(6)$. Then ask for $\text{inv}(\text{hilb}(6))$. How can these be different, when the computer never makes mistakes?
- 40 Use $\text{inv}(S)$ to invert MATLAB’s 4 by 4 symmetric matrix $S = \text{pascal}(4)$. Create Pascal’s lower triangular $A = \text{abs}(\text{pascal}(4,1))$ and test $\text{inv}(S) = \text{inv}(A') * \text{inv}(A)$.
- 41 If $A = \text{ones}(4,4)$ and $b = \text{rand}(4,1)$, how does MATLAB tell you that $Ax = b$ has no solution? If $b = \text{ones}(4,1)$, which solution to $Ax = b$ is found by $A \setminus b$?
- 42 If $AC = I$ and $AC^* = I$ (all square matrices) use 2I to prove that $C = C^*$.
- 43 Direct multiplication gives $MM^{-1} = I$, and I would recommend doing #3. M^{-1} shows the change in A^{-1} (useful to know) when a matrix is subtracted from A :

- 1 $M = I - uv$ and $M^{-1} = I + uv/(1 - vu)$
- 2 $M = A - uv$ and $M^{-1} = A^{-1} + A^{-1}uvA^{-1}/(1 - vA^{-1}u)$
- 3 $M = I - UV$ and $M^{-1} = I_n + U(I_m - VU)^{-1}V$
- 4 $M = A - UW^{-1}V$ and $M^{-1} = A^{-1} + A^{-1}U(W - VA^{-1}U)^{-1}VA^{-1}$

The Woodbury-Morrison formula 4 is the “matrix inversion lemma” in engineering. The four identities come from the 1, 1 block when inverting these matrices (v is 1 by n , u is n by 1, V is m by n , U is n by m , $m \leq n$):

$$\begin{bmatrix} I & u \\ v & 1 \end{bmatrix} \quad \begin{bmatrix} A & u \\ v & 1 \end{bmatrix} \quad \begin{bmatrix} I_n & U \\ V & I_m \end{bmatrix} \quad \begin{bmatrix} A & U \\ V & W \end{bmatrix}$$

ELIMINATION = FACTORIZATION: $A = LU$ ■ 2.6

Students often say that mathematics courses are too theoretical. Well, not this section. It is almost purely practical. The goal is to describe Gaussian elimination in the most useful way. Many key ideas of linear algebra, when you look at them closely, are really *factorizations* of a matrix. The original matrix A becomes the product of two or three special matrices. The first factorization—also the most important in practice—comes now from elimination. *The factors are triangular matrices. The factorization that comes from elimination is $A = LU$.*

We already know U , the upper triangular matrix with the pivots on its diagonal. The elimination steps take A to U . We will show how reversing those steps (taking U back to A) is achieved by a lower triangular L . *The entries of L are exactly the multipliers ℓ_{ij} —which multiplied row j when it was subtracted from row i .*

Start with a 2 by 2 example. The matrix A contains 2, 1, 6, 8. The number to eliminate is 6. *Subtract 3 times row 1 from row 2.* That step is E_{21} in the forward direction. The return step from U to A is $L = E_{21}^{-1}$ (an addition using +3):

$$\begin{aligned} \text{Forward from } A \text{ to } U : \quad E_{21}A &= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U \\ \text{Back from } U \text{ to } A : \quad E_{21}^{-1}U &= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A. \end{aligned}$$

The second line is our factorization. Instead of $E_{21}^{-1}U = A$ we write $LU = A$. Move now to larger matrices with many E 's. *Then L will include all their inverses.*

Each step from A to U multiplies by a matrix E_{ij} to produce zero in the (i, j) position. To keep this clear, we stay with the most frequent case—*when no row exchanges are involved.* If A is 3 by 3, we multiply by E_{21} and E_{31} and E_{32} . The multipliers ℓ_{ij} produce zeros in the (2, 1) and (3, 1) and (3, 2) positions—all below the diagonal. Elimination ends with the upper triangular U .

Now move those E 's onto the other side, *where their inverses multiply U :*

$$(E_{32}E_{31}E_{21})A = U \quad \text{becomes} \quad A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U \quad \text{which is} \quad A = LU. \quad (1)$$

The inverses go in opposite order, as they must. That product of three inverses is L . *We have reached $A = LU$.* Now we stop to understand it.

Explanation and Examples

First point: Every inverse matrix E_{ij}^{-1} is *lower triangular*. Its off-diagonal entry is ℓ_{ij} , to undo the subtraction with $-\ell_{ij}$. The main diagonals of E and E^{-1} contain 1's. Our example above had $\ell_{21} = 3$ and $E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ and $E^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

Second point: Equation (1) shows a lower triangular matrix (the product of E_{ij}) multiplying A . It also shows a lower triangular matrix (the product of E_{ij}^{-1}) multiplying U to bring back A . **This product of inverses is L .**

One reason for working with the inverses is that we want to factor A , not U . The “inverse form” gives $A = LU$. The second reason is that we get something extra, almost more than we deserve. This is the third point, showing that L is exactly right.

Third point: Each multiplier ℓ_{ij} goes directly into its i, j position—*unchanged*—in the product of inverses which is L . Usually matrix multiplication will mix up all the numbers. Here that doesn’t happen. The order is right for the inverse matrices, to keep the ℓ ’s unchanged. The reason is given below in equation (3).

Since each E^{-1} has 1’s down its diagonal, the final good point is that L does too.

2J ($A = LU$) *This is elimination without row exchanges.* The upper triangular U has the pivots on its diagonal. The lower triangular L has all 1's on its diagonal. **The multipliers ℓ_{ij} are below the diagonal of L .**

Example 1 The matrix A has 1, 2, 1 on its diagonals. Elimination subtracts $\frac{1}{2}$ times row 1 from row 2. The last step subtracts $\frac{2}{3}$ times row 2 from row 3. The lower triangular L has $\ell_{21} = \frac{1}{2}$ and $\ell_{32} = \frac{2}{3}$. Multiplying LU produces A :

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix} = LU.$$

The (3, 1) multiplier is zero because the (3, 1) entry in A is zero. No operation needed.

Example 2 Change the top left entry from 2 to 1. The pivots all become 1. The multipliers are all 1. That pattern continues when A is 4 by 4:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & \\ 1 & 1 & 1 & \\ 1 & & & 1 \end{bmatrix}.$$

These LU examples are showing something extra, which is very important in practice. Assume no row exchanges. When can we predict zeros in L and U ?

When a row of A starts with zeros, so does that row of L .

When a column of A starts with zeros, so does that column of U .

If a row starts with zero, we don't need an elimination step. L has a zero, which saves computer time. Similarly, zeros at the *start* of a column survive into U . But please realize: Zeros in the *middle* of a matrix are likely to be filled in, while elimination sweeps forward. We now explain why L has the multipliers ℓ_{ij} in position, with no mix-up.

The key reason why A equals LU : Ask yourself about the pivot rows that are subtracted from lower rows. Are they the original rows of A ? *No*, elimination probably changed them. Are they rows of U ? *Yes*, the pivot rows never change again. When computing the third row of U , we subtract multiples of earlier rows of U (*not rows of A !*):

$$\text{Row 3 of } U = (\text{Row 3 of } A) - \ell_{31}(\text{Row 1 of } U) - \ell_{32}(\text{Row 2 of } U). \quad (2)$$

Rewrite this equation to see that the row $[\ell_{31} \ \ell_{32} \ 1]$ is multiplying U :

$$(\text{Row 3 of } A) = \ell_{31}(\text{Row 1 of } U) + \ell_{32}(\text{Row 2 of } U) + 1(\text{Row 3 of } U). \quad (3)$$

This is exactly row 3 of $A = LU$. All rows look like this, whatever the size of A . With no row exchanges, we have $A = LU$.

Remark The LU factorization is “unsymmetric” because U has the pivots on its diagonal where L has 1's. This is easy to change. **Divide U by a diagonal matrix D that contains the pivots.** That leaves a new matrix with 1's on the diagonal:

$$\begin{array}{c} \text{Split } U \text{ into} \\ \left[\begin{array}{cccc} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{array} \right] \end{array} \left[\begin{array}{cccc} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \vdots \\ & & & 1 \end{array} \right].$$

It is convenient (but a little confusing) to keep the same letter U for this new upper triangular matrix. It has 1's on the diagonal (like L). Instead of the normal LU , the new form has D in the middle: **Lower triangular L times diagonal D times upper triangular U .**

The triangular factorization can be written $A = LU$ or $A = LDU$.

Whenever you see LDU , it is understood that U has 1's on the diagonal. *Each row is divided by its first nonzero entry—the pivot.* Then L and U are treated evenly in LDU :

$$\left[\begin{array}{cc} 1 & 0 \\ 3 & 1 \end{array} \right] \left[\begin{array}{cc} 2 & 8 \\ 0 & 5 \end{array} \right] \text{ splits further into } \left[\begin{array}{cc} 1 & 0 \\ 3 & 1 \end{array} \right] \left[\begin{array}{cc} 2 & \\ 5 & \end{array} \right] \left[\begin{array}{cc} 1 & 4 \\ 0 & 1 \end{array} \right]. \quad (4)$$

The pivots 2 and 5 went into D . Dividing the rows by 2 and 5 left the rows $[1 \ 4]$ and $[0 \ 1]$ in the new U . The multiplier 3 is still in L .

My own lectures sometimes stop at this point. The next paragraphs show how elimination codes are organized, and how long they take. If MATLAB (or any software) is available, I strongly recommend the last problems 32 to 35. You can measure the computing time by just counting the seconds!

One Square System = Two Triangular Systems

The matrix L contains our memory of Gaussian elimination. It holds the numbers that multiplied the pivot rows, before subtracting them from lower rows. When do we need this record and how do we use it?

We need L as soon as there is a *right side* b . The factors L and U were completely decided by the left side (the matrix A). On the right side of $Ax = b$, we use *Solve*:

- 1 *Factor* (into L and U , by forward elimination on A)
- 2 *Solve* (forward elimination on b using L , then back substitution using U).

Earlier, we worked on b while we were working on A . No problem with that—just augment A by an extra column b . But most computer codes keep the two sides separate. The memory of forward elimination is held in L and U , at no extra cost in storage. Then we process b whenever we want to. The User's Guide to LINPACK remarks that "This situation is so common and the savings are so important that no provision has been made for solving a single system with just one subroutine."

How does *Solve* work on b ? First, apply forward elimination to the right side (the multipliers are stored in L , use them now). This changes b to a new right side c —we are really solving $Lc = b$. Then back substitution solves $Ux = c$ as always. The original system $Ax = b$ is factored into *two triangular systems*:

$$\text{Solve } Lc = b \quad \text{and then solve } Ux = c. \quad (5)$$

To see that x is correct, multiply $Ux = c$ by L . Then $LUX = Lc$ is just $Ax = b$.

To emphasize: There is *nothing new* about those steps. This is exactly what we have done all along. We were really solving the triangular system $Lc = b$ as elimination went forward. Then back substitution produced x . An example shows it all.

Example 3 Forward elimination on $Ax = b$ ends at $Ux = c$:

$$\begin{array}{l} u + 2v = 5 \\ 4u + 9v = 21 \end{array} \quad \text{becomes} \quad \begin{array}{l} u + 2v = 5 \\ v = 1. \end{array}$$

The multiplier was 4, which is saved in L . The right side used it to find c :

$$Lc = b \quad \text{The lower triangular system} \quad \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \quad \text{gives} \quad c = \begin{bmatrix} 5 \\ 1 \end{bmatrix}.$$

$$Ux = c \quad \text{The upper triangular system} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \quad \text{gives} \quad x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

It is satisfying that L and U can take the n^2 storage locations that originally held A . The ℓ 's go below the diagonal. The whole discussion is only looking to see what elimination actually did.

The Cost of Elimination

A very practical question is cost—or computing time. Can we solve 1000 equations on a PC? What if $n = 10,000$? Large systems come up all the time in scientific computing, where a three-dimensional problem can easily lead to a million unknowns. We can let the calculation run overnight, but we can't leave it for 100 years.

The first stage of elimination, on column 1, produces zeros below the first pivot. To find each new entry below the pivot row requires one multiplication and one subtraction. *We will count this first stage as n^2 multiplications and n^2 subtractions.* It is actually less, $n^2 - n$, because row 1 does not change.

The next stage clears out the second column below the second pivot. The working matrix is now of size $n - 1$. Estimate this stage by $(n - 1)^2$ multiplications and subtractions. The matrices are getting smaller as elimination goes forward. The rough count to reach U is the sum of squares $n^2 + (n - 1)^2 + \dots + 2^2 + 1^2$.

There is an exact formula $\frac{1}{3}n(n + \frac{1}{2})(n + 1)$ for this sum of squares. When n is large, the $\frac{1}{2}$ and the 1 are not important. *The number that matters is $\frac{1}{3}n^3$.* The sum of squares is like the integral of x^2 ! The integral from 0 to n is $\frac{1}{3}n^3$:

Elimination on A requires about $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions.

What about the right side \mathbf{b} ? Going forward, we subtract multiples of b_1 from the lower components b_2, \dots, b_n . This is $n - 1$ steps. The second stage takes only $n - 2$ steps, because b_1 is not involved. The last stage of forward elimination takes one step.

Now start back substitution. Computing x_n uses one step (divide by the last pivot). The next unknown uses two steps. When we reach x_1 it will require n steps ($n - 1$ substitutions of the other unknowns, then division by the first pivot). The total count on the right side, from \mathbf{b} to \mathbf{x} —*forward to the bottom and back to the top*—is exactly n^2 :

$$[(n - 1) + (n - 2) + \dots + 1] + [1 + 2 + \dots + (n - 1) + n] = n^2. \quad (6)$$

To see that sum, pair off $(n - 1)$ with 1 and $(n - 2)$ with 2. The pairings leave n terms, each equal to n . That makes n^2 . The right side costs a lot less than the left side!

Each right side needs n^2 multiplications and n^2 subtractions.

Here are the MATLAB codes to factor A into LU and to solve $A\mathbf{x} = \mathbf{b}$. The program `slu` stops right away if a number smaller than the tolerance “*tol*” appears in a pivot

position. Later the program **plu** will look down the column for a pivot, to execute a row exchange and continue solving. These Teaching Codes are on web.mit.edu/18.06/www.

```
function [L, U] = slu(A)
%     Square LU factorization with no row exchanges!
[n, n] = size(A); tol = 1.e - 6;
for k = 1 : n
    if abs(A(k, k)) < tol
        end      % Cannot proceed without a row exchange: stop
    L(k, k) = 1;
    for i = k + 1 : n % Multipliers for column k are put into L
        L(i, k) = A(i, k)/A(k, k);
        for j = k + 1 : n % Elimination beyond row k and column k
            A(i, j) = A(i, j) - L(i, k) * A(k, j); % Matrix still called A
        end
    end
    for j = k : n
        U(k, j) = A(k, j); % row k is settled, now name it U
    end
end
```

```
function x = sly(A, b)
%     Solve Ax = b using L and U from slu(A). No row exchanges!
[L, U] = slu(A);
for k = 1 : n
    for j = 1 : k - 1
        s = s + L(k, j) * c(j);
    end
    c(k) = b(k) - s; % Forward elimination to solve Lc = b
end
for k = n : -1 : 1 % Going backwards from x(n) to x(1)
    for j = k + 1 : n % Back substitution
        t = t + U(k, j) * x(j);
    end
    x(k) = (c(k) - t)/U(k, k); % Divide by pivot
end
x = x'; % Transpose to column vector
```

How long does it take to solve $Ax = b$? For a random matrix of order $n = 1000$, we tried the MATLAB command `tic; A\b; toc`. The time on my PC was 3 seconds. For $n = 2000$ the time was 20 seconds, which is approaching the n^3 rule. The time is multiplied by about 8 when n is multiplied by 2.

According to this n^3 rule, matrices that are 10 times as large (order 10,000) will take thousands of seconds. Matrices of order 100,000 will take millions of seconds.

This is too expensive without a supercomputer, but remember that these matrices are full. Most matrices in practice are sparse (many zero entries). In that case $A = LU$ is much faster. For tridiagonal matrices of order 10,000, storing only the nonzeros, solving $Ax = b$ is a breeze.

■ REVIEW OF THE KEY IDEAS ■

1. Gaussian elimination (with no row exchanges) factors A into L times U .
2. The lower triangular L contains the numbers that multiply pivot rows, going from A to U . The product LU adds those rows back to recover A .
3. On the right side we solve $Lc = b$ (forward) and $Ux = c$ (backwards).
4. There are $\frac{1}{3}(n^3 - n)$ multiplications and subtractions on the left side.
5. There are n^2 multiplications and subtractions on the right side.

■ WORKED EXAMPLES ■

2.6 A The lower triangular Pascal matrix P_L was in the worked example **2.5 B**. (It contains the “Pascal triangle” and Gauss-Jordan found its inverse.) This problem connects P_L to the symmetric Pascal matrix P_S and the upper triangular P_U . The symmetric P_S has Pascal’s triangle tilted, so each entry is the sum of the entry above and the entry to the left. The n by n symmetric P_S is `pascal(n)` in MATLAB.

Problem: Establish the amazing lower-upper factorization $P_S = P_L P_U$:

$$\text{pascal}(4) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P_L P_U.$$

Then predict and check the next row and column for 5 by 5 Pascal matrices.

Solution You could multiply $P_L P_U$ to get P_S . Better to start with the symmetric P_S and reach the upper triangular P_U by elimination:

$$P_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = P_U.$$

The multipliers ℓ_{ij} that entered these steps go perfectly into P_L . Then $P_S = P_L P_U$ is a particularly neat example of $A = LU$. Notice that every pivot is 1! The pivots are

on the diagonal of P_U . The next section will show how symmetry produces a special relationship between the triangular L and U . You see P_U as the “transpose” of P_L .

You might expect the MATLAB command `lu(pascal(4))` to produce these factors P_L and P_U . That doesn’t happen because the `lu` subroutine chooses the largest available pivot in each column (it will exchange rows so the second pivot is 3). But a different command `chol` factors without row exchanges. Then $[L, U] = \text{chol}(\text{pascal}(4))$ produces the triangular Pascal matrices as L and U . Try it.

In the 5 by 5 case the new fifth rows do maintain $P_S = P_L P_U$:

$$\begin{array}{ll} \text{Next Row} & 1 \ 5 \ 15 \ 35 \ 70 \text{ for } P_S \end{array} \quad \begin{array}{ll} 1 \ 4 \ 6 \ 4 \ 1 \text{ for } P_L \end{array}$$

I will only check that this fifth row of P_L times the (same) fifth column of P_U gives $1^2 + 4^2 + 6^2 + 4^2 + 1^2 = 70$ in the fifth row of P_S . The full proof of $P_S = P_L P_U$ is quite fascinating—this factorization can be reached in at least four different ways. I am going to put these proofs on the course web page web.mit.edu/18.06/www, which is also available through MIT’s *OpenCourseWare* at ocw.mit.edu.

These Pascal matrices P_S, P_L, P_U have so many remarkable properties—we will see them again. You could locate them using the Index at the end of the book.

2.6 B The problem is: Solve $P_S \mathbf{x} = \mathbf{b} = (1, 0, 0, 0)$. This special right side means that \mathbf{x} will be the first column of P_S^{-1} . That is Gauss-Jordan, matching the columns of $P_S P_S^{-1} = I$. We already know the triangular P_L and P_U from **2.6 A**, so we solve

$$P_L \mathbf{c} = \mathbf{b} \text{ (forward substitution)} \quad P_U \mathbf{x} = \mathbf{c} \text{ (back substitution).}$$

Use MATLAB to find the full inverse matrix P_S^{-1} .

Solution The lower triangular system $P_L \mathbf{c} = \mathbf{b}$ is solved *top to bottom*:

$$\begin{array}{lll} c_1 & = 1 & c_1 = +1 \\ c_1 + c_2 & = 0 & c_2 = -1 \\ c_1 + 2c_2 + c_3 & = 0 & c_3 = +1 \\ c_1 + 3c_2 + 3c_3 + c_4 & = 0 & c_4 = -1 \end{array} \quad \text{gives}$$

Forward elimination is multiplication by P_L^{-1} . It produces the upper triangular system $P_U \mathbf{x} = \mathbf{c}$. The solution \mathbf{x} comes as always by back substitution, *bottom to top*:

$$\begin{array}{lll} x_1 + x_2 + x_3 + x_4 = 1 & x_1 = +4 \\ x_2 + 2x_3 + 3x_4 = -1 & x_2 = -6 \\ x_3 + 3x_4 = 1 & x_3 = +4 \\ x_4 = -1 & x_4 = -1 \end{array} \quad \text{gives}$$

The complete inverse matrix P_S^{-1} has that \mathbf{x} in its first column:

$$\text{inv}(\text{pascal}(4)) = \begin{bmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Problem Set 2.6

Problems 1–14 compute the factorization $A = LU$ (and also $A = LDU$).

- 1 (Important) Forward elimination changes $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}x = b$ to a triangular $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}x = c$:

$$\begin{array}{lcl} x + y = 5 & \longrightarrow & x + y = 5 \\ x + 2y = 7 & & y = 2 \end{array} \quad \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 7 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

That step subtracted $\ell_{21} = \underline{\hspace{2cm}}$ times row 1 from row 2. The reverse step adds ℓ_{21} times row 1 to row 2. The matrix for that reverse step is $L = \underline{\hspace{2cm}}$. Multiply this L times the triangular system $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ to get $\underline{\hspace{2cm}} = \underline{\hspace{2cm}}$. In letters, L multiplies $Ux = c$ to give $\underline{\hspace{2cm}}$.

- 2 (Move to 3 by 3) Forward elimination changes $Ax = b$ to a triangular $Ux = c$:

$$\begin{array}{lll} x + y + z = 5 & x + y + z = 5 & x + y + z = 5 \\ x + 2y + 3z = 7 & y + 2z = 2 & y + 2z = 2 \\ x + 3y + 6z = 11 & 2y + 5z = 6 & z = 2 \end{array}$$

The equation $z = 2$ in $Ux = c$ comes from the original $x + 3y + 6z = 11$ in $Ax = b$ by subtracting $\ell_{31} = \underline{\hspace{2cm}}$ times equation 1 and $\ell_{32} = \underline{\hspace{2cm}}$ times the final equation 2. Reverse that to recover $[1 \ 3 \ 6 \ 11]$ in A and b from the final $[1 \ 1 \ 1 \ 5]$ and $[0 \ 1 \ 2 \ 2]$ and $[0 \ 0 \ 1 \ 2]$ in U and c :

$$\text{Row 3 of } [A \ b] = (\ell_{31} \text{ Row 1} + \ell_{32} \text{ Row 2} + 1 \text{ Row 3}) \text{ of } [U \ c].$$

In matrix notation this is multiplication by L . So $A = LU$ and $b = Lc$.

- 3 Write down the 2 by 2 triangular systems $Lc = b$ and $Ux = c$ from Problem 1. Check that $c = (5, 2)$ solves the first one. Find x that solves the second one.
- 4 What are the 3 by 3 triangular systems $Lc = b$ and $Ux = c$ from Problem 2? Check that $c = (5, 2, 2)$ solves the first one. Which x solves the second one?
- 5 What matrix E puts A into triangular form $EA = U$? Multiply by $E^{-1} = L$ to factor A into LU :

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

- 6 What two elimination matrices E_{21} and E_{32} put A into upper triangular form $E_{32}E_{21}A = U$? Multiply by E_{32}^{-1} and E_{21}^{-1} to factor A into $LU = E_{21}^{-1}E_{32}^{-1}U$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0 \end{bmatrix}.$$

- 7 What three elimination matrices E_{21}, E_{31}, E_{32} put A into upper triangular form $E_{32}E_{31}E_{21}A = U$? Multiply by E_{32}^{-1} , E_{31}^{-1} and E_{21}^{-1} to factor A into LU where $L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$. Find L and U :

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix}.$$

- 8 Suppose A is already lower triangular with 1's on the diagonal. Then $U = I$!

$$A = L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}.$$

The elimination matrices E_{21}, E_{31}, E_{32} contain $-a$ then $-b$ then $-c$.

- (a) Multiply $E_{32}E_{31}E_{21}$ to find the single matrix E that produces $EA = I$.
 (b) Multiply $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$ to bring back L (nicer than E).
 9 When zero appears in a pivot position, $A = LU$ is not possible! (We are requiring nonzero pivots in U .) Show directly why these are both impossible:

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ l & 1 & \\ m & n & 1 \end{bmatrix} \begin{bmatrix} d & e & g \\ f & h & i \end{bmatrix}.$$

This difficulty is fixed by a row exchange. That needs a “permutation” P .

- 10 Which number c leads to zero in the second pivot position? A row exchange is needed and $A = LU$ is not possible. Which c produces zero in the third pivot position? Then a row exchange can't help and elimination fails:

$$A = \begin{bmatrix} 1 & c & 0 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

- 11 What are L and D for this matrix A ? What is U in $A = LU$ and what is the new U in $A = LDU$?

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 0 & 3 & 9 \\ 0 & 0 & 7 \end{bmatrix}.$$

- 12 A and B are symmetric across the diagonal (because $4 = 4$). Find their triple factorizations LDU and say how U is related to L for these symmetric matrices:

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix}.$$

- 13** (Recommended) Compute L and U for the symmetric matrix

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find four conditions on a, b, c, d to get $A = LU$ with four pivots.

- 14** Find L and U for the nonsymmetric matrix

$$A = \begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix}.$$

Find the four conditions on a, b, c, d, r, s, t to get $A = LU$ with four pivots.

Problems 15-16 use L and U (without needing A) to solve $Ax = b$.

- 15** Solve the triangular system $Lc = b$ to find c . Then solve $Ux = c$ to find x :

$$L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 11 \end{bmatrix}.$$

For safety find $A = LU$ and solve $Ax = b$ as usual. Circle c when you see it.

- 16** Solve $Lc = b$ to find c . Then solve $Ux = c$ to find x . What was A ?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

- 17** (a) When you apply the usual elimination steps to L , what matrix do you reach?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}.$$

(b) When you apply the same steps to I , what matrix do you get?

(c) When you apply the same steps to LU , what matrix do you get?

- 18** If $A = LDU$ and also $A = L_1D_1U_1$ with all factors invertible, then $L = L_1$ and $D = D_1$ and $U = U_1$. “The factors are unique.”

Derive the equation $L_1^{-1}LD = D_1U_1U^{-1}$. Are the two sides triangular or diagonal? Deduce $L = L_1$ and $U = U_1$ (they all have diagonal 1's). Then $D = D_1$.

- 19** *Tridiagonal matrices* have zero entries except on the main diagonal and the two adjacent diagonals. Factor these into $A = LU$ and $A = LDL^T$:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

- 20** When T is tridiagonal, its L and U factors have only two nonzero diagonals. How would you take advantage of the zeros in T in a computer code for Gaussian elimination? Find L and U .

$$T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}.$$

- 21** If A and B have nonzeros in the positions marked by x , which zeros (marked by 0) are still zero in their factors L and U ?

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} x & x & x & 0 \\ x & x & 0 & x \\ x & 0 & x & x \\ 0 & x & x & x \end{bmatrix}.$$

- 22** After elimination has produced zeros below the first pivot, put x 's to show which blank entries are known in the final L and U :

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} 0 & & \\ 0 & & \\ 0 & & \end{bmatrix}.$$

- 23** Suppose you eliminate upwards (almost unheard of). Use the last row to produce zeros in the last column (the pivot is 1). Then use the second row to produce zero above the second pivot. Find the factors in $A = UL(?)$:

$$A = \begin{bmatrix} 5 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- 24** Collins uses elimination in both directions, meeting at the center. Substitution goes out from the center. After eliminating both 2's in A , one from above and one from below, what 4 by 4 matrix is left? Solve $Ax = b$ his way.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 5 \\ 8 \\ 8 \\ 2 \end{bmatrix}.$$

- 25 (Important) If A has pivots 2, 7, 6 with no row exchanges, what are the pivots for the upper left 2 by 2 submatrix B (without row 3 and column 3)? Explain why.
- 26 Starting from a 3 by 3 matrix A with pivots 2, 7, 6, add a fourth row and column to produce M . What are the first three pivots for M , and why? What fourth row and column are sure to produce 9 as the fourth pivot?
- 27 Use `chol(pascal(5))` to find the triangular Pascal factors as in Worked Example 2.6 A. Show how row exchanges in $[L, U] = \text{lu}(\text{pascal}(5))$ spoil Pascal's pattern!
- 28 (Careful review) For which numbers c is $A = LU$ impossible—with three pivots?

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & c & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- 29 Change the program `slu(A)` into `sldu(A)`, so that it produces L , D , and U . Put L , D , U into the n^2 storage locations that held the original A . The extra storage used for L is not required.
- 30 Explain in words why $x(k)$ is $(c(k) - t)/U(k, k)$ at the end of `slv(A, b)`.
- 31 Write a program that multiplies a two-diagonal L times a two-diagonal U . Don't loop from 1 to n when you know there are zeros! L times U should undo `slu`.
- 32 I just learned MATLAB's `tic`-`toc` command, which measures computing time. Previously I counted seconds until the answer appeared, which required very large problems—now $A = \text{rand}(1000)$ and $b = \text{rand}(1000,1)$ is large enough. How much faster is `tic`; $A \setminus b$; `toc` for elimination than `tic`; $\text{inv}(A) * b$; `toc` which computes A^{-1} ?
- 33 Compare `tic`; $\text{inv}(A)$; `toc` for $A = \text{rand}(500)$ and $A = \text{rand}(1000)$. The n^3 operation count says that doubling n should multiply computing time by 8.
- 34 $I = \text{eye}(1000)$; $A = \text{rand}(1000)$; $B = \text{triu}(A)$; produces a random *triangular* matrix B . Compare the times for $\text{inv}(B)$ and $B \setminus I$. Backslash is engineered to use the zeros in B , while inv uses the zeros in I when reducing $[B \ I]$ by Gauss-Jordan. (Compare also with $\text{inv}(A)$ and $A \setminus I$ for the full matrix A .)
- 35 Estimate the time difference for each new right side b when $n = 800$. Create $A = \text{rand}(800)$ and $b = \text{rand}(800,1)$ and $B = \text{rand}(800,9)$. Compare `tic`; $A \setminus b$; `toc` and `tic`; $A \setminus B$; `toc` (which solves for 9 right sides).

- 36 Show that L^{-1} has entries j/i on and below its main diagonal:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix} \quad \text{and} \quad L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}.$$

I think this pattern continues for $L = \text{eye}(5) - \text{diag}(1:5)\backslash\text{diag}(1:4, -1)$ and $\text{inv}(L)$.

TRANSPOSES AND PERMUTATIONS ■ 2.7

We need one more matrix, and fortunately it is much simpler than the inverse. It is the “*transpose*” of A , which is denoted by A^T . *The columns of A^T are the rows of A .*

When A is an m by n matrix, the transpose is n by m :

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \quad \text{then} \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}.$$

You can write the rows of A into the columns of A^T . Or you can write the columns of A into the rows of A^T . The matrix “flips over” its main diagonal. The entry in row i , column j of A^T comes from row j , column i of the original A :

$$(A^T)_{ij} = A_{ji}.$$

The transpose of a lower triangular matrix is upper triangular. (But the inverse is still lower triangular.) The transpose of A^T is A .

Note MATLAB’s symbol for the transpose of A is A' . Typing $[1 \ 2 \ 3]$ gives a row vector and the column vector is $v = [1 \ 2 \ 3]'$. To enter a matrix M with second column $w = [4 \ 5 \ 6]'$ you could define $M = [v \ w]$. Quicker to enter by rows and then transpose the whole matrix: $M = [1 \ 2 \ 3; 4 \ 5 \ 6]'$.

The rules for transposes are very direct. We can transpose $A+B$ to get $(A+B)^T$. Or we can transpose A and B separately, and then add $A^T + B^T$ —same result. The serious questions are about the transpose of a product AB and an inverse A^{-1} :

$$\text{The transpose of } A+B \text{ is } A^T + B^T. \tag{1}$$

$$\text{The transpose of } AB \text{ is } (AB)^T = B^T A^T. \tag{2}$$

$$\text{The transpose of } A^{-1} \text{ is } (A^{-1})^T = (A^T)^{-1}. \tag{3}$$

Notice especially how $B^T A^T$ comes in reverse order. For inverses, this reverse order was quick to check: $B^{-1} A^{-1}$ times AB produces I . To understand $(AB)^T = B^T A^T$, start with $(Ax)^T = x^T A^T$:

A x combines the columns of A while $x^T A^T$ combines the rows of A^T .

It is the same combination of the same vectors! In A they are columns, in A^T they are rows. So the transpose of the column Ax is the row $x^T A^T$. That fits our formula $(Ax)^T = x^T A^T$. Now we can prove the formula for $(AB)^T$.

When $B = [x_1 \ x_2]$ has two columns, apply the same idea to each column. The columns of AB are Ax_1 and Ax_2 . Their transposes are the rows of $B^T A^T$:

$$\text{Transposing } AB = \begin{bmatrix} A & x_1 & A & x_2 & \cdots \end{bmatrix} \text{ gives } \begin{bmatrix} x_1^T A^T \\ x_2^T A^T \\ \vdots \end{bmatrix} \text{ which is } B^T A^T. \quad (4)$$

The right answer $B^T A^T$ comes out a row at a time. There is also a “*transparent proof*” by looking through the page at the end of the problem set. Here are numbers!

$$AB = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 9 & 1 \end{bmatrix} \quad \text{and} \quad B^T A^T = \begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 0 & 1 \end{bmatrix}.$$

The reverse order rule extends to three or more factors: $(ABC)^T$ equals $C^T B^T A^T$.

If $A = LDU$ then $A^T = U^T D^T L^T$. The pivot matrix has $D = D^T$.

Now apply this product rule to both sides of $A^{-1}A = I$. On one side, I^T is I . We confirm the rule that $(A^{-1})^T$ is the inverse of A^T :

$$A^{-1}A = I \quad \text{is transposed to} \quad A^T(A^{-1})^T = I. \quad (5)$$

Similarly $AA^{-1} = I$ leads to $(A^{-1})^T A^T = I$. We can invert the transpose or transpose the inverse. Notice especially: A^T is *invertible exactly when A is invertible*.

Example 1 The inverse of $A = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$. The transpose is $A^T = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$.

$$(A^{-1})^T \quad \text{and} \quad (A^T)^{-1} \quad \text{are both equal to} \quad \begin{bmatrix} 1 & -6 \\ 0 & 1 \end{bmatrix}.$$

Before leaving these rules, we call attention to dot products. The following statement looks extremely simple, but it actually contains the deep purpose for the transpose. For any vectors x and y ,

$$(Ax)^T y \quad \text{equals} \quad x^T A^T y \quad \text{equals} \quad x^T (A^T y). \quad (6)$$

When A moves from one side of a dot product to the other side, it becomes A^T .

Here are two quick applications to electrical engineering and mechanical engineering (with more in Chapter 8). The same A and A^T appear in both applications.

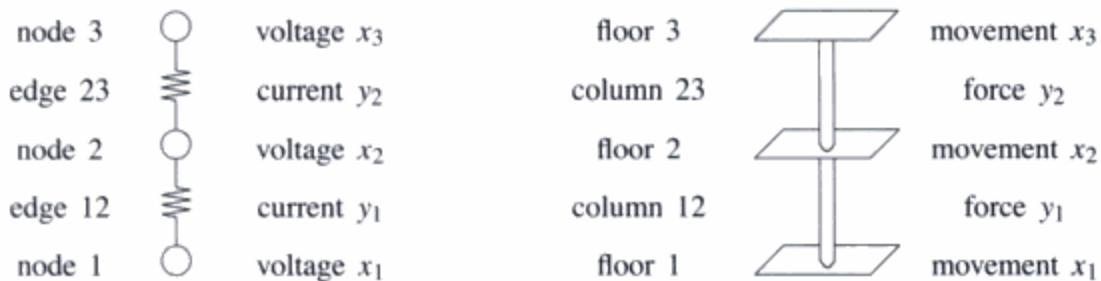


Figure 2.9 A line of resistors and a structure, both governed by A and A^T .

Electrical Networks The vector $\mathbf{x} = (x_1, x_2, x_3)$ gives voltages at 3 nodes, and $A\mathbf{x}$ gives the voltage differences across 2 edges. The “difference matrix” A is 2 by 3:

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix} = \text{voltage differences.}$$

The vector $\mathbf{y} = (y_1, y_2)$ gives currents on those edges (node 1 to 2, and node 2 to 3). Look how $A^T\mathbf{y}$ finds the total currents leaving each node in Kirchhoff’s Current Law:

$$A^T\mathbf{y} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} \text{current leaving node 1} \\ \text{out minus in at node 2} \\ \text{current leaving node 3} \end{bmatrix}.$$

Section 8.2 studies networks in detail. Here we look at the energy $\mathbf{x}^T A^T \mathbf{y}$ lost as heat:

$$\text{Energy (voltages } \mathbf{x}) \cdot (\text{inputs } A^T \mathbf{y}) = \text{Heat loss (voltage drops } A\mathbf{x}) \cdot (\text{currents } \mathbf{y}).$$

Forces on a Structure The vector $\mathbf{x} = (x_1, x_2, x_3)$ gives the movement of each floor under the weight of the floors above. The matrix A takes differences of the x ’s to give the strains $A\mathbf{x}$, the movements between floors:

$$A\mathbf{x} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \end{bmatrix} = \begin{bmatrix} \text{movement between 1 and 2} \\ \text{movement between 2 and 3} \end{bmatrix}.$$

The vector $\mathbf{y} = (y_1, y_2)$ gives the stresses (internal forces from the columns that resist the movement and save the structure). Then $A^T\mathbf{y}$ gives the forces that balance the weight:

$$A^T\mathbf{y} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 - y_1 \\ -y_2 \end{bmatrix} \quad \text{balances} \quad \begin{bmatrix} \text{weight of floor 1} \\ \text{weight of floor 2} \\ \text{weight of floor 3} \end{bmatrix}$$

In resistors, the relation of \mathbf{y} to $A\mathbf{x}$ is Ohm’s Law (current proportional to voltage difference). For elastic structures this is Hooke’s Law (stress proportional to strain). The

catastrophe on September 11 came when the fires in the World Trade Center weakened the steel columns. Hooke's Law eventually failed. The internal forces couldn't balance the weight of the tower. After the first columns buckled, the columns below couldn't take the extra weight.

For a linearly elastic structure, the work balance equation is $(Ax)^T y = x^T (A^T y)$:

$$\text{Internal work (strain } Ax \text{) } \cdot \text{ (stress } y \text{)} = \text{ External work (movement } x \text{) } \cdot \text{ (force } A^T y \text{).}$$

Symmetric Matrices

For a *symmetric* matrix—these are the most important matrices—transposing A to A^T produces no change. Then $A^T = A$. The matrix is symmetric across the main diagonal. A symmetric matrix is necessarily square. Its (j, i) and (i, j) entries are equal.

DEFINITION A *symmetric matrix* has $A^T = A$. This means that $a_{ji} = a_{ij}$.

Example 2 $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A^T$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = D^T$.

A is symmetric because of the 2's on opposite sides of the diagonal. The rows agree with the columns. In D those 2's are zeros. Every diagonal matrix is symmetric.

The inverse of a symmetric matrix is also symmetric. (We have to add: “If A is invertible.”) The transpose of A^{-1} is $(A^{-1})^T = (A^T)^{-1} = A^{-1}$, so A^{-1} is symmetric:

$$A^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Now we show that *multiplying any matrix R by R^T gives a symmetric matrix*.

Symmetric Products $R^T R$ and RR^T and LDL^T

Choose any matrix R , probably rectangular. Multiply R^T times R . Then the product $R^T R$ is automatically a square symmetric matrix:

$$\text{The transpose of } R^T R \text{ is } R^T (R^T)^T \text{ which is } R^T R. \quad (7)$$

That is a quick proof of symmetry for $R^T R$. We could also look at the (i, j) entry of $R^T R$. It is the dot product of row i of R^T (column i of R) with column j of R . The (j, i) entry is the same dot product, column j with column i . So $R^T R$ is symmetric.

The matrix RR^T is also symmetric. (The shapes of R and R^T allow multiplication.) But RR^T is a different matrix from $R^T R$. In our experience, most scientific problems that start with a rectangular matrix R end up with $R^T R$ or RR^T or both.

Example 3 $R = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$ and $R^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ produce $R^T R = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $RR^T = [5]$.

The product $R^T R$ is n by n . In the opposite order, RR^T is m by m . Even if $m = n$, it is not very likely that $R^T R = RR^T$. Equality can happen, but it is abnormal.

When elimination is applied to a symmetric matrix, $A^T = A$ is an advantage. The smaller matrices stay symmetric as elimination proceeds, and we can work with half the matrix! It is true that the upper triangular U cannot be symmetric. ***The symmetry is in LDU.*** Remember how the diagonal matrix D of pivots can be divided out, to leave 1's on the diagonal of both L and U :

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} && (L U \text{ misses the symmetry}) \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} && (LDU \text{ captures the symmetry}) \\ &&& \text{Now } U \text{ is the transpose of } L. \end{aligned}$$

When A is symmetric, the usual form $A = LDU$ becomes $A = LDL^T$. The final U (with 1's on the diagonal) is the transpose of L (also with 1's on the diagonal). The diagonal D —the matrix of pivots—is symmetric by itself.

2K If $A = A^T$ can be factored into LDU with no row exchanges, then $U = L^T$.
The symmetric factorization of a symmetric matrix is $A = LDL^T$.

Notice that the transpose of LDL^T is automatically $(L^T)^T D^T L^T$ which is LDL^T again. The work of elimination is cut in half, from $n^3/3$ multiplications to $n^3/6$. The storage is also cut essentially in half. We only keep L and D , not U .

Permutation Matrices

The transpose plays a special role for a *permutation matrix*. This matrix P has a single “1” in every row and every column. Then P^T is also a permutation matrix—maybe the same or maybe different. Any product $P_1 P_2$ is again a permutation matrix. We now create every P from the identity matrix, by reordering the rows of I .

The simplest permutation matrix is $P = I$ (*no exchanges*). The next simplest are the row exchanges P_{ij} . Those are constructed by exchanging two rows i and j of I . Other permutations reorder more rows. By doing all possible row exchanges to I , we get all possible permutation matrices:

DEFINITION A *permutation matrix* P has the rows of I in any order.

Example 4 There are six 3 by 3 permutation matrices. Here they are without the zeros:

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} 1 & 1 & \\ 1 & & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} & 1 & \\ 1 & & 1 \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & 1 & \\ 1 & 1 & \\ 1 & & \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.$$

There are $n!$ permutation matrices of order n . The symbol $n!$ means “ n factorial,” the product of the numbers $(1)(2) \cdots (n)$. Thus $3! = (1)(2)(3)$ which is 6. There will be 24 permutation matrices of order $n = 4$. And 120 permutations of order 5.

There are only two permutation matrices of order 2, namely $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Important: P^{-1} is also a permutation matrix. Among the six 3 by 3 P 's displayed above, the four matrices on the left are their own inverses. The two matrices on the right are inverses of each other. In all cases, a single row exchange is its own inverse. If we repeat the exchange we are back to I . But for $P_{32}P_{21}$, the inverses go in opposite order (of course). The inverse is $P_{21}P_{32}$.

More important: P^{-1} is always the same as P^T . The two matrices on the right are transposes—and inverses—of each other. When we multiply PP^T , the “1” in the first row of P hits the “1” in the first column of P^T (since the first row of P is the first column of P^T). It misses the ones in all the other columns. So $PP^T = I$.

Another proof of $P^T = P^{-1}$ looks at P as a product of row exchanges. A row exchange is its own transpose and its own inverse. P^T and P^{-1} both come from the product of row exchanges in the opposite order. So P^T and P^{-1} are the same.

Symmetric matrices led to $A = LDL^T$. Now permutations lead to $PA = LU$.

The LU Factorization with Row Exchanges

We sure hope you remember $A = LU$. It started with $A = (E_{21}^{-1} \cdots E_{ij}^{-1} \cdots)U$. Every elimination step was carried out by an E_{ij} and it was inverted by E_{ij}^{-1} . Those inverses were compressed into one matrix L , bringing U back to A . The lower triangular L has 1's on the diagonal, and the result is $A = LU$.

This is a great factorization, but it doesn't always work! Sometimes row exchanges are needed to produce pivots. Then $A = (E^{-1} \cdots P^{-1} \cdots E^{-1} \cdots P^{-1} \cdots)U$. Every row exchange is carried out by a P_{ij} and inverted by that P_{ij} . We now compress those row exchanges into a single permutation matrix P . This gives a factorization for every invertible matrix A —which we naturally want.

The main question is where to collect the P_{ij} 's. There are two good possibilities—do all the exchanges before elimination, or do them after the E_{ij} 's. The first way gives $PA = LU$. The second way has a permutation matrix P_1 in the middle.

1. The row exchanges can be done in advance. Their product P puts the rows of A in the right order, so that no exchanges are needed for PA . **Then $PA = LU$.**

2. If we hold row exchanges until *after elimination*, the pivot rows are in a strange order. P_1 puts them in the correct triangular order in U_1 . **Then $A = L_1 P_1 U_1$.**

$PA = LU$ is constantly used in almost all computing (and always in MATLAB). **We will concentrate on this form $PA = LU$.** The factorization $A = L_1 P_1 U_1$ might be more elegant. If we mention both, it is because the difference is not well known. Probably you will not spend a long time on either one. Please don't. The most important case has $P = I$, when A equals LU with no exchanges.

For this matrix A , exchange rows 1 and 2 to put the first pivot in its usual place. Then go through elimination on PA :

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}_A \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}_{PA} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}_{\ell_{31}=2} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}_{\ell_{32}=3}.$$

The matrix PA is in good order, and it factors as usual into LU :

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = LU. \quad (8)$$

We started with A and ended with U . *The only requirement is invertibility of A .*

2L If A is invertible, a permutation P will put its rows in the right order to factor $PA = LU$. There must be a full set of pivots, after row exchanges.

In the MATLAB code, $A([r \ k], :) = A([k \ r], :)$ exchanges row k with row r below it (where the k th pivot has been found). Then we update L and P and the sign of P :

$$\begin{aligned} A([r \ k], :) &= A([k \ r], :); \\ L([r \ k], 1:k-1) &= L([k \ r], 1:k-1); \\ P([r \ k], :) &= P([k \ r], :); \\ \text{sign} &= -\text{sign} \end{aligned}$$

The “**sign**” of P tells whether the number of row exchanges is even ($\text{sign} = +1$) or odd ($\text{sign} = -1$). At the start, P is I and $\text{sign} = +1$. When there is a row exchange, the sign is reversed. The final value of sign is the **determinant of P** and it does not depend on the order of the row exchanges.

For PA we get back to the familiar LU . This is the usual factorization. In reality, MATLAB might not use the first available pivot. Mathematically we can accept a small pivot—anything but zero. It is better if the computer looks down the column for the largest pivot. (Section 9.1 explains why this “**partial pivoting**” reduces the round-off error.) P may contain row exchanges that are not algebraically necessary. Still $PA = LU$.

Our advice is to understand permutations but let MATLAB do the computing. Calculations of $A = LU$ are enough to do by hand, without P . The Teaching Code **splu** (A) factors $PA = LU$ and **splv** (A, b) solves $Ax = b$ for any invertible A . The program **splu** stops if no pivot can be found in column k . That fact is printed.

■ REVIEW OF THE KEY IDEAS ■

1. The transpose puts the rows of A into the columns of A^T . Then $(A^T)_{ij} = A_{ji}$.
2. The transpose of AB is B^TA^T . The transpose of A^{-1} is the inverse of A^T .
3. The dot product $(Ax)^T y$ equals the dot product $x^T (A^T y)$.
4. When A is symmetric ($A^T = A$), its LDU factorization is symmetric: $A = LDL^T$.
5. A permutation matrix P has a 1 in each row and column, and $P^T = P^{-1}$.
6. If A is invertible then a permutation P will reorder its rows for $PA = LU$.

■ WORKED EXAMPLES ■

2.7 A Applying the permutation P to the rows of A destroys its symmetry:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \quad PA = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

What permutation matrix Q applied to the columns of PA will recover symmetry in PAQ ? The numbers 1, 2, 3 must come back to the main diagonal (not necessarily in order). How is Q related to P , when symmetry is saved by PAQ ?

Solution To recover symmetry and put “2” on the diagonal, column 2 of PA must move to column 1. Column 3 of PA (containing “3”) must move to column 2. Then the “1” moves to the 3, 3 position. The matrix that permutes columns is Q :

$$PA = \begin{bmatrix} 4 & 2 & 6 \\ 5 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad PAQ = \begin{bmatrix} 2 & 6 & 4 \\ 6 & 3 & 5 \\ 4 & 5 & 1 \end{bmatrix} \text{ is symmetric.}$$

The matrix Q is P^T . This choice always recovers symmetry, because PAP^T is guaranteed to be symmetric. (Its transpose is again PAP^T .) The matrix Q is also P^{-1} , because the inverse of every permutation matrix is its transpose.

If we look only at the main diagonal D of A , we are finding that PDP^T is guaranteed diagonal. When P moves row 1 down to row 3, P^T on the right will move column 1 to column 3. The $(1, 1)$ entry moves down to $(3, 1)$ and over to $(3, 3)$.

2.7 B Find the symmetric factorization $A = LDL^T$ for the matrix A above. Is A invertible? Find also the $PQ = LU$ factorization for Q , which needs row exchanges.

Solution To factor A into LDL^T we eliminate below the pivots:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & -14 & -22 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 \\ 0 & -14 & -14 \\ 0 & 0 & -8 \end{bmatrix} = U.$$

The multipliers were $\ell_{21} = 4$ and $\ell_{31} = 5$ and $\ell_{32} = 1$. The pivots $1, -14, -8$ go into D . When we divide the rows of U by those pivots, L^T should appear:

$$A = LDL^T = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -14 & \\ & & -8 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix A is invertible because it has three pivots. Its inverse is $(L^T)^{-1}D^{-1}L^{-1}$ and it is also symmetric. The numbers 14 and 8 will turn up in the denominators of A^{-1} . The “determinant” of A is the product of the pivots $(1)(-14)(-8) = 112$.

The matrix Q is certainly invertible. But elimination needs two row exchanges:

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rows } 1 \leftrightarrow 2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{rows } 2 \leftrightarrow 3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Then $L = I$ and $U = I$ are the LU factors. We only need the permutation P that put the rows of Q into their right order in I . Well, P must be Q^{-1} . It is the same P as above! We could find it as a product of the two row exchanges, $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$:

$$P = P_{23}P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{reorders } Q \text{ into } PQ = I.$$

Problem Set 2.7

Questions 1–7 are about the rules for transpose matrices.

1 Find A^T and A^{-1} and $(A^{-1})^T$ and $(A^T)^{-1}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix} \quad \text{and also} \quad A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}.$$

- 2** Verify that $(AB)^T$ equals B^TA^T but those are different from A^TB^T :

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

In case $AB = BA$ (not generally true!) how do you prove that $B^TA^T = A^TB^T$?

- 3** (a) The matrix $((AB)^{-1})^T$ comes from $(A^{-1})^T$ and $(B^{-1})^T$. In what order?
 (b) If U is upper triangular then $(U^{-1})^T$ is ____ triangular.
4 Show that $A^2 = 0$ is possible but $A^TA = 0$ is not possible (unless $A =$ zero matrix).
5 (a) The row vector x^T times A times the column y produces what number?

$$x^TAy = [0 \ 1] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \text{_____}.$$

- (b) This is the row $x^TA = \text{_____}$ times the column $y = (0, 1, 0)$.
 (c) This is the row $x^T = [0 \ 1]$ times the column $Ay = \text{_____}$.
6 When you transpose a block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ the result is $M^T = \text{_____}$. Test it. Under what conditions on A, B, C, D is the block matrix symmetric?
7 True or false:

- (a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric.
 (b) If A and B are symmetric then their product AB is symmetric.
 (c) If A is not symmetric then A^{-1} is not symmetric.
 (d) When A, B, C are symmetric, the transpose of ABC is CBA .

Questions 8–15 are about permutation matrices.

- 8** Why are there $n!$ permutation matrices of order n ?
9 If P_1 and P_2 are permutation matrices, so is P_1P_2 . This still has the rows of I in some order. Give examples with $P_1P_2 \neq P_2P_1$ and $P_3P_4 = P_4P_3$.
10 There are 12 “even” permutations of $(1, 2, 3, 4)$, with an even number of exchanges. Two of them are $(1, 2, 3, 4)$ with no exchanges and $(4, 3, 2, 1)$ with two exchanges. List the other ten. Instead of writing each 4 by 4 matrix, use the numbers 4, 3, 2, 1 to give the position of the 1 in each row.
11 (Try this question) Which permutation makes PA upper triangular? Which permutations make P_1AP_2 lower triangular? **Multiplying A on the right by P_2 exchanges the _____ of A.**

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}.$$

12 Explain why the dot product of \mathbf{x} and \mathbf{y} equals the dot product of $P\mathbf{x}$ and $P\mathbf{y}$. Then from $(P\mathbf{x})^T(P\mathbf{y}) = \mathbf{x}^T\mathbf{y}$ deduce that $P^T P = I$ for any permutation. With $\mathbf{x} = (1, 2, 3)$ and $\mathbf{y} = (1, 4, 2)$ choose P to show that $P\mathbf{x} \cdot \mathbf{y}$ is not always equal to $\mathbf{x} \cdot P\mathbf{y}$.

13 Find a 3 by 3 permutation matrix with $P^3 = I$ (but not $P = I$). Find a 4 by 4 permutation \widehat{P} with $\widehat{P}^4 \neq I$.

14 If you take powers of a permutation matrix, why is some P^k eventually equal to I ?

Find a 5 by 5 permutation P so that the smallest power to equal I is P^6 . (This is a challenge question. Combine a 2 by 2 block with a 3 by 3 block.)

15 Row exchange matrices are symmetric: $P^T = P$. Then $P^T P = I$ becomes $P^2 = I$. Some other permutation matrices are also symmetric.

(a) If P sends row 1 to row 4, then P^T sends row _____ to row _____.

When $P^T = P$ the row exchanges come in pairs with no overlap.

(b) Find a 4 by 4 example with $P^T = P$ that moves all four rows.

Questions 16–21 are about symmetric matrices and their factorizations.

16 If $A = A^T$ and $B = B^T$, which of these matrices are certainly symmetric?

(a) $A^2 - B^2$

(b) $(A + B)(A - B)$

(c) ABA

(d) $ABAB$.

17 Find 2 by 2 symmetric matrices $A = A^T$ with these properties:

(a) A is not invertible.

(b) A is invertible but cannot be factored into $L U$ (row exchanges needed).

(c) A can be factored into LDL^T but not into LL^T (because of negative D).

18 (a) How many entries of A can be chosen independently, if $A = A^T$ is 5 by 5?

(b) How do L and D (still 5 by 5) give the same number of choices?

(c) How many entries can be chosen if A is skew-symmetric? ($A^T = -A$).

19 Suppose R is rectangular (m by n) and A is symmetric (m by m).

(a) Transpose $R^T A R$ to show its symmetry. What shape is this matrix?

(b) Show why $R^T R$ has no negative numbers on its diagonal.

- 20** Factor these symmetric matrices into $A = LDL^T$. The pivot matrix D is diagonal:

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

- 21** After elimination clears out column 1 below the first pivot, find the symmetric 2 by 2 matrix that appears in the lower right corner:

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

Questions 22–30 are about the factorizations $PA = LU$ and $A = L_1P_1U_1$.

- 22** Find the $PA = LU$ factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- 23** Find a 3 by 3 permutation matrix (call it A) that needs two row exchanges to reach the end of elimination. For this matrix, what are its factors P , L , and U ?

- 24** Factor the following matrix into $PA = LU$. Factor it also into $A = L_1P_1U_1$ (hold the exchange of row 3 until 3 times row 1 is subtracted from row 2):

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix}.$$

- 25** Write out P after each step of the MATLAB code `splu`, when

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix}.$$

- 26** Write out P and L after each step of the code `splu` when

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 5 & 4 \end{bmatrix}.$$

- 27** Extend the MATLAB code `splu` to a code `spldu` which factors PA into LDU .

- 28** What is the matrix L_1 in $A = L_1P_1U_1$?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{bmatrix} = P_1U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}.$$

- 29** Prove that the identity matrix cannot be the product of three row exchanges (or five). It can be the product of two exchanges (or four).
- 30** (a) Choose E_{21} to remove the 3 below the first pivot. Then multiply $E_{21}AE_{21}^T$ to remove both 3's:

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 11 & 4 \\ 0 & 4 & 9 \end{bmatrix} \text{ is going toward } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (b) Choose E_{32} to remove the 4 below the second pivot. Then A is reduced to D by $E_{32}E_{21}AE_{21}^TE_{32}^T = D$. Invert the E 's to find L in $A = LDL^T$.

The next questions are about applications of the identity $(Ax)^T y = x^T(A^T y)$.

- 31** Wires go between Boston, Chicago, and Seattle. Those cities are at voltages x_B , x_C , x_S . With unit resistances between cities, the currents between cities are in y :

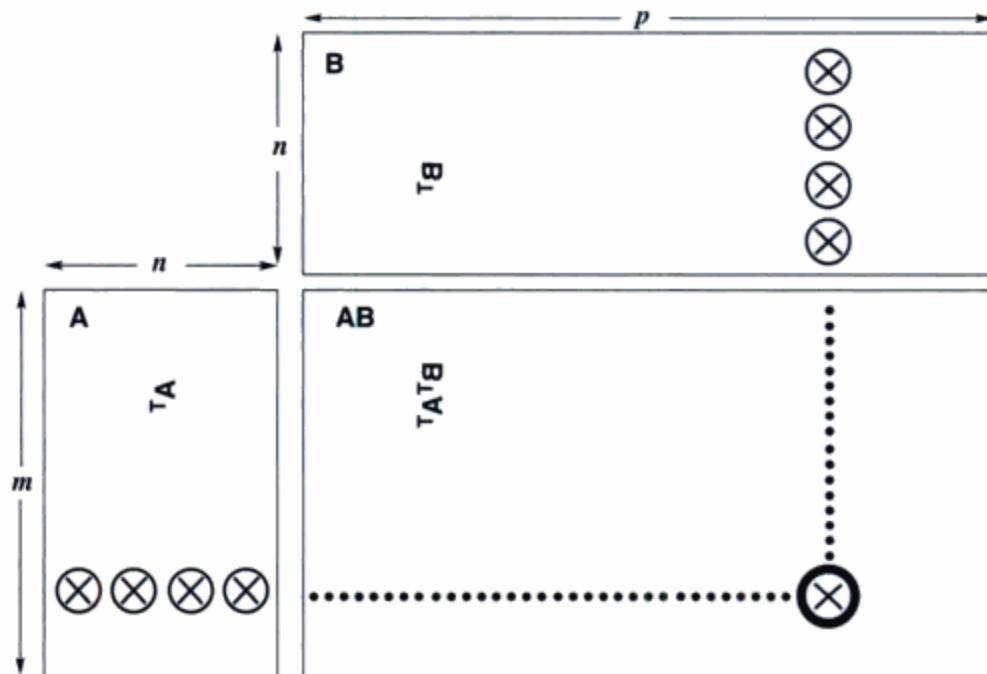
$$y = Ax \text{ is } \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_B \\ x_C \\ x_S \end{bmatrix}.$$

- (a) Find the total currents $A^T y$ out of the three cities.
 (b) Verify that $(Ax)^T y$ agrees with $x^T(A^T y)$ —six terms in both.
- 32** Producing x_1 trucks and x_2 planes needs $x_1 + 50x_2$ tons of steel, $40x_1 + 1000x_2$ pounds of rubber, and $2x_1 + 50x_2$ months of labor. If the unit costs y_1 , y_2 , y_3 are \$700 per ton, \$3 per pound, and \$3000 per month, what are the values of one truck and one plane? Those are the components of $A^T y$.
- 33** Ax gives the amounts of steel, rubber, and labor to produce x in Problem 32. Find A . Then $Ax \cdot y$ is the _____ of inputs while $x \cdot A^T y$ is the value of _____.
- 34** The matrix P that multiplies (x, y, z) to give (z, x, y) is also a rotation matrix. Find P and P^3 . The rotation axis $a = (1, 1, 1)$ doesn't move, it equals Pa . What is the angle of rotation from $v = (2, 3, -5)$ to $Pv = (-5, 2, 3)$?
- 35** Write $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$ as the product EH of an elementary row operation matrix E and a symmetric matrix H .
- 36** Here is a new factorization of A into triangular times symmetric:

Start from $A = LDU$. Then $A = L(U^T)^{-1}$ times $U^T DU$.

- Why is $L(U^T)^{-1}$ triangular? Its diagonal is all 1's. Why is $U^T DU$ symmetric?
- 37** A group of matrices includes AB and A^{-1} if it includes A and B . “Products and inverses stay in the group.” Which of these sets are groups? Lower triangular matrices L with 1's on the diagonal, symmetric matrices S , positive matrices M , diagonal invertible matrices D , permutation matrices P , matrices with $Q^T = Q^{-1}$. Invent two more matrix groups.

- 38 If every row of a 4 by 4 matrix contains the numbers 0, 1, 2, 3 in some order, can the matrix be symmetric?
- 39 Prove that no reordering of rows and reordering of columns can transpose a typical matrix.
- 40 A square **northwest matrix** B is zero in the southeast corner, below the antidiagonal that connects $(1, n)$ to $(n, 1)$. Will B^T and B^2 be northwest matrices? Will B^{-1} be northwest or southeast? What is the shape of $BC = \text{northwest times southeast}$? OK to combine permutations with the usual L and U (southwest and northeast).
- 41 If P has 1's on the antidiagonal from $(1, n)$ to $(n, 1)$, describe PAP .



* **Transparent proof that $(AB)^T = B^T A^T$.** Matrices can be transposed by looking through the page from the other side. Hold up to the light and practice with B . Its column with four entries \otimes becomes a row, when you look from the back and the symbol B^T is upright.

The three matrices are in position for matrix multiplication: the row of A times the column of B gives the entry in AB . Looking from the reverse side, the row of B^T times the column of A^T gives the correct entry in $B^T A^T = (AB)^T$.

3

VECTOR SPACES AND SUBSPACES

SPACES OF VECTORS ■ 3.1

To a newcomer, matrix calculations involve a lot of numbers. To you, they involve vectors. The columns of Ax and AB are linear combinations of n vectors—the columns of A . This chapter moves from numbers and vectors to a third level of understanding (the highest level). Instead of individual columns, we look at “spaces” of vectors. Without seeing *vector spaces* and especially their *subspaces*, you haven’t understood everything about $Ax = b$.

Since this chapter goes a little deeper, it may seem a little harder. That is natural. We are looking inside the calculations, to find the mathematics. The author’s job is to make it clear. These pages go to the heart of linear algebra.

We begin with the most important vector spaces. They are denoted by \mathbf{R}^1 , \mathbf{R}^2 , \mathbf{R}^3 , \mathbf{R}^4 , Each space \mathbf{R}^n consists of a whole collection of vectors. \mathbf{R}^5 contains all column vectors with five components. This is called “5-dimensional space.”

DEFINITION *The space \mathbf{R}^n consists of all column vectors v with n components.*

The components of v are real numbers, which is the reason for the letter \mathbf{R} . A vector whose n components are complex numbers lies in the space \mathbf{C}^n .

The vector space \mathbf{R}^2 is represented by the usual xy plane. Each vector v in \mathbf{R}^2 has two components. The word “space”, asks us to think of all those vectors—the whole plane. Each vector gives the x and y coordinates of a point in the plane.

Similarly the vectors in \mathbf{R}^3 correspond to points (x, y, z) in three-dimensional space. The one-dimensional space \mathbf{R}^1 is a line (like the x axis). As before, we print vectors as a column between brackets, or along a line using commas and parentheses:

$$\begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \text{ is in } \mathbf{R}^3, \quad (1, 1, 0, 1, 1) \text{ is in } \mathbf{R}^5, \quad \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \text{ is in } \mathbf{C}^2.$$

The great thing about linear algebra is that it deals easily with five-dimensional space. We don't draw the vectors, we just need the five numbers (or n numbers). To multiply \mathbf{v} by 7, multiply every component by 7. Here 7 is a "scalar." To add vectors in \mathbf{R}^5 , add them a component at a time. The two essential vector operations go on *inside the vector space*:

We can add any vectors in \mathbf{R}^n , and we can multiply any vector by any scalar.

"Inside the vector space" means that *the result stays in the space*. If \mathbf{v} is the vector in \mathbf{R}^4 with components 1, 0, 0, 1, then $2\mathbf{v}$ is the vector in \mathbf{R}^4 with components 2, 0, 0, 2. (In this case 2 is the scalar.) A whole series of properties can be verified in \mathbf{R}^n . The commutative law is $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$; the distributive law is $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$. There is a unique "zero vector" satisfying $\mathbf{0} + \mathbf{v} = \mathbf{v}$. Those are three of the eight conditions listed at the start of the problem set.

These eight conditions are required of every vector space. There are vectors other than column vectors, and vector spaces other than \mathbf{R}^n , and they have to obey the eight reasonable rules.

A *real vector space* is a set of "vectors" together with rules for vector addition and for multiplication by real numbers. The addition and the multiplication must produce vectors that are in the space. And the eight conditions must be satisfied (which is usually no problem). Here are three vector spaces other than \mathbf{R}^n :

- M The vector space of *all real 2 by 2 matrices*.
- F The vector space of *all real functions* $f(x)$.
- Z The vector space that consists only of a *zero vector*.

In M the "vectors" are really matrices. In F the vectors are functions. In Z the only addition is $\mathbf{0} + \mathbf{0} = \mathbf{0}$. In each case we can add: matrices to matrices, functions to functions, zero vector to zero vector. We can multiply a matrix by 4 or a function by 4 or the zero vector by 4. The result is still in M or F or Z. The eight conditions are all easily checked.

The space Z is zero-dimensional (by any reasonable definition of dimension). It is the smallest possible vector space. We hesitate to call it \mathbf{R}^0 , which means no components—you might think there was no vector. The vector space Z contains exactly *one vector* (zero). No space can do without that zero vector. Each space has its own zero vector—the zero matrix, the zero function, the vector $(0, 0, 0)$ in \mathbf{R}^3 .

Subspaces

At different times, we will ask you to think of matrices and functions as vectors. But at all times, the vectors that we need most are ordinary column vectors. They are vectors with n components—but *maybe not all* of the vectors with n components. There are important vector spaces *inside \mathbf{R}^n* .

Start with the usual three-dimensional space \mathbf{R}^3 . Choose a plane through the origin $(0, 0, 0)$. *That plane is a vector space in its own right.* If we add two vectors

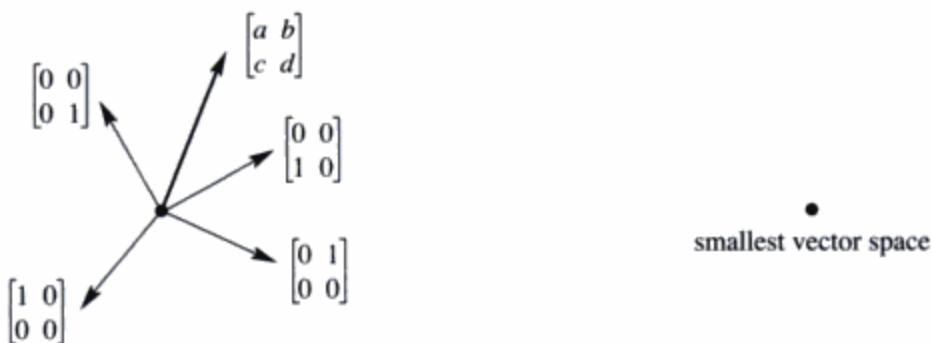


Figure 3.1 “Four-dimensional” matrix space \mathbf{M} . The “zero-dimensional” space \mathbf{Z} .

in the plane, their sum is in the plane. If we multiply an in-plane vector by 2 or -5 , it is still in the plane. The plane is not \mathbf{R}^2 (even if it looks like \mathbf{R}^2). The vectors have three components and they belong to \mathbf{R}^3 . The plane is a vector space inside \mathbf{R}^3 .

This illustrates one of the most fundamental ideas in linear algebra. The plane is a *subspace* of the full vector space \mathbf{R}^3 .

DEFINITION A *subspace* of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies two requirements: *If v and w are vectors in the subspace and c is any scalar, then* (i) $v + w$ is in the subspace and (ii) cv is in the subspace.

In other words, the set of vectors is “closed” under addition $v + w$ and multiplication cv (and cw). Those operations leave us in the subspace. We can also subtract, because $-w$ is in the subspace and its sum with v is $v - w$. In short, *all linear combinations stay in the subspace*.

All these operations follow the rules of the host space, so the eight required conditions are automatic. We just have to check the requirements (i) and (ii) for a subspace.

First fact: *Every subspace contains the zero vector*. The plane in \mathbf{R}^3 has to go through $(0, 0, 0)$. We mention this separately, for extra emphasis, but it follows directly from rule (ii). Choose $c = 0$, and the rule requires $0v$ to be in the subspace.

Planes that don’t contain the origin fail those tests. When v is on such a plane, $-v$ and $0v$ are *not* on the plane. A plane that misses the origin is not a subspace.

Lines through the origin are also subspaces. When we multiply by 5, or add two vectors on the line, we stay on the line. But the line must go through $(0, 0, 0)$.

Another subspace is all of \mathbf{R}^3 . The whole space is a subspace (*of itself*). Here is a list of all the possible subspaces of \mathbf{R}^3 :

- | | |
|-----------------------------------|-----------------------------------|
| (L) Any line through $(0, 0, 0)$ | (R ³) The whole space |
| (P) Any plane through $(0, 0, 0)$ | (Z) The single vector $(0, 0, 0)$ |

If we try to keep only *part* of a plane or line, the requirements for a subspace don't hold. Look at these examples in \mathbf{R}^2 .

Example 1 Keep only the vectors (x, y) whose components are positive or zero (this is a quarter-plane). The vector $(2, 3)$ is included but $(-2, -3)$ is not. So rule (ii) is violated when we try to multiply by $c = -1$. **The quarter-plane is not a subspace.**

Example 2 Include also the vectors whose components are both negative. Now we have two quarter-planes. Requirement (ii) is satisfied; we can multiply by any c . But rule (i) now fails. The sum of $v = (2, 3)$ and $w = (-3, -2)$ is $(-1, 1)$, which is outside the quarter-planes. **Two quarter-planes don't make a subspace.**

Rules (i) and (ii) involve vector addition $v + w$ and multiplication by scalars like c and d . The rules can be combined into a single requirement—the *rule for subspaces*:

A subspace containing v and w must contain all linear combinations $cv + dw$.

Example 3 Inside the vector space \mathbf{M} of all 2 by 2 matrices, here are two subspaces:

$$\text{(U) All upper triangular matrices } \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad \text{(D) All diagonal matrices } \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}.$$

Add any two matrices in **U**, and the sum is in **U**. Add diagonal matrices, and the sum is diagonal. In this case **D** is also a subspace of **U**! Of course the zero matrix is in these subspaces, when a , b , and d all equal zero.

To find a smaller subspace of diagonal matrices, we could require $a = d$. The matrices are multiples of the identity matrix I . The sum $2I + 3I$ is in this subspace, and so is 3 times $4I$. It is a “line of matrices” inside \mathbf{M} and **U** and **D**.

Is the matrix I a subspace by itself? Certainly not. Only the zero matrix is. Your mind will invent more subspaces of 2 by 2 matrices—write them down for Problem 5.

The Column Space of A

The most important subspaces are tied directly to a matrix A . We are trying to solve $Ax = b$. If A is not invertible, the system is solvable for some b and not solvable for other b . We want to describe the good right sides b —the vectors that *can* be written as A times some vector x .

Remember that Ax is a combination of the columns of A . To get every possible b , we use every possible x . So start with the columns of A , and *take all their linear combinations*. This produces the column space of A . It is a vector space made up of column vectors—not just the n columns of A , but all their combinations Ax .

DEFINITION The *column space* consists of *all linear combinations of the columns*. The combinations are all possible vectors Ax . They fill the column space $C(A)$.

This column space is crucial to the whole book, and here is why. **To solve $Ax = b$ is to express b as a combination of the columns.** The right side b has to be in the column space produced by A on the left side, or no solution!

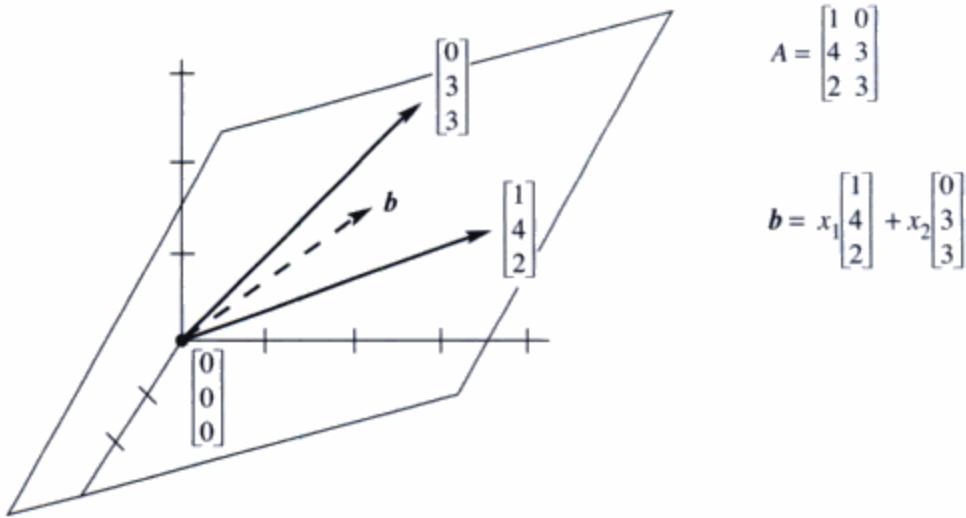


Figure 3.2 The column space $C(A)$ is a plane containing the two columns. $Ax = b$ is solvable when b is on that plane. Then b is a combination of the columns.

3A The system $Ax = b$ is solvable if and only if b is in the column space of A .

When b is in the column space, it is a combination of the columns. The coefficients in that combination give us a solution x to the system $Ax = b$.

Suppose A is an m by n matrix. Its columns have m components (not n). So the columns belong to \mathbf{R}^m . **The column space of A is a subspace of \mathbf{R}^m (not \mathbf{R}^n)**. The set of all column combinations Ax satisfies rules (i) and (ii) for a subspace: When we add linear combinations or multiply by scalars, we still produce combinations of the columns. The word “subspace” is justified by *taking all linear combinations*.

Here is a 3 by 2 matrix A , whose column space is a subspace of \mathbf{R}^3 . It is a plane.

Example 4

$$Ax \text{ is } \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ which is } x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$

The column space consists of all combinations of the two columns—any x_1 times the first column plus any x_2 times the second column. *Those combinations fill up a plane in \mathbf{R}^3* (Figure 3.2). If the right side b lies on that plane, then it is one of the combinations and (x_1, x_2) is a solution to $Ax = b$. The plane has zero thickness, so it is more likely that b is not in the column space. Then there is no solution to our 3 equations in 2 unknowns.

Of course $(0, 0, 0)$ is in the column space. The plane passes through the origin. There is certainly a solution to $Ax = \mathbf{0}$. That solution, always available, is $x = \underline{\hspace{2cm}}$.

To repeat, the attainable right sides \mathbf{b} are exactly the vectors in the column space. One possibility is the first column itself—take $x_1 = 1$ and $x_2 = 0$. Another combination is the second column—take $x_1 = 0$ and $x_2 = 1$. The new level of understanding is to see *all* combinations—the whole subspace is generated by those two columns.

Notation The column space of A is denoted by $C(A)$. Start with the columns and take all their linear combinations. We might get the whole \mathbf{R}^m or only a subspace.

Example 5 Describe the column spaces (they are subspaces of \mathbf{R}^2) for

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Solution The column space of I is the *whole space* \mathbf{R}^2 . Every vector is a combination of the columns of I . In vector space language, $C(I)$ is \mathbf{R}^2 .

The column space of A is only a line. The second column $(2, 4)$ is a multiple of the first column $(1, 2)$. Those vectors are different, but our eye is on vector *spaces*. The column space contains $(1, 2)$ and $(2, 4)$ and all other vectors $(c, 2c)$ along that line. The equation $A\mathbf{x} = \mathbf{b}$ is only solvable when \mathbf{b} is on the line.

The third matrix (with three columns) places no restriction on \mathbf{b} . The column space $C(B)$ is all of \mathbf{R}^2 . Every \mathbf{b} is attainable. The vector $\mathbf{b} = (5, 4)$ is column 2 plus column 3, so \mathbf{x} can be $(0, 1, 1)$. The same vector $(5, 4)$ is also 2(column 1) + column 3, so another possible \mathbf{x} is $(2, 0, 1)$. This matrix has the same column space as I —any \mathbf{b} is allowed. But now \mathbf{x} has extra components and there are more solutions.

The next section creates another vector space, to describe all the solutions of $A\mathbf{x} = \mathbf{0}$. This section created the column space, to describe all the attainable right sides \mathbf{b} .

■ REVIEW OF THE KEY IDEAS ■

1. \mathbf{R}^n contains all column vectors with n real components.
2. \mathbf{M} (2 by 2 matrices) and \mathbf{F} (functions) and \mathbf{Z} (zero vector alone) are vector spaces.
3. A subspace containing \mathbf{v} and \mathbf{w} must contain all their combinations $c\mathbf{v} + d\mathbf{w}$.
4. The combinations of the columns of A form the *column space* $C(A)$.
5. $A\mathbf{x} = \mathbf{b}$ has a solution exactly when \mathbf{b} is in the column space of A .

■ WORKED EXAMPLES ■

3.1 A We are given three different vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. Construct a matrix so that the equations $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ are solvable, but $A\mathbf{x} = \mathbf{b}_3$ is not solvable. How can you decide if this is possible? How could you construct A ?

Solution We want to have \mathbf{b}_1 and \mathbf{b}_2 in the column space of A . Then $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$ will be solvable. The quickest way is to make \mathbf{b}_1 and \mathbf{b}_2 the two columns of A . Then the solutions are $\mathbf{x} = (1, 0)$ and $\mathbf{x} = (0, 1)$.

Also, we don't want $A\mathbf{x} = \mathbf{b}_3$ to be solvable. So don't make the column space any larger! Keeping only the columns of \mathbf{b}_1 and \mathbf{b}_2 , the question is:

$$\text{Is } A\mathbf{x} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_3 \text{ solvable? Is } \mathbf{b}_3 \text{ a combination of } \mathbf{b}_1 \text{ and } \mathbf{b}_2?$$

If the answer is *no*, we have the desired matrix A . If the answer is *yes*, then it is *not possible* to construct A . When the column space contains \mathbf{b}_1 and \mathbf{b}_2 , it will have to contain all their linear combinations. So \mathbf{b}_3 would necessarily be in that column space and $A\mathbf{x} = \mathbf{b}_3$ would necessarily be solvable.

3.1 B Describe a subspace \mathbf{S} of each vector space \mathbf{V} , and then a subspace \mathbf{SS} of \mathbf{S} .

\mathbf{V}_1 = all combinations of $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$ and $(1, 1, 1, 1)$

\mathbf{V}_2 = all vectors perpendicular to $\mathbf{u} = (1, 2, 2, 1)$

\mathbf{V}_3 = all symmetric 2 by 2 matrices

\mathbf{V}_4 = all solutions to the equation $d^4y/dx^4 = 0$.

Describe each \mathbf{V} two ways: *All combinations of . . . , all solutions of the equations . . .*

Solution A subspace \mathbf{S} of \mathbf{V}_1 comes from all combinations of the first two vectors $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$. A subspace \mathbf{SS} of \mathbf{S} comes from all multiples $(c, c, 0, 0)$ of the first vector.

A subspace \mathbf{S} of \mathbf{V}_2 comes from all combinations of two vectors $(1, 0, 0, -1)$ and $(0, 1, -1, 0)$ that are perpendicular to \mathbf{u} . The vector $\mathbf{x} = (1, 1, -1, -1)$ is in \mathbf{S} and all its multiples $c\mathbf{x}$ give a subspace \mathbf{SS} .

The diagonal matrices are a subspace \mathbf{S} of the symmetric matrices. The multiples cI are a subspace \mathbf{SS} of the diagonal matrices.

\mathbf{V}_4 contains all cubic polynomials $y = a + bx + cx^2 + dx^3$. The quadratic polynomials give a subspace \mathbf{S} . The linear polynomials are one choice of \mathbf{SS} . The constants could be \mathbf{SSS} .

In all four parts we could have chosen $\mathbf{S} = \mathbf{V}$ itself, and $\mathbf{SS} =$ the zero subspace \mathbf{Z} .

Each \mathbf{V} can be described as all combinations of . . . and as all solutions of . . . :

\mathbf{V}_1 = all combinations of the 3 vectors = all solutions of $v_1 - v_2 = 0$

\mathbf{V}_2 = all combinations of $(1, 0, 0, -1), (0, 1, -1, 0), (2, -1, 0, 0)$
= all solutions of $\mathbf{u}^T \mathbf{v} = 0$

\mathbf{V}_3 = all combinations of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ = all solutions $\begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ of $b = c$

\mathbf{V}_4 = all combinations of $1, x, x^2, x^3$ = all solutions to $d^4y/dx^4 = 0$.

Problem Set 3.1

The first problems 1–8 are about vector spaces in general. The vectors in those spaces are not necessarily column vectors. In the definition of a *vector space*, vector addition $x + y$ and scalar multiplication cx must obey the following eight rules:

- (1) $x + y = y + x$
- (2) $x + (y + z) = (x + y) + z$
- (3) There is a unique “zero vector” such that $x + \mathbf{0} = x$ for all x
- (4) For each x there is a unique vector $-x$ such that $x + (-x) = \mathbf{0}$
- (5) 1 times x equals x
- (6) $(c_1 c_2)x = c_1(c_2x)$
- (7) $c(x + y) = cx + cy$
- (8) $(c_1 + c_2)x = c_1x + c_2x.$

- 1 Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_1 + y_2, x_2 + y_1)$. With the usual multiplication $cx = (cx_1, cx_2)$, which of the eight conditions are not satisfied?
- 2 Suppose the multiplication cx is defined to produce $(cx_1, 0)$ instead of (cx_1, cx_2) . With the usual addition in \mathbf{R}^2 , are the eight conditions satisfied?
- 3 (a) Which rules are broken if we keep only the positive numbers $x > 0$ in \mathbf{R}^1 ? Every c must be allowed. The half-line is not a subspace.
 (b) The positive numbers with $x + y$ and cx redefined to equal the usual xy and x^c do satisfy the eight rules. Test rule 7 when $c = 3, x = 2, y = 1$. (Then $x + y = 2$ and $cx = 8$.) Which number acts as the “zero vector”?
- 4 The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a “vector” in the space \mathbf{M} of all 2 by 2 matrices. Write down the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A ?
- 5 (a) Describe a subspace of \mathbf{M} that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
 (b) If a subspace of \mathbf{M} contains A and B , must it contain I ?
 (c) Describe a subspace of \mathbf{M} that contains no nonzero diagonal matrices.
- 6 The functions $f(x) = x^2$ and $g(x) = 5x$ are “vectors” in \mathbf{F} . This is the vector space of all real functions. (The functions are defined for $-\infty < x < \infty$.) The combination $3f(x) - 4g(x)$ is the function $h(x) = \underline{\hspace{2cm}}$.

- 7 Which rule is broken if multiplying $f(x)$ by c gives the function $f(cx)$? Keep the usual addition $f(x) + g(x)$.
- 8 If the sum of the “vectors” $f(x)$ and $g(x)$ is defined to be the function $f(g(x))$, then the “zero vector” is $g(x) = x$. Keep the usual scalar multiplication $cf(x)$ and find two rules that are broken.

Questions 9–18 are about the “subspace requirements”: $x + y$ and cx (and then all linear combinations $cx + dy$) must stay in the subspace.

- 9 One requirement can be met while the other fails. Show this by finding
- A set of vectors in \mathbf{R}^2 for which $x + y$ stays in the set but $\frac{1}{2}x$ may be outside.
 - A set of vectors in \mathbf{R}^2 (other than two quarter-planes) for which every cx stays in the set but $x + y$ may be outside.
- 10 Which of the following subsets of \mathbf{R}^3 are actually subspaces?
- The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.
 - The plane of vectors with $b_1 = 1$.
 - The vectors with $b_1 b_2 b_3 = 0$.
 - All linear combinations of $v = (1, 4, 0)$ and $w = (2, 2, 2)$.
 - All vectors that satisfy $b_1 + b_2 + b_3 = 0$.
 - All vectors with $b_1 \leq b_2 \leq b_3$.
- 11 Describe the smallest subspace of the matrix space \mathbf{M} that contains
- $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
 - $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
 - $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- 12 Let P be the plane in \mathbf{R}^3 with equation $x + y - 2z = 4$. The origin $(0, 0, 0)$ is not in P ! Find two vectors in P and check that their sum is not in P .
- 13 Let \mathbf{P}_0 be the plane through $(0, 0, 0)$ parallel to the previous plane P . What is the equation for \mathbf{P}_0 ? Find two vectors in \mathbf{P}_0 and check that their sum is in \mathbf{P}_0 .
- 14 The subspaces of \mathbf{R}^3 are planes, lines, \mathbf{R}^3 itself, or \mathbf{Z} containing only $(0, 0, 0)$.
- Describe the three types of subspaces of \mathbf{R}^2 .
 - Describe the five types of subspaces of \mathbf{R}^4 .
- 15 (a) The intersection of two planes through $(0, 0, 0)$ is probably a _____ but it could be a _____. It can't be \mathbf{Z} !

- (b) The intersection of a plane through $(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a _____ but it could be a _____.
- (c) If \mathbf{S} and \mathbf{T} are subspaces of \mathbf{R}^5 , prove that their intersection $\mathbf{S} \cap \mathbf{T}$ (vectors in both subspaces) is a subspace of \mathbf{R}^5 . Check the requirements on $\mathbf{x} + \mathbf{y}$ and $c\mathbf{x}$.
- 16** Suppose \mathbf{P} is a plane through $(0, 0, 0)$ and \mathbf{L} is a line through $(0, 0, 0)$. The smallest vector space containing both \mathbf{P} and \mathbf{L} is either _____ or _____.
- 17** (a) Show that the set of *invertible* matrices in \mathbf{M} is not a subspace.
 (b) Show that the set of *singular* matrices in \mathbf{M} is not a subspace.
- 18** True or false (check addition in each case by an example):
 (a) The symmetric matrices in \mathbf{M} (with $A^T = A$) form a subspace.
 (b) The skew-symmetric matrices in \mathbf{M} (with $A^T = -A$) form a subspace.
 (c) The unsymmetric matrices in \mathbf{M} (with $A^T \neq A$) form a subspace.

Questions 19–27 are about column spaces $C(A)$ and the equation $\mathbf{Ax} = \mathbf{b}$.

- 19** Describe the column spaces (lines or planes) of these particular matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

- 20** For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

$$(a) \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

- 21** Adding row 1 of A to row 2 produces B . Adding column 1 to column 2 produces C . A combination of the columns of _____ is also a combination of the columns of A . Which two matrices have the same column _____?

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$

- 22** For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

- 23 (Recommended) If we add an extra column \mathbf{b} to a matrix A , then the column space gets larger unless _____. Give an example where the column space gets larger and an example where it doesn't. Why is $A\mathbf{x} = \mathbf{b}$ solvable exactly when the column space *doesn't* get larger—it is the same for A and $[A \ \mathbf{b}]$?
- 24 The columns of AB are combinations of the columns of A . This means: *The column space of AB is contained in* (possibly equal to) *the column space of A .* Give an example where the column spaces of A and AB are not equal.
- 25 Suppose $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b}^*$ are both solvable. Then $A\mathbf{z} = \mathbf{b} + \mathbf{b}^*$ is solvable. What is \mathbf{z} ? This translates into: If \mathbf{b} and \mathbf{b}^* are in the column space $C(A)$, then $\mathbf{b} + \mathbf{b}^*$ is in $C(A)$.
- 26 If A is any 5 by 5 invertible matrix, then its column space is _____. Why?
- 27 True or false (with a counterexample if false):
- The vectors \mathbf{b} that are not in the column space $C(A)$ form a subspace.
 - If $C(A)$ contains only the zero vector, then A is the zero matrix.
 - The column space of $2A$ equals the column space of A .
 - The column space of $A - I$ equals the column space of A .
- 28 Construct a 3 by 3 matrix whose column space contains $(1, 1, 0)$ and $(1, 0, 1)$ but not $(1, 1, 1)$. Construct a 3 by 3 matrix whose column space is only a line.
- 29 If the 9 by 12 system $A\mathbf{x} = \mathbf{b}$ is solvable for every \mathbf{b} , then $C(A) = _____$.

THE NULLSPACE OF A : SOLVING $AX = \mathbf{0}$ ■ 3.2

This section is about the space of solutions to $Ax = \mathbf{0}$. The matrix A can be square or rectangular. *One immediate solution is $x = \mathbf{0}$.* For invertible matrices this is the only solution. For other matrices, not invertible, there are nonzero solutions to $Ax = \mathbf{0}$. *Each solution x belongs to the nullspace of A .*

Elimination will find all solutions and identify this very important subspace.

DEFINITION *The nullspace of A consists of all solutions to $Ax = \mathbf{0}$.* These solution vectors x are in \mathbf{R}^n . The nullspace containing all solutions is denoted by $N(A)$.

Check that the solution vectors form a subspace. Suppose x and y are in the nullspace (this means $Ax = \mathbf{0}$ and $Ay = \mathbf{0}$). The rules of matrix multiplication give $A(x + y) = \mathbf{0} + \mathbf{0}$. The rules also give $A(cx) = c\mathbf{0}$. The right sides are still zero. Therefore $x + y$ and cx are also in the nullspace $N(A)$. Since we can add and multiply without leaving the nullspace, it is a subspace.

To repeat: The solution vectors x have n components. They are vectors in \mathbf{R}^n , so *the nullspace is a subspace of \mathbf{R}^n* . The column space $C(A)$ is a subspace of \mathbf{R}^m .

If the right side b is not zero, the solutions of $Ax = b$ do *not* form a subspace. The vector $x = \mathbf{0}$ is only a solution if $b = \mathbf{0}$. When the set of solutions does not include $x = \mathbf{0}$, it cannot be a subspace. Section 3.4 will show how the solutions to $Ax = b$ (if there are any solutions) are shifted away from the origin by one particular solution.

Example 1 The equation $x + 2y + 3z = 0$ comes from the 1 by 3 matrix $A = [1 \ 2 \ 3]$. This equation produces a plane through the origin. The plane is a subspace of \mathbf{R}^3 . *It is the nullspace of A .*

The solutions to $x + 2y + 3z = 6$ also form a plane, but not a subspace.

Example 2 Describe the nullspace of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution Apply elimination to the linear equations $Ax = \mathbf{0}$:

$$\left[\begin{array}{l} x_1 + 2x_2 = 0 \\ 3x_1 + 6x_2 = 0 \end{array} \right] \rightarrow \left[\begin{array}{l} x_1 + 2x_2 = 0 \\ 0 = 0 \end{array} \right]$$

There is really only one equation. The second equation is the first equation multiplied by 3. In the row picture, the line $x_1 + 2x_2 = 0$ is the same as the line $3x_1 + 6x_2 = 0$. That line is the nullspace $N(A)$.

To describe this line of solutions, here is an efficient way. Choose one point on the line (one “**special solution**”). Then all points on the line are multiples of this one. We choose the second component to be $x_2 = 1$ (a special choice). From the equation $x_1 + 2x_2 = 0$, the first component must be $x_1 = -2$. The special solution is $(-2, 1)$:

The nullspace $N(A)$ contains all multiples of $s = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

This is the best way to describe the nullspace, by computing special solutions to $Ax = \mathbf{0}$. **The nullspace consists of all combinations of those special solutions.** This example has one special solution and the nullspace is a line.

The plane $x + 2y + 3z = 0$ in Example 1 had *two* special solutions:

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{0} \text{ has the special solutions } s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Those vectors s_1 and s_2 lie on the plane $x + 2y + 3z = 0$, which is the nullspace of $A = [1 \ 2 \ 3]$. All vectors on the plane are combinations of s_1 and s_2 .

Notice what is special about s_1 and s_2 . They have ones and zeros in the last two components. *Those components are “free” and we choose them specially.* Then the first components -2 and -3 are determined by the equation $Ax = \mathbf{0}$.

The first column of $A = [1 \ 2 \ 3]$ contains the *pivot*, so the first component of \mathbf{x} is *not free*. The free components correspond to columns without pivots. This description of special solutions will be completed after one more example.

The special choice (one or zero) is only for the free variables.

Example 3 Describe the nullspaces of these three matrices A, B, C :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad C = \begin{bmatrix} A & 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Solution The equation $Ax = \mathbf{0}$ has only the zero solution $\mathbf{x} = \mathbf{0}$. *The nullspace is \mathbf{Z} .* It contains only the single point $\mathbf{x} = \mathbf{0}$ in \mathbf{R}^2 . This comes from elimination:

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields } \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 = 0 \\ x_2 = 0 \end{bmatrix}.$$

A is invertible. There are no special solutions. All columns have pivots.

The rectangular matrix B has the same nullspace \mathbf{Z} . The first two equations in $Bx = \mathbf{0}$ again require $\mathbf{x} = \mathbf{0}$. The last two equations would also force $\mathbf{x} = \mathbf{0}$. When we add extra equations, the nullspace certainly cannot become larger. The extra rows impose more conditions on the vectors \mathbf{x} in the nullspace.

The rectangular matrix C is different. It has extra columns instead of extra rows. The solution vector \mathbf{x} has *four* components. Elimination will produce pivots in the first two columns of C , but the last two columns are “free”. They don’t have pivots:

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \text{ becomes } U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

↑ ↑ ↑ ↑

pivot columns free columns

For the free variables x_3 and x_4 , we make special choices of ones and zeros. First $x_3 = 1$, $x_4 = 0$ and second $x_3 = 0$, $x_4 = 1$. The pivot variables x_1 and x_2 are

determined by the equation $Ux = \mathbf{0}$. We get two special solutions in the nullspace of C (and also the nullspace of U). The special solutions are:

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{pivot} \\ \leftarrow \text{variables} \\ \leftarrow \text{free} \\ \leftarrow \text{variables} \end{array}$$

One more comment to anticipate what is coming soon. Elimination will not stop at the upper triangular U ! We can continue to make this matrix simpler, in two ways:

1. *Produce zeros above the pivots*, by eliminating upward.
2. *Produce ones in the pivots*, by dividing the whole row by its pivot.

Those steps don't change the zero vector on the right side of the equation. The nullspace stays the same. This nullspace becomes easiest to see when we reach the **reduced row echelon form** R . It has I in the pivot columns:

$$U = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \quad \begin{array}{c} \uparrow \uparrow \\ \text{pivot columns contain } I \end{array}$$

I subtracted row 2 of U from row 1, and then multiplied row 2 by $\frac{1}{2}$. The original two equations have simplified to $x_1 + 2x_3 = 0$ and $x_2 + 2x_4 = 0$.

The first special solution is still $s_1 = (-2, 0, 1, 0)$, and s_2 is unchanged. Special solutions are much easier to find from the reduced system $Rx = \mathbf{0}$.

Before moving to m by n matrices A and their nullspaces $N(A)$ and special solutions, allow me to repeat one comment. For many matrices, the only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$. Their nullspaces $N(A) = \mathbb{Z}$ contain only that one vector. The only combination of the columns that produces $b = \mathbf{0}$ is then the “zero combination” or “trivial combination”. The solution is trivial (just $x = \mathbf{0}$) but the idea is not trivial.

This case of a zero nullspace \mathbb{Z} is of the greatest importance. It says that the columns of A are independent. No combination of columns gives the zero vector (except the zero combination). All columns have pivots and no columns are free. You will see this idea of independence again ...

Solving $Ax = \mathbf{0}$ by Elimination

This is important. **A is rectangular and we still use elimination.** We solve m equations in n unknowns when $b = \mathbf{0}$. After A is simplified by row operations, we read off the solution (or solutions). Remember the two stages in solving $Ax = \mathbf{0}$:

1. Forward elimination from A to a triangular U (or its reduced form R).
2. Back substitution in $Ux = \mathbf{0}$ or $Rx = \mathbf{0}$ to find x .

You will notice a difference in back substitution, when A and U have fewer than n pivots. We are allowing all matrices in this chapter, not just the nice ones (which are square matrices with inverses).

Pivots are still nonzero. The columns below the pivots are still zero. But it might happen that a column has no pivot. In that case, don't stop the calculation. **Go on to the next column.** The first example is a 3 by 4 matrix with two pivots:

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}.$$

Certainly $a_{11} = 1$ is the first pivot. Clear out the 2 and 3 below that pivot:

$$A \rightarrow \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 4 & 4 \end{bmatrix} \quad \begin{array}{l} (\text{subtract } 2 \times \text{ row 1}) \\ (\text{subtract } 3 \times \text{ row 1}) \end{array}$$

The second column has a zero in the pivot position. We look below the zero for a nonzero entry, ready to do a row exchange. *The entry below that position is also zero.* Elimination can do nothing with the second column. This signals trouble, which we expect anyway for a rectangular matrix. There is no reason to quit, and we go on to the third column.

The second pivot is 4 (but it is in the third column). Subtracting row 2 from row 3 clears out that column below the pivot. We arrive at

Triangular U : $U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(only two pivots)
(the last equation became $0 = 0$)

The fourth column also has a zero in the pivot position—but nothing can be done. There is no row below it to exchange, and forward elimination is complete. The matrix has three rows, four columns, and *only two pivots*. The original $Ax = \mathbf{0}$ seemed to involve three different equations, but the third equation is the sum of the first two. It is automatically satisfied ($0 = 0$) when the first two equations are satisfied. Elimination reveals the inner truth about a system of equations. Soon we push on from U to R .

Now comes back substitution, to find all solutions to $Ux = \mathbf{0}$. With four unknowns and only two pivots, there are many solutions. The question is how to write them all down. A good method is to separate the *pivot variables* from the *free variables*.

- P** The *pivot* variables are x_1 and x_3 , since columns 1 and 3 contain pivots.
F The *free* variables are x_2 and x_4 , because columns 2 and 4 have no pivots.

The free variables x_2 and x_4 can be given any values whatsoever. Then back substitution finds the pivot variables x_1 and x_3 . (In Chapter 2 no variables were free. When A is invertible, all variables are pivot variables.) The simplest choices for the free variables are ones and zeros. Those choices give the *special solutions*.

Special Solutions to $x_1 + x_2 + 2x_3 + 3x_4 = 0$ and $4x_3 + 4x_4 = 0$

- Set $x_2 = 1$ and $x_4 = 0$. By back substitution $x_3 = 0$. Then $x_1 = -1$.
- Set $x_2 = 0$ and $x_4 = 1$. By back substitution $x_3 = -1$. Then $x_1 = -1$.

These special solutions solve $U\mathbf{x} = \mathbf{0}$ and therefore $A\mathbf{x} = \mathbf{0}$. They are in the nullspace. The good thing is that *every solution is a combination of the special solutions*.

Complete Solution $\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -x_2 - x_4 \\ x_2 \\ -x_4 \\ x_4 \end{bmatrix}$ (1)

special special complete

Please look again at that answer. It is the main goal of this section. The vector $s_1 = (-1, 1, 0, 0)$ is the special solution when $x_2 = 1$ and $x_4 = 0$. The second special solution has $x_2 = 0$ and $x_4 = 1$. **All solutions are linear combinations of s_1 and s_2** . The special solutions are in the nullspace $N(A)$, and their combinations fill out the whole nullspace.

The MATLAB code **nullbasis** computes these special solutions. They go into the columns of a **nullspace matrix** N . The complete solution to $A\mathbf{x} = \mathbf{0}$ is a combination of those columns. Once we have the special solutions, we have the whole nullspace.

There is a special solution for each free variable. If no variables are free—this means there are n pivots—then the only solution to $U\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{0}$ is the trivial solution $\mathbf{x} = \mathbf{0}$. All variables are pivot variables. In that case the nullspaces of A and U contain only the zero vector. With no free variables, and pivots in every column, the output from **nullbasis** is an empty matrix.

Example 4 Find the nullspace of $U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix}$.

The second column of U has no pivot. So x_2 is free. The special solution has $x_2 = 1$. Back substitution into $9x_3 = 0$ gives $x_3 = 0$. Then $x_1 + 5x_2 = 0$ or $x_1 = -5$. The solutions to $U\mathbf{x} = \mathbf{0}$ are multiples of one special solution:

$$\mathbf{x} = x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{The nullspace of } U \text{ is a line in } \mathbf{R}^3. \\ \text{It contains multiples of the special solution.} \\ \text{One variable is free, and } N = \text{nullbasis}(U) \text{ has one column.} \end{array}$$

In a minute we will continue elimination on U , to get *zeros above the pivots and ones in the pivots*. The 7 is eliminated and the pivot changes from 9 to 1. The final result of this elimination will be the reduced row echelon matrix R :

$$U = \begin{bmatrix} 1 & 5 & 7 \\ 0 & 0 & 9 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This makes it even clearer that the special solution (column of N) is $s = (-5, 1, 0)$.

Echelon Matrices

Forward elimination goes from A to U . The process starts with an m by n matrix A . It acts by row operations, including row exchanges. It goes on to the next column when no pivot is available in the current column. The m by n “staircase,” U is an **echelon matrix**.

Here is a 4 by 7 echelon matrix with the three pivots highlighted in boldface:

$$U = \begin{bmatrix} x & x & x & x & x & x & x \\ 0 & \mathbf{x} & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & \mathbf{x} & x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \text{Three pivot variables } x_1, x_2, x_6 \\ \text{Four free variables } x_3, x_4, x_5, x_7 \\ \text{Four special solutions in } N(U) \end{array}$$

Question What are the column space and the nullspace for this matrix?

Answer The columns have four components so they lie in \mathbb{R}^4 . (Not in \mathbb{R}^3 !) The fourth component of every column is zero. Every combination of the columns—every vector in the column space—has fourth component zero. *The column space $C(U)$ consists of all vectors of the form $(b_1, b_2, b_3, 0)$.* For those vectors we can solve $Ux = \mathbf{b}$ by back substitution. These vectors \mathbf{b} are all possible combinations of the seven columns.

The nullspace $N(U)$ is a subspace of \mathbb{R}^7 . The solutions to $Ux = \mathbf{0}$ are all the combinations of the four special solutions—one for each free variable:

1. Columns 3, 4, 5, 7 have no pivots. So the free variables are x_3, x_4, x_5, x_7 .
2. Set one free variable to 1 and set the other free variables to zero.
3. Solve $Ux = \mathbf{0}$ for the pivot variables x_1, x_2, x_6 .
4. This gives one of the four special solutions in the nullspace matrix N .

The nonzero rows of an echelon matrix go down in a staircase pattern. The pivots are the first nonzero entries in those rows. There is a column of zeros below every pivot.

Counting the pivots leads to an extremely important theorem. Suppose A has more columns than rows. **With $n > m$ there is at least one free variable.** The system $Ax = \mathbf{0}$ has at least one special solution. This solution is *not zero!*

3B If $Ax = \mathbf{0}$ has more unknowns than equations ($n > m$, more columns than rows), then it has nonzero solutions. There must be free columns, without pivots.

In other words, a short wide matrix ($n > m$) always has nonzero vectors in its nullspace. There must be at least $n - m$ free variables, since the number of pivots cannot exceed m . (The matrix only has m rows, and a row never has two pivots.) Of course a row might have *no* pivot—which means an extra free variable. But here is the point: When there is a free variable, it can be set to 1. Then the equation $Ax = \mathbf{0}$ has a nonzero solution.

To repeat: There are at most m pivots. With $n > m$, the system $Ax = \mathbf{0}$ has a nonzero solution. Actually there are infinitely many solutions, since any multiple $c\mathbf{x}$ is also a solution. The nullspace contains at least a line of solutions. With two free variables, there are two special solutions and the nullspace is even larger.

The nullspace is a subspace. Its “dimension” is the number of free variables. This central idea—the **dimension** of a subspace—is defined and explained in this chapter.

The Reduced Row Echelon Matrix R

>From an echelon matrix U we can go one more step. Continue with our example

$$U = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We can divide the second row by 4. Then both pivots equal 1. We can subtract 2 times this new row $[0 \ 0 \ 1 \ 1]$ from the row above. The **reduced row echelon matrix R** has zeros above the pivots as well as below:

$$R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

R has 1's as pivots. Zeros above pivots come from **upward elimination**.

If A is invertible, its reduced row echelon form is the identity matrix $R = I$. This is the ultimate in row reduction. Of course the nullspace is then \mathbf{Z} .

The zeros in R make it easy to find the special solutions (the same as before):

1. Set $x_2 = 1$ and $x_4 = 0$. Solve $R\mathbf{x} = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = 0$.

Those numbers -1 and 0 are sitting in column 2 of R (with plus signs).

2. Set $x_2 = 0$ and $x_4 = 1$. Solve $R\mathbf{x} = \mathbf{0}$. Then $x_1 = -1$ and $x_3 = -1$.

Those numbers -1 and -1 are sitting in column 4 (with plus signs).

By reversing signs we can read off the special solutions directly from R . The nullspace $N(A) = N(U) = N(R)$ contains all combinations of the special solutions:

$$x = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = (\text{complete solution of } Ax = \mathbf{0}).$$

The next section of the book moves firmly from U to R . The MATLAB command $[R, \text{pivcol}] = \text{rref}(A)$ produces R and also a list of the pivot columns.

■ REVIEW OF THE KEY IDEAS ■

1. The nullspace $N(A)$, a subspace of \mathbf{R}^n , contains all solutions to $Ax = \mathbf{0}$.
2. Elimination produces an echelon matrix U , and then a row reduced R , with pivot columns and free columns.
3. Every free column of U or R leads to a special solution. The free variable equals 1 and the other free variables equal 0. Back substitution solves $Ax = \mathbf{0}$.
4. The complete solution to $Ax = \mathbf{0}$ is a combination of the special solutions.
5. If $n > m$ then A has at least one column without pivots, giving a special solution. So there are nonzero vectors x in the nullspace of this rectangular A .

■ WORKED EXAMPLES ■

3.2 A Create a 3 by 4 matrix whose special solutions to $Ax = \mathbf{0}$ are s_1 and s_2 :

$$s_1 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad s_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix} \quad \begin{array}{l} \text{pivot columns 1 and 3} \\ \text{free variables } x_2 \text{ and } x_4 \end{array}$$

You could create the matrix A in row reduced form R . Then describe all possible matrices A with the required nullspace $N(A) = \text{all combinations of } s_1 \text{ and } s_2$.

Solution The reduced matrix R has pivots = 1 in columns 1 and 3. There is no third pivot, so the third row of R is all zeros. The free columns 2 and 4 will be combinations of the pivot columns:

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{has} \quad Rs_1 = \mathbf{0} \quad \text{and} \quad Rs_2 = \mathbf{0}.$$

The entries 3, 2, 6 are the negatives of $-3, -2, -6$ in the special solutions!

R is only one matrix (one possible A) with the required nullspace. We could do any elementary operations on R —exchange rows, multiply a row by any $c \neq 0$, subtract any multiple of one row from another. R can be multiplied by *any invertible matrix*, without changing the row space and nullspace.

Every 3 by 4 matrix has at least one special solution. *These A 's have two.*

3.2 B Find the special solutions and describe the *complete solution* to $Ax = \mathbf{0}$ for

$$A_1 = 3 \text{ by } 4 \text{ zero matrix} \quad A_2 = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix} \quad A_3 = [A_2 \ A_2]$$

Which are the pivot columns? Which are the free variables? What is R in each case?

Solution $A_1x = \mathbf{0}$ has four special solutions. They are the columns s_1, s_2, s_3, s_4 of the 4 by 4 identity matrix. The nullspace is all of \mathbf{R}^4 . The complete solution is any $x = c_1s_1 + c_2s_2 + c_3s_3 + c_4s_4$ in \mathbf{R}^4 . There are no pivot columns; all variables are free; the reduced R is the same zero matrix as A_1 .

$A_2x = \mathbf{0}$ has only one special solution $s = (-2, 1)$. The multiples $x = cs$ give the complete solution. The first column of A_2 is its pivot column, and x_2 is the free variable. The row reduced matrices R_2 for A_2 and R_3 for $A_3 = [A_2 \ A_2]$ have 1's in the pivot:

$$R_2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad R_3 = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notice that R_3 has only one pivot column (the first column). All the variables x_2, x_3, x_4 are free. There are three special solutions to $A_3x = \mathbf{0}$ (and also $R_3x = \mathbf{0}$):

$$s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{Complete solution } x = c_1s_1 + c_2s_2 + c_3s_3.$$

With r pivots, A has $n - r$ free variables and $Ax = \mathbf{0}$ has $n - r$ special solutions.

Problem Set 3.2

Questions 1–4 and 5–8 are about the matrices in Problems 1 and 5.

- 1 Reduce these matrices to their ordinary echelon forms U :

$$(a) \quad A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \quad (b) \quad B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix}.$$

Which are the free variables and which are the pivot variables?

- 2** For the matrices in Problem 1, find a special solution for each free variable. (Set the free variable to 1. Set the other free variables to zero.)
- 3** By combining the special solutions in Problem 2, describe every solution to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. The nullspace contains only $x = \mathbf{0}$ when there are no ____.
- 4** By further row operations on each U in Problem 1, find the reduced echelon form R . *True or false:* The nullspace of R equals the nullspace of U .
- 5** By row operations reduce each matrix to its echelon form U . Write down a 2 by 2 lower triangular L such that $B = LU$.

$$(a) A = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 10 \end{bmatrix} \quad (b) B = \begin{bmatrix} -1 & 3 & 5 \\ -2 & 6 & 7 \end{bmatrix}.$$

- 6** Find the special solutions to $Ax = \mathbf{0}$ and $Bx = \mathbf{0}$. For an m by n matrix, the number of pivot variables plus the number of free variables is ____.
- 7** In Problem 5, describe the nullspaces of A and B in two ways. Give the equations for the plane or the line, and give all vectors x that satisfy those equations as combinations of the special solutions.
- 8** Reduce the echelon forms U in Problem 5 to R . For each R draw a box around the identity matrix that is in the pivot rows and pivot columns.

Questions 9–17 are about free variables and pivot variables.

- 9** True or false (with reason if true or example to show it is false):
- A square matrix has no free variables.
 - An invertible matrix has no free variables.
 - An m by n matrix has no more than n pivot variables.
 - An m by n matrix has no more than m pivot variables.
- 10** Construct 3 by 3 matrices A to satisfy these requirements (if possible):
- A has no zero entries but $U = I$.
 - A has no zero entries but $R = I$.
 - A has no zero entries but $R = U$.
 - $A = U = 2R$.
- 11** Put as many 1's as possible in a 4 by 7 echelon matrix U whose pivot variables are
- 2, 4, 5
 - 1, 3, 6, 7
 - 4 and 6.

- 12 Put as many 1's as possible in a 4 by 8 *reduced echelon* matrix R so that the free variables are
- 2, 4, 5, 6
 - 1, 3, 6, 7, 8.
- 13 Suppose column 4 of a 3 by 5 matrix is all zero. Then x_4 is certainly a ____ variable. The special solution for this variable is the vector $\mathbf{x} = \text{_____}$.
- 14 Suppose the first and last columns of a 3 by 5 matrix are the same (not zero). Then _____ is a free variable. Find the special solution for this variable.
- 15 Suppose an m by n matrix has r pivots. The number of special solutions is _____. The nullspace contains only $\mathbf{x} = \mathbf{0}$ when $r = \text{_____}$. The column space is all of \mathbf{R}^m when $r = \text{_____}$.
- 16 The nullspace of a 5 by 5 matrix contains only $\mathbf{x} = \mathbf{0}$ when the matrix has _____ pivots. The column space is \mathbf{R}^5 when there are _____ pivots. Explain why.
- 17 The equation $x - 3y - z = 0$ determines a plane in \mathbf{R}^3 . What is the matrix A in this equation? Which are the free variables? The special solutions are $(3, 1, 0)$ and _____.
- 18 (Recommended) The plane $x - 3y - z = 12$ is parallel to the plane $x - 3y - z = 0$ in Problem 17. One particular point on this plane is $(12, 0, 0)$. All points on the plane have the form (fill in the first components)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

- 19 Prove that U and $A = LU$ have the same nullspace when L is invertible:
 If $U\mathbf{x} = \mathbf{0}$ then $LUX = \mathbf{0}$. If $LUX = \mathbf{0}$, how do you know $U\mathbf{x} = \mathbf{0}$?
- 20 Suppose column 1 + column 3 + column 5 = $\mathbf{0}$ in a 4 by 5 matrix with four pivots. Which column is sure to have no pivot (and which variable is free)? What is the special solution? What is the nullspace?

Questions 21–28 ask for matrices (if possible) with specific properties.

- 21 Construct a matrix whose nullspace consists of all combinations of $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$.
- 22 Construct a matrix whose nullspace consists of all multiples of $(4, 3, 2, 1)$.
- 23 Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

- 24** Construct a matrix whose column space contains $(1, 1, 0)$ and $(0, 1, 1)$ and whose nullspace contains $(1, 0, 1)$ and $(0, 0, 1)$.
- 25** Construct a matrix whose column space contains $(1, 1, 1)$ and whose nullspace is the line of multiples of $(1, 1, 1, 1)$.
- 26** Construct a 2 by 2 matrix whose nullspace equals its column space. This is possible.
- 27** Why does no 3 by 3 matrix have a nullspace that equals its column space?
- 28** If $AB = \mathbf{0}$ then the column space of B is contained in the _____ of A . Give an example of A and B .
- 29** The reduced form R of a 3 by 3 matrix with randomly chosen entries is almost sure to be _____. What R is virtually certain if the random A is 4 by 3?
- 30** Show by example that these three statements are generally *false*:
- A and A^T have the same nullspace.
 - A and A^T have the same free variables.
 - If R is the reduced form $\text{rref}(A)$ then R^T is $\text{rref}(A^T)$.
- 31** If the nullspace of A consists of all multiples of $x = (2, 1, 0, 1)$, how many pivots appear in U ? What is R ?
- 32** If the special solutions to $Rx = \mathbf{0}$ are in the columns of these N , go backward to find the nonzero rows of the reduced matrices R :
- $$N = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \end{bmatrix} \quad (\text{empty 3 by 1}).$$
- 33** (a) What are the five 2 by 2 reduced echelon matrices R whose entries are all 0's and 1's?
(b) What are the eight 1 by 3 matrices containing only 0's and 1's? Are all eight of them reduced echelon matrices R ?
- 34** Explain why A and $-A$ always have the same reduced echelon form R .

THE RANK AND THE ROW REDUCED FORM ■ 3.3

This section completes the step from A to its reduced row echelon form R . The matrix A is m by n (completely general). The matrix R is also m by n , but each pivot column has only one nonzero entry (the pivot which is always 1). This example is 3 by 5:

Reduced Row Echelon Form

$$R = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

You see zero above the second pivot as well as below. R is the final result of elimination, and MATLAB uses the command **rref**. The Teaching Code **elim** for this book has **rref** built into it. Of course **rref**(R) would give R again!

MATLAB: $[R, pivcol] = \text{rref}(A)$ Teaching Code: $[E, R] = \text{elim}(A)$

The extra output *pivcol* gives the numbers of the pivot columns. They are the same in A and R . The extra output E is the m by m **elimination matrix** that puts the original A (whatever it was) into its row reduced form R :

$$EA = R. \quad (1)$$

The square matrix E is the product of elementary matrices E_{ij} and P_{ij} and D^{-1} . Now we allow $j > i$, when E_{ij} subtracts a multiple of row j from row i . P_{ij} exchanges these rows. D^{-1} divides rows by their pivots to produce 1's.

If we want E , we can apply row reduction to the matrix $[A \ I]$ with $n+m$ columns. All the elementary matrices that multiply A (to produce R) will also multiply I (to produce E). The whole augmented matrix is being multiplied by E :

$$E [A \ I] = [R \ E] \quad (2)$$

This is exactly what “Gauss-Jordan” did in Chapter 2 to compute A^{-1} . **When A is square and invertible, its reduced row echelon form is $R = I$.** Then $EA = R$ becomes $EA = I$. In this invertible case, E is A^{-1} . This chapter is going further, to any (rectangular) matrix A and its reduced form R . The matrix E that multiplies A is still square and invertible, but the best it can do is to produce R . The pivot columns are reduced to ones and zeros.

The Rank of a Matrix

The numbers m and n give the size of a matrix—but not necessarily the *true size* of a linear system. An equation like $0 = 0$ should not count. If there are two identical rows in A , the second one disappears in R . Also if row 3 is a combination of rows 1 and 2, then row 3 will become all zeros in R . We don't want to count rows of zeros. **The true size of A is given by its rank:**

DEFINITION The rank of A is the number of pivots. This number is r .

The matrix R at the start of this section has rank $r = 2$. It has two pivots and two pivot columns. So does the unknown matrix A that produced R . This number $r = 2$ will be crucial to the theory, but its first definition is entirely computational. To execute the command $r = \text{rank}(A)$, the computer just counts the pivots. When pivot gives a list of the pivot columns, the length of that list is r .

Actually the computer has a hard time to decide whether a small number is really zero. When it subtracts 3 times $\frac{1}{3}$ from 1, does it obtain zero? Our Teaching Codes treat numbers below the tolerance 10^{-6} as zero.

We know right away that $r \leq m$ and $r \leq n$. The number of pivots can't be greater than the number of rows. It can't be greater than the number of columns. The cases $r = m$ and $r = n$ of “full row rank” and “full column rank” will be especially important. We mention them here and come back to them soon:

- *A has full row rank if every row has a pivot: $r = m$. No zero rows in R .*
- *A has full column rank if every column has a pivot: $r = n$. No free variables.*

A square invertible matrix has $r = m = n$. Then R is the same as I .

At the other extreme are the matrices of **rank one**. There is only *one* pivot. When elimination clears out the first column, it clears out all the columns. *Every row is a multiple of the pivot row.* At the same time, every column is a multiple of the pivot column!

$$\text{Rank one matrix} \quad A = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \quad \rightarrow \quad R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The column space of a rank one matrix is “one-dimensional”. Here all columns are on the line through $\mathbf{u} = (1, 2, 3)$. The columns of A are \mathbf{u} and $3\mathbf{u}$ and $10\mathbf{u}$. Put those numbers into the row $\mathbf{v}^T = [1 \ 3 \ 10]$ and you have the special rank one form $A = \mathbf{u}\mathbf{v}^T$:

$$A = \text{column times row} = \mathbf{u}\mathbf{v}^T \quad \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [1 \ 3 \ 10]. \quad (3)$$

Example 1 When all rows are multiples of one pivot row, the rank is $r = 1$:

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 2 \end{bmatrix} \text{ and } [6] \text{ all have rank 1.}$$

The reduced row echelon forms $R = \text{rref}(A)$ can be checked by eye:

$$R = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } [1] \text{ have only one pivot.}$$

Our second definition of rank is coming at a higher level. It deals with entire rows and entire columns—vectors and not just numbers. The matrices A and U and R

have r independent rows (the pivot rows). They also have r independent columns (the pivot columns). Section 3.5 says what it means for rows or columns to be independent.

A third definition of rank, at the top level of linear algebra, will deal with *spaces* of vectors. The rank r is the “dimension” of the column space. It is also the dimension of the row space. The great thing is that r also reveals the dimension of the nullspace.

The Pivot Columns

The pivot columns of R have 1's in the pivots and 0's everywhere else. The r pivot columns taken together contain an r by r identity matrix I . It sits above $m - r$ rows of zeros. The numbers of the pivot columns are in the list *pivcol*.

The pivot columns of A are probably *not* obvious from A itself. But their column numbers are given by the same list *pivcol*. The r columns of A that eventually have pivots (in U and R) are the pivot columns. The first matrix R in this section is the row reduced echelon form of this matrix A , with *pivcol* = (1, 3):

$$\begin{array}{ll} \textbf{Pivot} & A = \begin{bmatrix} 1 & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \text{ yields } R = \begin{bmatrix} \mathbf{1} & 3 & \mathbf{0} & 2 & -1 \\ \mathbf{0} & 0 & \mathbf{1} & 4 & -3 \\ \mathbf{0} & 0 & \mathbf{0} & 0 & 0 \end{bmatrix}. \end{array}$$

The column spaces of R and A can be different! All columns of this R end with zeros. E subtracts rows 1 and 2 of A from row 3 (to produce that zero row in R):

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The r pivot columns of A are also the first r columns of E^{-1} . The reason is that each column of A is E^{-1} times a column of R . The r by r identity matrix inside R just picks out the first r columns of E^{-1} .

One more fact about pivot columns. Their definition has been purely computational, based on R . Here is a direct mathematical description of the pivot columns of A :

3C *The pivot columns are not combinations of earlier columns.* The free columns are combinations of earlier columns. These combinations are the special solutions!

A pivot column of R (with 1 in the pivot row) cannot be a combination of earlier columns (with 0's in that row). The same column of A can't be a combination of earlier columns, because $Ax = \mathbf{0}$ exactly when $Rx = \mathbf{0}$. Now we look at the special solution x from each free column.

The Special Solutions

Each special solution to $Ax = \mathbf{0}$ and $Rx = \mathbf{0}$ has one free variable equal to 1. The other free variables are all zero. The solutions come directly from the echelon form R :

$$\begin{array}{ll} \text{Free columns} & Rx = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \\ \text{Free variables} & \end{array}$$

The free variables are in boldface. Set the first free variable to $x_2 = 1$ with $x_4 = x_5 = 0$. The equations give the pivot variables $x_1 = -3$ and $x_3 = 0$. This says that column 2 (a free column) is 3 times column 1. The special solution is $s_1 = (-3, 1, 0, 0, 0)$.

The next special solution has $x_4 = 1$. The other free variables are $x_2 = x_5 = 0$. The solution is $s_2 = (-2, 0, -4, 1, 0)$. Notice -2 and -4 in R , with plus signs.

The third special solution has $x_5 = 1$. With $x_2 = 0$ and $x_4 = 0$ we find $s_3 = (1, 0, 3, 0, 1)$. The numbers $x_1 = 1$ and $x_3 = 3$ are in column 5 of R , again with opposite signs. This is a general rule as we soon verify. The nullspace matrix N contains the three special solutions in its columns:

$$\begin{array}{ll} \text{Nullspace matrix} & N = \begin{bmatrix} -3 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \text{not free} \\ \text{free} \\ \text{not free} \\ \text{free} \\ \text{free} \end{array} \end{array}$$

The linear combinations of these three columns give all vectors in the nullspace. This is the complete solution to $Ax = \mathbf{0}$ (and $Rx = \mathbf{0}$). Where R had the identity matrix (2 by 2) in its pivot columns, N has the identity matrix (3 by 3) in its free rows.

There is a special solution for every free variable. Since r columns have pivots, that leaves $n - r$ free variables. This is the key to $Ax = \mathbf{0}$.

3D $Ax = \mathbf{0}$ has $n - r$ free variables and special solutions: n columns minus r pivot columns. The nullspace matrix N has $n - r$ columns (the special solutions).

When we introduce the idea of “independent” vectors, we will show that the special solutions are independent. You can see in N that no column is a combination of the other columns. The beautiful thing is that the count is exactly right:

$Ax = \mathbf{0}$ has r independent equations so $n - r$ independent solutions.

To complete this section, look again at the special solutions. Suppose for simplicity that the first r columns are the pivot columns, and the last $n - r$ columns are free (no pivots). Then the reduced row echelon form looks like

$$R = \left[\begin{array}{cc} I & F \\ 0 & 0 \end{array} \right] \quad \begin{matrix} r \text{ pivot rows} \\ m - r \text{ zero rows} \end{matrix} \quad (4)$$

r pivot columns $n - r$ free columns

3E The pivot variables in the $n - r$ special solutions come by changing F to $-F$:

Nullspace matrix $N = \left[\begin{array}{c} -F \\ I \end{array} \right] \quad \begin{matrix} r \text{ pivot variables} \\ n - r \text{ free variables} \end{matrix} \quad (5)$

Check $RN = 0$. The first block row of RN is (I times $-F$) + (F times I) = zero. The columns of N solve $Rx = \mathbf{0}$. When the free part of $Rx = \mathbf{0}$ moves to the right side, the left side just holds the identity matrix:

$$I \left[\begin{array}{c} \text{pivot} \\ \text{variables} \end{array} \right] = -F \left[\begin{array}{c} \text{free} \\ \text{variables} \end{array} \right]. \quad (6)$$

In each special solution, the free variables are a column of I . Then the pivot variables are a column of $-F$. Those special solutions give the nullspace matrix N .

The idea is still true if the pivot columns are mixed in with the free columns. Then I and F are mixed together. You can still see $-F$ in the solutions. Here is an example where $I = [1]$ comes first and $F = [2 \ 3]$ comes last.

Example 2 The special solutions of $Rx = x_1 + 2x_2 + 3x_3 = 0$ are the columns of N :

$$R = \left[\begin{array}{ccc} 1 & 2 & 3 \end{array} \right] \quad N = \left[\begin{array}{cc} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{array} \right].$$

The rank is one. There are $n - r = 3 - 1$ special solutions $(-2, 1, 0)$ and $(-3, 0, 1)$.

Final Note How can I write confidently about R when I don't know which steps MATLAB will take? A could be reduced to R in different ways. Very likely you and Mathematica and Maple would do the elimination differently. The key point is that **the final matrix R is always the same**. The original A completely determines the I and F and zero rows in R , according to **3C**:

The pivot columns *are not* combinations of earlier columns of A .

The free columns *are* combinations of earlier columns (F tells the combinations).

A small example with rank one will show two E 's that produce the correct $EA = R$:

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and no other } R.$$

You could multiply row 1 of A by $\frac{1}{2}$, and subtract row 1 from row 2:

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} = E.$$

Or you could exchange rows in A , and then subtract 2 times row 1 from row 2:

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = E_{\text{new}}.$$

Multiplication gives $EA = R$ and also $E_{\text{new}}A = R$. *Different E 's but the same R .*

■ REVIEW OF THE KEY IDEAS ■

1. The rank of A is the number of pivots (which are 1's in R).
2. The r pivot columns of A and R are in the same list $pivcol$.
3. Those r pivot columns are not combinations of earlier columns.
4. The $n - r$ free columns are combinations of earlier columns.
5. Those combinations (using $-F$ taken from R) give the $n - r$ special solutions to $Ax = \mathbf{0}$ and $Rx = \mathbf{0}$. They are the $n - r$ columns of the nullspace matrix N .

■ WORKED EXAMPLES ■

3.3 A Factor these rank one matrices into $A = \mathbf{u}\mathbf{v}^T = \text{row times column}$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (\text{find } d \text{ from } a^{-1}, b, c)$$

Split this rank two matrix into $\mathbf{u}_1\mathbf{v}_1^T + \mathbf{u}_2\mathbf{v}_2^T = (3 \text{ by } 2) \text{ times } (2 \text{ by } 4)$ using E^{-1} and R :

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E^{-1}R.$$

Solution For the 3 by 3 matrix A , all rows are multiples of $\mathbf{v}^T = [1 \ 2 \ 3]$. All columns are multiples of the column $\mathbf{u} = (1, 2, 3)$. This symmetric matrix has $\mathbf{u} = \mathbf{v}$ and A is $\mathbf{u}\mathbf{u}^T$. Every rank one symmetric matrix will have this form or else $-\mathbf{u}\mathbf{u}^T$.

If the 2 by 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has rank one, it must be singular. In Chapter 5, its determinant is $ad - bc = 0$. In this chapter, row 2 is a multiple of row 1. That multiple is $\frac{c}{a}$ (the problem assumes $a \neq 0$). Rank one always produces column times row:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 \\ c/a \end{bmatrix} [a \ b] = \begin{bmatrix} a & b \\ c & bc/a \end{bmatrix}. \text{ So } d = \frac{bc}{a}.$$

The 3 by 4 matrix of rank two is a sum of *two matrices of rank one*. All columns of A are combinations of the pivot columns 1 and 2. All rows are combinations of the nonzero rows of R . The pivot columns are \mathbf{u}_1 and \mathbf{u}_2 and those nonzero rows are \mathbf{v}_1^T and \mathbf{v}_2^T . Then A is $\mathbf{u}_1\mathbf{v}_1^T + \mathbf{u}_2\mathbf{v}_2^T$, multiplying columns of E^{-1} times rows of R :

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 0 & 3 \\ 2 & 3 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} [1 \ 0 \ 0 \ 1] + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} [0 \ 1 \ 0 \ 1]$$

3.3 B Find the row reduced form R and the rank r of A (*those depend on c*). Which are the pivot columns of A ? Which variables are free? What are the special solutions and the nullspace matrix N (always depending on c)?

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} c & c \\ c & c \end{bmatrix}.$$

Solution The 3 by 3 matrix A has rank $r = 2$ *except if $c = 4$* . The pivots are in columns 1 and 3. The second variable x_2 is free. Notice the form of R :

$$c \neq 4 \quad R = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad c = 4 \quad R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

When $c = 4$, the only pivot is in column 1 (one pivot column). Columns 2 and 3 are multiples of column 1 (so rank = 1). The second and third variables are free, producing two special solutions:

$$c \neq 4 \quad \text{Special solution with } x_2 = 1 \text{ goes into } N = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}.$$

$$c = 4 \quad \text{Another special solution goes into } N = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The 2 by 2 matrix $\begin{bmatrix} c & c \\ c & c \end{bmatrix}$ has rank $r = 1$ *except if* $c = 0$, when the rank is zero!

$$c \neq 0 \quad R = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad c = 0 \quad R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The first column is the pivot column if $c \neq 0$, and the second variable is free (one special solution in N). The matrix has *no pivot columns* if $c = 0$, and both variables are free:

$$c \neq 0 \quad N = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad c = 0 \quad N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem Set 3.3

- 1 Which of these rules gives a correct definition of the *rank* of A ?
 - (a) The number of nonzero rows in R .
 - (b) The number of columns minus the total number of rows.
 - (c) The number of columns minus the number of free columns.
 - (d) The number of 1's in the matrix R .
- 2 Find the reduced row echelon forms R and the rank of these matrices:
 - (a) The 3 by 4 matrix of all ones.
 - (b) The 3 by 4 matrix with $a_{ij} = i + j - 1$.
 - (c) The 3 by 4 matrix with $a_{ij} = (-1)^j$.
- 3 Find R for each of these (block) matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad B = \begin{bmatrix} A & A \end{bmatrix} \quad C = \begin{bmatrix} A & A \\ A & 0 \end{bmatrix}$$
- 4 Suppose all the pivot variables come *last* instead of first. Describe all four blocks in the reduced echelon form (the block B should be r by r):

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

What is the nullspace matrix N containing the special solutions?

- 5 (Silly problem) Describe all 2 by 3 matrices A_1 and A_2 , with row echelon forms R_1 and R_2 , such that $R_1 + R_2$ is the row echelon form of $A_1 + A_2$. Is it true that $R_1 = A_1$ and $R_2 = A_2$ in this case?

- 6 If A has r pivot columns, how do you know that A^T has r pivot columns? Give a 3 by 3 example for which the column numbers are different.
- 7 What are the special solutions to $Rx = \mathbf{0}$ and $y^T R = \mathbf{0}$ for these R ?

$$R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problems 8–11 are about matrices of rank $r = 1$.

- 8 Fill out these matrices so that they have rank 1:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & & \\ 4 & & \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & & \\ 1 & & \\ 2 & 6 & -3 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a & b \\ c & \end{bmatrix}.$$

- 9 If A is an m by n matrix with $r = 1$, its columns are multiples of one column and its rows are multiples of one row. The column space is a _____ in \mathbf{R}^m . The nullspace is a _____ in \mathbf{R}^n . Also the column space of A^T is a _____ in \mathbf{R}^n .

- 10 Choose vectors u and v so that $A = uv^T$ = column times row:

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix}.$$

$A = uv^T$ is the natural form for every matrix that has rank $r = 1$.

- 11 If A is a rank one matrix, the second row of U is _____. Do an example.

Problems 12–14 are about r by r invertible matrices inside A .

- 12 If A has rank r , then it has an r by r submatrix S that is invertible. Remove $m - r$ rows and $n - r$ columns to find an invertible submatrix S inside each A (you could keep the pivot rows and pivot columns of A):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- 13 Suppose P is the submatrix of A containing only the pivot columns. Explain why this m by r submatrix P has rank r .
- 14 In Problem 13, we can transpose P and find the r pivot columns of P^T . Transposing back, we have an r by r invertible submatrix S inside P :

$$\text{For } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 4 & 7 \end{bmatrix} \text{ find } P \text{ (3 by 2) and then } S \text{ (2 by 2).}$$

Problems 15–20 show that $\text{rank}(AB)$ is not greater than $\text{rank}(A)$ or $\text{rank}(B)$.

- 15 Find the ranks of AB and AM (rank one matrix times rank one matrix):

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1.5 & 6 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1 & b \\ c & bc \end{bmatrix}.$$

- 16 The rank one matrix uv^T times the rank one matrix wz^T is uz^T times the number _____. This has rank one unless _____ = 0.
- 17 (a) Suppose column j of B is a combination of previous columns of B . Show that column j of AB is the same combination of previous columns of AB . Then **rank**(AB) \leq **rank**(B), because AB cannot have new pivot columns.
 (b) Find A_1 and A_2 so that $\text{rank}(A_1B) = 1$ and $\text{rank}(A_2B) = 0$ for $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.
- 18 Problem 17 proved that $\text{rank}(AB) \leq \text{rank}(B)$. Then the same reasoning gives $\text{rank}(B^TA^T) \leq \text{rank}(A^T)$. How do you deduce that **rank**(AB) \leq **rank** A ?
- 19 (Important) Suppose A and B are n by n matrices, and $AB = I$. Prove from $\text{rank}(AB) \leq \text{rank}(A)$ that the rank of A is n . So A is invertible and B must be its two-sided inverse (Section 2.5). Therefore $BA = I$ (which is not so obvious!).
- 20 If A is 2 by 3 and B is 3 by 2 and $AB = I$, show from its rank that $BA \neq I$. Give an example of A and B . For $m < n$, a right inverse is not a left inverse.

- 21 Suppose A and B have the *same* reduced row echelon form R .
- (a) Show that A and B have the same nullspace and the same row space.
 (b) We know $E_1A = R$ and $E_2B = R$. So A equals an _____ matrix times B .
- 22 Every m by n matrix of rank r reduces to (m by r) times (r by n):

$$A = (\text{pivot columns of } A) (\text{first } r \text{ rows of } R) = (\text{COL})(\text{ROW})^T.$$

Write the 3 by 5 matrix A at the start of this section as the product of the 3 by 2 matrix from the pivot columns and the 2 by 5 matrix from R .

- 23 $A = (\text{COL})(\text{ROW})^T$ is a sum of r rank one matrices (multiply columns times rows). Express A and B as the sum of two rank one matrices:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 4 \\ 1 & 1 & 8 \end{bmatrix} \quad B = [A \quad A].$$

- 24 Suppose A is an m by n matrix of rank r . Its reduced echelon form is R . Describe exactly the matrix Z (its shape and all its entries) that comes from *transposing the reduced row echelon form of R'* (prime means transpose):

$$R = \text{rref}(A) \quad \text{and} \quad Z = (\text{rref}(R'))'.$$

- 25 Instead of transposing R (Problem 24) we could transpose A first. Explain in one line why $Y = Z$:

$$V = \text{rref}(A') \quad \text{and} \quad Y = \text{rref}(V').$$

- 26 Answer the same questions as in Worked Example 3.3 B for

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 1 & c & 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1-c & 2 \\ 0 & 2-c \end{bmatrix}.$$

- 27 What is the nullspace matrix N (containing the special solutions) for A, B, C ?

$$A = [I \ I] \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C = [I \ I \ I].$$

THE COMPLETE SOLUTION TO $AX = B$ ■ 3.4

The last section totally solved $Ax = 0$. Elimination converted the problem to $Rx = 0$. The free variables were given special values (one and zero). Then the pivot variables were found by back substitution. We paid no attention to the right side b because it started and ended as zero. The solution x was in the nullspace of A .

Now b is not zero. Row operations on the left side must act also on the right side. One way to organize that is to ***add b as an extra column of the matrix***. We keep the same example A as before. But we “augment” A with the right side $(b_1, b_2, b_3) = (1, 6, 7)$:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \text{ has the augmented matrix } \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{bmatrix} = [A \ b].$$

The augmented matrix is just $[A \ b]$. When we apply the usual elimination steps to A , we also apply them to b . In this example we subtract row 1 from row 3 and then subtract row 2 from row 3. This produces a *complete row of zeros*:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix} \text{ has the augmented matrix } \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [R \ d].$$

That very last zero is crucial. It means that the equations can be solved; the third equation has become $0 = 0$. In the original matrix A , the first row plus the second row equals the third row. If the equations are consistent, this must be true on the right side of the equations also! The all-important property on the right side was $1 + 6 = 7$.

Here are the same augmented matrices for a general $\mathbf{b} = (b_1, b_2, b_3)$:

$$\left[\begin{array}{ccccc} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 1 & 3 & 1 & 6 & b_3 \end{array} \right] \rightarrow \left[\begin{array}{ccccc} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array} \right].$$

Now we get $0 = 0$ in the third equation provided $b_3 - b_1 - b_2 = 0$. This is $b_1 + b_2 = b_3$.

One Particular Solution

Choose the free variables to be $x_2 = x_4 = 0$. Then the equations give the pivot variables $x_1 = 1$ and $x_3 = 6$. They are in the last column \mathbf{d} of the reduced augmented matrix. The code $\mathbf{x} = \text{partic}(A, \mathbf{b})$ gives this particular solution (call it \mathbf{x}_p) to $A\mathbf{x} = \mathbf{b}$. First A and \mathbf{b} reduce to R and \mathbf{d} . Zero rows in R must also be zero in \mathbf{d} . Then the r pivot variables in \mathbf{x} are taken directly from \mathbf{d} , because the pivot columns in R contain the identity matrix. After row reduction we are just solving $I\mathbf{x} = \mathbf{d}$.

Notice how we *choose* the free variables (as zero) and *solve* for the pivot variables. After the row reduction to R , those steps are quick. When the free variables are zero, the pivot variables for \mathbf{x}_p are in the extra column:

The particular solution solves	$A\mathbf{x}_p = \mathbf{b}$
The $n - r$ special solutions solve	$A\mathbf{x}_n = \mathbf{0}$.

In this example the particular solution is $(1, 0, 6, 0)$. The two special (nullspace) solutions to $R\mathbf{x} = \mathbf{0}$ come from the two free columns of R , by reversing signs of 3, 2, and 4. **Please notice how I write the complete solution $\mathbf{x}_p + \mathbf{x}_n$ to $A\mathbf{x} = \mathbf{b}$:**

Complete solution: $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}.$

Question Suppose A is a square invertible matrix, $m = n = r$. What are \mathbf{x}_p and \mathbf{x}_n ?

Answer The particular solution is the one and *only* solution $A^{-1}\mathbf{b}$. There are no special solutions or free variables. $R = I$ has no zero rows. The only vector in the nullspace is $\mathbf{x}_n = \mathbf{0}$. The complete solution is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = A^{-1}\mathbf{b} + \mathbf{0}$.

This was the situation in Chapter 2. We didn't mention the nullspace in that chapter. $N(A)$ contained only the zero vector. Reduction goes from $[A \ \mathbf{b}]$ to $[I \ A^{-1}\mathbf{b}]$. The original $A\mathbf{x} = \mathbf{b}$ is reduced all the way to $\mathbf{x} = A^{-1}\mathbf{b}$. This is a special case here, but square invertible matrices are the ones we see most often in practice. So they got their own chapter at the start of the book.

For small examples we can put $[A \ \mathbf{b}]$ into reduced row echelon form. For a large matrix, MATLAB can do it better. Here is a small example with *full column rank*. Both columns have pivots.

Example 1 Find the condition on (b_1, b_2, b_3) for $Ax = b$ to be solvable, if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This condition puts \mathbf{b} in the column space of A . Find the complete $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$.

Solution Use the augmented matrix, with its extra column \mathbf{b} . Elimination subtracts row 1 from row 2, and adds 2 times row 1 to row 3:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ 0 & 0 & b_3 + b_1 + b_2 \end{bmatrix}.$$

The last equation is $0 = 0$ provided $b_3 + b_1 + b_2 = 0$. This is the condition to put \mathbf{b} in the column space; then the system is solvable. The rows of A add to the zero row. So for consistency (these are equations!) the entries of \mathbf{b} must also add to zero.

This example has no free variables and no special solutions. The nullspace solution is $\mathbf{x}_n = \mathbf{0}$. The (only) particular solution \mathbf{x}_p is at the top of the augmented column:

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \\ 0 \end{bmatrix}.$$

If $b_3 + b_1 + b_2$ is not zero, there is no solution to $Ax = b$ (\mathbf{x}_p doesn't exist).

This example is typical of the extremely important case when A has *full column rank*. Every column has a pivot. *The rank is $r = n$* . The matrix is tall and thin ($m \geq n$). Row reduction puts I at the top, when A is reduced to R :

$$\text{Full column rank } R = \begin{bmatrix} n \text{ by } n \text{ identity matrix} \\ m - n \text{ rows of zeros} \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}. \quad (1)$$

There are no free columns or free variables. The nullspace matrix is empty!

We will collect together the different ways of recognizing this type of matrix.

3F Every matrix A with **full column rank** ($r = n$) has all these properties:

1. All columns of A are pivot columns.
2. There are no free variables or special solutions.
3. The nullspace $N(A)$ contains only the zero vector $\mathbf{x} = \mathbf{0}$.
4. If $Ax = b$ has a solution (it might not) then it has only *one solution*.

In the language of the next section, this A has *independent columns*. In Chapter 4 we will add one more fact to the list: *The square matrix $A^T A$ is invertible*.

In this case the nullspace of A (and R) has shrunk to the zero vector. The solution to $Ax = b$ is *unique* (if it exists). There will be $m - n$ (here $3 - 2$) zero rows in R . So there are $m - n$ (here 1 condition) conditions on b in order to have $0 = 0$ in those rows. If $b_3 + b_1 + b_2 = 0$ is satisfied, $Ax = b$ has exactly one solution.

The Complete Solution

The other extreme case is full row rank. Now $Ax = b$ either has one or infinitely many solutions. In this case A is *short and wide* ($m \leq n$). The number of unknowns is at least the number of equations. A matrix has **full row rank** if $r = m$. The nullspace of A^T shrinks to the zero vector. Every row has a pivot, and here is an example.

Example 2 There are $n = 3$ unknowns but only two equations. The rank is $r = m = 2$:

$$\begin{array}{rcl} x & + & y & + & z & = & 3 \\ x & + & 2y & - & z & = & 4 \end{array}$$

These are two planes in xyz space. The planes are not parallel so they intersect in a line. This line of solutions is exactly what elimination will find. **The particular solution will be one point on the line. Adding the nullspace vectors x_n will move us along the line.** Then $x = x_p + x_n$ gives the whole line of solutions.

We find x_p and x_n by elimination. Subtract row 1 from row 2 and then subtract row 2 from row 1:

$$\left[\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{array} \right] = [R \ d].$$

The particular solution has free variable $x_3 = 0$. The special solution has $x_3 = 1$:

$x_{\text{particular}}$ comes directly from d the right side: $x_p = (2, 1, 0)$

x_{special} comes from the third column (free column F) of R : $s = (-3, 2, 1)$

It is wise to check that x_p and s satisfy the original equations $Ax_p = b$ and $As = 0$:

$$\begin{array}{rcl} 2+1 & = & 3 \\ 2+2 & = & 4 \end{array} \quad \begin{array}{rcl} -3+2+1 & = & 0 \\ -3+4-1 & = & 0 \end{array}$$

The nullspace solution x_n is any multiple of s . It moves along the line of solutions, starting at $x_{\text{particular}}$. **Please notice again how to write the answer:**

Complete Solution: $x = x_p + x_n = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$

This line is drawn in Figure 3.3. Any point on the line *could* have been chosen as the particular solution; we chose the point with $x_3 = 0$. The particular solution is *not* multiplied by an arbitrary constant! The special solution is, and you understand why.

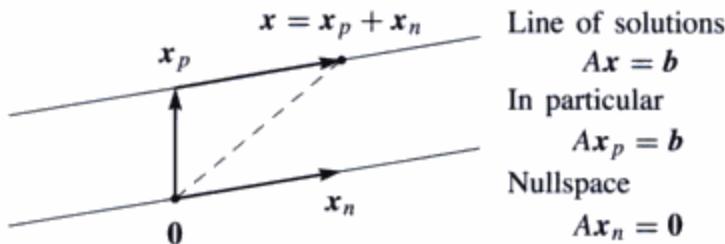


Figure 3.3 The complete solution is *one* particular solution plus *all* nullspace solutions.

Now we summarize this short wide case ($m \leq n$) of *full row rank*:

3G Every matrix A with *full row rank* ($r = m$) has all these properties:

1. All rows have pivots, and R has no zero rows.
2. $Ax = b$ has a solution for every right side b .
3. The column space is the whole space \mathbf{R}^m .
4. There are $n - r = n - m$ special solutions in the nullspace of A .

In this case with m pivots, the rows are “linearly independent”. In other words, the columns of A^T are linearly independent. We are more than ready for the definition of linear independence, as soon as we summarize the four possibilities—which depend on the rank. Notice how r, m, n are the critical numbers!

The four possibilities for linear equations depend on the rank r :

$r = m$	and	$r = n$	<i>Square and invertible</i>	$Ax = b$	has 1 solution
$r = m$	and	$r < n$	<i>Short and wide</i>	$Ax = b$	has ∞ solutions
$r < m$	and	$r = n$	<i>Tall and thin</i>	$Ax = b$	has 0 or 1 solution
$r < m$	and	$r < n$	<i>Unknown shape</i>	$Ax = b$	has 0 or ∞ solutions

The reduced R will fall in the same category as the matrix A . In case the pivot columns happen to come first, we can display these four possibilities for R :

$$R = [I] \quad [I \ F] \quad \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

$r = m = n \quad r = m < n \quad r = n < m \quad r < m, r < n$

Cases 1 and 2 have full row rank $r = m$. Cases 1 and 3 have full column rank $r = n$. Case 4 is the most general in theory and the least common in practice.

Note In the first edition of this textbook, we generally stopped at U before reaching R . Instead of reading the complete solution directly from $Rx = d$, we found it by back substitution from $Ux = c$. That combination of reduction to U and back substitution for x is slightly faster. Now we prefer the complete reduction: a single “1” in each pivot column. We find that everything is so much clearer in R (and the computer should do the hard work anyway) that we reduce all the way.

■ REVIEW OF THE KEY IDEAS ■

1. The rank r is the number of pivots. The matrix R has $m - r$ zero rows.
2. $Ax = b$ is solvable if and only if the last $m - r$ equations reduce to $0 = 0$.
3. One particular solution x_p has all free variables equal to zero.
4. The pivot variables are determined after the free variables are chosen.
5. Full column rank $r = n$ means no free variables: one solution or none.
6. Full row rank $r = m$ means one solution if $m = n$ or infinitely many if $m < n$.

■ WORKED EXAMPLES ■

3.4 A This question connects elimination-pivot columns-back substitution to column space-nullspace-rank-solvability (the full picture). The 3 by 3 matrix A has rank 2:

$$Ax = b \text{ is } \begin{array}{l} x_1 + 2x_2 + 3x_3 + 5x_4 = b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 = b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3 \end{array}$$

1. Reduce $[A \ b]$ to $[U \ c]$, so that $Ax = b$ becomes a triangular system $Ux = c$.
2. Find the condition on b_1, b_2, b_3 for $Ax = b$ to have a solution.
3. Describe the column space of A . Which plane in \mathbb{R}^3 ?
4. Describe the nullspace of A . Which special solutions in \mathbb{R}^4 ?
5. Find a particular solution to $Ax = (0, 6, -6)$ and then the complete solution.
6. Reduce $[U \ c]$ to $[R \ d]$: Special solutions from R , particular solution from d .

Solution

1. The multipliers in elimination are 2 and 3 and -1 . They take $[A \ b]$ into $[U \ c]$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array} \right]$$

2. The last equation shows the solvability condition $b_3 + b_2 - 5b_1 = 0$. Then $0 = 0$.
3. **First description:** The column space is the plane containing all combinations of the pivot columns $(1, 2, 3)$ and $(3, 8, 7)$, since the pivots are in columns 1 and 3. **Second description:** The column space contains all vectors with $b_3 + b_2 - 5b_1 = 0$. That makes $Ax = \mathbf{b}$ solvable, so \mathbf{b} is in the column space. *All columns of A pass this test $b_3 + b_2 - 5b_1 = 0$. This is the equation for the plane in the first description.*
4. The special solutions have free variables $x_2 = 1, x_4 = 0$ and then $x_2 = 0, x_4 = 1$:

Special solutions to $Ax = \mathbf{0}$ $s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $s_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$
Back substitution in $Ux = \mathbf{0}$

The nullspace $N(A)$ in \mathbb{R}^4 contains all $\mathbf{x}_n = c_1 s_1 + c_2 s_2 = (-2c_1 - 2c_2, c_1, -c_2, c_2)$.

5. One particular solution \mathbf{x}_p has free variables = zero. Back substitute in $Ux = \mathbf{c}$:

Particular solution to $Ax_p = (0, 6, -6)$ $x_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$
This vector \mathbf{b} satisfies $b_3 + b_2 - 5b_1 = 0$

The complete solution to $Ax = (0, 6, -6)$ is $\mathbf{x} = \mathbf{x}_p + \text{all } \mathbf{x}_n$.

6. In the reduced form R , the third column changes from $(3, 2, 0)$ in U to $(0, 1, 0)$. The right side $\mathbf{c} = (0, 6, 0)$ now becomes $\mathbf{d} = (-9, 3, 0)$ showing -9 and 3 in \mathbf{x}_p :

$$[U \ \mathbf{c}] = \begin{bmatrix} 1 & 2 & 3 & 5 & \mathbf{0} \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \rightarrow [R \ \mathbf{d}] = \begin{bmatrix} 1 & 2 & 0 & 2 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

3.4 B If you have this information about the solutions to $Ax = \mathbf{b}$ for a specific \mathbf{b} , what does that tell you about the shape of A (and A itself)? And possibly about \mathbf{b} .

1. There is exactly one solution.
2. All solutions to $Ax = \mathbf{b}$ have the form $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
3. There are no solutions.
4. All solutions to $Ax = \mathbf{b}$ have the form $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
5. There are infinitely many solutions.

Solution In case 1, with exactly one solution, A must have full column rank $r = n$. The nullspace of A contains only the zero vector. Necessarily $m \geq n$.

In case 2, A must have $n = 2$ columns (and m is arbitrary). With $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in the nullspace of A , column 2 is the negative of column 1. With $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a solution, $\mathbf{b} = (\text{column 1}) + 2(\text{column 2}) = \text{column 2}$. The columns can't be zero vectors.

In case 3 we only know that b is not in the column space of A . The rank of A must be less than m . I guess we know $b \neq 0$, otherwise $x = 0$ would be a solution.

In case 4, A must have $n = 3$ columns. With $(1, 0, 1)$ in the nullspace of A , column 3 is the negative of column 1. Column 2 must *not* be a multiple of column 1, or the nullspace would contain another special solution. So the rank of A is $3 - 1 = 2$. Necessarily A has $m \geq 2$ rows. The right side b is column 1 + column 2.

In case 5 with infinitely many solutions, the nullspace must contain nonzero vectors. The rank r must be less than n (not full column rank), and b must be in the column space of A . We don't know if *every* b is in the column space, so we don't know if $r = m$.

3.4 C Find the complete solution $x = x_p + x_n$ by forward elimination on $[A \ b]$:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}.$$

Find numbers y_1, y_2, y_3 so that y_1 (row 1) + y_2 (row 2) + y_3 (row 3) = **zero row**. Check that $b = (4, 2, 10)$ satisfies the condition $y_1 b_1 + y_2 b_2 + y_3 b_3 = 0$. Why is this the condition for the equations to be solvable and b to be in the column space?

Solution Forward elimination on $[A \ b]$ produces a zero row in $[U \ c]$. The third equation becomes $0 = 0$ and the equations are consistent (and solvable):

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 2 & 8 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1 and 3 contain pivots. The variables x_2 and x_4 are free. If we set those to zero we can solve (back substitution) for the particular solution $x_p = (7, 0, -3, 0)$. We see 7 and -3 again if elimination continues all the way to $[R \ d]$:

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -4 & 7 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the nullspace part x_n with $b = 0$, set the free variables x_2, x_4 to 1, 0 and also 0, 1:

Special solutions $s_1 = (-2, 1, 0, 0)$ and $s_2 = (4, 0, -4, 1)$

Then the complete solution to $Ax = b$ (and $Rx = d$) is $x_{\text{complete}} = x_p + c_1 s_1 + c_2 s_2$.

The rows of A produced the zero row from $2(\text{row 1}) + (\text{row 2}) - (\text{row 3}) = (0, 0, 0, 0)$. The same combination for $b = (4, 2, 10)$ gives $2(4) + (2) - (10) = 0$. If a combination of the rows (on the left side) gives the zero row, then the same combination must give zero on the right side. Of course! Otherwise no solution.

Later we will say this again in different words: If every column of A is perpendicular to $y = (2, 1, -1)$, then any combination b of those columns must also be perpendicular to y . Otherwise b is not in the column space and $Ax = b$ is not solvable.

And again: If y is in the nullspace of A^T then y must be perpendicular to every b in the column space. Just looking ahead ...

Problem Set 3.4

- 1 (Recommended) Execute the six steps of Worked Example 3.4 A to describe the column space and nullspace of A and the complete solution to $Ax = b$:

$$A = \begin{bmatrix} 2 & 4 & 6 & 4 \\ 2 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$$

- 2 Carry out the same six steps for this matrix A with rank one. You will find *two* conditions on b_1, b_2, b_3 for $Ax = b$ to be solvable. Together these two conditions put b into the _____ space (two planes give a line):

$$A = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} [2 \ 1 \ 3] = \begin{bmatrix} 2 & 1 & 3 \\ 6 & 3 & 9 \\ 4 & 2 & 6 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 30 \\ 20 \end{bmatrix}$$

Questions 3–15 are about the solution of $Ax = b$. Follow the steps in the text to x_p and x_n . Use the augmented matrix with last column b .

- 3 Write the complete solution as x_p plus any multiple of s in the nullspace:

$$\begin{aligned} x + 3y + 3z &= 1 \\ 2x + 6y + 9z &= 5 \\ -x - 3y + 3z &= 5. \end{aligned}$$

- 4 Find the complete solution (also called the *general solution*) to

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

- 5 Under what condition on b_1, b_2, b_3 is this system solvable? Include b as a fourth column in elimination. Find all solutions when that condition holds:

$$\begin{aligned} x + 2y - 2z &= b_1 \\ 2x + 5y - 4z &= b_2 \\ 4x + 9y - 8z &= b_3. \end{aligned}$$

- 6 What conditions on b_1, b_2, b_3, b_4 make each system solvable? Find \mathbf{x} in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

- 7 Show by elimination that (b_1, b_2, b_3) is in the column space if $b_3 - 2b_2 + 4b_1 = 0$.

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 8 & 2 \\ 2 & 4 & 0 \end{bmatrix}.$$

What combination of the rows of A gives the zero row?

- 8 Which vectors (b_1, b_2, b_3) are in the column space of A ? Which combinations of the rows of A give zero?

(a) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 6 & 3 \\ 0 & 2 & 5 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix}$

- 9 (a) The Worked Example 3.4 A reached $[U \ c]$ from $[A \ b]$. Put the multipliers into L and verify that LU equals A and Lc equals b .
 (b) Combine the pivot columns of A with the numbers -9 and 3 in the particular solution \mathbf{x}_p . What is that linear combination and why?

- 10 Construct a 2 by 3 system $A\mathbf{x} = \mathbf{b}$ with particular solution $\mathbf{x}_p = (2, 4, 0)$ and homogeneous solution $\mathbf{x}_n = \text{any multiple of } (1, 1, 1)$.

- 11 Why can't a 1 by 3 system have $\mathbf{x}_p = (2, 4, 0)$ and $\mathbf{x}_n = \text{any multiple of } (1, 1, 1)$?

- 12 (a) If $A\mathbf{x} = \mathbf{b}$ has two solutions \mathbf{x}_1 and \mathbf{x}_2 , find two solutions to $A\mathbf{x} = \mathbf{0}$.
 (b) Then find another solution to $A\mathbf{x} = \mathbf{0}$ and another solution to $A\mathbf{x} = \mathbf{b}$.

- 13 Explain why these are all false:

- (a) The complete solution is any linear combination of \mathbf{x}_p and \mathbf{x}_n .
 (b) A system $A\mathbf{x} = \mathbf{b}$ has at most one particular solution.
 (c) The solution \mathbf{x}_p with all free variables zero is the shortest solution (minimum length $\|\mathbf{x}\|$). Find a 2 by 2 counterexample.
 (d) If A is invertible there is no solution \mathbf{x}_n in the nullspace.

- 14 Suppose column 5 of U has no pivot. Then x_5 is a _____ variable. The zero vector (is) (is not) the only solution to $A\mathbf{x} = \mathbf{0}$. If $A\mathbf{x} = \mathbf{b}$ has a solution, then it has _____ solutions.

- 15 Suppose row 3 of U has no pivot. Then that row is _____. The equation $Ux = c$ is only solvable provided _____. The equation $Ax = b$ (is) (is not) (might not be) solvable.

Questions 16–20 are about matrices of “full rank” $r = m$ or $r = n$.

- 16 The largest possible rank of a 3 by 5 matrix is _____. Then there is a pivot in every _____ of U and R . The solution to $Ax = b$ (always exists) (is unique). The column space of A is _____. An example is $A =$ _____.
- 17 The largest possible rank of a 6 by 4 matrix is _____. Then there is a pivot in every _____ of U and R . The solution to $Ax = b$ (always exists) (is unique). The nullspace of A is _____. An example is $A =$ _____.
- 18 Find by elimination the rank of A and also the rank of A^T :

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \quad (\text{rank depends on } q).$$

- 19 Find the rank of A and also of A^TA and also of AA^T :

$$A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

- 20 Reduce A to its echelon form U . Then find a triangular L so that $A = LU$.

$$A = \begin{bmatrix} 3 & 4 & 1 & 0 \\ 6 & 5 & 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 6 & 5 & 4 \end{bmatrix}.$$

- 21 Find the complete solution in the form $x_p + x_n$ to these full rank systems:

- (a) $x + y + z = 4$ (b) $\begin{aligned} x + y + z &= 4 \\ x - y + z &= 4. \end{aligned}$
- 22 If $Ax = b$ has infinitely many solutions, why is it impossible for $Ax = B$ (new right side) to have only one solution? Could $Ax = B$ have no solution?
- 23 Choose the number q so that (if possible) the ranks are (a) 1, (b) 2, (c) 3:

$$A = \begin{bmatrix} 6 & 4 & 2 \\ -3 & -2 & -1 \\ 9 & 6 & q \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 1 & 3 \\ q & 2 & q \end{bmatrix}.$$

- 24** Give examples of matrices A for which the number of solutions to $Ax = b$ is
- 0 or 1, depending on b
 - ∞ , regardless of b
 - 0 or ∞ , depending on b
 - 1, regardless of b .
- 25** Write down all known relations between r and m and n if $Ax = b$ has
- no solution for some b
 - infinitely many solutions for every b
 - exactly one solution for some b , no solution for other b
 - exactly one solution for every b .

Questions 26–33 are about Gauss-Jordan elimination (upwards as well as downwards) and the reduced echelon matrix R .

- 26** Continue elimination from U to R . Divide rows by pivots so the new pivots are all 1. Then produce zeros *above* those pivots to reach R :

$$U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

- 27** Suppose U is square with n pivots (an invertible matrix). Explain why $R = I$.
- 28** Apply Gauss-Jordan elimination to $Ux = \mathbf{0}$ and $Ux = c$. Reach $Rx = \mathbf{0}$ and $Rx = d$:

$$\begin{bmatrix} U & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 0 & 4 & 8 \end{bmatrix}.$$

Solve $Rx = \mathbf{0}$ to find x_n (its free variable is $x_2 = 1$). Solve $Rx = d$ to find x_p (its free variable is $x_2 = 0$).

- 29** Apply Gauss-Jordan elimination to reduce to $Rx = \mathbf{0}$ and $Rx = d$:

$$\begin{bmatrix} U & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & \mathbf{0} \\ 0 & 0 & 2 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U & c \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 & 9 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Solve $Ux = \mathbf{0}$ or $Rx = \mathbf{0}$ to find x_n (free variable = 1). What are the solutions to $Rx = d$?

- 30 Reduce to $Ux = c$ (Gaussian elimination) and then $Rx = d$ (Gauss-Jordan):

$$Ax = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = b.$$

Find a particular solution x_p and all homogeneous solutions x_n .

- 31 Find matrices A and B with the given property or explain why you can't: The only solution of $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The only solution of $Bx = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

- 32 Find the LU factorization of A and the complete solution to $Ax = b$:

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 5 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 5 \end{bmatrix} \quad \text{and then} \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

- 33 The complete solution to $Ax = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find A .
- 34 Suppose you know that the 3 by 4 matrix A has the vector $s = (2, 3, 1, 0)$ as a basis for its nullspace
- (a) What is the *rank* of A and the complete solution to $Ax = \mathbf{0}$?
 - (b) What is the exact row reduced echelon form R of A ?

INDEPENDENCE, BASIS AND DIMENSION ■ 3.5

This important section is about the true size of a subspace. There are n columns in an m by n matrix, and each column has m components. But the true “dimension” of the column space is not necessarily m or n . The dimension is measured by counting *independent columns*—and we have to say what that means. We will see that *the true dimension of the column space is the rank r* .

The idea of independence applies to any vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in any vector space. Most of this section concentrates on the subspaces that we know and use—especially the column space in \mathbf{R}^m and the nullspace in \mathbf{R}^n . In the last part we also study “vectors” that are not column vectors. They can be matrices and functions; they can be linearly independent (or not). First come the key examples using column vectors.

The final goal is to understand a *basis* for a vector space. A basis contains independent vectors that “span the space”. We are at the heart of our subject, and we cannot go on without a basis. The four essential ideas in this section (with first hints at their meaning) are:

1. Independent vectors (*not too many*)
2. Spanning a space (*not too few*)
3. Basis for a space (*not too many or too few*)
4. Dimension of a space (*the right number of vectors*).

Linear Independence

Our first definition of independence is not so conventional, but you are ready for it.

DEFINITION The columns of A are *linearly independent* when the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. *No other combination $A\mathbf{x}$ of the columns gives the zero vector.*

With linearly independent columns, the nullspace $N(A)$ contains only the zero vector. Let me illustrate linear independence (and linear dependence) with three vectors in \mathbf{R}^3 :

1. If three vectors are *not* in the same plane, they are independent. No combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in Figure 3.4 gives zero except $0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$.
2. If three vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are *in the same plane*, they are dependent.

This idea of independence applies to 7 vectors in 12-dimensional space. If they are the columns of A , and independent, the nullspace only contains $\mathbf{x} = \mathbf{0}$. Now we choose different words to express the same idea. The following definition of independence will apply to any sequence of vectors in any vector space. When the vectors are the columns of A , the two definitions say exactly the same thing.

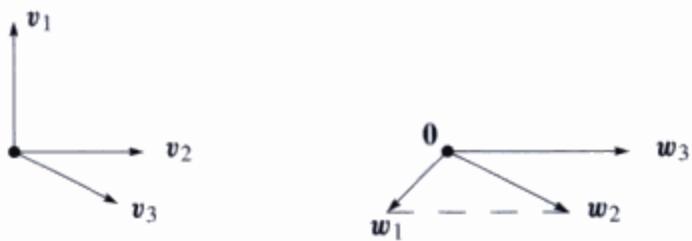


Figure 3.4 Independent vectors v_1, v_2, v_3 . Dependent vectors w_1, w_2, w_3 . The combination $w_1 - w_2 + w_3$ is $(0, 0, 0)$.

DEFINITION The sequence of vectors v_1, \dots, v_n is *linearly independent* if the only combination that gives the zero vector is $0v_1 + 0v_2 + \dots + 0v_n$. Thus linear independence means that

$$x_1 v_1 + x_2 v_2 + \dots + x_n v_n = \mathbf{0} \quad \text{only happens when all } x\text{'s are zero.} \quad (1)$$

If a combination gives $\mathbf{0}$, when the x 's are not all zero, the vectors are *dependent*.

Correct language: “The sequence of vectors is linearly independent.” Acceptable shortcut: “The vectors are independent.” Unacceptable: “The matrix is independent.”

A sequence of vectors is either dependent or independent. They can be combined to give the zero vector (with nonzero x 's) or they can't. So the key question is: Which combinations of the vectors give zero? We begin with some small examples in \mathbb{R}^2 :

- (a) The vectors $(1, 0)$ and $(0, 1)$ are independent.
- (b) The vectors $(1, 1)$ and $(1, 0.00001)$ are independent.
- (c) The vectors $(1, 1)$ and $(2, 2)$ are *dependent*.
- (d) The vectors $(1, 1)$ and $(0, 0)$ are *dependent*.

Geometrically, $(1, 1)$ and $(2, 2)$ are on a line through the origin. They are not independent. To use the definition, find numbers x_1 and x_2 so that $x_1(1, 1) + x_2(2, 2) = (0, 0)$. This is the same as solving $Ax = \mathbf{0}$:

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{for } x_1 = 2 \text{ and } x_2 = -1.$$

The columns are dependent exactly when *there is a nonzero vector in the nullspace*.

If one of the v 's is the zero vector, independence has no chance. Why not?

Now move to three vectors in \mathbb{R}^3 . If one of them is a multiple of another one, these vectors are dependent. But the complete test involves all three vectors at once. We put them in a matrix and try to solve $Ax = \mathbf{0}$.

Example 1 The columns of A are dependent. $Ax = \mathbf{0}$ has a nonzero solution:

$$Ax = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \quad \text{is} \quad -3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The rank of A is only $r = 2$. Independent columns would give full column rank $r = n = 3$.

In that matrix the rows are also dependent. Row 1 minus row 3 is the zero row. For a *square matrix*, we will show that dependent columns imply dependent rows (and vice versa).

Question How do you find that solution to $Ax = \mathbf{0}$? The systematic way is elimination.

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 5 \\ 1 & 0 & 3 \end{bmatrix} \text{ reduces to } R = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The solution $x = (-3, 1, 1)$ was exactly the special solution. It shows how the free column (column 3) is a combination of the pivot columns. That kills independence!

3H The columns of A are independent exactly when the rank is $r = n$. There are n pivots and no free variables. Only $x = \mathbf{0}$ is in the nullspace.

One case is of special importance because it is clear from the start. Suppose seven columns have five components each ($m = 5$ is less than $n = 7$). Then the columns *must be dependent*. Any seven vectors from \mathbb{R}^5 are dependent. The rank of A cannot be larger than 5. There cannot be more than five pivots in five rows. The system $Ax = \mathbf{0}$ has at least $7-5 = 2$ free variables, so it has nonzero solutions—which means that the columns are dependent.

3I Any set of n vectors in \mathbb{R}^m must be linearly dependent if $n > m$.

The matrix has more columns than rows—it is short and wide. The columns are certainly dependent if $n > m$, because $Ax = \mathbf{0}$ has a nonzero solution. The columns might be dependent or might be independent if $n \leq m$. Elimination will reveal the pivot columns. *It is those pivot columns that are independent.*

Note Another way to describe linear independence is this: “*One vector is a combination of the other vectors.*” That sounds clear. Why don’t we say this from the start? Our definition was longer: “*Some combination gives the zero vector, other than the trivial combination with every $x = 0$.*” We must rule out the easy way to get the zero vector. That trivial combination of zeros gives every author a headache. If one vector is a combination of the others, that vector has coefficient $x = 1$.

The point is, our definition doesn't pick out one particular vector as guilty. All columns of A are treated the same. We look at $Ax = \mathbf{0}$, and it has a nonzero solution or it hasn't. In the end that is better than asking if the last column (or the first, or a column in the middle) is a combination of the others.

Vectors that Span a Subspace

The first subspace in this book was the column space. Starting with columns v_1, \dots, v_n , the subspace was filled out by including all combinations $x_1v_1 + \dots + x_nv_n$. *The column space consists of all combinations Ax of the columns.* We now introduce the single word “span” to describe this: The column space is **spanned** by the columns.

DEFINITION A set of vectors *spans* a space if their linear combinations fill the space.

Example 2 $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full two-dimensional space \mathbf{R}^2 .

Example 3 $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ also span the full space \mathbf{R}^2 .

Example 4 $w_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ only span a line in \mathbf{R}^2 . So does w_1 by itself. So does w_2 by itself.

Think of two vectors coming out from $(0, 0, 0)$ in 3-dimensional space. Generally they span a plane. Your mind fills in that plane by taking linear combinations. Mathematically you know other possibilities: two vectors spanning a line, three vectors spanning all of \mathbf{R}^3 , three vectors spanning only a plane. It is even possible that three vectors span only a line, or ten vectors span only a plane. They are certainly not independent!

The columns span the column space. Here is a new subspace—which is spanned by the rows. *The combinations of the rows produce the “row space”.*

DEFINITION The *row space* of a matrix is the subspace of \mathbf{R}^n spanned by the rows.

The rows of an m by n matrix have n components. They are vectors in \mathbf{R}^n —or they would be if they were written as column vectors. There is a quick way to fix that: *Transpose the matrix.* Instead of the rows of A , look at the columns of A^T . Same numbers, but now in columns.

The row space of A is $C(A^T)$. It is the column space of A^T . It is a subspace of \mathbf{R}^n . The vectors that span it are the columns of A^T , which are the rows of A .

Example 5

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 5 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 5 \end{bmatrix}. \text{ Here } m = 3 \text{ and } n = 2.$$

The column space of A is spanned by the two columns of A . It is a plane in \mathbf{R}^3 . The row space of A is spanned by the three rows of A (which are columns of A^T). This row space is all of \mathbf{R}^2 . Remember: The rows are in \mathbf{R}^n . The columns are in \mathbf{R}^m . Same numbers, different vectors, different spaces.

A Basis for a Vector Space

In the xy plane, a set of independent vectors could be quite small—just one vector. A set that spans the xy plane could be large—three vectors, or four, or infinitely many. One vector won't span the plane. Three vectors won't be independent. A “*basis*” is just right. We want *enough independent vectors to span the space*.

DEFINITION A *basis* for a vector space is a sequence of vectors that has two properties at once:

1. The vectors are *linearly independent*.
2. The vectors *span the space*.

This combination of properties is fundamental to linear algebra. Every vector v in the space is a combination of the basis vectors, because they span the space. More than that, the combination that produces v is *unique*, because the basis vectors v_1, \dots, v_n are independent:

There is one and only one way to write v as a combination of the basis vectors.

Reason: Suppose $v = a_1v_1 + \dots + a_nv_n$ and also $v = b_1v_1 + \dots + b_nv_n$. By subtraction $(a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n$ is the zero vector. From the independence of the v 's, each $a_i - b_i = 0$. Hence $a_i = b_i$.

Example 6 The columns of $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ produce the “standard basis” for \mathbf{R}^2 .

The basis vectors $i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are independent. They span \mathbf{R}^2 .

Everybody thinks of this basis first. The vector i goes across and j goes straight up. The columns of the 3 by 3 identity matrix are the standard basis i, j, k . The columns of the n by n identity matrix give the “standard basis” for \mathbf{R}^n . Now we find other bases.

Example 7 (Important) The columns of *any invertible n by n matrix* give a basis for \mathbf{R}^n :

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{but not} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

When A is invertible, its columns are independent. The only solution to $Ax = \mathbf{0}$ is $x = \mathbf{0}$. The columns span the whole space \mathbf{R}^n —because every vector b is a combination of the columns. $Ax = b$ can always be solved by $x = A^{-1}b$. Do you see how everything comes together for invertible matrices? Here it is in one sentence:

3J The vectors v_1, \dots, v_n are a *basis for \mathbf{R}^n* exactly when they are *the columns of an n by n invertible matrix*. Thus \mathbf{R}^n has infinitely many different bases.

When any matrix has independent columns, they are a basis for its column space. When the columns are dependent, we keep only the *pivot columns*—the r columns with pivots. They are independent and they span the column space.

3K *The pivot columns of A are a basis for its column space.* The pivot rows of A are a basis for its row space. So are the pivot rows of its echelon form R .

Example 8 This matrix is not invertible. Its columns are not a basis for anything!

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \quad \text{which reduces to } R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Column 1 of A is the pivot column. That column alone is a basis for its column space. The second column of A would be a different basis. So would any nonzero multiple of that column. There is no shortage of bases! So we often make a definite choice: the pivot columns.

Notice that the pivot column of this R ends in zero. That column is a basis for the column space of R , but it is not even a member of the column space of A . The column spaces of A and R are different. Their bases are different.

The row space of A is the *same* as the row space of R . It contains $(2, 4)$ and $(1, 2)$ and all other multiples of those vectors. As always, there are infinitely many bases to choose from. I think the most natural choice is to pick the nonzero rows of R (rows with a pivot). So this matrix A with rank one has only one vector in the basis:

$$\text{Basis for the column space: } \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \quad \text{Basis for the row space: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The next chapter will come back to these bases for the column space and row space. We are happy first with examples where the situation is clear (and the idea of a basis is still new). The next example is larger but still clear.

Example 9 Find bases for the column and row spaces of a rank two matrix:

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Columns 1 and 3 are the pivot columns. They are a basis for the column space (of R !). The vectors in that column space all have the form $\mathbf{b} = (x, y, 0)$. The column space of R is the “xy plane” inside the full 3-dimensional xyz space. That plane is not \mathbf{R}^2 , it is a subspace of \mathbf{R}^3 . Columns 2 and 3 are a basis for the same column space. So are columns 1 and 4, and also columns 2 and 4. Which pairs of columns of R are *not* a basis for its column space?

The row space of R is a subspace of \mathbf{R}^4 . The simplest basis for that row space is the two nonzero rows of R . The third row (the zero vector) is in the row space too. But it is not in a *basis* for the row space. The basis vectors must be independent.

Question Given five vectors in \mathbf{R}^7 , how do you find a basis for the space they span?

First answer Make them the rows of A , and eliminate to find the nonzero rows of R .

Second answer Put the five vectors into the columns of A . Eliminate to find the pivot columns (of A not R !). The program **colbasis** uses the column numbers from **pivcol**.

Could another basis have more vectors, or fewer? This is a crucial question with a good answer. **All bases for a vector space contain the same number of vectors. This number is the “dimension” of the space.**

Dimension of a Vector Space

We have to prove what was just stated. There are many choices for the basis vectors, but the *number* of basis vectors doesn’t change.

3L If $\mathbf{v}_1, \dots, \mathbf{v}_m$ and $\mathbf{w}_1, \dots, \mathbf{w}_n$ are both bases for the same vector space, then $m = n$.

Proof Suppose that there are more \mathbf{w} ’s than \mathbf{v} ’s. From $n > m$ we want to reach a contradiction. The \mathbf{v} ’s are a basis, so \mathbf{w}_1 must be a combination of the \mathbf{v} ’s. If \mathbf{w}_1 equals $a_{11}\mathbf{v}_1 + \dots + a_{m1}\mathbf{v}_m$, this is the first column of a matrix multiplication VA :

$$W = \begin{bmatrix} \mathbf{w}_1 \mathbf{w}_2 \dots \mathbf{w}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_m \end{bmatrix} \begin{bmatrix} a_{11} & & \\ \vdots & & \\ a_{m1} & & \end{bmatrix} = VA.$$

We don’t know each a_{ij} , but we know the shape of A (it is m by n). The second vector \mathbf{w}_2 is also a combination of the \mathbf{v} ’s. The coefficients in that combination fill

the second column of A . The key is that A has a row for every v and a column for every w . A is a short wide matrix, since $n > m$. **There is a nonzero solution to $Ax = \mathbf{0}$.** Then $VAx = \mathbf{0}$ which is $Wx = \mathbf{0}$. A combination of the w 's gives zero! The w 's could not be a basis—so we cannot have $n > m$.

If $m > n$ we exchange the v 's and w 's and repeat the same steps. The only way to avoid a contradiction is to have $m = n$. This completes the proof that $m = n$.

The number of basis vectors depends on the space—not on a particular basis. The number is the same for every basis, and it tells how many “degrees of freedom” the vector space allows. The dimension of \mathbf{R}^n is n . We now introduce the important word **dimension** for other vector spaces too.

DEFINITION The *dimension of a space* is the number of vectors in every basis.

This matches our intuition. The line through $v = (1, 5, 2)$ has dimension one. It is a subspace with one vector v in its basis. Perpendicular to that line is the plane $x + 5y + 2z = 0$. This plane has dimension 2. To prove it, we find a basis $(-5, 1, 0)$ and $(-2, 0, 1)$. The dimension is 2 because the basis contains two vectors.

The plane is the nullspace of the matrix $A = [1 \ 5 \ 2]$, which has two free variables. Our basis vectors $(-5, 1, 0)$ and $(-2, 0, 1)$ are the “special solutions” to $Ax = \mathbf{0}$. The next section shows that the $n - r$ special solutions always give a basis for the nullspace. So $N(A)$ has dimension $n - r$. Here we emphasize only this: All bases for a space contain the same number of vectors.

Note about the language of linear algebra We never say “the rank of a space” or “the dimension of a basis” or “the basis of a matrix”. Those terms have no meaning. It is the **dimension of the column space** that equals the **rank of the matrix**.

Bases for Matrix Spaces and Function Spaces

The words “independence” and “basis” and “dimension” are not at all restricted to column vectors. We can ask whether three 3 by 4 matrices A_1, A_2, A_3 are independent. They are members of the space of all 3 by 4 matrices; some combination might give the zero matrix. We can also ask the dimension of that matrix space (it is 12).

In differential equations, the space of solutions to $d^2y/dx^2 = y$ contains functions. One basis is $y = e^x$ and $y = e^{-x}$. Counting the basis functions gives the dimension 2 (for the space of all solutions).

We think matrix spaces and function spaces are optional. Your class can go past this page—no problem. But in some way, you haven't got the ideas of basis and dimension straight until you can apply them to “vectors” other than column vectors.

Matrix spaces The vector space \mathbf{M} contains all 2 by 2 matrices. Its dimension is 4.

One basis is $A_1, A_2, A_3, A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Those matrices are linearly independent. We are not looking at their columns, but at the whole matrix. Combinations of those four matrices can produce any matrix in \mathbf{M} , so they span the space:

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.$$

This is zero only if the c 's are all zero—which proves independence.

The matrices A_1, A_2, A_4 are a basis for a subspace—the upper triangular matrices. Its dimension is 3. A_1 and A_4 are a basis for the diagonal matrices. What is a basis for the symmetric matrices? Keep A_1 and A_4 , and throw in $A_2 + A_3$.

To push this further, think about the space of all n by n matrices. For a basis, choose matrices that have only a single nonzero entry (that entry is 1). There are n^2 positions for that 1, so there are n^2 basis matrices:

The dimension of the whole n by n matrix space is n^2 .

The dimension of the subspace of *upper triangular* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$.

The dimension of the subspace of *diagonal* matrices is n .

The dimension of the subspace of *symmetric* matrices is $\frac{1}{2}n^2 + \frac{1}{2}n$.

Function spaces The equations $d^2y/dx^2 = 0$ and $d^2y/dx^2 = -y$ and $d^2y/dx^2 = y$ involve the second derivative. In calculus we solve to find the functions $y(x)$:

- $y'' = 0$ is solved by any linear function $y = cx + d$
- $y'' = -y$ is solved by any combination $y = c \sin x + d \cos x$
- $y'' = y$ is solved by any combination $y = ce^x + de^{-x}$.

The second solution space has two basis functions: $\sin x$ and $\cos x$. The third solution space has basis functions e^x and e^{-x} . The first space has x and 1. It is the “nullspace” of the second derivative! The dimension is 2 in each case (these are second-order equations).

The solutions of $y'' = 2$ don't form a subspace—the right side $b = 2$ is not zero. A particular solution is $y(x) = x^2$. The complete solution is $y(x) = x^2 + cx + d$. All those functions satisfy $y'' = 2$. Notice the particular solution plus any function $cx + d$ in the nullspace. A linear differential equation is like a linear matrix equation $Ax = b$. But we solve it by calculus instead of linear algebra.

We end here with the space \mathbf{Z} that contains only the zero vector. The dimension of this space is *zero*. *The empty set* (containing no vectors at all) is a basis. We can never allow the zero vector into a basis, because then linear independence is lost.

■ REVIEW OF THE KEY IDEAS ■

1. The columns of A are *independent* if $\mathbf{x} = \mathbf{0}$ is the only solution to $A\mathbf{x} = \mathbf{0}$.
2. The vectors $\mathbf{v}_1, \dots, \mathbf{v}_r$ *span* a space if their combinations fill that space.
3. A *basis* consists of linearly independent vectors that span the space. Every vector is a *unique* combination of the basis vectors.
4. All bases for a space have the same number of vectors. This number is the *dimension* of the space.
5. The pivot columns are a basis for the column space and the dimension is r .

■ WORKED EXAMPLES ■

3.5 A Start with the vectors $\mathbf{v}_1 = (1, 2, 0)$ and $\mathbf{v}_2 = (2, 3, 0)$. (a) Are they linearly independent? (b) Are they a basis for any space? (c) What space V do they span? (d) What is the dimension of that space? (e) Which matrices A have V as their column space? (f) Which matrices have V as their nullspace? (g) Describe all vectors \mathbf{v}_3 that complete a basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for \mathbf{R}^3 .

Solution

- (a) \mathbf{v}_1 and \mathbf{v}_2 are independent—the only combination to give $\mathbf{0}$ is $0\mathbf{v}_1 + 0\mathbf{v}_2$.
- (b) Yes, they are a basis for whatever space V they span.
- (c) That space V contains all vectors $(x, y, 0)$. It is the xy plane in \mathbf{R}^3 .
- (d) The dimension of V is 2 since the basis contains two vectors.
- (e) This V is the column space of any 3 by n matrix A of rank 2, if every column is a combination of \mathbf{v}_1 and \mathbf{v}_2 . In particular A could just have columns \mathbf{v}_1 and \mathbf{v}_2 .
- (f) This V is the nullspace of any m by 3 matrix B of rank 1, if every row is a multiple of $(0, 0, 1)$. In particular take $B = [0 \ 0 \ 1]$. Then $B\mathbf{v}_1 = \mathbf{0}$ and $B\mathbf{v}_2 = \mathbf{0}$.
- (g) Any third vector $\mathbf{v}_3 = (a, b, c)$ will complete a basis for \mathbf{R}^3 provided $c \neq 0$.

3.5 B Start with three independent vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Take combinations of those vectors to produce $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Write the combinations in matrix form as $V = WM$:

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{w}_1 + \mathbf{w}_2 \\ \mathbf{v}_2 &= \mathbf{w}_1 + 2\mathbf{w}_2 + \mathbf{w}_3 \quad \text{which is} \\ \mathbf{v}_3 &= \mathbf{w}_2 + c\mathbf{w}_3\end{aligned}\quad \left[\begin{array}{ccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{array} \right] = \left[\begin{array}{ccc} \mathbf{w}_1 & \mathbf{w}_2 & \mathbf{w}_3 \end{array} \right] \left[\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & c \end{array} \right]$$

What is the test on a matrix V to see if its columns are linearly independent? If $c \neq 1$ show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. If $c = 1$ show that the \mathbf{v} 's are linearly dependent.

Solution The test on V for independence of its columns was in our first definition: *The nullspace of V must contain only the zero vector.* Then $\mathbf{x} = (0, 0, 0)$ is the only combination of the columns that gives $V\mathbf{x} =$ zero vector.

In $c = 1$ in our problem, we can see dependence in two ways. First, $\mathbf{v}_1 + \mathbf{v}_3$ will be the same as \mathbf{v}_2 . (If you add $\mathbf{w}_1 + \mathbf{w}_2$ to $\mathbf{w}_2 + \mathbf{w}_3$ you get $\mathbf{w}_1 + 2\mathbf{w}_2 + \mathbf{w}_3$ which is \mathbf{v}_2 .) In other words $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ —which says that the \mathbf{v} 's are not independent.

The other way is to look at the nullspace of M . If $c = 1$, the vector $\mathbf{x} = (1, -1, 1)$ is in that nullspace, and $M\mathbf{x} = \mathbf{0}$. Then certainly $WM\mathbf{x} = \mathbf{0}$ which is the same as $V\mathbf{x} = 0$. So the \mathbf{v} 's are dependent. This specific $\mathbf{x} = (1, -1, 1)$ from the nullspace tells us again that $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$.

Now suppose $c \neq 1$. Then the matrix M is invertible. So if \mathbf{x} is any nonzero vector we know that $M\mathbf{x}$ is nonzero. Since the \mathbf{w} 's are given as independent, we further know that $WM\mathbf{x}$ is nonzero. Since $V = WM$, this says that \mathbf{x} is not in the nullspace of V . In other words $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent.

The general rule is “independent \mathbf{v} 's from independent \mathbf{w} 's when M is invertible”. And if these vectors are in \mathbf{R}^3 , they are not only independent—they are a basis for \mathbf{R}^3 . “Basis of \mathbf{v} 's from basis of \mathbf{w} 's when the change of basis matrix M is invertible.”

3.5 C Suppose $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a basis for \mathbf{R}^n and the n by n matrix A is invertible. Show that $A\mathbf{v}_1, \dots, A\mathbf{v}_n$ is also a basis for \mathbf{R}^n .

Solution In matrix language: Put the basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ in the columns of an invertible(!) matrix V . Then $A\mathbf{v}_1, \dots, A\mathbf{v}_n$ are the columns of AV . Since A is invertible, so is AV and its columns give a basis.

In vector language: Suppose $c_1A\mathbf{v}_1 + \dots + c_nA\mathbf{v}_n = \mathbf{0}$. This is $A\mathbf{v} = \mathbf{0}$ with $\mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$. Multiply by A^{-1} to get $\mathbf{v} = \mathbf{0}$. By linear independence of the \mathbf{v} 's, all $c_i = 0$. So the $A\mathbf{v}$'s are independent.

To show that the $A\mathbf{v}$'s span \mathbf{R}^n , solve $c_1A\mathbf{v}_1 + \dots + c_nA\mathbf{v}_n = \mathbf{b}$ which is the same as $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n = A^{-1}\mathbf{b}$. Since the \mathbf{v} 's are a basis, this must be solvable.

Problem Set 3.5

Questions 1–10 are about linear independence and linear dependence.

- 1 Show that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent but $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are dependent:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solve either $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ or $A\mathbf{x} = \mathbf{0}$. The \mathbf{v} 's go in the columns of A .

- 2 (Recommended) Find the largest possible number of independent vectors among

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

- 3 Prove that if $a = 0$ or $d = 0$ or $f = 0$ (3 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

- 4 If a, d, f in Question 3 are all nonzero, show that the only solution to $U\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Then U has independent columns.

- 5 Decide the dependence or independence of

- (a) the vectors $(1, 3, 2)$ and $(2, 1, 3)$ and $(3, 2, 1)$
 (b) the vectors $(1, -3, 2)$ and $(2, 1, -3)$ and $(-3, 2, 1)$.

- 6 Choose three independent columns of U . Then make two other choices. Do the same for A .

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

- 7 If $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are independent vectors, show that the differences $\mathbf{v}_1 = \mathbf{w}_2 - \mathbf{w}_3$ and $\mathbf{v}_2 = \mathbf{w}_1 - \mathbf{w}_3$ and $\mathbf{v}_3 = \mathbf{w}_1 - \mathbf{w}_2$ are *dependent*. Find a combination of the \mathbf{v} 's that gives zero.

- 8 If $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are independent vectors, show that the sums $\mathbf{v}_1 = \mathbf{w}_2 + \mathbf{w}_3$ and $\mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_3$ and $\mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2$ are *independent*. (Write $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ in terms of the \mathbf{w} 's. Find and solve equations for the c 's.)

- 9 Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are vectors in \mathbf{R}^3 .

- (a) These four vectors are dependent because ____.
 (b) The two vectors \mathbf{v}_1 and \mathbf{v}_2 will be dependent if ____.
 (c) The vectors \mathbf{v}_1 and $(0, 0, 0)$ are dependent because ____.

- 10 Find two independent vectors on the plane $x + 2y - 3z - t = 0$ in \mathbf{R}^4 . Then find three independent vectors. Why not four? This plane is the nullspace of what matrix?

Questions 11–15 are about the space *spanned* by a set of vectors. Take all linear combinations of the vectors.

- 11 Describe the subspace of \mathbf{R}^3 (is it a line or plane or \mathbf{R}^3 ?) spanned by
- the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$
 - the three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$
 - the columns of a 3 by 5 echelon matrix with 2 pivots
 - all vectors with positive components.
- 12 The vector b is in the subspace spanned by the columns of A when there is a solution to _____. The vector c is in the row space of A when there is a solution to _____.

True or false: If the zero vector is in the row space, the rows are dependent.

- 13 Find the dimensions of these 4 spaces. Which two of the spaces are the same?
 (a) column space of A , (b) column space of U , (c) row space of A , (d) row space of U :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

- 14 Choose $x = (x_1, x_2, x_3, x_4)$ in \mathbf{R}^4 . It has 24 rearrangements like (x_2, x_1, x_3, x_4) and (x_4, x_3, x_1, x_2) . Those 24 vectors, including x itself, span a subspace S . Find specific vectors x so that the dimension of S is: (a) zero, (b) one, (c) three, (d) four.
- 15 $v + w$ and $v - w$ are combinations of v and w . Write v and w as combinations of $v + w$ and $v - w$. The two pairs of vectors _____ the same space. When are they a basis for the same space?

Questions 16–26 are about the requirements for a basis.

- 16 If v_1, \dots, v_n are linearly independent, the space they span has dimension _____. These vectors are a _____ for that space. If the vectors are the columns of an m by n matrix, then m is _____ than n .
- 17 Find a basis for each of these subspaces of \mathbf{R}^4 :
- All vectors whose components are equal.
 - All vectors whose components add to zero.
 - All vectors that are perpendicular to $(1, 1, 0, 0)$ and $(1, 0, 1, 1)$.
 - The column space (in \mathbf{R}^2) and nullspace (in \mathbf{R}^5) of $U = [\begin{smallmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{smallmatrix}]$.

- 18** Find three different bases for the column space of U above. Then find two different bases for the row space of U .
- 19** Suppose v_1, v_2, \dots, v_6 are six vectors in \mathbf{R}^4 .
- Those vectors (do)(do not)(might not) span \mathbf{R}^4 .
 - Those vectors (are)(are not)(might be) linearly independent.
 - Any four of those vectors (are)(are not)(might be) a basis for \mathbf{R}^4 .
- 20** The columns of A are n vectors from \mathbf{R}^m . If they are linearly independent, what is the rank of A ? If they span \mathbf{R}^m , what is the rank? If they are a basis for \mathbf{R}^m , what then?
- 21** Find a basis for the plane $x - 2y + 3z = 0$ in \mathbf{R}^3 . Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.
- 22** Suppose the columns of a 5 by 5 matrix A are a basis for \mathbf{R}^5 .
- The equation $Ax = \mathbf{0}$ has only the solution $x = \mathbf{0}$ because ____.
 - If b is in \mathbf{R}^5 then $Ax = b$ is solvable because ____.
- Conclusion: A is invertible. Its rank is 5.
- 23** Suppose S is a 5-dimensional subspace of \mathbf{R}^6 . True or false:
- Every basis for S can be extended to a basis for \mathbf{R}^6 by adding one more vector.
 - Every basis for \mathbf{R}^6 can be reduced to a basis for S by removing one vector.
- 24** U comes from A by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces.

- 25** True or false (give a good reason):
- If the columns of a matrix are dependent, so are the rows.
 - The column space of a 2 by 2 matrix is the same as its row space.
 - The column space of a 2 by 2 matrix has the same dimension as its row space.
 - The columns of a matrix are a basis for the column space.

- 26** For which numbers c and d do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

Questions 27–32 are about spaces where the “vectors” are matrices.

- 27** Find a basis for each of these subspaces of 3 by 3 matrices:

- (a) All diagonal matrices.
- (b) All symmetric matrices ($A^T = A$).
- (c) All skew-symmetric matrices ($A^T = -A$).

- 28** Construct six linearly independent 3 by 3 echelon matrices U_1, \dots, U_6 .

- 29** Find a basis for the space of all 2 by 3 matrices whose columns add to zero. Find a basis for the subspace whose rows also add to zero.

- 30** Show that the six 3 by 3 permutation matrices (Section 2.6) are linearly dependent.

- 31** What subspace of 3 by 3 matrices is spanned by

- (a) all invertible matrices?
- (b) all echelon matrices?
- (c) the identity matrix?

- 32** Find a basis for the space of 2 by 3 matrices whose nullspace contains $(2, 1, 1)$.

Questions 33–37 are about spaces where the “vectors” are functions.

- 33** (a) Find all functions that satisfy $\frac{dy}{dx} = 0$.

- (b) Choose a particular function that satisfies $\frac{dy}{dx} = 3$.

- (c) Find all functions that satisfy $\frac{dy}{dx} = 3$.

- 34** The cosine space \mathbf{F}_3 contains all combinations $y(x) = A \cos x + B \cos 2x + C \cos 3x$. Find a basis for the subspace with $y(0) = 0$.

- 35** Find a basis for the space of functions that satisfy

- (a) $\frac{dy}{dx} - 2y = 0$

- (b) $\frac{dy}{dx} - \frac{y}{x} = 0$.

- 36** Suppose $y_1(x), y_2(x), y_3(x)$ are three different functions of x . The vector space they span could have dimension 1, 2, or 3. Give an example of y_1, y_2, y_3 to show each possibility.

- 37 Find a basis for the space of polynomials $p(x)$ of degree ≤ 3 . Find a basis for the subspace with $p(1) = 0$.
- 38 Find a basis for the space S of vectors (a, b, c, d) with $a+c+d = 0$ and also for the space T with $a+b = 0$ and $c = 2d$. What is the dimension of the intersection $S \cap T$?
- 39 Write the 3 by 3 identity matrix as a combination of the other five permutation matrices! Then show that those five matrices are linearly independent. (Assume a combination gives zero, and check entries to prove each term is zero.) The five permutations are a basis for the subspace of 3 by 3 matrices with row and column sums all equal.
- 40 If $AS = SA$ for the *shift matrix* S , show that A must have this special form:

$$\text{If } \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \text{ then } A = \begin{bmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{bmatrix}.$$

“The subspace of matrices that commute with the shift S has dimension ____.”

- 41 Which of the following are bases for \mathbf{R}^3 ?
- $(1, 2, 0)$ and $(0, 1, -1)$
 - $(1, 1, -1), (2, 3, 4), (4, 1, -1), (0, 1, -1)$
 - $(1, 2, 2), (-1, 2, 1), (0, 8, 0)$
 - $(1, 2, 2), (-1, 2, 1), (0, 8, 6)$
- 42 Suppose A is 5 by 4 with rank 4. Show that $Ax = b$ has no solution when the 5 by 5 matrix $[A \ b]$ is invertible. Show that $Ax = b$ is solvable when $[A \ b]$ is singular.

DIMENSIONS OF THE FOUR SUBSPACES ■ 3.6

The main theorem in this chapter connects **rank** and **dimension**. The **rank** of a matrix is the number of pivots. The **dimension** of a subspace is the number of vectors in a basis. We count pivots or we count basis vectors. *The rank of A reveals the dimensions of all four fundamental subspaces.* Here are the subspaces, including the new one.

Two subspaces come directly from A , and the other two from A^T :

Four Fundamental Subspaces

1. The **row space** is $C(A^T)$, a subspace of \mathbf{R}^n .
2. The **column space** is $C(A)$, a subspace of \mathbf{R}^m .
3. The **nullspace** is $N(A)$, a subspace of \mathbf{R}^n .
4. The **left nullspace** is $N(A^T)$, a subspace of \mathbf{R}^m . This is our new space.

In this book the column space and nullspace came first. We know $C(A)$ and $N(A)$ pretty well. Now the other two subspaces come forward. The row space contains all combinations of the rows. *This is the column space of A^T .*

For the left nullspace we solve $A^T \mathbf{y} = \mathbf{0}$ —that system is n by m . *This is the nullspace of A^T .* The vectors \mathbf{y} go on the *left* side of A when the equation is written as $\mathbf{y}^T A = \mathbf{0}^T$. The matrices A and A^T are usually different. So are their column spaces and their nullspaces. But those spaces are connected in an absolutely beautiful way.

Part 1 of the Fundamental Theorem finds the dimensions of the four subspaces. One fact stands out: *The row space and column space have the same dimension r* (the rank of the matrix). The other important fact involves the two nullspaces: *Their dimensions are $n - r$ and $m - r$, to make up the full dimensions n and m .*

Part 2 of the Fundamental Theorem will describe how the four subspaces fit together (two in \mathbf{R}^n and two in \mathbf{R}^m). That completes the “right way” to understand $A\mathbf{x} = \mathbf{b}$. Stay with it—you are doing real mathematics.

The Four Subspaces for R

Suppose A is reduced to its row echelon form R . For that special form, the four subspaces are easy to identify. We will find a basis for each subspace and check its dimension. Then we watch how the subspaces change (or don’t change!) as we look back at A . The main point is that *the four dimensions are the same for A and R*.

As a specific 3 by 5 example, look at the four subspaces for the echelon matrix R :

$$\begin{array}{lll} m = 3 & \left[\begin{array}{ccccc} 1 & 3 & 5 & 0 & 9 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] & \text{pivot rows 1 and 2} \\ n = 5 & & \\ r = 2 & & \text{pivot columns 1 and 4} \end{array}$$

The rank of this matrix R is $r = 2$ (*two pivots*). Take the subspaces in order:

1. The *row space* of R has dimension 2, matching the rank.

Reason: The first two rows are a basis. The row space contains combinations of all three rows, but the third row (the zero row) adds nothing new. So rows 1 and 2 span the row space.

The pivot rows 1 and 2 are also independent. That is obvious for this example, and it is always true. If we look only at the pivot columns, we see the r by r identity matrix. There is no way to combine its rows to give the zero row (except by the combination with all coefficients zero). So the r pivot rows are independent and the dimension is r .

The dimension of the row space is r . The nonzero rows of R form a basis.

2. The *column space* of R also has dimension $r = 2$.

Reason: The pivot columns 1 and 4 form a basis. They are independent because they start with the r by r identity matrix. No combination of those pivot columns can give the zero column (except the combination with all coefficients zero). And they also span the column space. Every other (free) column is a combination of the pivot columns. Actually the combinations we need are the three special solutions:

Column 2 is 3 (column 1). The special solution is $(-3, 1, 0, 0, 0)$.

Column 3 is 5 (column 1). The special solution is $(-5, 0, 1, 0, 0)$.

Column 5 is 9 (column 1) + 8 (column 4). That solution is $(-9, 0, 0, -8, 1)$.

The pivot columns are independent, and they span, so they are a basis for $C(A)$.

The dimension of the column space is r . The pivot columns form a basis.

3. The *nullspace* has dimension $n - r = 5 - 2$. There are $n - r = 3$ free variables. Here x_2, x_3, x_5 are free (no pivots in those columns). They yield the three special solutions to $Rx = \mathbf{0}$. Set a free variable to 1, and solve for x_1 and x_4 :

$$s_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad s_3 = \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad s_5 = \begin{bmatrix} -9 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix} \quad \begin{aligned} Rx = \mathbf{0} &\text{ has the} \\ &\text{complete solution} \\ &x = x_2 s_2 + x_3 s_3 + x_5 s_5 \end{aligned}$$

There is a special solution for each free variable. With n variables and r pivot variables, that leaves $n - r$ free variables and special solutions:

The nullspace has dimension $n - r$. The special solutions form a basis.

The special solutions are independent, because they contain the identity matrix in rows 2, 3, 5. All solutions are combinations of special solutions, $\mathbf{x} = x_2 s_2 + x_3 s_3 + x_5 s_5$, because this gets x_2 , x_3 and x_5 in the correct positions. Then the pivot variables x_1 and x_4 are totally determined by the equations $R\mathbf{x} = \mathbf{0}$.

4. The *nullspace* of R^T has dimension $m - r = 3 - 2$.

Reason: The equation $R^T \mathbf{y} = \mathbf{0}$ looks for combinations of the columns of R^T (*the rows of R*) that produce zero. This equation $R^T \mathbf{y} = \mathbf{0}$ or $\mathbf{y}^T R = \mathbf{0}^T$ is

$$\begin{array}{r} y_1 [1, 3, 5, 0, 9] \\ + y_2 [0, 0, 0, 1, 8] \\ + y_3 [0, 0, 0, 0, 0] \\ \hline [0, 0, 0, 0, 0] \end{array} \quad (1)$$

The solutions y_1, y_2, y_3 are pretty clear. We need $y_1 = 0$ and $y_2 = 0$. The variable y_3 is free (it can be anything). The nullspace of R^T contains all vectors $\mathbf{y} = (0, 0, y_3)$. It is the line of all multiples of the basis vector $(0, 0, 1)$.

In all cases R ends with $m - r$ zero rows. Every combination of these $m - r$ rows gives zero. These are the *only* combinations of the rows of R that give zero, because the pivot rows are linearly independent. So we can identify the left nullspace of R , which is the nullspace of R^T :

$$R^T \mathbf{y} = \mathbf{0} : \quad \begin{aligned} & \text{The left nullspace has dimension } m - r. \\ & \text{The solutions are } \mathbf{y} = (0, \dots, 0, y_{r+1}, \dots, y_m). \end{aligned}$$

To produce a zero combination, \mathbf{y} must start with r zeros. This leaves dimension $m - r$.

Why is this a “left nullspace”? The reason is that $R^T \mathbf{y} = \mathbf{0}$ can be transposed to $\mathbf{y}^T R = \mathbf{0}^T$. Now \mathbf{y}^T is a row vector to the *left* of R . You see the \mathbf{y} ’s in equation (1) multiplying the rows. This subspace came fourth, and some linear algebra books omit it—but that misses the beauty of the whole subject.

In \mathbb{R}^n the row space and nullspace have dimensions r and $n - r$ (adding to n).

In \mathbb{R}^m the column space and left nullspace have dimensions r and $m - r$ (total m).

So far this is proved for echelon matrices R . Figure 3.5 shows the same for A .

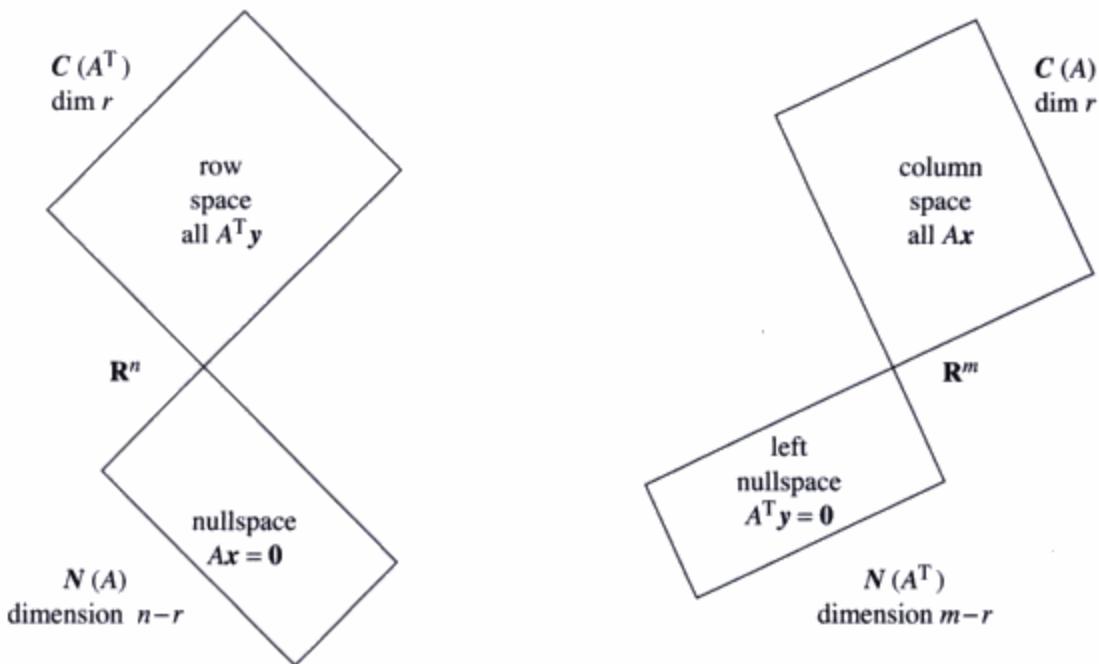


Figure 3.5 The dimensions of the four fundamental subspaces (for R and for A).

The Four Subspaces for A

We have a small job still to do. *The subspace dimensions for A are the same as for R .* The job is to explain why. Remember that those matrices are connected by an invertible matrix E (the product of all the elementary matrices that reduce A to R):

$$EA = R \quad \text{and} \quad A = E^{-1}R \quad (2)$$

1 *A has the same row space as R .* Same dimension r and same basis.

Reason: Every row of A is a combination of the rows of R . Also every row of R is a combination of the rows of A . In one direction the combinations are given by E^{-1} , in the other direction by E . Elimination changes the rows, but the row spaces are identical.

Since A has the same row space as R , we can choose the first r rows of R as a basis. Or we could choose r suitable rows of the original A . They might not always be the *first* r rows of A , because those could be dependent. The good r rows of A are the ones that end up as pivot rows in R .

2 *The column space of A has dimension r .* For every matrix this is essential: *The number of independent columns equals the number of independent rows.*

Wrong reason: “ A and R have the same column space.” This is false. The columns of R often end in zeros. The columns of A don’t often end in zeros. The column spaces are different, but their *dimensions* are the same—equal to r .

Right reason: The same combinations of the columns are zero, for A and R . Say that another way: $Ax = \mathbf{0}$ exactly when $Rx = \mathbf{0}$. So the r independent columns match.

Conclusion The r pivot columns of A are a basis for its column space.

3 A has the same nullspace as R. Same dimension $n - r$ and same basis.

Reason: The elimination steps don't change the solutions. The special solutions are a basis for this nullspace. There are $n - r$ free variables, so the dimension is $n - r$. Notice that $r + (n - r)$ equals n :

$$\text{(dimension of column space)} + \text{(dimension of nullspace)} = \text{dimension of } \mathbf{R}^n.$$

4 The left nullspace of A (the nullspace of } A^T \text{) has dimension } m - r.

Reason: A^T is just as good a matrix as A . When we know the dimensions for every A , we also know them for A^T . Its column space was proved to have dimension r . Since A^T is n by m , the "whole space" is now \mathbf{R}^m . The counting rule for A was $r + (n - r) = n$. The counting rule for A^T is $r + (m - r) = m$. So the nullspace of A^T has dimension $m - r$. We now have all details of the main theorem:

Fundamental Theorem of Linear Algebra, Part 1

The column space and row space both have dimension r .
The nullspaces have dimensions $n - r$ and $m - r$.

By concentrating on *spaces* of vectors, not on individual numbers or vectors, we get these clean rules. You will soon take them for granted—eventually they begin to look obvious. But if you write down an 11 by 17 matrix with 187 nonzero entries, we don't think most people would see why these facts are true:

$$\text{dimension of } C(A) = \text{dimension of } C(A^T)$$

$$\text{dimension of } C(A) + \text{dimension of } N(A) = 17.$$

Example 1 $A = [1 \ 2 \ 3]$ has $m = 1$ and $n = 3$ and rank $r = 1$.

The row space is a line in \mathbf{R}^3 . The nullspace is the plane $Ax = x_1 + 2x_2 + 3x_3 = 0$. This plane has dimension 2 (which is $3 - 1$). The dimensions add to $1 + 2 = 3$.

The columns of this 1 by 3 matrix are in \mathbf{R}^1 ! The column space is all of \mathbf{R}^1 . The left nullspace contains only the zero vector. The only solution to $A^T y = \mathbf{0}$ is $y = \mathbf{0}$, the only combination of the row that gives the zero row. Thus $N(A^T)$ is \mathbf{Z} , the zero space with dimension 0 (which is $m - r$). In \mathbf{R}^m the dimensions add to $1 + 0 = 1$.

Example 2 $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ has $m = 2$ with $n = 3$ and rank $r = 1$.

The row space is the same line through $(1, 2, 3)$. The nullspace is the same plane $x_1 + 2x_2 + 3x_3 = 0$. Their dimensions still add to $1 + 2 = 3$.

The columns are multiples of the first column $(1, 1)$. But there is more than the zero vector in the left nullspace. The first row minus the second row is the zero row. Therefore $A^T y = \mathbf{0}$ has the solution $y = (1, -1)$. The column space and left nullspace are perpendicular lines in \mathbf{R}^2 . Their dimensions are 1 and 1, adding to 2:

$$\text{column space} = \text{line through } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{left nullspace} = \text{line through } \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If A has three equal rows, its rank is _____. What are two of the y 's in its left nullspace? *The y's combine the rows to give the zero row.*

Matrices of Rank One

That last example had rank $r = 1$ —and rank one matrices are special. We can describe them all. You will see again that dimension of row space = dimension of column space. When $r = 1$, every row is a multiple of the same row:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{equals} \quad \begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix} \quad \text{times} \quad [1 \ 2 \ 3].$$

A column times a row (4 by 1 times 1 by 3) produces a matrix (4 by 3). All rows are multiples of the row $(1, 2, 3)$. All columns are multiples of the column $(1, 2, -3, 0)$. The row space is a line in \mathbf{R}^n , and the column space is a line in \mathbf{R}^m .

Every rank one matrix has the special form $A = u v^T = \text{column times row}$.

The columns are multiples of u . The rows are multiples of v^T . *The nullspace is the plane perpendicular to v .* ($Ax = \mathbf{0}$ means that $u(v^T x) = \mathbf{0}$ and then $v^T x = 0$.) It is this perpendicularity of the subspaces that will be Part 2 of the Fundamental Theorem.

■ REVIEW OF THE KEY IDEAS ■

1. The r pivot rows of R are a basis for the row spaces of R and A (same space).
2. The r pivot columns of A (!) are a basis for its column space.
3. The $n-r$ special solutions are a basis for the nullspaces of A and R (same space).
4. The last $m-r$ rows of I are a basis for the left nullspace of R .
5. The last $m-r$ rows of E are a basis for the left nullspace of A .

■ WORKED EXAMPLES ■

3.6 A Find bases and dimensions for all four fundamental subspaces if you know that

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E^{-1}R.$$

By changing only *one* of those numbers, change the dimensions of all four subspaces.

Solution This matrix has pivots in columns 1 and 3. Its rank is $r = 2$.

Row space: Basis $(1, 3, 0, 5)$ and $(0, 0, 1, 6)$ from R . Dimension 2.

Column space: Basis $(1, 2, 5)$ and $(0, 1, 0)$ from E^{-1} . Dimension 2.

Nullspace: Basis $(-3, 1, 0, 0)$ and $(-5, 0, -6, 1)$ from R . Dimension 2.

Nullspace of A^T : Basis $(-5, 0, 1)$ from row 3 of E . Dimension $3 - 2 = 1$.

We need to comment on that left nullspace $N(A^T)$. $EA = R$ says that the last row of E combines the three rows of A into the zero row of R . So that last row of E is a basis vector for the left nullspace. If R had *two* zero rows, then the last *two* rows of E would be a basis for the left nullspace (which combines rows of A to give zero rows).

To change all these dimensions we need to change the rank r . The way to do that is to change an entry (any entry) in the last row of R .

3.6 B Suppose you have to put four 1's into a 5 by 6 matrix (all other entries are zero). Describe all the ways to make the dimension of its *row space* as small as possible. Describe all the ways to make the dimension of its *column space* as small as possible. Describe all the ways to make the dimension of its *nullspace* as small as possible. What are those smallest dimensions? What to do if you want the *sum of the dimensions of all four subspaces* as small as possible?

Solution The rank is 1 if the four 1's go into the same row, or into the same column, or into *two rows and two columns* (so $a_{ii} = a_{ij} = a_{ji} = a_{jj} = 1$). Since the column space and row space always have the same dimensions, this answers the first two questions: Dimension 1.

The nullspace has its smallest possible dimension $6 - 4 = 2$ when the rank is $r = 4$. To achieve rank 4, the four 1's must go into four different rows and four different columns. You can't do anything about the sum $r + (n - r) + r + (m - r) = n + m$. It will be $6 + 5 = 11$ no matter how the 1's are placed. The sum is 11 even if there aren't any 1's...

If all the other entries of A are 2's instead of 0's, how do these answers change?

Problem Set 3.6

- 1 (a) If a 7 by 9 matrix has rank 5, what are the dimensions of the four subspaces? What is the sum of all four dimensions?
 (b) If a 3 by 4 matrix has rank 3, what are its column space and left nullspace?
- 2 Find bases for the four subspaces associated with A and B :

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}.$$

- 3 Find a basis for each of the four subspaces associated with

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- 4 Construct a matrix with the required property or explain why this is impossible:
- (a) Column space contains $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, row space contains $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.
 (b) Column space has basis $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, nullspace has basis $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.
 (c) Dimension of nullspace = 1 + dimension of left nullspace.
 (d) Left nullspace contains $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, row space contains $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
 (e) Row space = column space, nullspace \neq left nullspace.
- 5 If \mathbf{V} is the subspace spanned by $(1, 1, 1)$ and $(2, 1, 0)$, find a matrix A that has \mathbf{V} as its row space and a matrix B that has \mathbf{V} as its nullspace.
- 6 Without elimination, find dimensions and bases for the four subspaces for

$$A = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}.$$

- 7 Suppose the 3 by 3 matrix A is invertible. Write down bases for the four subspaces for A , and also for the 3 by 6 matrix $B = [A \ A]$.
- 8 What are the dimensions of the four subspaces for A , B , and C , if I is the 3 by 3 identity matrix and 0 is the 3 by 2 zero matrix?

$$A = [I \ 0] \quad \text{and} \quad B = \begin{bmatrix} I & I \\ 0^T & 0^T \end{bmatrix} \quad \text{and} \quad C = [0].$$

- 9 Which subspaces are the same for these matrices of different sizes?

$$(a) [A] \text{ and } \begin{bmatrix} A \\ A \end{bmatrix} \quad (b) \begin{bmatrix} A \\ A \end{bmatrix} \text{ and } \begin{bmatrix} A & A \\ A & A \end{bmatrix}.$$

Prove that all three matrices have the same rank r .

- 10** If the entries of a 3 by 3 matrix are chosen randomly between 0 and 1, what are the most likely dimensions of the four subspaces? What if the matrix is 3 by 5?
- 11** (Important) A is an m by n matrix of rank r . Suppose there are right sides \mathbf{b} for which $A\mathbf{x} = \mathbf{b}$ has *no solution*.
- (a) What are all inequalities ($<$ or \leq) that must be true between m , n , and r ?
 - (b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?
- 12** Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?
- 13** True or false:
- (a) If $m = n$ then the row space of A equals the column space.
 - (b) The matrices A and $-A$ share the same four subspaces.
 - (c) If A and B share the same four subspaces then A is a multiple of B .
- 14** Without computing A , find bases for the four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

- 15** If you exchange the first two rows of A , which of the four subspaces stay the same? If $\mathbf{v} = (1, 2, 3, 4)$ is in the column space of A , write down a vector in the column space of the new matrix.
- 16** Explain why $\mathbf{v} = (1, 0, -1)$ cannot be a row of A and also be in the nullspace.
- 17** Describe the four subspaces of \mathbf{R}^3 associated with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

- 18** (Left nullspace) Add the extra column \mathbf{b} and reduce A to echelon form:

$$[A \ \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of the rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right side.) Which vectors are in the nullspace of A^T and which are in the nullspace of A ?

- 19** Following the method of Problem 18, reduce A to echelon form and look at zero rows. The \mathbf{b} column tells which combinations you have taken of the rows:

$$(a) \begin{bmatrix} 1 & 2 & b_1 \\ 3 & 4 & b_2 \\ 4 & 6 & b_3 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 2 & b_1 \\ 2 & 3 & b_2 \\ 2 & 4 & b_3 \\ 2 & 5 & b_4 \end{bmatrix}$$

>From the \mathbf{b} column after elimination, read off $m - r$ basis vectors in the left nullspace of A (combinations of rows that give zero).

- 20** (a) Describe all solutions to $A\mathbf{x} = \mathbf{0}$ if

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- (b) How many independent solutions are there to $A^T\mathbf{y} = \mathbf{0}$?
 (c) Give a basis for the column space of A .
21 Suppose A is the sum of two matrices of rank one: $A = \mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{z}^T$.
 (a) Which vectors span the column space of A ?
 (b) Which vectors span the row space of A ?
 (c) The rank is less than 2 if ____ or if ____.
 (d) Compute A and its rank if $\mathbf{u} = \mathbf{z} = (1, 0, 0)$ and $\mathbf{v} = \mathbf{w} = (0, 0, 1)$.
22 Construct $A = \mathbf{u}\mathbf{v}^T + \mathbf{w}\mathbf{z}^T$ whose column space has basis $(1, 2, 4), (2, 2, 1)$ and whose row space has basis $(1, 0, 0), (0, 1, 1)$.
23 Without multiplying matrices, find bases for the row and column spaces of A :

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 0 & 3 \\ 1 & 1 & 2 \end{bmatrix}.$$

How do you know from these shapes that A is not invertible?

- 24** $A^T\mathbf{y} = \mathbf{d}$ is solvable when the right side \mathbf{d} is in which of the four subspaces? The solution is unique when the ____ contains only the zero vector.
25 True or false (with a reason or a counterexample):
 (a) A and A^T have the same number of pivots.
 (b) A and A^T have the same left nullspace.
 (c) If the row space equals the column space then $A^T = A$.
 (d) If $A^T = -A$ then the row space of A equals the column space.

- 26** (*Rank of AB*) If $AB = C$, the rows of C are combinations of the rows of _____. So the rank of C is not greater than the rank of _____. Since $B^T A^T = C^T$, the rank of C is also not greater than the rank of _____.
- 27** If a, b, c are given with $a \neq 0$, how would you choose d so that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has rank one? Find a basis for the row space and nullspace.
- 28** Find the ranks of the 8 by 8 checkerboard matrix B and chess matrix C :

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \cdot & \cdot \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} r & n & b & q & k & b & n & r \\ p & p & p & p & p & p & p & p \\ & & & & & & \text{four zero rows} & \\ p & p & p & p & p & p & p & p \\ r & n & b & q & k & b & n & r \end{bmatrix}$$

The numbers r, n, b, q, k, p are all different. Find bases for the row space and left nullspace of B and C . Challenge problem: Find a basis for the nullspace of C .

- 29** Can tic-tac-toe be completed (5 ones and 4 zeros in A) so that $\text{rank}(A) = 2$ but neither side passed up a winning move?

4

ORTHOGONALITY

ORTHOGONALITY OF THE FOUR SUBSPACES ■ 4.1

Two vectors are orthogonal when their dot product is zero: $\mathbf{v} \cdot \mathbf{w} = 0$ or $\mathbf{v}^T \mathbf{w} = 0$. This chapter moves up a level, from orthogonal vectors to *orthogonal subspaces*. Orthogonal means the same as perpendicular.

Subspaces entered Chapter 3 with a specific purpose—to throw light on $A\mathbf{x} = \mathbf{b}$. Right away we needed the column space (for \mathbf{b}) and the nullspace (for \mathbf{x}). Then the light turned onto A^T , uncovering two more subspaces. Those four fundamental subspaces reveal what a matrix really does.

A matrix multiplies a vector: A times \mathbf{x} . At the first level this is only numbers. At the second level $A\mathbf{x}$ is a combination of column vectors. The third level shows subspaces. But I don't think you have seen the whole picture until you study Figure 4.1. It fits the subspaces together, to show the hidden reality of A times \mathbf{x} . The 90° angles between subspaces are new—and we have to say what they mean.

The row space is perpendicular to the nullspace. Every row of A is perpendicular to every solution of $A\mathbf{x} = \mathbf{0}$. That gives the 90° angle on the left side of the figure. This perpendicularity of subspaces is Part 2 of the Fundamental Theorem of Linear Algebra.

May we add a word about the left nullspace? It is never reached by $A\mathbf{x}$, so it might seem useless. But when \mathbf{b} is outside the column space—when we want to solve $A\mathbf{x} = \mathbf{b}$ and can't do it—then this nullspace of A^T comes into its own. It contains the error in the “least-squares” solution. That is the key application of linear algebra in this chapter.

Part 1 of the Fundamental Theorem gave the dimensions of the subspaces. The row and column spaces have the same dimension r (they are drawn the same size). The two nullspaces have the remaining dimensions $n - r$ and $m - r$. Now we will show that *the row space and nullspace are orthogonal subspaces inside \mathbf{R}^n* .

DEFINITION Two subspaces V and W of a vector space are *orthogonal* if every vector \mathbf{v} in V is perpendicular to every vector \mathbf{w} in W :

$$\mathbf{v} \cdot \mathbf{w} = 0 \text{ or } \mathbf{v}^T \mathbf{w} = 0 \text{ for all } \mathbf{v} \text{ in } V \text{ and all } \mathbf{w} \text{ in } W.$$

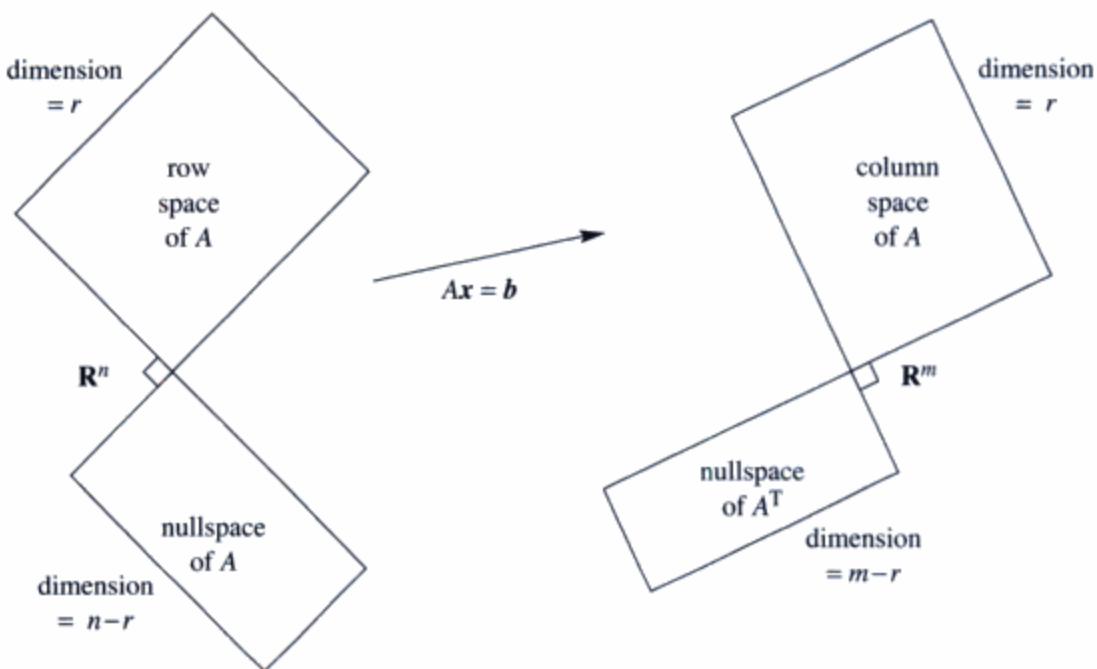


Figure 4.1 Two pairs of orthogonal subspaces. Dimensions add to n and add to m .

Example 1 The floor of your room (extended to infinity) is a subspace V . The line where two walls meet is a subspace W (one-dimensional). Those subspaces are orthogonal. Every vector up the meeting line is perpendicular to every vector in the floor. The origin $(0, 0, 0)$ is in the corner. We assume you don't live in a tent.

Example 2 Suppose V is still the floor but W is a wall (a two-dimensional space). The wall and floor look like orthogonal subspaces but they are not! You can find vectors in V and W that are not perpendicular. In fact a vector running along the bottom of the wall is also in the floor. This vector is in both V and W —and it is not perpendicular to itself.

When a vector is in two orthogonal subspaces, it *must* be zero. It is perpendicular to itself. It is v and it is w , so $v^T v = 0$. This has to be the zero vector.

The crucial examples for linear algebra come from the fundamental subspaces. Zero is the only point where the nullspace meets the row space. The spaces meet at 90° .

4A Every vector x in the nullspace of A is perpendicular to every row of A , because $Ax = \mathbf{0}$. *The nullspace and row space are orthogonal subspaces.*

To see why \mathbf{x} is perpendicular to the rows, look at $A\mathbf{x} = \mathbf{0}$. Each row multiplies \mathbf{x} :

$$A\mathbf{x} = \begin{bmatrix} & \text{row 1} \\ & \vdots \\ & \text{row } m \end{bmatrix} \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (1)$$

The first equation says that row 1 is perpendicular to \mathbf{x} . The last equation says that row m is perpendicular to \mathbf{x} . Every row has a zero dot product with \mathbf{x} . Then \mathbf{x} is perpendicular to every combination of the rows. The whole row space $C(A^T)$ is orthogonal to the whole nullspace $N(A)$.

Here is a second proof of that orthogonality for readers who like matrix shorthand. The vectors in the row space are combinations $A^T\mathbf{y}$ of the rows. Take the dot product of $A^T\mathbf{y}$ with any \mathbf{x} in the nullspace. These vectors are perpendicular:

$$\mathbf{x}^T(A^T\mathbf{y}) = (A\mathbf{x})^T\mathbf{y} = \mathbf{0}^T\mathbf{y} = 0. \quad (2)$$

We like the first proof. You can see those rows of A multiplying \mathbf{x} to produce zeros in equation (1). The second proof shows why A and A^T are both in the Fundamental Theorem. A^T goes with \mathbf{y} and A goes with \mathbf{x} . At the end we used $A\mathbf{x} = \mathbf{0}$.

Example 3 The rows of A are perpendicular to $\mathbf{x} = (1, 1, -1)$ in the nullspace:

$$A\mathbf{x} = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives the dot products} \quad \begin{array}{l} 1 + 3 - 4 = 0 \\ 5 + 2 - 7 = 0 \end{array}$$

Now we turn to the other two subspaces. In this example, the column space is all of \mathbb{R}^2 . The nullspace of A^T is only the zero vector. Those two subspaces are also orthogonal.

4B Every vector \mathbf{y} in the nullspace of A^T is perpendicular to every column of A . *The left nullspace and the column space are orthogonal in \mathbb{R}^m .*

Apply the original proof to A^T . Its nullspace is orthogonal to its row space—and the row space of A^T is the column space of A . Q.E.D.

For a visual proof, look at $A^T\mathbf{y} = \mathbf{0}$. Each column of A multiplies \mathbf{y} to give 0:

$$A^T\mathbf{y} = \begin{bmatrix} (\text{column 1})^T \\ \cdots \\ (\text{column } n)^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 0 \end{bmatrix}. \quad (3)$$

The dot product of \mathbf{y} with every column of A is zero. Then \mathbf{y} in the left nullspace is perpendicular to each column—and to the whole column space.

Orthogonal Complements

Very Important The fundamental subspaces are more than just orthogonal (in pairs). Their dimensions are also right. Two lines could be perpendicular in \mathbf{R}^3 , but they *could not be* the row space and nullspace of a 3 by 3 matrix. The lines have dimensions 1 and 1, adding to 2. The correct dimensions r and $n - r$ must add to $n = 3$. The fundamental subspaces have dimensions 2 and 1, or 3 and 0. The subspaces are not only orthogonal, they are *orthogonal complements*.

DEFINITION The *orthogonal complement* of V contains every vector that is perpendicular to V . This orthogonal subspace is denoted by V^\perp (pronounced “ V perp”).

By this definition, the nullspace is the orthogonal complement of the row space. Every x that is perpendicular to the rows satisfies $Ax = \mathbf{0}$.

The reverse is also true (automatically). If v is orthogonal to the nullspace, it must be in the row space. Otherwise we could add this v as an extra row of the matrix, without changing its nullspace. The row space would grow, which breaks the law $r + (n - r) = n$. We conclude that $N(A)^\perp$ is exactly the row space $C(A^T)$.

The left nullspace and column space are not only orthogonal in \mathbf{R}^m , they are also orthogonal complements. Their dimensions add to the full dimension m .

Fundamental Theorem of Linear Algebra, Part 2

The nullspace is the orthogonal complement of the row space (in \mathbf{R}^n).

The left nullspace is the orthogonal complement of the column space (in \mathbf{R}^m).

Part 1 gave the dimensions of the subspaces. Part 2 gives the 90° angles between them. The point of “complements” is that every x can be split into a *row space component* x_r and a *nullspace component* x_n . When A multiplies $x = x_r + x_n$, Figure 4.2 shows what happens:

The nullspace component goes to zero: $Ax_n = \mathbf{0}$.

The row space component goes to the column space: $Ax_r = Ax$.

Every vector goes to the column space! Multiplying by A cannot do anything else. But more than that: *Every vector in the column space comes from one and only one vector x_r in the row space.* Proof: If $Ax_r = Ax'_r$, the difference $x_r - x'_r$ is in the nullspace. It is also in the row space, where x_r and x'_r came from. This difference must be the zero vector, because the spaces are perpendicular. Therefore $x_r = x'_r$.

There is an r by r invertible matrix hiding inside A , if we throw away the two nullspaces. From the row space to the column space, A is invertible. The “pseudoinverse” will invert it in Section 7.4.

Example 4 Every diagonal matrix has an r by r invertible submatrix:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{contains} \quad \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}.$$

The rank is $r = 2$. The other eleven zeros are responsible for the nullspaces.

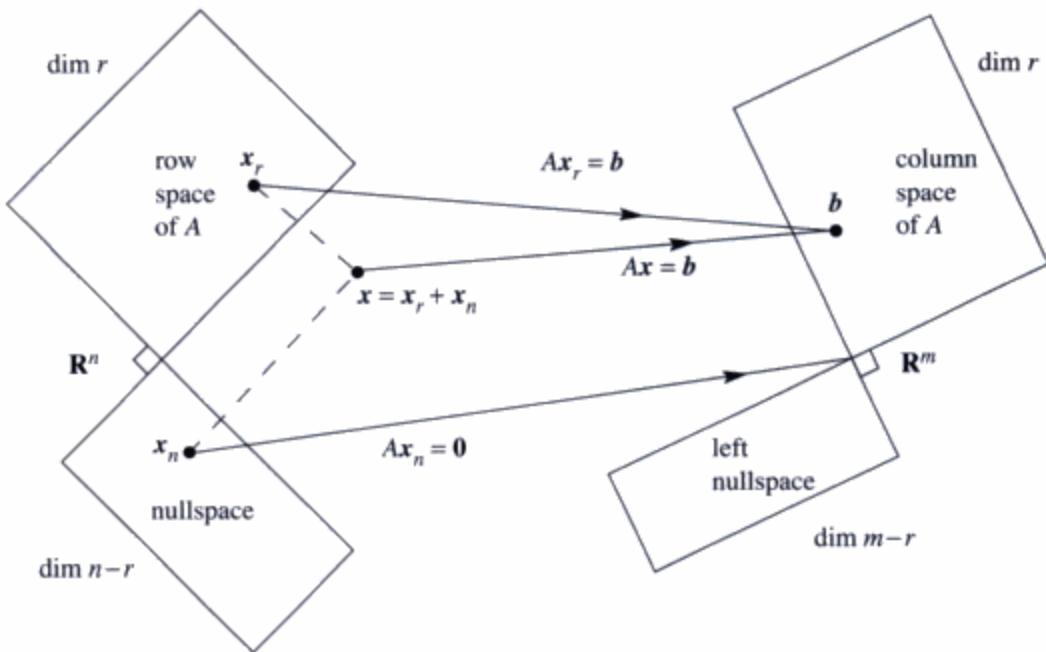


Figure 4.2 The true action of A times $x = x_r + x_n$. Row space vector x_r to column space, nullspace vector x_n to zero.

Section 7.4 will show how every A becomes a diagonal matrix, when we choose the right bases for \mathbf{R}^n and \mathbf{R}^m . This **Singular Value Decomposition** is a part of the theory that has become extremely important in applications.

Combining Bases from Subspaces

What follows are some valuable facts about bases. They were saved until now—when we are ready to use them. After a week you have a clearer sense of what a basis is (*independent* vectors that *span* the space). Normally we have to check both of these properties. When the count is right, one property implies the other:

4C Any n linearly independent vectors in \mathbf{R}^n must span \mathbf{R}^n . They are a basis. Any n vectors that span \mathbf{R}^n must be independent. They are a basis.

Starting with the correct number of vectors, one property of a basis produces the other. This is true in any vector space, but we care most about \mathbf{R}^n . When the vectors go into the columns of an n by n *square* matrix A , here are the same two facts:

4D If the n columns of A are independent, they span \mathbf{R}^n . So $Ax = b$ is solvable. If the n columns span \mathbf{R}^n , they are independent. So $Ax = b$ has only one solution.

Uniqueness implies existence and existence implies uniqueness. *Then A is invertible.*

If there are no free variables (uniqueness), there must be n pivots. Then back substitution solves $Ax = b$ (existence). Starting in the opposite direction, suppose $Ax = b$ can always be solved (existence of solutions). Then elimination produced no zero rows. There are n pivots and no free variables. The nullspace contains only $x = \mathbf{0}$ (uniqueness of solutions).

With a basis for the row space and a basis for the nullspace, we have $r + (n - r) = n$ vectors—the right number. Those n vectors are independent.² *Therefore they span \mathbf{R}^n .* They are a basis:

Each x in \mathbf{R}^n is the sum $x_r + x_n$ of a row space vector x_r and a nullspace vector x_n .

This confirms the splitting in Figure 4.2. It is the key point of orthogonal complements—the dimensions add to n and all vectors are fully accounted for.

Example 5 For $A = [I \ I] = [\begin{smallmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{smallmatrix}]$, write any vector x as $x_r + x_n$.

$(1, 0, 1, 0)$ and $(0, 1, 0, 1)$ are a basis for the row space. $(1, 0, -1, 0)$ and $(0, 1, 0, -1)$ are a basis for the nullspace of A . Those four vectors are a basis for \mathbf{R}^4 . Any $x = (a, b, c, d)$ can be split into x_r in the row space and x_n in the nullspace:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \frac{a+c}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{b+d}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{a-c}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \frac{b-d}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

■ REVIEW OF THE KEY IDEAS ■

1. Subspaces V and W are orthogonal if every v in V is orthogonal to every w in W .
2. V and W are “orthogonal complements” if W contains all vectors perpendicular to V (and vice versa). Inside \mathbf{R}^n , the dimensions of V and W add to n .
3. The nullspace $N(A)$ and the row space $C(A^T)$ are orthogonal complements, from $Ax = \mathbf{0}$. Similarly $N(A^T)$ and $C(A)$ are orthogonal complements.

²If a combination of the vectors gives $x_r + x_n = \mathbf{0}$, then $x_r = -x_n$ is in both subspaces. It is orthogonal to itself and must be zero. All coefficients of the row space basis and nullspace basis must be zero—which proves independence of the n vectors together.

4. Any n independent vectors in \mathbf{R}^n will span \mathbf{R}^n .
5. Every x in \mathbf{R}^n has a nullspace component x_n and a row space component x_r .

■ WORKED EXAMPLES ■

4.1 A Suppose S is a six-dimensional subspace of \mathbf{R}^9 . What are the possible dimensions of subspaces orthogonal to S ? What are the possible dimensions of the orthogonal complement S^\perp of S ? What is the smallest possible size of a matrix A that has row space S ? What is the shape of its nullspace matrix N ? How could you create a matrix B with extra rows but the same row space? Compare the nullspace matrix for B with the nullspace matrix for A .

Solution If S is six-dimensional in \mathbf{R}^9 , subspaces orthogonal to S can have dimensions 0, 1, 2, 3. The orthogonal complement S^\perp will be the largest orthogonal subspace, with dimension 3. The smallest matrix A must have 9 columns and 6 rows (its rows are a basis for the 6-dimensional row space S). Its nullspace matrix N will be 9 by 3, since its columns contain a basis for S^\perp .

If row 7 of B is a combination of the six rows of A , then B has the same row space as A . It also has the same nullspace matrix N . (The special solutions s_1, s_2, s_3 will be the same. Elimination will change row 7 of B to all zeros.)

4.1 B The equation $x - 4y - 5z = 0$ describes a plane P in \mathbf{R}^3 (actually a subspace). The plane P is the nullspace $N(A)$ of what 1 by 3 matrix A ? Find a basis s_1, s_2 of special solutions of $x - 3y - 4z = 0$ (these would be the columns of the nullspace matrix N). Also find a basis for the line P^\perp that is perpendicular to P . Then split $v = (6, 4, 5)$ into its nullspace component v_n in P and its row space component v_r in P^\perp .

Solution The equation $x - 3y - 4z = 0$ is $Ax = \mathbf{0}$ for the 1 by 3 matrix $A = [1 \ -3 \ -4]$. Columns 2 and 3 are free (no pivots). The special solutions with free variables “1 and 0” are $s_1 = (3, 1, 0)$ and $s_2 = (4, 0, 1)$. The row space of A (which is the line P^\perp) certainly has basis $z = (1, -3, -4)$. This is perpendicular to s_1 and s_2 and their plane P .

To split v into $v_n + v_r = (c_1 s_1 + c_2 s_2) + c_3 z$, solve for the numbers c_1, c_2, c_3 :

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} \quad \begin{array}{l} \text{leads to } c_1 = 1, c_2 = 1, c_3 = -1 \\ v_n = s_1 + s_2 = (7, 1, 1) \text{ is in } P = N(A) \\ v_r = -z = (-1, 3, 4) \text{ is in } P^\perp = C(A^T). \end{array}$$

Problem Set 4.1

Questions 1–12 grow out of Figures 4.1 and 4.2.

- 1 Construct any 2 by 3 matrix of rank one. Copy Figure 4.1 and put one vector in each subspace (two in the nullspace). Which vectors are orthogonal?
- 2 Redraw Figure 4.2 for a 3 by 2 matrix of rank $r = 2$. Which subspace is \mathbf{Z} (zero vector only)? The nullspace part of any vector x in \mathbb{R}^2 is $x_n = \underline{\hspace{2cm}}$.
- 3 Construct a matrix with the required property or say why that is impossible:
 - (a) Column space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
 - (b) Row space contains $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$, nullspace contains $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
 - (c) $Ax = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ has a solution and $A^T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
 - (d) Every row is orthogonal to every column (A is not the zero matrix)
 - (e) Columns add up to a column of zeros, rows add to a row of 1's.
- 4 If $AB = 0$ then the columns of B are in the $\underline{\hspace{2cm}}$ of A . The rows of A are in the $\underline{\hspace{2cm}}$ of B . Why can't A and B be 3 by 3 matrices of rank 2?
- 5 (a) If $Ax = b$ has a solution and $A^T y = \mathbf{0}$, then y is perpendicular to $\underline{\hspace{2cm}}$.
 (b) If $A^T y = c$ has a solution and $Ax = \mathbf{0}$, then x is perpendicular to $\underline{\hspace{2cm}}$.
- 6 This is a system of equations $Ax = b$ with *no solution*:

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply the equations so they add to $0 = 1$. You have found a vector y in which subspace? Its dot product $y^T b$ is 1.

- 7 Every system with no solution is like the one in Problem 6. There are numbers y_1, \dots, y_m that multiply the m equations so they add up to $0 = 1$. This is called

Fredholm's Alternative: Exactly one of these problems has a solution

$$Ax = b \quad \text{OR} \quad A^T y = \mathbf{0} \quad \text{with} \quad y^T b = 1.$$

If b is not in the column space of A , it is not orthogonal to the nullspace of A^T . Multiply the equations $x_1 - x_2 = 1$ and $x_2 - x_3 = 1$ and $x_1 - x_3 = 1$ by numbers y_1, y_2, y_3 chosen so that the equations add up to $0 = 1$.

- 8 In Figure 4.2, how do we know that Ax_r is equal to Ax ? How do we know that this vector is in the column space? If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ what is x_r ?
- 9 If Ax is in the nullspace of A^T then $Ax = \mathbf{0}$. Reason: Ax is also in the _____ of A and the spaces are _____. Conclusion: A^TA has the same nullspace as A .
- 10 Suppose A is a symmetric matrix ($A^T = A$).
- Why is its column space perpendicular to its nullspace?
 - If $Ax = \mathbf{0}$ and $Az = 5z$, which subspaces contain these “eigenvectors” x and z ? Symmetric matrices have perpendicular eigenvectors.
- 11 (Recommended) Draw Figure 4.2 to show each subspace correctly for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

- 12 Find the pieces x_r and x_n and draw Figure 4.2 properly if

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Questions 13–23 are about orthogonal subspaces.

- 13 Put bases for the subspaces V and W into the columns of matrices V and W . Explain why the test for orthogonal subspaces can be written $V^TW = \text{zero matrix}$. This matches $v^Tw = 0$ for orthogonal vectors.
- 14 The floor V and the wall W are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet). No planes V and W in \mathbb{R}^3 can be orthogonal! Find a vector in the column spaces of both matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{bmatrix}$$

This will be a vector Ax and also $B\hat{x}$. Think 3 by 4 with the matrix $[A \ B]$.

- 15 Extend problem 14 to a p -dimensional subspace V and a q -dimensional subspace W of \mathbb{R}^n . What inequality on $p+q$ guarantees that V intersects W in a nonzero vector? These subspaces cannot be orthogonal.
- 16 Prove that every y in $N(A^T)$ is perpendicular to every Ax in the column space, using the matrix shorthand of equation (2). Start from $A^Ty = \mathbf{0}$.
- 17 If S is the subspace of \mathbb{R}^3 containing only the zero vector, what is S^\perp ? If S is spanned by $(1, 1, 1)$, what is S^\perp ? If S is spanned by $(2, 0, 0)$ and $(0, 0, 3)$, what is S^\perp ?

- 18** Suppose S only contains two vectors $(1, 5, 1)$ and $(2, 2, 2)$ (not a subspace). Then S^\perp is the nullspace of the matrix $A = \underline{\hspace{2cm}}$. S^\perp is a subspace even if S is not.
- 19** Suppose L is a one-dimensional subspace (a line) in \mathbf{R}^3 . Its orthogonal complement L^\perp is the $\underline{\hspace{2cm}}$ perpendicular to L . Then $(L^\perp)^\perp$ is a $\underline{\hspace{2cm}}$ perpendicular to L^\perp . In fact $(L^\perp)^\perp$ is the same as $\underline{\hspace{2cm}}$.
- 20** Suppose V is the whole space \mathbf{R}^4 . Then V^\perp contains only the vector $\underline{\hspace{2cm}}$. Then $(V^\perp)^\perp$ is $\underline{\hspace{2cm}}$. So $(V^\perp)^\perp$ is the same as $\underline{\hspace{2cm}}$.
- 21** Suppose S is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span S^\perp . This is the same as solving $Ax = \mathbf{0}$ for which A ?
- 22** If P is the plane of vectors in \mathbf{R}^4 satisfying $x_1 + x_2 + x_3 + x_4 = 0$, write a basis for P^\perp . Construct a matrix that has P as its nullspace.
- 23** If a subspace S is contained in a subspace V , prove that S^\perp contains V^\perp .

Questions 24–30 are about perpendicular columns and rows.

- 24** Suppose an n by n matrix is invertible: $AA^{-1} = I$. Then the first column of A^{-1} is orthogonal to the space spanned by which rows of A ?
- 25** Find $A^T A$ if the columns of A are unit vectors, all mutually perpendicular.
- 26** Construct a 3 by 3 matrix A with no zero entries whose columns are mutually perpendicular. Compute $A^T A$. Why is it a diagonal matrix?
- 27** The lines $3x + y = b_1$ and $6x + 2y = b_2$ are $\underline{\hspace{2cm}}$. They are the same line if $\underline{\hspace{2cm}}$. In that case (b_1, b_2) is perpendicular to the vector $\underline{\hspace{2cm}}$. The nullspace of the matrix is the line $3x + y = \underline{\hspace{2cm}}$. One particular vector in that nullspace is $\underline{\hspace{2cm}}$.
- 28** Why is each of these statements false?
- $(1, 1, 1)$ is perpendicular to $(1, 1, -2)$ so the planes $x + y + z = 0$ and $x + y - 2z = 0$ are orthogonal subspaces.
 - The subspace spanned by $(1, 1, 0, 0, 0)$ and $(0, 0, 0, 1, 1)$ is the orthogonal complement of the subspace spanned by $(1, -1, 0, 0, 0)$ and $(2, -2, 3, 4, -4)$.
 - If two subspaces meet only in the zero vector, the subspaces are orthogonal.
- 29** Find a matrix with $v = (1, 2, 3)$ in the row space and column space. Find another matrix with v in the nullspace and column space. Which pairs of subspaces can v *not* be in?
- 30** Suppose A is 3 by 4 and B is 4 by 5 and $AB = 0$. Prove $\text{rank}(A) + \text{rank}(B) \leq 4$.
- 31** The command $N = \text{null}(A)$ will produce a basis for the nullspace of A . Then the command $B = \text{null}(N')$ will produce a basis for the $\underline{\hspace{2cm}}$ of A .

PROJECTIONS ■ 4.2

May we start this section with two questions? (In addition to that one.) The first question aims to show that projections are easy to visualize. The second question is about “projection matrices”:

- 1 What are the projections of $\mathbf{b} = (2, 3, 4)$ onto the z axis and the xy plane?
- 2 What matrices produce those projections onto a line and a plane?

When \mathbf{b} is projected onto a line, *its projection p is the part of \mathbf{b} along that line*. If \mathbf{b} is projected onto a plane, p is the part in that plane. *The projection p is $P\mathbf{b}$.*

There is a projection matrix P that multiplies \mathbf{b} to give p . This section finds p and P .

The projection onto the z axis we call \mathbf{p}_1 . The second projection drops straight down to the xy plane. The picture in your mind should be Figure 4.3. Start with $\mathbf{b} = (2, 3, 4)$. One projection gives $\mathbf{p}_1 = (0, 0, 4)$ and the other gives $\mathbf{p}_2 = (2, 3, 0)$. Those are the parts of \mathbf{b} along the z axis and in the xy plane.

The projection matrices P_1 and P_2 are 3 by 3. They multiply \mathbf{b} with 3 components to produce p with 3 components. Projection onto a line comes from a rank one matrix. Projection onto a plane comes from a rank two matrix:

$$\text{Onto the } z \text{ axis: } P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Onto the } xy \text{ plane: } P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

P_1 picks out the z component of every vector. P_2 picks out the x and y components. To find \mathbf{p}_1 and \mathbf{p}_2 , multiply \mathbf{b} by P_1 and P_2 (small p for the vector, capital P for the matrix that produces it):

$$\mathbf{p}_1 = P_1 \mathbf{b} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \quad \mathbf{p}_2 = P_2 \mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}.$$

In this case the projections P_1 and P_2 are perpendicular. The xy plane and the z axis are *orthogonal subspaces*, like the floor of a room and the line between two walls.

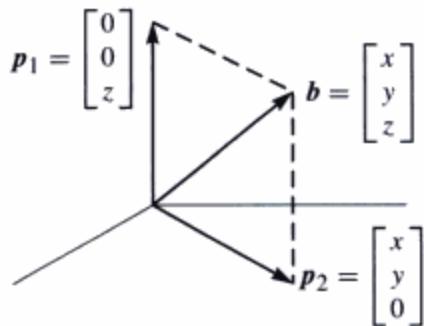


Figure 4.3 The projections of \mathbf{b} onto the z axis and the xy plane.

More than that, the line and plane are orthogonal **complements**. Their dimensions add to $1 + 2 = 3$ —every vector \mathbf{b} in the whole space is the sum of its parts in the two subspaces. The projections \mathbf{p}_1 and \mathbf{p}_2 are exactly those parts:

$$\text{The vectors give } \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{b}. \quad \text{The matrices give } P_1 + P_2 = I. \quad (1)$$

This is perfect. Our goal is reached—for this example. We have the same goal for any line and any plane and any n -dimensional subspace. The object is to find the part \mathbf{p} in each subspace, and the projection matrix P that produces that part $\mathbf{p} = Pb$. Every subspace of \mathbf{R}^m has its own m by m projection matrix. To compute P , we absolutely need a good description of the subspace that it projects onto.

The best description of a subspace is a basis. We put the basis vectors into the columns of A . **Now we are projecting onto the column space of A !** Certainly the z axis is the column space of the 3 by 1 matrix A_1 . The xy plane is the column space of A_2 . That plane is also the column space of A_3 (a subspace has many bases):

$$A_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}.$$

Our problem is **to project onto the column space of any m by n matrix**. Start with a line (dimension $n = 1$). The matrix A has only one column. Call it \mathbf{a} .

Projection Onto a Line

We are given a line through the origin, in the direction of $\mathbf{a} = (a_1, \dots, a_m)$. Along that line, we want the point \mathbf{p} closest to $\mathbf{b} = (b_1, \dots, b_m)$. The key to projection is orthogonality: **The line from \mathbf{b} to \mathbf{p} is perpendicular to the vector \mathbf{a}** . This is the dotted line marked e in Figure 4.4—which we now compute by algebra.

The projection \mathbf{p} is some multiple of \mathbf{a} . Call it $\mathbf{p} = \hat{x}\mathbf{a}$ = “ x hat” times \mathbf{a} . Our first step is to compute this unknown number \hat{x} . That will give the vector \mathbf{p} . Then from the formula for \mathbf{p} , we read off the projection matrix P . These three steps will lead to all projection matrices: *find \hat{x} , then find the vector \mathbf{p} , then find the matrix P* .

The dotted line $\mathbf{b} - \mathbf{p}$ is $\mathbf{b} - \hat{x}\mathbf{a}$. It is perpendicular to \mathbf{a} —this will determine \hat{x} . Use the fact that two vectors are perpendicular when their dot product is zero:

$$\mathbf{a} \cdot (\mathbf{b} - \hat{x}\mathbf{a}) = 0 \quad \text{or} \quad \mathbf{a} \cdot \mathbf{b} - \hat{x}\mathbf{a} \cdot \mathbf{a} = 0 \quad \text{or} \quad \hat{x} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}}. \quad (2)$$

For vectors the multiplication $\mathbf{a}^T \mathbf{b}$ is the same as $\mathbf{a} \cdot \mathbf{b}$. Using the transpose is better, because it applies also to matrices. (We will soon meet $A^T \mathbf{b}$.) Our formula for \hat{x} immediately gives the formula for \mathbf{p} :

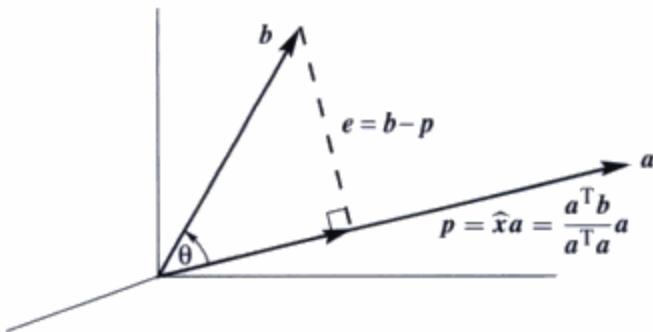


Figure 4.4 The projection p , perpendicular to e , has length $\|b\| \cos \theta$.

4E The projection of b onto the line through a is the vector $p = \hat{x}a = \frac{a^T b}{a^T a}a$.

Special case 1: If $b = a$ then $\hat{x} = 1$. The projection of a onto a is itself.

Special case 2: If b is perpendicular to a then $a^T b = 0$. The projection is $p = 0$.

Example 1 Project $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ onto $a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ to find $p = \hat{x}a$ in Figure 4.4.

Solution The number \hat{x} is the ratio of $a^T b = 5$ to $a^T a = 9$. So the projection is $p = \frac{5}{9}a$. The error vector between b and p is $e = b - p$. Those vectors p and e will add to b :

$$p = \frac{5}{9}a = \left(\frac{5}{9}, \frac{10}{9} \right) \text{ and } e = b - p = \left(\frac{4}{9}, -\frac{1}{9}, -\frac{1}{9} \right).$$

The error e should be perpendicular to $a = (1, 2, 2)$ and it is: $e^T a = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$.

Look at the right triangle of b , p , and e . The vector b is split into two parts—its component along the line is p , its perpendicular part is e . Those two sides of a right triangle have length $\|b\| \cos \theta$ and $\|b\| \sin \theta$. Trigonometry matches the dot product:

$$p = \frac{a^T b}{a^T a}a \text{ so its length is } \|p\| = \frac{\|a\| \|b\| \cos \theta}{\|a\|^2} \|a\| = \|b\| \cos \theta. \quad (3)$$

The dot product is a lot simpler than getting involved with $\cos \theta$ and the length of b . The example has square roots in $\cos \theta = 5/3\sqrt{3}$ and $\|b\| = \sqrt{3}$. There are no square roots in the projection $p = \frac{5}{9}a$.

Now comes the **projection matrix**. In the formula for p , what matrix is multiplying b ? You can see it better if the number \hat{x} is on the right side of a :

$$p = a\hat{x} = a \frac{a^T b}{a^T a} = Pb \quad \text{when the matrix is } P = \frac{aa^T}{a^T a}.$$

P is a column times a row! The column is \mathbf{a} , the row is \mathbf{a}^T . Then divide by the number $\mathbf{a}^T \mathbf{a}$. The projection matrix P is m by m , but *its rank is one*. We are projecting onto a one-dimensional subspace, the line through \mathbf{a} .

Example 2 Find the projection matrix $P = \frac{\mathbf{aa}^T}{\mathbf{a}^T \mathbf{a}}$ onto the line through $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

Solution Multiply column times row and divide by $\mathbf{a}^T \mathbf{a} = 9$:

$$P = \frac{\mathbf{aa}^T}{\mathbf{a}^T \mathbf{a}} = \frac{1}{9} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix}.$$

This matrix projects *any* vector \mathbf{b} onto \mathbf{a} . Check $\mathbf{p} = P\mathbf{b}$ for the particular $\mathbf{b} = (1, 1, 1)$ in Example 1:

$$\mathbf{p} = P\mathbf{b} = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} \quad \text{which is correct.}$$

If the vector \mathbf{a} is doubled, the matrix P stays the same. It still projects onto the same line. If the matrix is squared, P^2 equals P . *Projecting a second time doesn't change anything*, so $P^2 = P$. The diagonal entries of P add up to $\frac{1}{9}(1 + 4 + 4) = 1$.

The matrix $I - P$ should be a projection too. It produces the other side \mathbf{e} of the triangle—the perpendicular part of \mathbf{b} . Note that $(I - P)\mathbf{b}$ equals $\mathbf{b} - \mathbf{p}$ which is \mathbf{e} . *When P projects onto one subspace, $I - P$ projects onto the perpendicular subspace*. Here $I - P$ projects onto the plane perpendicular to \mathbf{a} .

Now we move beyond projection onto a line. Projecting onto an n -dimensional subspace of \mathbf{R}^m takes more effort. The crucial formulas will be collected in equations (5)–(6)–(7). Basically you need to remember them.

Projection Onto a Subspace

Start with n vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$ in \mathbf{R}^m . Assume that these \mathbf{a} 's are linearly independent. **Problem:** *Find the combination $\hat{x}_1 \mathbf{a}_1 + \dots + \hat{x}_n \mathbf{a}_n$ that is closest to a given vector \mathbf{b}* . We are projecting each \mathbf{b} in \mathbf{R}^m onto the subspace spanned by the \mathbf{a} 's.

With $n = 1$ (only one vector \mathbf{a}_1) this is projection onto a line. The line is the column space of A , which has just one column. In general the matrix A has n columns $\mathbf{a}_1, \dots, \mathbf{a}_n$. Their combinations in \mathbf{R}^m are the vectors Ax in the column space. We are looking for the particular combination $\mathbf{p} = A\hat{x}$ (*the projection*) that is closest to \mathbf{b} . The hat over \hat{x} indicates the *best* choice, to give the closest vector in the column space. That choice is $\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}$ when $n = 1$. For $n > 1$, the best \hat{x} is to be found.

We solve this problem for an n -dimensional subspace in three steps: *Find the vector \hat{x} , find the projection $\mathbf{p} = A\hat{x}$, find the matrix P* .

The key is in the geometry! The dotted line in Figure 4.5 goes from \mathbf{b} to the nearest point $A\hat{\mathbf{x}}$ in the subspace. **This error vector $\mathbf{b} - A\hat{\mathbf{x}}$ is perpendicular to the subspace.** The error $\mathbf{b} - A\hat{\mathbf{x}}$ makes a right angle with all the vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$. That gives the n equations we need to find $\hat{\mathbf{x}}$:

$$\begin{array}{l} \mathbf{a}_1^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \\ \vdots \\ \mathbf{a}_n^T(\mathbf{b} - A\hat{\mathbf{x}}) = 0 \end{array} \quad \text{or} \quad \left[\begin{array}{c} -\mathbf{a}_1^T- \\ \vdots \\ -\mathbf{a}_n^T- \end{array} \right] \left[\begin{array}{c} \mathbf{b} - A\hat{\mathbf{x}} \end{array} \right] = \left[\begin{array}{c} \mathbf{0} \end{array} \right]. \quad (4)$$

The matrix in those equations is A^T . The n equations are exactly $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$.

Rewrite $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$ in its famous form $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$. This is the equation for $\hat{\mathbf{x}}$, and the coefficient matrix is A^TA . Now we can find $\hat{\mathbf{x}}$ and \mathbf{p} and P :

4F The combination $\hat{x}_1\mathbf{a}_1 + \dots + \hat{x}_n\mathbf{a}_n = A\hat{\mathbf{x}}$ that is closest to \mathbf{b} comes from

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0} \quad \text{or} \quad A^TA\hat{\mathbf{x}} = A^T\mathbf{b}. \quad (5)$$

The symmetric matrix A^TA is n by n . It is invertible if the \mathbf{a} 's are independent. The solution is $\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$. The **projection** of \mathbf{b} onto the subspace is the vector

$$\mathbf{p} = A\hat{\mathbf{x}} = A(A^TA)^{-1}A^T\mathbf{b}. \quad (6)$$

This formula shows the n by n **projection matrix** that produces $\mathbf{p} = Pb$:

$$P = A(A^TA)^{-1}A^T. \quad (7)$$

Compare with projection onto a line, when the matrix A has only one column \mathbf{a} :

$$\hat{\mathbf{x}} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}} \quad \text{and} \quad \mathbf{p} = \mathbf{a} \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}} \quad \text{and} \quad P = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$$

Those formulas are identical with (5) and (6) and (7)! The number $\mathbf{a}^T\mathbf{a}$ becomes the matrix A^TA . When it is a number, we divide by it. When it is a matrix, we invert it. The new formulas contain $(A^TA)^{-1}$ instead of $1/\mathbf{a}^T\mathbf{a}$. The linear independence of the columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ will guarantee that this inverse matrix exists.

The key step was $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$. We used geometry (e is perpendicular to all the \mathbf{a} 's). Linear algebra gives this “normal equation” too, in a very quick way:

1. Our subspace is the column space of A .
2. The error vector $\mathbf{b} - A\hat{\mathbf{x}}$ is perpendicular to that column space.
3. Therefore $\mathbf{b} - A\hat{\mathbf{x}}$ is in the left nullspace. This means $A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}$.

The left nullspace is important in projections. This nullspace of A^T contains the error vector $\mathbf{e} = \mathbf{b} - A\hat{\mathbf{x}}$. The vector \mathbf{b} is being split into the projection \mathbf{p} and the error $\mathbf{e} = \mathbf{b} - \mathbf{p}$. Figure 4.5 shows the right triangle with sides \mathbf{p} , \mathbf{e} , and \mathbf{b} .

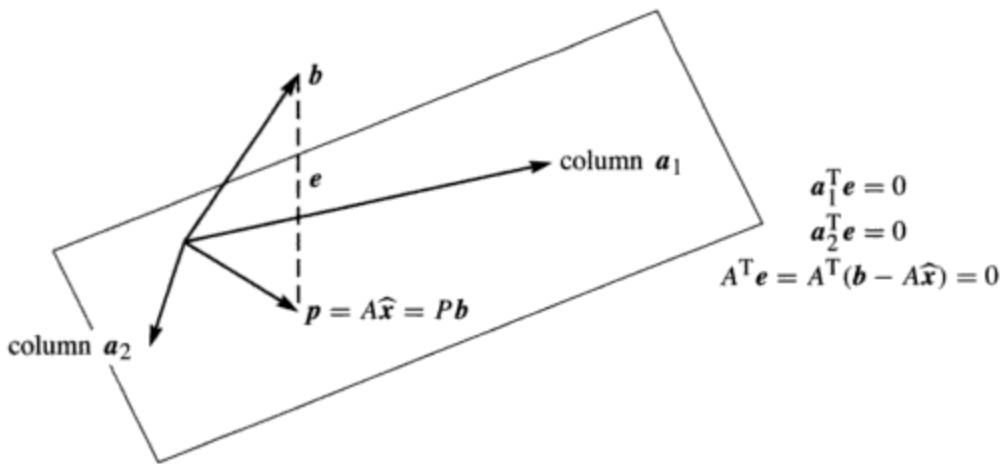


Figure 4.5 The projection p is the nearest point to b in the column space of A . The perpendicular error e must be in the nullspace of A^T .

Example 3 If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ find \hat{x} and p and P .

Solution Compute the square matrix $A^T A$ and also the vector $A^T b$:

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \quad \text{and} \quad A^T b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}.$$

Now solve the normal equation $A^T A \hat{x} = A^T b$ to find \hat{x} :

$$\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} \quad \text{gives} \quad \hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}. \quad (8)$$

The combination $p = A\hat{x}$ is the projection of b onto the column space of A :

$$p = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}. \quad \text{The error is } e = b - p = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}. \quad (9)$$

That solves the problem for one particular b . To solve it for every b , compute the matrix $P = A(A^T A)^{-1} A^T$. The determinant of $A^T A$ is $15 - 9 = 6$; $(A^T A)^{-1}$ is easy. Then multiply A times $(A^T A)^{-1}$ times A^T to reach P :

$$(A^T A)^{-1} = \frac{1}{6} \begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} \quad \text{and} \quad P = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}. \quad (10)$$

Two checks on the calculation. First, the error $e = (1, -2, 1)$ is perpendicular to both columns $(1, 1, 1)$ and $(0, 1, 2)$. Second, the final P times $b = (6, 0, 0)$ correctly gives $p = (5, 2, -1)$. We must also have $P^2 = P$, because a second projection doesn't change the first projection.

Warning The matrix $P = A(A^T A)^{-1} A^T$ is deceptive. You might try to split $(A^T A)^{-1}$ into A^{-1} times $(A^T)^{-1}$. If you make that mistake, and substitute it into P , you will find $P = AA^{-1}(A^T)^{-1}A^T$. Apparently everything cancels. This looks like $P = I$, the identity matrix. We want to say why this is wrong.

The matrix A is rectangular. It has no inverse matrix. We cannot split $(A^T A)^{-1}$ into A^{-1} times $(A^T)^{-1}$ because there is no A^{-1} in the first place.

In our experience, a problem that involves a rectangular matrix almost always leads to $A^T A$. We cannot split up its inverse, since A^{-1} and $(A^T)^{-1}$ don't exist. What does exist is the inverse of the square matrix $A^T A$. This fact is so crucial that we state it clearly and give a proof.

4G $A^T A$ is invertible if and only if A has linearly independent columns.

Proof $A^T A$ is a square matrix (n by n). For every matrix A , we will now show that $A^T A$ has the same nullspace as A . When the columns of A are linearly independent, its nullspace contains only the zero vector. Then $A^T A$, with this same nullspace, is invertible.

Let A be any matrix. If x is in its nullspace, then $Ax = \mathbf{0}$. Multiplying by A^T gives $A^T Ax = \mathbf{0}$. So x is also in the nullspace of $A^T A$.

Now start with the nullspace of $A^T A$. From $A^T Ax = \mathbf{0}$ we must prove that $Ax = \mathbf{0}$. We can't multiply by $(A^T)^{-1}$, which generally doesn't exist. Just multiply by x^T :

$$(x^T) A^T A x = 0 \quad \text{or} \quad (Ax)^T (Ax) = 0 \quad \text{or} \quad \|Ax\|^2 = 0.$$

The vector Ax has length zero. Therefore $Ax = \mathbf{0}$. Every vector x in one nullspace is in the other nullspace. If A has dependent columns, so does $A^T A$. If A has independent columns, so does $A^T A$. This is the good case:

When A has independent columns, $A^T A$ is **square**, **symmetric**, and **invertible**.

To repeat for emphasis: $A^T A$ is (n by m) times (m by n). It is square (n by n). It is symmetric, because its transpose is $(A^T A)^T = A^T (A^T)^T$ which equals $A^T A$. We just proved that $A^T A$ is invertible—provided A has independent columns. Watch the difference between dependent and independent columns:

$$\begin{array}{ccc} A^T & A & A^T A \\ \left[\begin{matrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{matrix} \right] \left[\begin{matrix} 1 & 2 \\ 1 & 2 \\ 0 & 0 \end{matrix} \right] & = \left[\begin{matrix} 2 & 4 \\ 4 & 8 \end{matrix} \right] & \left[\begin{matrix} 1 & 1 & 0 \\ 2 & 2 & 1 \end{matrix} \right] \left[\begin{matrix} 1 & 2 \\ 1 & 2 \\ 0 & 1 \end{matrix} \right] = \left[\begin{matrix} 2 & 4 \\ 4 & 9 \end{matrix} \right] \\ \text{dependent} & \text{singular} & \text{indep.} \quad \text{invertible} \end{array}$$

Very brief summary To find the projection $p = \hat{x}_1\mathbf{a}_1 + \dots + \hat{x}_n\mathbf{a}_n$, solve $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$. This gives $\hat{\mathbf{x}}$. The projection is $A \hat{\mathbf{x}}$ and the error is $e = \mathbf{b} - p = \mathbf{b} - A \hat{\mathbf{x}}$. The projection matrix $P = A(A^T A)^{-1} A^T$ gives $p = Pb$.

This matrix satisfies $P^2 = P$. The distance from \mathbf{b} to the subspace is $\|e\|$.

■ REVIEW OF THE KEY IDEAS ■

1. The projection of \mathbf{b} onto the line through \mathbf{a} is $p = \mathbf{a}\hat{x} = \mathbf{a}(\mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a})$.
2. The rank one projection matrix $P = \mathbf{a}\mathbf{a}^T / \mathbf{a}^T \mathbf{a}$ multiplies \mathbf{b} to produce p .
3. Projecting \mathbf{b} onto a subspace leaves $e = \mathbf{b} - p$ perpendicular to the subspace.
4. When the columns of A are a basis, the equation $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$ leads to $\hat{\mathbf{x}}$ and $p = A \hat{\mathbf{x}}$.
5. The projection matrix $P = A(A^T A)^{-1} A^T$ has $P^T = P$ and $P^2 = P$. Another projection leaves $p = Pb$ unchanged so $P^2 = P$.

■ WORKED EXAMPLES ■

4.2 A Project the vector $\mathbf{b} = (3, 4, 4)$ onto the line through $\mathbf{a} = (2, 2, 1)$ and then onto the plane that also contains $\mathbf{a}^* = (1, 0, 0)$. Check that the first error vector $\mathbf{b} - p$ is perpendicular to \mathbf{a} , and the second error vector $\mathbf{b} - p^*$ is also perpendicular to \mathbf{a}^* . Find the 3 by 3 projection matrix P onto that plane. Find a vector \mathbf{e}^* whose projection onto the plane of \mathbf{a} and \mathbf{a}^* is the zero vector.

Solution The projection of $\mathbf{b} = (3, 4, 4)$ onto the line through $\mathbf{a} = (2, 2, 1)$ is $2\mathbf{a}$:

$$p = \frac{\mathbf{b}^T \mathbf{a}}{\mathbf{a}^T \mathbf{a}} \mathbf{a} = \frac{18}{9} (2, 2, 1) = (4, 4, 2).$$

The error vector $e = \mathbf{b} - p = (-1, 0, 2)$ is perpendicular to \mathbf{a} . So p is correct.

The plane containing $\mathbf{a} = (2, 2, 1)$ and $\mathbf{a}^* = (1, 0, 0)$ is the column space of A :

$$A = \begin{bmatrix} 2 & 1 \\ 2 & 0 \\ 1 & 0 \end{bmatrix} \quad A^T A = \begin{bmatrix} 9 & 2 \\ 2 & 1 \end{bmatrix} \quad (A^T A)^{-1} = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 9 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & .4 \\ 0 & .4 & .2 \end{bmatrix}$$

Then $p^* = Pb = (3, 4.8, 2.4)$ and $e^* = \mathbf{b} - p^* = (0, -.8, 1.6)$ is perpendicular to \mathbf{a} and \mathbf{a}^* . This vector \mathbf{e}^* is in the nullspace of P and its projection is zero! Note $P^2 = P$.

4.2 B Suppose your pulse is measured at $x = 70$ beats per minute, then at $x = 80$, then at $x = 120$. Those three equations $Ax = b$ in one unknown have $A^T = [1 \ 1 \ 1]$ and $b = (70, 80, 120)$. *The best \hat{x} is the _____ of 70, 80, 120.* Use calculus and projection:

- 1 Minimize $E = (x - 70)^2 + (x - 80)^2 + (x - 120)^2$ by solving $dE/dx = 0$.
- 2 Project $b = (70, 80, 120)$ onto $a = (1, 1, 1)$ to find $\hat{x} = a^T b / a^T a$.

In *recursive* least squares, a new fourth measurement 130 changes \hat{x}_{old} to \hat{x}_{new} . Compute \hat{x}_{new} and verify the *update formula* $\hat{x}_{\text{new}} = \hat{x}_{\text{old}} + \frac{1}{4}(130 - \hat{x}_{\text{old}})$. Going from 999 to 1000 measurements, $\hat{x}_{\text{new}} = \hat{x}_{\text{old}} + \frac{1}{1000}(b_{1000} - \hat{x}_{\text{old}})$ would only need \hat{x}_{old} and the latest value b_{1000} . We don't have to average all 1000 numbers!

Solution The closest horizontal line to the heights 70, 80, 120 is the *average* $\hat{x} = 90$:

$$\text{Calculus : } \frac{dE}{dx} = 2(x - 70) + 2(x - 80) + 2(x - 120) = 0 \text{ gives } \hat{x} = \frac{70 + 80 + 120}{3}$$

$$\text{Projection : } \hat{x} = \frac{a^T b}{a^T a} = \frac{(1, 1, 1)^T (70, 80, 120)}{(1, 1, 1)^T (1, 1, 1)} = \frac{70 + 80 + 120}{3}$$

$A^T A \hat{x} = A^T b$ is 1 by 1 because A has only one column $(1, 1, 1)$. The new measurement $b_4 = 130$ adds a fourth equation and \hat{x} is updated to 100, either by averaging b_1, b_2, b_3, b_4 or by recursively using the old average of b_1, b_2, b_3 :

$$\hat{x}_{\text{new}} = \frac{70 + 80 + 120 + 130}{4} = 100 \quad \text{is also} \quad \hat{x}_{\text{old}} + \frac{1}{4}(b_4 - \hat{x}_{\text{old}}) = 90 + \frac{1}{4}(40).$$

The update from 999 to 1000 measurements shows the “gain matrix” $\frac{1}{1000}$ in a **Kalman filter** multiplying the prediction error $b_{\text{new}} - \hat{x}_{\text{old}}$. Notice $\frac{1}{1000} = \frac{1}{999} - \frac{1}{999000}$:

$$\hat{x}_{\text{new}} = \frac{b_1 + \dots + b_{1000}}{1000} = \frac{b_1 + \dots + b_{999}}{999} + \frac{1}{1000} \left(b_{1000} - \frac{b_1 + \dots + b_{999}}{999} \right).$$

Problem Set 4.2

Questions 1–9 ask for projections onto lines. Also errors $e = b - p$ and matrices P .

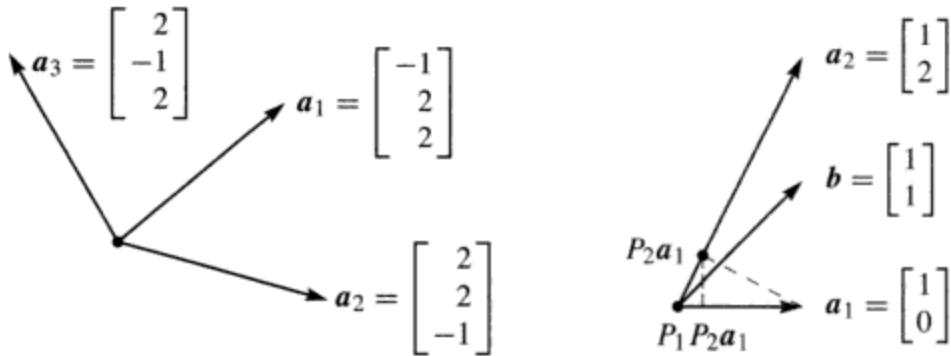
- 1 Project the vector b onto the line through a . Check that e is perpendicular to a :

$$(a) \quad b = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (b) \quad b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}.$$

- 2 Draw the projection of \mathbf{b} onto \mathbf{a} and also compute it from $\mathbf{p} = \hat{x}\mathbf{a}$:

$$(a) \quad \mathbf{b} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (b) \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

- 3 In Problem 1, find the projection matrix $P = \mathbf{aa}^T/\mathbf{a}^T\mathbf{a}$ onto the line through each vector \mathbf{a} . Verify in both cases that $P^2 = P$. Multiply $P\mathbf{b}$ in each case to compute the projection \mathbf{p} .
- 4 Construct the projection matrices P_1 and P_2 onto the lines through the \mathbf{a} 's in Problem 2. Is it true that $(P_1 + P_2)^2 = P_1 + P_2$? This would be true if $P_1 P_2 = 0$.
- 5 Compute the projection matrices $\mathbf{aa}^T/\mathbf{a}^T\mathbf{a}$ onto the lines through $\mathbf{a}_1 = (-1, 2, 2)$ and $\mathbf{a}_2 = (2, 2, -1)$. Multiply those projection matrices and explain why their product $P_1 P_2$ is what it is.
- 6 Project $\mathbf{b} = (1, 0, 0)$ onto the lines through \mathbf{a}_1 and \mathbf{a}_2 in Problem 5 and also onto $\mathbf{a}_3 = (2, -1, 2)$. Add up the three projections $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$.
- 7 Continuing Problems 5–6, find the projection matrix P_3 onto $\mathbf{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. The basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is orthogonal!



Questions 5–6–7

Questions 8–9–10

- 8 Project the vector $\mathbf{b} = (1, 1)$ onto the lines through $\mathbf{a}_1 = (1, 0)$ and $\mathbf{a}_2 = (1, 2)$. Draw the projections \mathbf{p}_1 and \mathbf{p}_2 and add $\mathbf{p}_1 + \mathbf{p}_2$. The projections do not add to \mathbf{b} because the \mathbf{a} 's are not orthogonal.
- 9 In Problem 8, the projection of \mathbf{b} onto the plane of \mathbf{a}_1 and \mathbf{a}_2 will equal \mathbf{b} . Find $P = A(A^T A)^{-1} A^T$ for $A = [\mathbf{a}_1 \ \mathbf{a}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.
- 10 Project $\mathbf{a}_1 = (1, 0)$ onto $\mathbf{a}_2 = (1, 2)$. Then project the result back onto \mathbf{a}_1 . Draw these projections and multiply the projection matrices $P_1 P_2$: Is this a projection?

Questions 11–20 ask for projections, and projection matrices, onto subspaces.

- 11 Project \mathbf{b} onto the column space of A by solving $A^T A \hat{x} = A^T \mathbf{b}$ and $\mathbf{p} = A \hat{x}$:

$$(a) \quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad (b) \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}.$$

Find $\mathbf{e} = \mathbf{b} - \mathbf{p}$. It should be perpendicular to the columns of A .

- 12 Compute the projection matrices P_1 and P_2 onto the column spaces in Problem 11. Verify that $P_1\mathbf{b}$ gives the first projection \mathbf{p}_1 . Also verify $P_2^2 = P_2$.
- 13 (Quick and Recommended) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b} = (1, 2, 3, 4)$ onto the column space of A . What shape is the projection matrix P and what is P ?
- 14 Suppose \mathbf{b} equals 2 times the first column of A . What is the projection of \mathbf{b} onto the column space of A ? Is $P = I$ for sure in this case? Compute \mathbf{p} and P when $\mathbf{b} = (0, 2, 4)$ and the columns of A are $(0, 1, 2)$ and $(1, 2, 0)$.
- 15 If A is doubled, then $P = 2A(4A^TA)^{-1}2A^T$. This is the same as $A(A^TA)^{-1}A^T$. The column space of $2A$ is the same as _____. Is $\hat{\mathbf{x}}$ the same for A and $2A$?
- 16 What linear combination of $(1, 2, -1)$ and $(1, 0, 1)$ is closest to $\mathbf{b} = (2, 1, 1)$?
- 17 (Important) If $P^2 = P$ show that $(I - P)^2 = I - P$. When P projects onto the column space of A , $I - P$ projects onto the _____.
- 18 (a) If P is the 2 by 2 projection matrix onto the line through $(1, 1)$, then $I - P$ is the projection matrix onto _____.
 (b) If P is the 3 by 3 projection matrix onto the line through $(1, 1, 1)$, then $I - P$ is the projection matrix onto _____.
- 19 To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in that plane and make them the columns of A . The plane should be the column space. Then compute $P = A(A^TA)^{-1}A^T$.
- 20 To find the projection matrix P onto the same plane $x - y - 2z = 0$, write down a vector \mathbf{e} that is perpendicular to that plane. Compute the projection $Q = \mathbf{e}\mathbf{e}^T/\mathbf{e}^T\mathbf{e}$ and then $P = I - Q$.

Questions 21–26 show that projection matrices satisfy $P^2 = P$ and $P^T = P$.

- 21 Multiply the matrix $P = A(A^TA)^{-1}A^T$ by itself. Cancel to prove that $P^2 = P$. Explain why $P(P\mathbf{b})$ always equals $P\mathbf{b}$: The vector $P\mathbf{b}$ is in the column space so its projection is _____.
- 22 Prove that $P = A(A^TA)^{-1}A^T$ is symmetric by computing P^T . Remember that the inverse of a symmetric matrix is symmetric.
- 23 If A is square and invertible, the warning against splitting $(A^TA)^{-1}$ does not apply. It is true that $AA^{-1}(A^T)^{-1}A^T = I$. *When A is invertible, why is $P = I$? What is the error e ?*

- 24 The nullspace of A^T is _____ to the column space $C(A)$. So if $A^T b = \mathbf{0}$, the projection of b onto $C(A)$ should be $p = \text{_____}$. Check that $P = A(A^T A)^{-1} A^T$ gives this answer.
- 25 The projection matrix P onto an n -dimensional subspace has rank $r = n$. **Reason:** The projections Pb fill the subspace S . So S is the _____ of P .
- 26 If an m by m matrix has $A^2 = A$ and its rank is m , prove that $A = I$.
- 27 The important fact in Theorem 4G is this: *If $A^T A x = \mathbf{0}$ then $A x = \mathbf{0}$* . The vector $A x$ is in the nullspace of _____. $A x$ is always in the column space of _____. To be in both perpendicular spaces, $A x$ must be zero.
- 28 Use $P^T = P$ and $P^2 = P$ to prove that the length squared of column 2 always equals the diagonal entry p_{22} . This number is $\frac{2}{6} = \frac{4}{36} + \frac{4}{36} + \frac{4}{36}$ for

$$P = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}.$$

- 29 If B has rank m (full row rank, independent rows) show that BB^T is invertible.
- 30 (a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix}$$

- (b) Find the 3 by 3 projection matrix P_R onto the row space of A . Multiply $B = P_C A P_R$. Your answer B should be a little surprising—can you explain it?

LEAST SQUARES APPROXIMATIONS ■ 4.3

It often happens that $Ax = b$ has no solution. The usual reason is: *too many equations*. The matrix has more rows than columns. There are more equations than unknowns (m is greater than n). The n columns span a small part of m -dimensional space. Unless all measurements are perfect, b is outside that column space. Elimination reaches an impossible equation and stops. But these are real problems and they need an answer.

To repeat: We cannot always get the error $e = b - Ax$ down to zero. When e is zero, x is an exact solution to $Ax = b$. When the length of e is as small as possible, \hat{x} is a least squares solution. Our goal in this section is to compute \hat{x} and use it.

The previous section emphasized p (the projection). This section emphasizes \hat{x} (the least squares solution). They are connected by $p = A\hat{x}$. The fundamental equation is still $A^T A \hat{x} = A^T b$. Here is a short unofficial way to derive it:

When the original $Ax = b$ has no solution, multiply by A^T and solve $A^T A \hat{x} = A^T b$.

Example 1 A crucial application of least squares is fitting a straight line to m points. Start with three points: *Find the closest line to the points $(0, 6)$, $(1, 0)$, and $(2, 0)$* .

No straight line goes through those points. We are asking for two numbers C and D that satisfy three equations. The line is $b = C + Dt$. Here are the equations at $t = 0, 1, 2$ to match the given values $b = 6, 0, 0$:

$$\text{The first point is on the line } b = C + Dt \text{ if } C + D \cdot 0 = 6$$

$$\text{The second point is on the line } b = C + Dt \text{ if } C + D \cdot 1 = 0$$

$$\text{The third point is on the line } b = C + Dt \text{ if } C + D \cdot 2 = 0.$$

This 3 by 2 system has no solution; $b = (6, 0, 0)$ is not a combination of the columns $(1, 1, 1)$ and $(0, 1, 2)$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \quad x = \begin{bmatrix} C \\ D \end{bmatrix} \quad b = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} \quad Ax = b \text{ is not solvable.}$$

The same numbers were in Example 3 in the last section. We computed $\hat{x} = (5, -3)$. Those numbers are the best C and D , so $5 - 3t$ is the best line for the 3 points.

In practical problems, there easily could be $m = 100$ points instead of $m = 3$. They don't exactly match any $C + Dt$. Our numbers $6, 0, 0$ exaggerate the error so you can see it clearly.

Minimizing the Error

How do we make the error $e = b - Ax$ as small as possible? This is an important question with a beautiful answer. The best x (called \hat{x}) can be found by geometry or algebra or calculus:

By geometry Every Ax lies in the plane of the columns $(1, 1, 1)$ and $(0, 1, 2)$. In that plane, we look for the point closest to b . *The nearest point is the projection p .*

The best choice for Ax is p . The smallest possible error is $e = b - p$. The three points at heights $p = (p_1, p_2, p_3)$ do lie on a line, because p is in the column space. $Ax = p$ has the same solution \hat{x} , the best choice for (C, D) .

By algebra Every vector b splits into two parts. The part in the column space is p . The perpendicular part in the left nullspace is e . There is an equation we cannot solve ($Ax = b$). There is an equation we do solve (by removing e):

$$Ax = b = p + e \text{ is impossible; } A\hat{x} = p \text{ is solvable.} \quad (1)$$

The solution $A\hat{x} = p$ makes the error as small as possible, because for any x :

$$\|Ax - b\|^2 = \|Ax - p\|^2 + \|e\|^2. \quad (2)$$

This is the law $c^2 = a^2 + b^2$ for a right triangle. The vector $Ax - p$ in the column space is perpendicular to e in the left nullspace. We reduce $Ax - p$ to zero by choosing x to be \hat{x} . That leaves the smallest possible errors, namely $e = (e_1, e_2, e_3)$.

Notice what “smallest” means. The *squared length* of $Ax - b$ is minimized:

The least squares solution \hat{x} makes $E = \|Ax - b\|^2$ as small as possible.

By calculus Most functions are minimized by calculus! The graph bottoms out and the derivative in every direction is zero. Here the error function to be minimized is a *sum of squares* $e_1^2 + e_2^2 + e_3^2$ (the square of the error in each equation 1, 2, 3):

$$E = \|Ax - b\|^2 = (C + D \cdot 0 - 6)^2 + (C + D \cdot 1)^2 + (C + D \cdot 2)^2. \quad (3)$$

The unknowns are C and D . With two unknowns there are *two derivatives*—both zero at the minimum. They are “partial derivatives” because $\partial E / \partial C$ treats D as constant and $\partial E / \partial D$ treats C as constant:

$$\frac{\partial E}{\partial C} = 2(C + D \cdot 0 - 6) + 2(C + D \cdot 1) + 2(C + D \cdot 2) = 0$$

$$\frac{\partial E}{\partial D} = 2(C + D \cdot 0 - 6)(0) + 2(C + D \cdot 1)(1) + 2(C + D \cdot 2)(2) = 0.$$

$\partial E / \partial D$ contains the extra factors 0, 1, 2 from the chain rule. (The derivative of $(4 + 5x)^2$ is 2 times $4 + 5x$ times an extra 5.) In the C derivative the corresponding factors are 1, 1, 1, because C is always multiplied by 1. It is no accident that 1, 1, 1 and 0, 1, 2 are the columns of A .

Now cancel 2 from every term above and collect all C 's and all D 's:

The C derivative $\frac{\partial E}{\partial C}$ is zero: $3C + 3D = 6$
 The D derivative $\frac{\partial E}{\partial D}$ is zero: $3C + 5D = 0$ This matrix $\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$ is $A^T A$ (4)

These equations are identical with $A^T A \hat{x} = A^T b$. The best C and D are the components of \hat{x} . The equations from calculus are the same as the “normal equations” from linear algebra. These are the key equations of least squares:

The partial derivatives of $\|Ax - b\|^2$ are zero when $A^T A \hat{x} = A^T b$.

The solution is $C = 5$ and $D = -3$. Therefore $b = 5 - 3t$ is the best line—it comes closest to the three points. At $t = 0, 1, 2$ this line goes through $p = 5, 2, -1$. It could not go through $b = 6, 0, 0$. The errors are $1, -2, 1$. This is the vector e !

Figure 4.6a shows the closest line. It misses by distances $e_1, e_2, e_3 = 1, -2, 1$. Those are vertical distances. The least squares line minimizes the total squared error $E = e_1^2 + e_2^2 + e_3^2$.

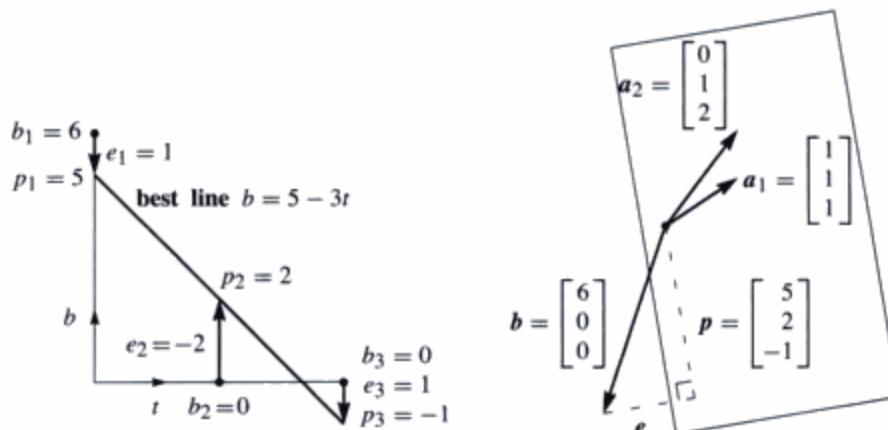


Figure 4.6 Best line and projection: Two pictures, same problem. The line has heights $p = (5, 2, -1)$ with errors $e = (1, -2, 1)$. The equations $A^T A \hat{x} = A^T b$ give $\hat{x} = (5, -3)$. The best line is $b = 5 - 3t$ and the projection is $p = 5a_1 - 3a_2$.

Figure 4.6b shows the same problem in 3-dimensional space (bpe space). The vector b is not in the column space of A . That is why we could not solve $Ax = b$ and put a line through the three points. The smallest possible error is the perpendicular vector e . This is $e = b - A\hat{x}$, the vector of errors $(1, -2, 1)$ in the three equations—and the distances from the best line. Behind both figures is the fundamental equation $A^T A \hat{x} = A^T b$.

Notice that the errors $1, -2, 1$ add to zero. The error $e = (e_1, e_2, e_3)$ is perpendicular to the first column $(1, 1, 1)$ in A . The dot product gives $e_1 + e_2 + e_3 = 0$.

The Big Picture

The key figure of this book shows the four subspaces and the true action of a matrix. The vector \mathbf{x} on the left side of Figure 4.2 went to $\mathbf{b} = A\mathbf{x}$ on the right side. In that figure \mathbf{x} was split into $\mathbf{x}_r + \mathbf{x}_n$. There were many solutions to $A\mathbf{x} = \mathbf{b}$.

In this section the situation is just the opposite. There are *no* solutions to $A\mathbf{x} = \mathbf{b}$. Instead of splitting up \mathbf{x} we are splitting up \mathbf{b} . Figure 4.7 shows the big picture for least squares. Instead of $A\mathbf{x} = \mathbf{b}$ we solve $A\hat{\mathbf{x}} = \mathbf{p}$. The error $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is unavoidable.

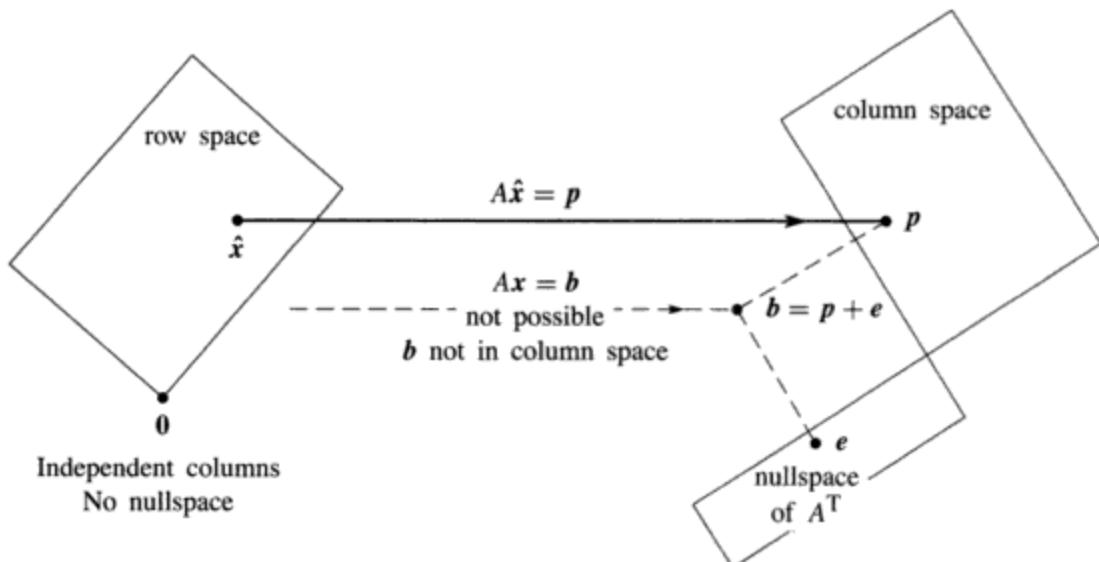


Figure 4.7 The projection $\mathbf{p} = A\hat{\mathbf{x}}$ is closest to \mathbf{b} , so $\hat{\mathbf{x}}$ minimizes $E = \|\mathbf{b} - A\mathbf{x}\|^2$.

Notice how the nullspace $N(A)$ is very small—just one point. With independent columns, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. Then A^TA is invertible. The equation $A^TA\hat{\mathbf{x}} = A^T\mathbf{b}$ fully determines the best vector $\hat{\mathbf{x}}$.

Chapter 7 will have the complete picture—all four subspaces included. Every \mathbf{x} splits into $\mathbf{x}_r + \mathbf{x}_n$, and every \mathbf{b} splits into $\mathbf{p} + \mathbf{e}$. The best solution is still $\hat{\mathbf{x}}$ (or $\hat{\mathbf{x}}_r$) in the row space. We can't help \mathbf{e} and we don't want \mathbf{x}_n —this leaves $A\hat{\mathbf{x}} = \mathbf{p}$.

Fitting a Straight Line

Fitting a line is the clearest application of least squares. It starts with $m > 2$ points—hopefully near a straight line. At times t_1, \dots, t_m those points are at heights b_1, \dots, b_m . Figure 4.6a shows the best line $b = C + Dt$, which misses the points by vertical distances e_1, \dots, e_m . No line is perfect, and the least squares line minimizes $E = e_1^2 + \dots + e_m^2$.

The first example in this section had three points. Now we allow m points (m can be large). The two components of $\hat{\mathbf{x}}$ are still C and D .

A line goes through the m points when we exactly solve $A\mathbf{x} = \mathbf{b}$. Generally we can't do it. Two unknowns C and D determine a line, so A has only $n = 2$ columns.

To fit the m points, we are trying to solve m equations (and we only want two!):

$$Ax = b \quad \text{is} \quad \begin{array}{l} C + Dt_1 = b_1 \\ C + Dt_2 = b_2 \\ \vdots \\ C + Dt_m = b_m \end{array} \quad \text{with} \quad A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix}. \quad (5)$$

The column space is so thin that almost certainly b is outside of it. When b happens to lie in the column space, the points happen to lie on a line. In that case $b = p$. Then $Ax = b$ is solvable and the errors are $e = (0, \dots, 0)$.

The closest line $C + Dt$ has heights p_1, \dots, p_m with errors e_1, \dots, e_m .

$A^T A \hat{x} = A^T b$ will give $\hat{x} = (C, D)$. The errors are $e_i = b_i - C - Dt_i$.

Fitting points by a straight line is so important that we give the two equations $A^T A \hat{x} = A^T b$, once and for all. The two columns of A are independent (unless all times t_i are the same). So we turn to least squares and solve $A^T A \hat{x} = A^T b$. The “dot-product matrix” $A^T A$ is 2 by 2:

$$A^T A = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}. \quad (6)$$

On the right side of the normal equation is the 2 by 1 vector $A^T b$:

$$A^T b = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}. \quad (7)$$

In a specific problem, all these numbers are given. A formula for C and D is coming.

4H The line $C + Dt$ which minimizes $e_1^2 + \dots + e_m^2$ is determined by $A^T A \hat{x} = A^T b$:

$$\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}. \quad (8)$$

The vertical errors at the m points on the line are the components of $e = b - p$. This error vector (the *residual*) $b - A\hat{x}$ is perpendicular to the columns of A (geometry). It is in the nullspace of A^T (linear algebra). The best $\hat{x} = (C, D)$ minimizes the total error E , the sum of squares:

$$E(\hat{x}) = \|Ax - b\|^2 = (C + Dt_1 - b_1)^2 + \dots + (C + Dt_m - b_m)^2.$$

When calculus sets the derivatives $\partial E / \partial C$ and $\partial E / \partial D$ to zero, it produces 4H.

Other least squares problems have more than two unknowns. Fitting by the best parabola has $n = 3$ coefficients C, D, E (see below). In general we are fitting m data points by n parameters x_1, \dots, x_n . The matrix A has n columns and $n < m$. The derivatives of $\|Ax - b\|^2$ give the n equations $A^T A \hat{x} = A^T b$. **The derivative of a square is linear**—this is why the method of least squares is so popular.

Example 2 A has *orthogonal columns* when the measurement times t_i add to zero. Suppose $b = 1, 2, 4$ at times $t = -2, 0, 2$. Those times add to zero. The dot product with the other column 1, 1, 1 is zero:

$$\begin{array}{lcl} C + D(-2) & = 1 \\ C + D(0) & = 2 \\ C + D(2) & = 4 \end{array} \quad \text{or} \quad A\mathbf{x} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Look at the matrix $A^T A$ in the least squares equation for \hat{x} :

$$A^T A \hat{x} = A^T b \quad \text{is} \quad \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}.$$

Main point: Now $A^T A$ is *diagonal*. We can solve separately for $C = \frac{7}{3}$ and $D = \frac{6}{8}$. The zeros in $A^T A$ are dot products of perpendicular columns in A . The diagonal matrix $A^T A$, with entries $m = 3$ and $t_1^2 + t_2^2 + t_3^2 = 8$, is virtually as good as the identity matrix.

Orthogonal columns are so helpful that it is worth moving the time origin to produce them. To do that, subtract away the average time $\hat{t} = (t_1 + \dots + t_m)/m$. The shifted times $T_i = t_i - \hat{t}$ add to zero. With the columns now orthogonal, $A^T A$ is diagonal. Its entries are m and $T_1^2 + \dots + T_m^2$. The best C and D have direct formulas:

$$C = \frac{b_1 + \dots + b_m}{m} \quad \text{and} \quad D = \frac{b_1 T_1 + \dots + b_m T_m}{T_1^2 + \dots + T_m^2}. \quad (9)$$

The best line is $C + DT$ or $C + D(t - \hat{t})$. The time shift that makes $A^T A$ diagonal is an example of the Gram-Schmidt process: *orthogonalize the columns in advance*.

Fitting by a Parabola

If we throw a ball, it would be crazy to fit the path by a straight line. A parabola $b = C + Dt + Et^2$ allows the ball to go up and come down again (b is the height at time t). The actual path is not a perfect parabola, but the whole theory of projectiles starts with that approximation.

When Galileo dropped a stone from the Leaning Tower of Pisa, it accelerated. The distance contains a quadratic term $\frac{1}{2}gt^2$. (Galileo's point was that the stone's mass is not involved.) Without that term we could never send a satellite into the right orbit. But even with a nonlinear function like t^2 , the unknowns C, D, E appear linearly. Choosing the best parabola is still a problem in linear algebra.

Problem Fit heights b_1, \dots, b_m at times t_1, \dots, t_m by a parabola $b = C + Dt + Et^2$.

With $m > 3$ points, the m equations for an exact fit are generally unsolvable:

$$\begin{aligned} C + Dt_1 + Et_1^2 &= b_1 \\ \vdots & \quad \text{has the } m \text{ by 3 matrix} \\ C + Dt_m + Et_m^2 &= b_m \end{aligned} \quad A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix}. \quad (10)$$

Least squares The best parabola chooses $\hat{x} = (C, D, E)$ to satisfy the three normal equations $A^T A \hat{x} = A^T b$.

May I ask you to convert this to a problem of projection? The column space of A has dimension _____. The projection of b is $p = A\hat{x}$, which combines the three columns using the coefficients C, D, E . The error at the first data point is $e_1 = b_1 - C - Dt_1 - Et_1^2$. The total squared error is $e_1^2 + _____$. If you prefer to minimize by calculus, take the partial derivatives of E with respect to _____, _____, _____. These three derivatives will be zero when $\hat{x} = (C, D, E)$ solves the 3 by 3 system of equations _____.

Section 8.5 has more least squares applications. The big one is Fourier series—approximating functions instead of vectors. The error to be minimized changes from a sum $e_1^2 + \dots + e_m^2$ to an integral. We will find the straight line closest to $f(x)$.

Example 3 For a parabola $b = C + Dt + Et^2$ to go through the three heights $b = 6, 0, 0$ when $t = 0, 1, 2$, the equations are

$$\begin{aligned} C + D \cdot 0 + E \cdot 0^2 &= 6 \\ C + D \cdot 1 + E \cdot 1^2 &= 0 \\ C + D \cdot 2 + E \cdot 2^2 &= 0. \end{aligned} \quad (11)$$

This is $Ax = b$. We can solve it exactly. Three data points give three equations and a square matrix. The solution is $x = (C, D, E) = (6, -9, 3)$. The parabola through the three points in Figure 4.8a is $b = 6 - 9t + 3t^2$.

What does this mean for projection? The matrix has three columns, which span the whole space \mathbf{R}^3 . The projection matrix is the identity matrix! The projection of b is b . The error is zero. We didn't need $A^T A \hat{x} = A^T b$, because we solved $Ax = b$. Of course we could multiply by A^T , but there is no reason to do it.

Figure 4.8a also shows a fourth point b_4 at time t_4 . If that falls on the parabola, the new $Ax = b$ (four equations) is still solvable. When the fourth point is not on the parabola, we turn to $A^T A \hat{x} = A^T b$. Will the least squares parabola stay the same, with all the error at the fourth point? Not likely!

The smallest error vector (e_1, e_2, e_3, e_4) is perpendicular to $(1, 1, 1, 1)$, the first column of A . Least squares balances out the four errors, and they add to zero.

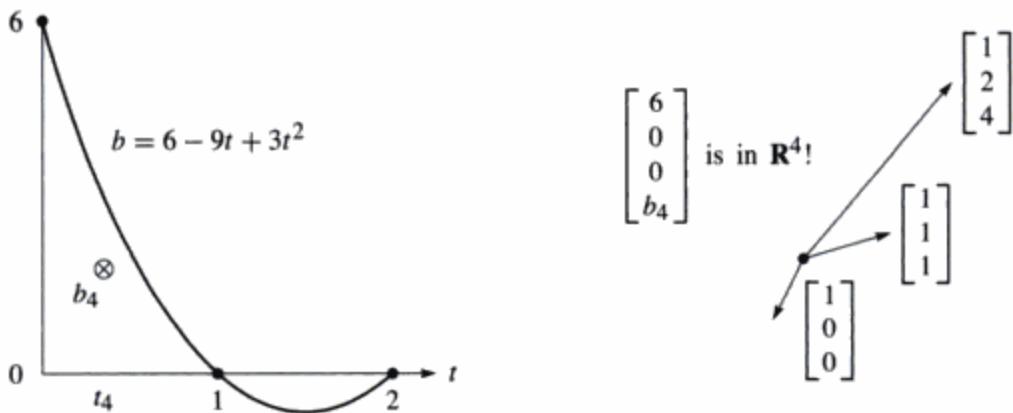


Figure 4.8 From Example 3: An exact fit of the parabola through three points means $p = b$ and $e = \mathbf{0}$. The fourth point will require least squares.

■ REVIEW OF THE KEY IDEAS ■

1. The least squares solution \hat{x} minimizes $E = \|Ax - b\|^2$. This is the sum of squares of the errors in the m equations ($m > n$).
2. The best \hat{x} comes from the normal equations $A^T A \hat{x} = A^T b$.
3. To fit m points by a line $b = C + Dt$, the two normal equations give C and D .
4. The heights of the best line are $p = (p_1, \dots, p_m)$. The vertical distances to the data points are the errors $e = (e_1, \dots, e_m)$.
5. If we try to fit m points by a combination of $n < m$ functions, the m equations are generally unsolvable. The n normal equations give the least squares solution.

■ WORKED EXAMPLES ■

4.3 A Start with nine measurements b_1 to b_9 , all zero, at times $t = 1, \dots, 9$. The tenth measurement $b_{10} = 40$ is an outlier. Find the best horizontal line $y = C$ to fit the ten points $(1, 0), (2, 0), \dots, (9, 0), (10, 40)$ using three measures for the error E : (1) Least squares $e_1^2 + \dots + e_{10}^2$ (2) Least maximum error $|e_{\max}|$ (3) Least sum of errors $|e_1| + \dots + |e_{10}|$.

Then find the least squares straight line $C + Dt$ through those ten points. What happens to C and D if you multiply the b_i by 3 and then add 30 to get $b_{\text{new}} = (30, 30, \dots, 150)$? What is the best line if you multiply the times $t_i = 1, \dots, 10$ by 2 and then add 10 to get $t_{\text{new}} = 12, 14, \dots, 30$?

Solution (1) The least squares fit to $0, 0, \dots, 0, 40$ by a horizontal line is the average $C = \frac{40}{10} = 4$. (2) The least maximum error requires $C = 20$, halfway between

0 and 40. (3) The least sum requires $C = 0$ (!!). The sum of errors $9|C| + |40 - C|$ would increase if C moves up from zero.

The least sum comes from the *median* measurement (the median of $0, \dots, 0, 40$ is zero). Changing the best $y = C = b_{\text{median}}$ increases half the errors and decreases half. Many statisticians feel that the least squares solution is too heavily influenced by outliers like $b_{10} = 40$, and they prefer least sum. But the equations become nonlinear.

The least squares straight line $C + Dt$ requires $A^T A$ and $A^T b$ with $t = 1, \dots, 10$:

$$A^T A = \begin{bmatrix} 10 & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} = \begin{bmatrix} 10 & 55 \\ 55 & 385 \end{bmatrix} \quad A^T b = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix} = \begin{bmatrix} 40 \\ 400 \end{bmatrix}$$

Those come from equation (9). Then $A^T A \hat{x} = A^T b$ gives $C = -8$ and $D = 24/11$. Linearity allows us to rescale the measurements $b = (0, 0, \dots, 40)$. Multiplying b by 3 will multiply C and D by 3. Adding 30 to all b_i will add 30 to C .

Multiplying the times t_i by 2 will *divide* D by 2 (so the line reaches the same heights at the new times). Adding 10 to all times will replace t by $t - 10$. The new line $C + D(\frac{t-10}{2})$ reaches the same heights at $t = 12, 14, \dots, 30$ (with the same errors) that it previously did at $t = 1, 2, \dots, 10$. In linear algebra language, these matrices A_{old} and A_{new} have the same column space (why?) so no change in the projection:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{bmatrix}^T \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 12 & 14 & 16 & 18 & 20 & 22 & 24 & 26 & 28 & 30 \end{bmatrix}^T$$

4.3 B Find the parabola $C + Dt + Et^2$ that comes closest (least squares error) to the values $b = (0, 0, 1, 0, 0)$ at the times $t = -2, -1, 0, 1, 2$. First write down the five equations $Ax = b$ in three unknowns $x = (C, D, E)$ for a parabola to go through the five points. No solution because no such parabola exists. Solve $Ax = b$ by least squares (using $A^T A \hat{x} = A^T b$).

I would predict that $D = 0$. Why should the best parabola be symmetric around $t = 0$? In $A^T A \hat{x} = A^T b$, equation 2 for D should uncouple from equations 1 and 3.

Solution The five equations $Ax = b$ and the 3 by 3 matrix $A^T A$ are

$$\begin{aligned} C + D(-2) + E(-2)^2 &= 0 \\ C + D(-1) + E(-1)^2 &= 0 \\ C + D(0) + E(0)^2 &= 0 \\ C + D(1) + E(1)^2 &= 0 \\ C + D(2) + E(2)^2 &= 0 \end{aligned} \quad A = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad A^T A = \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix}$$

Those zeros in $A^T A$ mean that column 2 of A is orthogonal to columns 1 and 3. We see this directly in A (because the times $-2, -1, 0, 1, 2$ are symmetric). The best C, D, E in the parabola $C + Dt + Et^2$ come from $A^T A \hat{x} = A^T b$, and equation 2 for D is uncoupled:

$$\begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{leads to} \quad \begin{aligned} C &= \frac{34}{70} = \frac{17}{35} \\ D &= 0 \text{ as predicted} \\ E &= -\frac{10}{70} = -\frac{1}{7} \end{aligned}$$

The symmetry of t 's uncoupled equation 2. The symmetry of $\mathbf{b} = (0, 0, 1, 0, 0)$ made its right side zero. Symmetric inputs produced a symmetric parabola $\frac{17}{35} - \frac{1}{7}t^2$.

Column 3 can be orthogonalized by subtracting its projection $(2, 2, 2, 2, 2)$ onto column 1:

$$A_{\text{new}} \hat{\mathbf{x}}_{\text{new}} = \mathbf{b} \quad \text{is} \quad \begin{bmatrix} 1 & -2 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} C + 2E \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{Notice new third column}$$

$$(A_{\text{new}}^T A_{\text{new}}) \hat{\mathbf{x}}_{\text{new}} = A_{\text{new}}^T \mathbf{b} \quad \text{is} \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} C + 2E \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

All equations are now uncoupled! A_{new} has orthogonal columns. Immediately $14E = -2$ and $E = -\frac{1}{7}$ and $D = 0$. Then $C + 2E = \frac{1}{5}$ gives $C = \frac{1}{5} + \frac{2}{7} = \frac{17}{35}$ as before. $A^T A$ becomes easy when the work of orthogonalization (which is Gram-Schmidt) is done first.

Problem Set 4.3

Problems 1–11 use four data points $\mathbf{b} = (0, 8, 8, 20)$ to bring out the key ideas.

- 1 With $\mathbf{b} = 0, 8, 8, 20$ at $t = 0, 1, 3, 4$, set up and solve the normal equations $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$. For the best straight line in Figure 4.9a, find its four heights p_i and four errors e_i . What is the minimum value $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$?
- 2 (Line $C + Dt$ does go through p 's) With $\mathbf{b} = 0, 8, 8, 20$ at times $t = 0, 1, 3, 4$, write down the four equations $A\mathbf{x} = \mathbf{b}$ (unsolvable). Change the measurements to $p = 1, 5, 13, 17$ and find an exact solution to $A\hat{\mathbf{x}} = p$.
- 3 Check that $\mathbf{e} = \mathbf{b} - \mathbf{p} = (-1, 3, -5, 3)$ is perpendicular to both columns of A . What is the shortest distance $\|\mathbf{e}\|$ from \mathbf{b} to the column space of A ?
- 4 (By calculus) Write down $E = \|A\mathbf{x} - \mathbf{b}\|^2$ as a sum of four squares—the last one is $(C + 4D - 20)^2$. Find the derivative equations $\partial E / \partial C = 0$ and $\partial E / \partial D = 0$. Divide by 2 to obtain the normal equations $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$.
- 5 Find the height C of the best *horizontal line* to fit $\mathbf{b} = (0, 8, 8, 20)$. An exact fit would solve the unsolvable equations $C = 0, C = 8, C = 8, C = 20$. Find the 4 by 1 matrix A in these equations and solve $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$. Draw the horizontal line at height $\hat{\mathbf{x}} = C$ and the four errors in \mathbf{e} .

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- 6** Project $\mathbf{b} = (0, 8, 8, 20)$ onto the line through $\mathbf{a} = (1, 1, 1, 1)$. Find $\hat{\mathbf{x}} = \mathbf{a}^T \mathbf{b} / \mathbf{a}^T \mathbf{a}$ and the projection $\mathbf{p} = \hat{\mathbf{x}} \mathbf{a}$. Check that $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is perpendicular to \mathbf{a} , and find the shortest distance $\|\mathbf{e}\|$ from \mathbf{b} to the line through \mathbf{a} .
- 7** Find the closest line $b = Dt$, *through the origin*, to the same four points. An exact fit would solve $D \cdot 0 = 0, D \cdot 1 = 8, D \cdot 3 = 8, D \cdot 4 = 20$. Find the 4 by 1 matrix and solve $\mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{b}$. Redraw Figure 4.9a showing the best line $b = Dt$ and the e 's.
- 8** Project $\mathbf{b} = (0, 8, 8, 20)$ onto the line through $\mathbf{a} = (0, 1, 3, 4)$. Find $\hat{\mathbf{x}} = D$ and $\mathbf{p} = \hat{\mathbf{x}} \mathbf{a}$. The best C in Problems 5–6 and the best D in Problems 7–8 do *not* agree with the best (C, D) in Problems 1–4. That is because $(1, 1, 1, 1)$ and $(0, 1, 3, 4)$ are _____ perpendicular.
- 9** For the closest parabola $b = C + Dt + Et^2$ to the same four points, write down the unsolvable equations $Ax = \mathbf{b}$ in three unknowns $x = (C, D, E)$. Set up the three normal equations $\mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{b}$ (solution not required). In Figure 4.9a you are now fitting a parabola to 4 points—what is happening in Figure 4.9b?
- 10** For the closest cubic $b = C + Dt + Et^2 + Ft^3$ to the same four points, write down the four equations $Ax = \mathbf{b}$. Solve them by elimination. In Figure 4.9a this cubic now goes exactly through the points. What are \mathbf{p} and \mathbf{e} ?
- 11** The average of the four times is $\hat{t} = \frac{1}{4}(0 + 1 + 3 + 4) = 2$. The average of the four b 's is $\hat{b} = \frac{1}{4}(0 + 8 + 8 + 20) = 9$.
 - Verify that the best line goes through the center point $(\hat{t}, \hat{b}) = (2, 9)$.
 - Explain why $C + D\hat{t} = \hat{b}$ comes from the first equation in $\mathbf{A}^T \mathbf{A} \hat{\mathbf{x}} = \mathbf{A}^T \mathbf{b}$.

Questions 12–16 introduce basic ideas of statistics—the foundation for least squares.

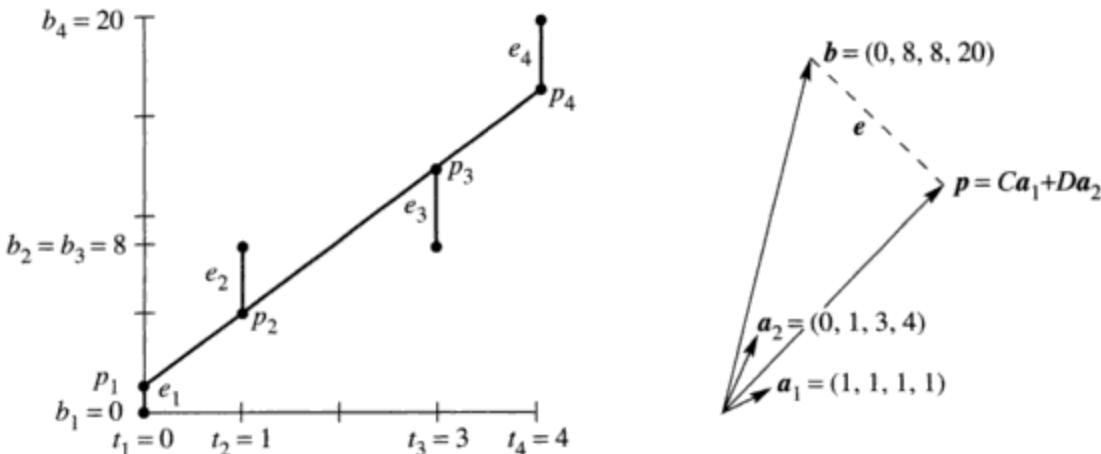


Figure 4.9 Problems 1–11: The closest line $C + Dt$ matches $Ca_1 + Da_2$ in \mathbb{R}^4 .

- 12 (Recommended) This problem projects $\mathbf{b} = (b_1, \dots, b_m)$ onto the line through $\mathbf{a} = (1, \dots, 1)$. We solve m equations $\mathbf{ax} = \mathbf{b}$ in 1 unknown (by least squares).
- Solve $\mathbf{a}^T \mathbf{a} \hat{\mathbf{x}} = \mathbf{a}^T \mathbf{b}$ to show that $\hat{\mathbf{x}}$ is the *mean* (the average) of the b 's.
 - Find $\mathbf{e} = \mathbf{b} - \mathbf{a} \hat{\mathbf{x}}$ and the *variance* $\|\mathbf{e}\|^2$ and the *standard deviation* $\|\mathbf{e}\|$.
 - The horizontal line $\hat{\mathbf{b}} = 3$ is closest to $\mathbf{b} = (1, 2, 6)$. Check that $\mathbf{p} = (3, 3, 3)$ is perpendicular to \mathbf{e} and find the matrix P .
- 13 First assumption behind least squares: Each measurement error has *mean zero*. Multiply the 8 error vectors $\mathbf{b} - \mathbf{Ax} = (\pm 1, \pm 1, \pm 1)$ by $(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ to show that the 8 vectors $\hat{\mathbf{x}} - \mathbf{x}$ also average to zero. The estimate $\hat{\mathbf{x}}$ is *unbiased*.
- 14 Second assumption behind least squares: The m errors \mathbf{e}_i are independent with variance σ^2 , so the average of $(\mathbf{b} - \mathbf{Ax})(\mathbf{b} - \mathbf{Ax})^T$ is $\sigma^2 I$. Multiply on the left by $(\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T$ and on the right by $\mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1}$ to show that the average of $(\hat{\mathbf{x}} - \mathbf{x})(\hat{\mathbf{x}} - \mathbf{x})^T$ is $\sigma^2 (\mathbf{A}^T \mathbf{A})^{-1}$. This is the *covariance matrix* for the error in $\hat{\mathbf{x}}$.
- 15 A doctor takes 4 readings of your heart rate. The best solution to $x = b_1, \dots, x = b_4$ is the average $\hat{\mathbf{x}}$ of b_1, \dots, b_4 . The matrix A is a column of 1's. Problem 14 gives the expected error $(\hat{\mathbf{x}} - \mathbf{x})^2$ as $\sigma^2 (\mathbf{A}^T \mathbf{A})^{-1} = \underline{\hspace{2cm}}$. By averaging, the variance drops from σ^2 to $\sigma^2/4$.
- 16 If you know the average \hat{x}_9 of 9 numbers b_1, \dots, b_9 , how can you quickly find the average \hat{x}_{10} with one more number b_{10} ? The idea of *recursive* least squares is to avoid adding 10 numbers. What coefficient correctly gives \hat{x}_{10} ?

$$\hat{x}_{10} = \frac{1}{10} b_{10} + \underline{\hspace{2cm}} \hat{x}_9 = \frac{1}{10} (b_1 + \dots + b_{10}).$$

Questions 17–25 give more practice with $\hat{\mathbf{x}}$ and \mathbf{p} and \mathbf{e} . Note Question 26.

- 17 Write down three equations for the line $b = C + Dt$ to go through $b = 7$ at $t = -1$, $b = 7$ at $t = 1$, and $b = 21$ at $t = 2$. Find the least squares solution $\hat{\mathbf{x}} = (C, D)$ and draw the closest line.
- 18 Find the projection $\mathbf{p} = \mathbf{A} \hat{\mathbf{x}}$ in Problem 17. This gives the three heights of the closest line. Show that the error vector is $\mathbf{e} = (2, -6, 4)$.
- 19 Suppose the measurements at $t = -1, 1, 2$ are the errors 2, -6, 4 in Problem 18. Compute $\hat{\mathbf{x}}$ and the closest line to these new measurements. Explain the answer: $\mathbf{b} = (2, -6, 4)$ is perpendicular to $\underline{\hspace{2cm}}$ so the projection is $\mathbf{p} = \mathbf{0}$.
- 20 Suppose the measurements at $t = -1, 1, 2$ are $\mathbf{b} = (5, 13, 17)$. Compute $\hat{\mathbf{x}}$ and the closest line and \mathbf{e} . The error is $\mathbf{e} = \mathbf{0}$ because this \mathbf{b} is $\underline{\hspace{2cm}}$.
- 21 Which of the four subspaces contains the error vector \mathbf{e} ? Which contains \mathbf{p} ? Which contains $\hat{\mathbf{x}}$? What is the nullspace of \mathbf{A} ?
- 22 Find the best line $C + Dt$ to fit $b = 4, 2, -1, 0, 0$ at times $t = -2, -1, 0, 1, 2$.

- 23 (Distance between lines) The points $P = (x, x, x)$ are on a line through $(1, 1, 1)$ and $Q = (y, 3y, -1)$ are on another line. Choose x and y to minimize the squared distance $\|P - Q\|^2$.
- 24 Is the error vector e orthogonal to b or p or e or \hat{x} ? Show that $\|e\|^2$ equals $e^T b$ which equals $b^T b - p^T b$. This is the smallest total error E .
- 25 The derivatives of $\|Ax\|^2$ with respect to the variables x_1, \dots, x_n fill the vector $2A^T Ax$. The derivatives of $2b^T Ax$ fill the vector $2A^T b$. So the derivatives of $\|Ax - b\|^2$ are zero when _____.
- 26 *What condition on $(t_1, b_1), (t_2, b_2), (t_3, b_3)$ puts those three points onto a straight line?* A column space answer is: (b_1, b_2, b_3) must be a combination of $(1, 1, 1)$ and (t_1, t_2, t_3) . Try to reach a specific equation connecting the t 's and b 's. I should have thought of this question sooner!
- 27 Find the *plane* that gives the best fit to the 4 values $b = (0, 1, 3, 4)$ at the corners $(1, 0)$ and $(0, 1)$ and $(-1, 0)$ and $(0, -1)$ of a square. The equations $C + Dx + Ey = b$ at those 4 points are $Ax = b$ with 3 unknowns $x = (C, D, E)$. At the center $(0, 0)$ of the square, show that $C + Dx + Ey = \text{average of the } b\text{'s}$.

ORTHOGONAL BASES AND GRAM-SCHMIDT ■ 4.4

This section has two goals. The first is to see how orthogonality can make calculations simpler. Dot products are zero—so $A^T A$ becomes a diagonal matrix. The second goal is to **construct orthogonal vectors**. We will pick combinations of the original vectors to produce right angles. Those original vectors are the columns of A , probably *not* orthogonal. The orthogonal vectors will be the columns of a new matrix Q .

You know from Chapter 3 what a basis consists of—*independent vectors that span the space*. The basis vectors could meet at any angle (except 0° and 180°). But every time we visualize axes, they are perpendicular. *In our imagination, the coordinate axes are practically always orthogonal.* This simplifies the picture and it greatly simplifies the computations.

The vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$ are **orthogonal** when their dot products $\mathbf{q}_i \cdot \mathbf{q}_j$ are zero. More exactly $\mathbf{q}_i^T \mathbf{q}_j = 0$ whenever $i \neq j$. With one more step—just divide each vector by its length—the vectors become **orthogonal unit vectors**. Their lengths are all 1. Then the basis is called **orthonormal**.

DEFINITION The vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$ are **orthonormal** if

$$\mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 0 & \text{when } i \neq j \quad (\text{orthogonal vectors}) \\ 1 & \text{when } i = j \quad (\text{unit vectors: } \|\mathbf{q}_i\| = 1) \end{cases}$$

A matrix with orthonormal columns is assigned the special letter Q .

The matrix Q is easy to work with because $Q^T Q = I$. This repeats in matrix language that the columns $\mathbf{q}_1, \dots, \mathbf{q}_n$ are orthonormal. It is equation (1) below, and Q is not required to be square.

When Q is square, $Q^T Q = I$ means that $Q^T = Q^{-1}$: **transpose = inverse**.

41 A matrix Q with orthonormal columns satisfies $Q^T Q = I$:

$$Q^T Q = \left[\begin{array}{c|c|c} -\mathbf{q}_1^T & - & \\ \hline -\mathbf{q}_2^T & - & \\ \hline -\mathbf{q}_n^T & - & \end{array} \right] \left[\begin{array}{c|c|c} | & | & | \\ \hline \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_n \\ \hline | & | & | \end{array} \right] = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \right] = I, \quad (1)$$

When row i of Q^T multiplies column j of Q , the dot product is $\mathbf{q}_i^T \mathbf{q}_j$. Off the diagonal ($i \neq j$) that dot product is zero by orthogonality. On the diagonal ($i = j$) the unit vectors give $\mathbf{q}_i^T \mathbf{q}_i = \|\mathbf{q}_i\|^2 = 1$.

If the columns are only orthogonal (not unit vectors), then $Q^T Q$ is a diagonal matrix (not the identity matrix). We wouldn't use the letter Q . But this matrix is almost as good. The important thing is orthogonality—then it is easy to produce unit vectors.

To repeat: $Q^T Q = I$ even when Q is rectangular. In that case Q^T is only an inverse from the left. For square matrices we also have $QQ^T = I$, so Q^T is the two-sided inverse of Q . The rows of a square Q are orthonormal like the columns. **The inverse is the transpose.** In this square case we call Q an **orthogonal matrix**.²

Here are three examples of orthogonal matrices—rotation and permutation and reflection. The quickest test is to check $Q^T Q = I$.

Example 1 (Rotation) Q rotates every vector in the plane through the angle θ :

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad Q^T = Q^{-1} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}.$$

The columns of Q are orthogonal (take their dot product). They are unit vectors because $\sin^2 \theta + \cos^2 \theta = 1$. Those columns give an **orthonormal basis** for the plane \mathbf{R}^2 . The standard basis vectors i and j are rotated through θ (see Figure 4.10a).

Q^{-1} rotates vectors back through $-\theta$. It agrees with Q^T , because the cosine of $-\theta$ is the cosine of θ , and $\sin(-\theta) = -\sin \theta$. We have $Q^T Q = I$ and $QQ^T = I$.

Example 2 (Permutation) These matrices change the order to (y, z, x) and (y, x) :

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ x \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$

All columns of these Q 's are unit vectors (their lengths are obviously 1). They are also orthogonal (the 1's appear in different places). *The inverse of a permutation matrix is its transpose.* The inverse puts the components back into their original order:

$$\text{Inverse = transpose: } \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Every permutation matrix is an orthogonal matrix.

Example 3 (Reflection) If u is any unit vector, set $Q = I - 2uu^T$. Notice that uu^T is a matrix while u^Tu is the number $\|u\|^2 = 1$. Then Q^T and Q^{-1} both equal Q :

$$Q^T = I - 2uu^T = Q \quad \text{and} \quad Q^T Q = I - 4uu^T + 4uu^Tuu^T = I. \quad (2)$$

²“Orthonormal matrix” would have been a better name for Q , but it's not used. Any matrix with orthonormal columns has the letter Q , but we only call it an **orthogonal matrix** when it is square.

Reflection matrices $I - 2uu^T$ are symmetric and also orthogonal. If you square them, you get the identity matrix: $Q^TQ = I$. Reflecting twice through a mirror brings back the original. Notice $u^Tu = 1$ inside $4uu^Tuu^T$ in equation (2).

As examples choose two unit vectors, $u = (1, 0)$ and then $u = (1/\sqrt{2}, -1/\sqrt{2})$. Compute $2uu^T$ (column times row) and subtract from I to get Q :

$$Q_1 = I - 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_2 = I - 2 \begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Q_1 reflects $(1, 0)$ across the y axis to $(-1, 0)$. Every vector (x, y) goes into its image $(-x, y)$, and the y axis is the mirror:

$$\text{Reflection from } Q_1: \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

Q_2 is reflection across the 45° line. Every (x, y) goes to (y, x) —this was the permutation in Example 2. A vector like $(3, 3)$ doesn't move when you exchange 3 and 3—it is on the mirror line. Figure 4.10b shows the 45° mirror.

Rotations preserve the length of a vector. So do reflections. So do permutations. So does multiplication by any orthogonal matrix—**lengths and angles don't change**.

4] If Q has orthonormal columns ($Q^TQ = I$), it leaves lengths unchanged:

$$\|Qx\| = \|x\| \text{ for every vector } x. \quad (3)$$

Q also preserves dot products: $(Qx)^T(Qy) = x^TQ^TQy = x^Ty$. Just use $Q^TQ = I$!

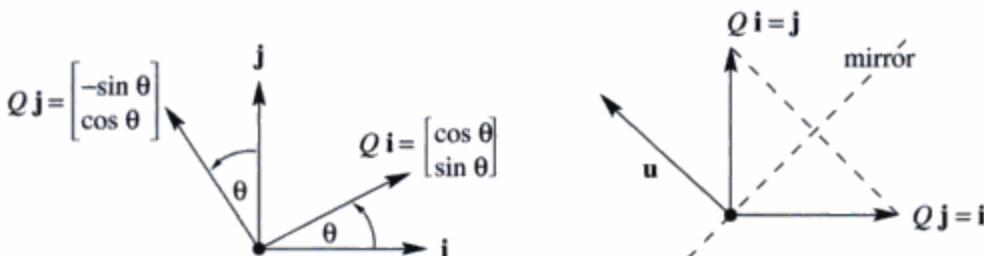


Figure 4.10 Rotation by $Q = [\begin{smallmatrix} c & -s \\ s & c \end{smallmatrix}]$ and reflection across 45° by $Q = [\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}]$.

Proof $\|Qx\|^2$ equals $\|x\|^2$ because $(Qx)^T(Qx) = x^T Q^T Q x = x^T I x = x^T x$. Orthogonal matrices are excellent for computations—numbers can never grow too large when lengths of vectors are fixed. Good computer codes use Q 's as much as possible. That makes them numerically stable.

Projections Using Orthogonal Bases: Q Replaces A

This chapter is about projections onto subspaces. We developed the equations for \hat{x} and p and P . When the columns of A were a basis for the subspace, all formulas involved $A^T A$. The entries of $A^T A$ are the dot products $a_i^T a_j$.

Suppose the basis vectors are actually orthonormal. The a 's become q 's. Then $A^T A$ simplifies to $Q^T Q = I$. Look at the improvements in \hat{x} and p and P . Instead of $Q^T Q$ we print a blank for the identity matrix:

$$\hat{x} = Q^T b \quad \text{and} \quad p = Q \hat{x} \quad \text{and} \quad P = Q \quad Q^T. \quad (4)$$

The least squares solution of $Qx = b$ is $\hat{x} = Q^T b$. The projection matrix is $P = Q Q^T$.

There are no matrices to invert. This is the point of an orthonormal basis. The best $\hat{x} = Q^T b$ just has dot products of b with the rows of Q^T , which are the q 's:

$$\hat{x} = \begin{bmatrix} - q_1^T - \\ \vdots \\ - q_n^T - \end{bmatrix} \begin{bmatrix} b \end{bmatrix} = \begin{bmatrix} q_1^T b \\ \vdots \\ q_n^T b \end{bmatrix} \quad (\text{dot products})$$

We have n separate 1-dimensional projections. The “coupling matrix” or “correlation matrix” $A^T A$ is now $Q^T Q = I$. There is no coupling. Here is $p = Q \hat{x}$:

$$\text{Projection} \quad p = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} q_1^T b \\ \vdots \\ q_n^T b \end{bmatrix} = q_1(q_1^T b) + \cdots + q_n(q_n^T b). \quad (5)$$

Important case: When Q is square and $m = n$, the subspace is the whole space. Then $Q^T = Q^{-1}$ and $\hat{x} = Q^T b$ is the same as $x = Q^{-1} b$. The solution is exact! The projection of b onto the whole space is b itself. In this case $P = Q Q^T = I$.

You may think that projection onto the whole space is not worth mentioning. But when $p = b$, our formula assembles b out of its 1-dimensional projections. If q_1, \dots, q_n is an orthonormal basis for the whole space, so Q is square, then every b is the sum of its components along the q 's:

$$b = q_1(q_1^T b) + q_2(q_2^T b) + \cdots + q_n(q_n^T b). \quad (6)$$

That is $Q Q^T = I$. It is the foundation of Fourier series and all the great “transforms” of applied mathematics. They break vectors or functions into perpendicular pieces. Then by adding the pieces, the inverse transform puts the function back together.

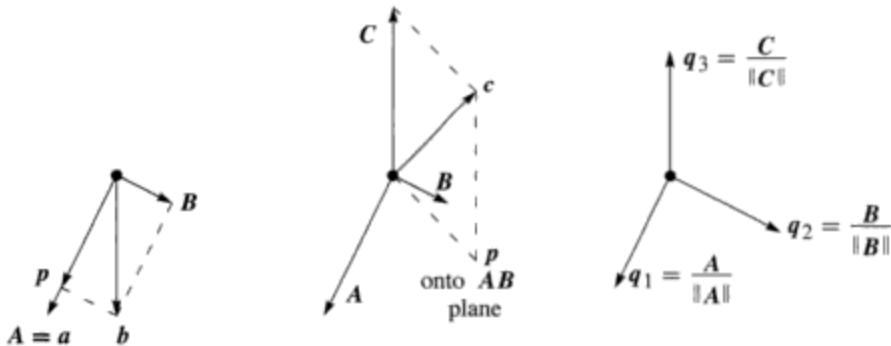


Figure 4.11 First project b onto the line through a and find B as $b - p$. Then project c onto the AB plane and find C as $c - p$. Then divide by $\|A\|$, $\|B\|$, and $\|C\|$.

Example 4 The columns of this matrix Q are orthonormal vectors q_1, q_2, q_3 :

$$Q = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \quad \text{has first column } q_1 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}.$$

The separate projections of $b = (0, 0, 1)$ onto q_1 and q_2 and q_3 are

$$q_1(q_1^T b) = \frac{2}{3}q_1 \quad \text{and} \quad q_2(q_2^T b) = \frac{2}{3}q_2 \quad \text{and} \quad q_3(q_3^T b) = -\frac{1}{3}q_3.$$

The sum of the first two is the projection of b onto the *plane* of q_1 and q_2 . The sum of all three is the projection of b onto the *whole space*—which is b itself:

$$\frac{2}{3}q_1 + \frac{2}{3}q_2 - \frac{1}{3}q_3 = \frac{1}{9} \begin{bmatrix} -2+4-2 \\ 4-2-2 \\ 4+4+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = b.$$

The Gram-Schmidt Process

The point of this section is that “orthogonal is good.” Projections and least squares always involve $A^T A$. When this matrix becomes $Q^T Q = I$, the inverse is no problem. The one-dimensional projections are uncoupled. The best \hat{x} is $Q^T b$ (n separate dot products). For this to be true, we had to say “*If* the vectors are orthonormal.” **Now we find a way to create orthonormal vectors.**

Start with three independent vectors a, b, c . We intend to construct three orthogonal vectors A, B, C . Then (at the end is easiest) we divide A, B, C by their lengths. That produces three orthonormal vectors $q_1 = A/\|A\|$, $q_2 = B/\|B\|$, $q_3 = C/\|C\|$.

Gram-Schmidt Begin by choosing $A = a$. This first direction is accepted. The next direction B must be perpendicular to A . **Start with b and subtract its projection along**

A. This leaves the perpendicular part, which is the orthogonal vector \mathbf{B} :

$$\text{Gram-Schmidt idea} \quad \mathbf{B} = \mathbf{b} - \frac{\mathbf{A}^T \mathbf{b}}{\mathbf{A}^T \mathbf{A}} \mathbf{A}. \quad (7)$$

\mathbf{A} and \mathbf{B} are orthogonal in Figure 4.11. Take the dot product with \mathbf{A} to verify that $\mathbf{A}^T \mathbf{B} = \mathbf{A}^T \mathbf{b} - \mathbf{A}^T \mathbf{b} = 0$. This vector \mathbf{B} is what we have called the error vector \mathbf{e} , perpendicular to \mathbf{A} . Notice that \mathbf{B} in equation (7) is not zero (otherwise \mathbf{a} and \mathbf{b} would be dependent). The directions \mathbf{A} and \mathbf{B} are now set.

The third direction starts with \mathbf{c} . This is not a combination of \mathbf{A} and \mathbf{B} (because \mathbf{c} is not a combination of \mathbf{a} and \mathbf{b}). But most likely \mathbf{c} is not perpendicular to \mathbf{A} and \mathbf{B} . So subtract off its components in those two directions to get \mathbf{C} :

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A}^T \mathbf{c}}{\mathbf{A}^T \mathbf{A}} \mathbf{A} - \frac{\mathbf{B}^T \mathbf{c}}{\mathbf{B}^T \mathbf{B}} \mathbf{B}. \quad (8)$$

This is the one and only idea of the Gram-Schmidt process. *Subtract from every new vector its projections in the directions already set.* That idea is repeated at every step.² If we also had a fourth vector \mathbf{d} , we would subtract its projections onto $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to get \mathbf{D} . At the end, divide the orthogonal vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ by their lengths. The resulting vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4$ are orthonormal.

Example 5 Suppose the independent non-orthogonal vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are

$$\mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}.$$

Then $\mathbf{A} = \mathbf{a}$ has $\mathbf{A}^T \mathbf{A} = 2$. Subtract from \mathbf{b} its projection along $\mathbf{A} = (1, -1, 0)$:

$$\mathbf{B} = \mathbf{b} - \frac{\mathbf{A}^T \mathbf{b}}{\mathbf{A}^T \mathbf{A}} \mathbf{A} = \mathbf{b} - \frac{2}{2} \mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

Check: $\mathbf{A}^T \mathbf{B} = 0$ as required. Now subtract two projections from \mathbf{c} to get \mathbf{C} :

$$\mathbf{C} = \mathbf{c} - \frac{\mathbf{A}^T \mathbf{c}}{\mathbf{A}^T \mathbf{A}} \mathbf{A} - \frac{\mathbf{B}^T \mathbf{c}}{\mathbf{B}^T \mathbf{B}} \mathbf{B} = \mathbf{c} - \frac{6}{2} \mathbf{A} + \frac{6}{6} \mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Check: $\mathbf{C} = (1, 1, 1)$ is perpendicular to \mathbf{A} and \mathbf{B} . Finally convert $\mathbf{A}, \mathbf{B}, \mathbf{C}$ to unit vectors (length 1, orthonormal). The lengths of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are $\sqrt{2}$ and $\sqrt{6}$ and $\sqrt{3}$. Divide by those lengths, for an orthonormal basis:

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{q}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Usually $\mathbf{A}, \mathbf{B}, \mathbf{C}$ contain fractions. Almost always $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ contain square roots.

²I think Gram had the idea. I don't really know where Schmidt came in.

The Factorization $A = QR$

We started with a matrix A , whose columns were $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We ended with a matrix Q , whose columns are $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$. How are those matrices related? Since the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are combinations of the \mathbf{q} 's (and vice versa), there must be a third matrix connecting A to Q . Call it R .

The first step was $\mathbf{q}_1 = \mathbf{a}/\|\mathbf{a}\|$ (other vectors not involved). The second step was equation (7), where \mathbf{b} is a combination of A and B . At that stage C and \mathbf{q}_3 were not involved. This non-involvement of later vectors is the key point of Gram-Schmidt:

- The vectors \mathbf{a} and A and \mathbf{q}_1 are all along a single line.
- The vectors \mathbf{a}, \mathbf{b} and A, B and $\mathbf{q}_1, \mathbf{q}_2$ are all in the same plane.
- The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and A, B, C and $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ are in one subspace (dimension 3).

At every step $\mathbf{a}_1, \dots, \mathbf{a}_k$ are combinations of $\mathbf{q}_1, \dots, \mathbf{q}_k$. Later \mathbf{q} 's are not involved. The connecting matrix R is *triangular*, and we have $A = QR$:

$$\begin{bmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \mathbf{q}_1^\top \mathbf{a} & \mathbf{q}_1^\top \mathbf{b} & \mathbf{q}_1^\top \mathbf{c} \\ \mathbf{q}_2^\top \mathbf{a} & \mathbf{q}_2^\top \mathbf{b} & \mathbf{q}_2^\top \mathbf{c} \\ \mathbf{q}_3^\top \mathbf{a} & \mathbf{q}_3^\top \mathbf{b} & \mathbf{q}_3^\top \mathbf{c} \end{bmatrix} \quad \text{or} \quad A = QR. \quad (9)$$

$A = QR$ is Gram-Schmidt in a nutshell. Multiply by Q^\top to see why $R = Q^\top A$.

4K (Gram-Schmidt) From independent vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$, Gram-Schmidt constructs orthonormal vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$. The matrices with these columns satisfy $A = QR$. Then $R = Q^\top A$ is *triangular* because later \mathbf{q} 's are orthogonal to earlier \mathbf{a} 's.

Here are the \mathbf{a} 's and \mathbf{q} 's from the example. The i, j entry of $R = Q^\top A$ is row i of Q^\top times column j of A . This is the dot product of \mathbf{q}_i with \mathbf{a}_j :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = QR.$$

The lengths of A, B, C are the numbers $\sqrt{2}, \sqrt{6}, \sqrt{3}$ on the diagonal of R . Because of the square roots, QR looks less beautiful than LU . Both factorizations are absolutely central to calculations in linear algebra.

Any m by n matrix A with independent columns can be factored into QR . The m by n matrix Q has orthonormal columns, and the square matrix R is upper triangular with positive diagonal. We must not forget why this is useful for least squares: $A^\top A \hat{x} = A^\top b$ simplifies to $R^\top R \hat{x} = R^\top b$.

$$R^\top R \hat{x} = R^\top Q^\top b \quad \text{or} \quad R \hat{x} = Q^\top b. \quad (10)$$

Instead of solving $Ax = b$, which is impossible, we solve $R\hat{x} = Q^T b$ by back substitution—which is very fast. The real cost is the mn^2 multiplications in the Gram-Schmidt process, which are needed to construct the orthogonal Q and the triangular R .

Here is an informal code. It executes equations (11) and (12), for $k = 1$ then $k = 2$ and eventually $k = n$. Equation (11) normalizes to unit vectors: For $k = 1, \dots, n$

$$r_{kk} = \left(\sum_{i=1}^m a_{ik}^2 \right)^{1/2} \quad \text{and} \quad q_{ik} = \frac{a_{ik}}{r_{kk}} \quad \text{for } i = 1, \dots, m. \quad (11)$$

Equation (12) subtracts from a_j its projection onto q_k : For $j = k + 1, \dots, n$

$$r_{kj} = \sum_{i=1}^m q_{ik} a_{ij} \quad \text{and} \quad a_{ij} = a_{ij} - q_{ik} r_{kj} \quad \text{for } i = 1, \dots, m. \quad (12)$$

Starting from $\mathbf{a}, \mathbf{b}, \mathbf{c} = \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ this code will construct $\mathbf{q}_1, \mathbf{B}, \mathbf{q}_2, \mathbf{C}, \mathbf{q}_3$:

- 1 $\mathbf{q}_1 = \mathbf{a}_1 / \|\mathbf{a}_1\|$ in (11)
- 2 $\mathbf{B} = \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2) \mathbf{q}_1$ and $\mathbf{C}^* = \mathbf{a}_3 - (\mathbf{q}_1^T \mathbf{a}_3) \mathbf{q}_1$ in (12)
- 3 $\mathbf{q}_2 = \mathbf{B} / \|\mathbf{B}\|$ in (11)
- 4 $\mathbf{C} = \mathbf{C}^* - (\mathbf{q}_2^T \mathbf{C}^*) \mathbf{q}_2$ in (12)
- 5 $\mathbf{q}_3 = \mathbf{C} / \|\mathbf{C}\|$ in (11)

Equation (12) subtracts off projections as soon as the new vector \mathbf{q}_k is found. This change to “subtract one projection at a time” is called **modified Gram-Schmidt**. It is numerically more stable than equation (8) which subtracts all projections at once.

■ REVIEW OF THE KEY IDEAS ■

1. If the orthonormal vectors $\mathbf{q}_1, \dots, \mathbf{q}_n$ are the columns of Q , then $\mathbf{q}_i^T \mathbf{q}_j = 0$ and $\mathbf{q}_i^T \mathbf{q}_i = 1$ translate into $Q^T Q = I$.
2. If Q is square (an *orthogonal matrix*) then $Q^T = Q^{-1}$.
3. The length of Qx equals the length of x : $\|Qx\| = \|x\|$.
4. The projection onto the column space spanned by the \mathbf{q} 's is $P = Q Q^T$.
5. If Q is square then $P = I$ and every $\mathbf{b} = \mathbf{q}_1(\mathbf{q}_1^T \mathbf{b}) + \dots + \mathbf{q}_n(\mathbf{q}_n^T \mathbf{b})$.
6. Gram-Schmidt produces orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ from independent $\mathbf{a}, \mathbf{b}, \mathbf{c}$. In matrix form this is the factorization $A = QR = (\text{orthogonal } Q)(\text{triangular } R)$.

■ WORKED EXAMPLES ■

4.4 A Add two more columns with all entries 1 or -1 , so the columns of this 4 by 4 “Hadamard matrix” are orthogonal. How do you turn H into an *orthogonal matrix* Q ?

$$H = \begin{bmatrix} 1 & 1 & x & x \\ 1 & 1 & x & x \\ 1 & -1 & x & x \\ 1 & -1 & x & x \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Why can't a 5 by 5 matrix have orthogonal columns of 1's and -1 's? Actually the next possible size is 8 by 8, constructed from four blocks:

The block matrix $H_8 = \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ is a Hadamard matrix with orthogonal columns. What is the product $H_8^T H_8$?

The projection of $\mathbf{b} = (6, 0, 0, 2)$ onto the first column of H is $\mathbf{p}_1 = (2, 2, 2, 2)$ and the projection onto the second column is $\mathbf{p}_2 = (1, 1, -1, -1)$. What is the projection $\mathbf{p}_{1,2}$ of \mathbf{b} onto the 2-dimensional space spanned by the first two columns, and why?

Solution Columns 3 and 4 of this H could be multiplied by -1 or exchanged:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \text{ has orthogonal columns. } Q = \frac{H}{2} \text{ has orthonormal columns.}$$

Dividing by 2 gives unit vectors in Q . Orthogonality for 5 by 5 is impossible because the dot product of columns would have five 1's and/or -1 's and could not add to zero. The 8 by 8 matrix H_8 does have orthogonal columns (of length $\sqrt{8}$). Then Q_8 will be $H_8/\sqrt{8}$:

$$H_8^T H_8 = \begin{bmatrix} H^T & H^T \\ H^T & -H^T \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \begin{bmatrix} 2H^T H & 0 \\ 0 & 2H^T H \end{bmatrix} = \begin{bmatrix} 8I & 0 \\ 0 & 8I \end{bmatrix}.$$

When columns are orthogonal, we can project $(6, 0, 0, 2)$ onto $(1, 1, 1, 1)$ and $(1, 1, -1, -1)$ and add:

$$\text{Projection } \mathbf{p}_{1,2} = \mathbf{p}_1 + \mathbf{p}_2 = (2, 2, 2, 2) + (1, 1, -1, -1) = (3, 3, 1, 1).$$

This is the value of orthogonal columns. A quick proof of $\mathbf{p}_{1,2} = \mathbf{p}_1 + \mathbf{p}_2$ is to check that columns 1 and 2 (call them \mathbf{a}_1 and \mathbf{a}_2) are perpendicular to the error $\mathbf{e} = \mathbf{b} - \mathbf{p}_1 - \mathbf{p}_2$:

$$\mathbf{e} = \mathbf{b} - \frac{\mathbf{a}_1^T \mathbf{b}}{\mathbf{a}_1^T \mathbf{a}_1} \mathbf{a}_1 - \frac{\mathbf{a}_2^T \mathbf{b}}{\mathbf{a}_2^T \mathbf{a}_2} \mathbf{a}_2 \quad \text{and} \quad \mathbf{a}_1^T \mathbf{e} = \mathbf{a}_1^T \mathbf{b} - \frac{\mathbf{a}_1^T \mathbf{b}}{\mathbf{a}_1^T \mathbf{a}_1} \mathbf{a}_1^T \mathbf{a}_1 = 0 \quad \text{and also} \quad \mathbf{a}_2^T \mathbf{e} = 0.$$

So $\mathbf{p}_1 + \mathbf{p}_2$ is in the space of \mathbf{a}_1 and \mathbf{a}_2 , and its error \mathbf{e} is perpendicular to that space.

The Gram-Schmidt process on those orthogonal columns \mathbf{a}_1 and \mathbf{a}_2 would be happy with their directions. It would only divide by their lengths. But if \mathbf{a}_1 and \mathbf{a}_2 are not orthogonal, the projection $\mathbf{p}_{1,2}$ is not generally $\mathbf{p}_1 + \mathbf{p}_2$. For example, if $\mathbf{b} = \mathbf{a}_1$ then $\mathbf{p}_1 = \mathbf{b}$ and $\mathbf{p}_{1,2} = \mathbf{b}$ but $\mathbf{p}_2 \neq \mathbf{0}$.

Problem Set 4.4

Problems 1–12 are about orthogonal vectors and orthogonal matrices.

- 1 Are these pairs of vectors orthonormal or only orthogonal or only independent?

(a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} .6 \\ .8 \end{bmatrix}$ and $\begin{bmatrix} .4 \\ -.3 \end{bmatrix}$ (c) $\begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ and $\begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$.

Change the second vector when necessary to produce orthonormal vectors.

- 2 The vectors $(2, 2, -1)$ and $(-1, 2, 2)$ are orthogonal. Divide them by their lengths to find orthonormal vectors \mathbf{q}_1 and \mathbf{q}_2 . Put those into the columns of Q and multiply $Q^T Q$ and QQ^T .
- 3 (a) If A has three orthogonal columns each of length 4, what is $A^T A$?
 (b) If A has three orthogonal columns of lengths 1, 2, 3, what is $A^T A$?
- 4 Give an example of each of the following:
- (a) A matrix Q that has orthonormal columns but $QQ^T \neq I$.
 (b) Two orthogonal vectors that are not linearly independent.
 (c) An orthonormal basis for \mathbf{R}^4 , where every component is $\frac{1}{2}$ or $-\frac{1}{2}$.
- 5 Find two orthogonal vectors in the plane $x+y+2z=0$. Make them orthonormal.
- 6 If Q_1 and Q_2 are orthogonal matrices, show that their product $Q_1 Q_2$ is also an orthogonal matrix. (Use $Q^T Q = I$.)
- 7 If Q has orthonormal columns, what is the least squares solution $\hat{\mathbf{x}}$ to $Q\mathbf{x} = \mathbf{b}$?
- 8 If \mathbf{q}_1 and \mathbf{q}_2 are orthonormal vectors in \mathbf{R}^5 , what combination $\underline{\quad} \mathbf{q}_1 + \underline{\quad} \mathbf{q}_2$ is closest to a given vector \mathbf{b} ?
- 9 (a) Compute $P = QQ^T$ when $\mathbf{q}_1 = (.8, .6, 0)$ and $\mathbf{q}_2 = (-.6, .8, 0)$. Verify that $P^2 = P$.
 (b) Prove that always $(QQ^T)(QQ^T) = QQ^T$ by using $Q^T Q = I$. Then $P = QQ^T$ is the projection matrix onto the column space of Q .
- 10 Orthonormal vectors are automatically linearly independent. Two proofs:

- (a) Vector proof: When $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3 = \mathbf{0}$, what dot product leads to $c_1 = 0$? Similarly $c_2 = 0$ and $c_3 = 0$. Thus the \mathbf{q} 's are independent.
- (b) Matrix proof: Show that $Q\mathbf{x} = \mathbf{0}$ leads to $\mathbf{x} = \mathbf{0}$. Since Q may be rectangular, you can use Q^T but not Q^{-1} .
- 11** (a) Find orthonormal vectors \mathbf{q}_1 and \mathbf{q}_2 in the plane of $\mathbf{a} = (1, 3, 4, 5, 7)$ and $\mathbf{b} = (-6, 6, 8, 0, 8)$.
- (b) Which vector in this plane is closest to $(1, 0, 0, 0, 0)$?
- 12** If $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is a basis for \mathbb{R}^3 , any vector \mathbf{b} can be written as

$$\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 \quad \text{or} \quad \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{b}.$$

- (a) Suppose the \mathbf{a} 's are orthonormal. Show that $x_1 = \mathbf{a}_1^T \mathbf{b}$.
- (b) Suppose the \mathbf{a} 's are orthogonal. Show that $x_1 = \mathbf{a}_1^T \mathbf{b} / \|\mathbf{a}_1\|$.
- (c) If the \mathbf{a} 's are independent, x_1 is the first component of _____ times \mathbf{b} .

Problems 13–25 are about the Gram-Schmidt process and $A = QR$.

- 13** What multiple of $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be subtracted from $\mathbf{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ to make the result \mathbf{B} orthogonal to \mathbf{a} ? Sketch a figure to show \mathbf{a} , \mathbf{b} , and \mathbf{B} .
- 14** Complete the Gram-Schmidt process in Problem 13 by computing $\mathbf{q}_1 = \mathbf{a}/\|\mathbf{a}\|$ and $\mathbf{q}_2 = \mathbf{B}/\|\mathbf{B}\|$ and factoring into QR :

$$\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{bmatrix} \begin{bmatrix} \|\mathbf{a}\| & ? \\ 0 & \|\mathbf{B}\| \end{bmatrix}.$$

- 15** (a) Find orthonormal vectors $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ such that $\mathbf{q}_1, \mathbf{q}_2$ span the column space of

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

- (b) Which of the four fundamental subspaces contains \mathbf{q}_3 ?
- (c) Solve $A\mathbf{x} = (1, 2, 7)$ by least squares.
- 16** What multiple of $\mathbf{a} = (4, 5, 2, 2)$ is closest to $\mathbf{b} = (1, 2, 0, 0)$? Find orthonormal vectors \mathbf{q}_1 and \mathbf{q}_2 in the plane of \mathbf{a} and \mathbf{b} .
- 17** Find the projection of \mathbf{b} onto the line through \mathbf{a} :

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{p} = ? \quad \text{and} \quad \mathbf{e} = \mathbf{b} - \mathbf{p} = ?$$

Compute the orthonormal vectors $\mathbf{q}_1 = \mathbf{a}/\|\mathbf{a}\|$ and $\mathbf{q}_2 = \mathbf{e}/\|\mathbf{e}\|$.

- 18** (Recommended) Find orthogonal vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ by Gram-Schmidt from $\mathbf{a}, \mathbf{b}, \mathbf{c}$:

$$\mathbf{a} = (1, -1, 0, 0) \quad \mathbf{b} = (0, 1, -1, 0) \quad \mathbf{c} = (0, 0, 1, -1).$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are bases for the vectors perpendicular to $\mathbf{d} = (1, 1, 1, 1)$.

- 19** If $A = QR$ then $A^T A = R^T R = \underline{\hspace{2cm}}$ triangular times $\underline{\hspace{2cm}}$ triangular. Gram-Schmidt on A corresponds to elimination on $A^T A$. Compare the pivots for $A^T A$ with $\|\mathbf{a}\|^2 = 3$ and $\|\mathbf{e}\|^2 = 8$ in Problem 17:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad A^T A = \begin{bmatrix} 3 & 9 \\ 9 & 35 \end{bmatrix}.$$

- 20** True or false (give an example in either case):

- (a) Q^{-1} is an orthogonal matrix when Q is an orthogonal matrix.
 (b) If Q (3 by 2) has orthonormal columns then $\|Q\mathbf{x}\|$ always equals $\|\mathbf{x}\|$.

- 21** Find an orthonormal basis for the column space of A :

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -4 \\ -3 \\ 3 \\ 0 \end{bmatrix}.$$

Then compute the projection of \mathbf{b} onto that column space.

- 22** Find orthogonal vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ by Gram-Schmidt from

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

- 23** Find $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$ (orthonormal) as combinations of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (independent columns). Then write A as QR :

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 5 \\ 0 & 3 & 6 \end{bmatrix}.$$

- 24** (a) Find a basis for the subspace S in \mathbf{R}^4 spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

- (b) Find a basis for the orthogonal complement S^\perp .
 (c) Find \mathbf{b}_1 in S and \mathbf{b}_2 in S^\perp so that $\mathbf{b}_1 + \mathbf{b}_2 = \mathbf{b} = (1, 1, 1, 1)$.

- 25 If $ad - bc > 0$, the entries in $A = QR$ are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{\begin{bmatrix} a & -c \\ c & a \end{bmatrix}}{\sqrt{a^2 + c^2}} \frac{\begin{bmatrix} a^2 + c^2 & ab + cd \\ 0 & ad - bc \end{bmatrix}}{\sqrt{a^2 + c^2}}.$$

Write $A = QR$ when $a, b, c, d = 2, 1, 1, 1$ and also $1, 1, 1, 1$. Which entry of R becomes zero when the columns are dependent and Gram-Schmidt breaks down?

Problems 26–29 use the QR code in equations (11–12). It executes Gram-Schmidt.

- 26 Show why C (found via C^* in the steps after (12)) is equal to C in equation (8).
- 27 Equation (8) subtracts from c its components along A and B . Why not subtract the components along a and along b ?
- 28 Write a working code and apply it to $a = (2, 2, -1)$, $b = (0, -3, 3)$, $c = (1, 0, 0)$. What are the q 's?
- 29 Where are the mn^2 multiplications in equations (11) and (12)?

Problems 30–35 involve orthogonal matrices that are special.

- 30 The first four *wavelets* are in the columns of this wavelet matrix W :

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}.$$

What is special about the columns? Find the inverse wavelet transform W^{-1} .

- 31 (a) Choose c so that Q is an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

- (b) Change the first row and column to all 1's and fill in another orthogonal Q .

- 32 Project $b = (1, 1, 1, 1)$ onto the first column in Problem 31(a). Then project b onto the plane of the first two columns.
- 33 If u is a unit vector, then $Q = I - 2uu^T$ is a Householder reflection matrix (Example 3). Find Q_1 from $u = (0, 1)$ and Q_2 from $u = (0, \sqrt{2}/2, \sqrt{2}/2)$. Draw the reflections when Q_1 and Q_2 multiply (x, y) and (x, y, z) .

- 34 $Q = I - 2\mathbf{u}\mathbf{u}^T$ is a reflection matrix when $\mathbf{u}^T\mathbf{u} = 1$.
- Show that $Q\mathbf{u} = -\mathbf{u}$. The mirror is perpendicular to \mathbf{u} .
 - Find $Q\mathbf{v}$ when $\mathbf{u}^T\mathbf{v} = 0$. The mirror contains \mathbf{v} . It reflects to itself.
- 35 (MATLAB) Factor $[Q, R] = \mathbf{qr}(A)$ for $A = \mathbf{eye}(4) - \mathbf{diag}([1 1 1], -1)$. Can you renormalize the orthogonal columns of Q to get nice integer components?
- 36 Find all matrices that are both orthogonal and lower triangular.

5

DETERMINANTS

THE PROPERTIES OF DETERMINANTS ■ 5.1

The determinant of a square matrix is a single number. That number contains an amazing amount of information about the matrix. It tells immediately whether the matrix is invertible. *The determinant is zero when the matrix has no inverse.* When A is invertible, the determinant of A^{-1} is $1/(\det A)$. If $\det A = 2$ then $\det A^{-1} = \frac{1}{2}$. In fact the determinant leads to a formula for every entry in A^{-1} .

This is one use for determinants—to find formulas for inverse matrices and pivots and solutions $A^{-1}\mathbf{b}$. For a matrix of numbers, we seldom use those formulas. (Or rather, we use elimination as the quickest way to the answer.) For a matrix with entries a, b, c, d , its determinant shows how A^{-1} changes as A changes:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (1)$$

Multiply those matrices to get I . The determinant of A is $ad - bc$. When $\det A = 0$, we are asked to divide by zero and we can't—then A has no inverse. (The rows are parallel when $a/c = b/d$. This gives $ad = bc$ and a zero determinant.) Dependent rows lead to $\det A = 0$.

The determinant is also connected to the pivots. For a 2 by 2 matrix the pivots are a and $d - (c/a)b$. *The product of the pivots is the determinant:*

$$a\left(d - \frac{c}{a}b\right) = ad - bc \quad \text{which is} \quad \det A.$$

After a row exchange the pivots are c and $b - (a/c)d$. Those pivots multiply to give $bc - ad$. The row exchange reversed the sign of the determinant.

Looking ahead The determinant of an n by n matrix can be found in three ways:

- | | |
|--|---|
| 1 Multiply the n pivots (times 1 or -1).
2 Add up $n!$ terms (times 1 or -1).
3 Combine n smaller determinants (times 1 or -1). | This is the pivot formula.
This is the “big” formula.
This is the cofactor formula. |
|--|---|

You see that plus or minus signs—the decisions between 1 and -1 —play a big part in determinants. That comes from the following rule for n by n matrices:

The determinant changes sign when two rows (or two columns) are exchanged.

The identity matrix has determinant +1. Exchange two rows and $\det P = -1$. Exchange two more rows and the new permutation has $\det P = +1$. Half of all permutations are *even* ($\det P = 1$) and half are *odd* ($\det P = -1$). Starting from I , half of the P 's involve an even number of exchanges and half require an odd number. In the 2 by 2 case, ad has a plus sign and bc has minus—coming from the row exchange:

$$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \quad \text{and} \quad \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1.$$

The other essential rule is linearity—but a warning comes first. Linearity does not mean that $\det(A + B) = \det A + \det B$. **This is absolutely false.** That kind of linearity is not even true when $A = I$ and $B = I$. The false rule would say that $\det 2I = 1 + 1 = 2$. The true rule is $\det 2I = 2^n$. Determinants are multiplied by 2^n (not just by 2) when matrices are multiplied by 2.

We don't intend to define the determinant by its formulas. It is better to start with its properties—*sign reversal and linearity*. The properties are simple (Section 5.1). They prepare for the formulas (Section 5.2). Then come the applications, including these three:

- (1) Determinants give A^{-1} and $A^{-1}\mathbf{b}$ (this formula is called *Cramer's Rule*).
- (2) When the edges of a box are the rows of A , the *volume* is $|\det A|$.
- (3) The numbers λ for which $A - \lambda I$ is singular and $\det(A - \lambda I) = 0$ are the *eigenvalues* of A . This is the most important application and it fills Chapter 6.

The Properties of the Determinant

There are three basic properties (rules 1, 2, 3). By using those rules we can compute the determinant of any square matrix A . **This number is written in two ways, $\det A$ and $|A|$.** Notice: Brackets for the matrix, straight bars for its determinant. When A is a 2 by 2 matrix, the three properties lead to the answer we expect:

$$\text{The determinant of } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We will check each rule against this 2 by 2 formula, but do not forget: The rules apply to any n by n matrix. When we prove that properties 4–10 follow from 1–3, the proof must apply to all square matrices.

Property 1 (the easiest rule) matches the determinant of I with the volume of a unit cube.

1 The determinant of the n by n identity matrix is 1.

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{vmatrix} = 1.$$

2 The determinant changes sign when two rows are exchanged (sign reversal):

$$\text{Check: } \begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad (\text{both sides equal } bc - ad).$$

Because of this rule, we can find $\det P$ for any permutation matrix. Just exchange rows of I until you reach P . Then $\det P = +1$ for an **even** number of row exchanges and $\det P = -1$ for an **odd** number.

The third rule has to make the big jump to the determinants of all matrices.

3 The determinant is a linear function of each row separately (all other rows stay fixed). If the first row is multiplied by t , the determinant is multiplied by t . If first rows are added, determinants are added. This rule only applies when the other rows do not change! Notice how c and d stay the same:

$$\begin{array}{l} \text{multiply row 1 by any number } t: \quad \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix} \\ \text{add row 1 of } A \text{ to row 1 of } A': \quad \begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}. \end{array}$$

In the first case, both sides are $tad - tbc$. Then t factors out. In the second case, both sides are $ad + a'd - bc - b'c$. These rules still apply when A is n by n , and the last $n - 1$ rows don't change. May we emphasize rule 3 with numbers:

$$\begin{vmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 5 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

By itself, rule 3 does not say what any of those determinants are. But with rule 1, the first determinant is 5 (and the second is 1).

Combining multiplication and addition, we get any linear combination in the first row: $t(\text{row 1 of } A) + t'(\text{row 1 of } A')$. With this combined row, the determinant is t times $\det A$ plus t' times $\det A'$. The other rows must stay the same.

This rule does not mean that $\det 2I = 2 \det I$. To obtain $2I$ we have to multiply *both* rows by 2, and the factor 2 comes out both times:

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 2^2 = 4 \quad \text{and} \quad \begin{vmatrix} t & 0 \\ 0 & t \end{vmatrix} = t^2.$$

This is just like area and volume. Expand a rectangle by 2 and its area increases by 4. Expand an n -dimensional box by t and its volume increases by t^n . The connection is no accident—we will see how *determinants equal volumes*.

Pay special attention to rules 1–3. They completely determine the number $\det A$ —but for a big matrix that fact is not obvious. We could stop here to find a formula for n by n determinants. It would be a little complicated—we prefer to go gradually. Instead we write down other properties which follow directly from the first three. These extra rules make determinants much easier to work with.

4 If two rows of A are equal, then $\det A = 0$.

$$\text{Check 2 by 2: } \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

Rule 4 follows from rule 2. (Remember we must use the rules and not the 2 by 2 formula.) *Exchange the two equal rows.* The determinant D is supposed to change sign. But also D has to stay the same, because the matrix is not changed. The only number with $-D = D$ is $D = 0$ —this must be the determinant. (Note: In Boolean algebra the reasoning fails, because $-1 = 1$. Then D is defined by rules 1, 3, 4.)

A matrix with two equal rows has no inverse. Rule 4 makes $\det A = 0$. But matrices can be singular and determinants can be zero without having equal rows! Rule 5 will be the key. We can do row operations without changing $\det A$.

5 Subtracting a multiple of one row from another row leaves $\det A$ unchanged.

$$\begin{vmatrix} a & b \\ c - \ell a & d - \ell b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Linearity splits the left side into the right side plus another term $-\ell \begin{vmatrix} a & b \\ a & b \end{vmatrix}$. This extra term is zero by rule 4. Therefore rule 5 is correct. Note how the second row changes while the first row stays the same—as required by rule 3.

Conclusion *The determinant is not changed by the usual elimination steps from A to U .* Thus $\det A$ equals $\det U$. If we can find determinants of triangular matrices U , we can find determinants of all matrices A . Every row exchange reverses the sign, so always $\det A = \pm \det U$. Rule 5 has narrowed the problem to triangular matrices.

6 A matrix with a row of zeros has $\det A = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0.$$

For an easy proof, add some other row to the zero row. The determinant is not changed (rule 5). But the matrix now has two equal rows. So $\det A = 0$ by rule 4.

7 If A is triangular then $\det A = a_{11}a_{22}\cdots a_{nn} = \text{product of diagonal entries}$.

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad \quad \text{and also} \quad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$$

Suppose all diagonal entries of A are nonzero. Eliminate the off-diagonal entries by the usual steps. (If A is lower triangular, subtract multiples of each row from lower

rows. If A is upper triangular, subtract from higher rows.) By rule 5 the determinant is not changed—and now the matrix is diagonal:

$$\text{We must still prove that } \det \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} = a_{11}a_{22} \cdots a_{nn}.$$

For this we apply rules 1 and 3. Factor a_{11} from the first row. Then factor a_{22} from the second row. Eventually factor a_{nn} from the last row. The determinant is a_{11} times a_{22} times \cdots times a_{nn} times $\det I$. Then rule 1 (used at last!) is $\det I = 1$.

What if a diagonal entry a_{ii} is zero? Then the triangular A is singular. Elimination produces a *zero row*. By rule 5 the determinant is unchanged, and by rule 6 a zero row means $\det A = 0$. Thus rule 7 is proved—triangular matrices have easy determinants.

8 If A is singular then $\det A = 0$. If A is invertible then $\det A \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is singular if and only if } ad - bc = 0.$$

Proof Elimination goes from A to U . If A is singular then U has a zero row. The rules give $\det A = \det U = 0$. If A is invertible then U has the pivots along its diagonal. The product of nonzero pivots (using rule 7) gives a nonzero determinant:

$$\det A = \pm \det U = \pm (\text{product of the pivots}). \quad (2)$$

The pivots of a 2 by 2 matrix (if $a \neq 0$) are a and $d - (bc/a)$:

$$\text{The determinant is } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - (bc/a) \end{vmatrix} = ad - bc.$$

This is the first formula for the determinant. MATLAB would use it to find $\det A$ from the pivots. The plus or minus sign depends on whether the number of row exchanges is even or odd. In other words, $+1$ or -1 is the determinant of the permutation matrix P that exchanges rows. With no row exchanges, the number zero is even and $P = I$ and $\det A = \det U = \text{product of pivots}$. Always $\det L = 1$, because L is triangular with 1's on the diagonal. What we have is this:

$$\text{If } PA = LU \text{ then } \det P \det A = \det L \det U. \quad (3)$$

Again, $\det P = \pm 1$ and $\det A = \pm \det U$. Equation (3) is our first case of rule 9.

9 The determinant of AB is $\det A$ times $\det B$: $|AB| = |A||B|$.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} p & q \\ r & s \end{vmatrix} = \begin{vmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{vmatrix}.$$

When the matrix B is A^{-1} , this rule says that the determinant of A^{-1} is $1/\det A$:

$$AA^{-1} = I \text{ so } (\det A)(\det A^{-1}) = \det I = 1.$$

This product rule is the most intricate so far. We could check the 2 by 2 case by algebra:

$$|A||B| = (ad - bc)(ps - qr) = (ap + br)(cq + ds) - (aq + bs)(cp + dr) = |AB|.$$

For the n by n case, here is a snappy proof that $|AB| = |A||B|$. When $|B|$ is not zero, consider the ratio $D(A) = |AB|/|B|$. If this ratio has properties 1,2,3—which we now check—it has to be the determinant $|A|$.

Property 1 (Determinant of I) If $A = I$ then the ratio becomes $|B|/|B| = 1$.

Property 2 (Sign reversal) When two rows of A are exchanged, so are the same two rows of AB . Therefore $|AB|$ changes sign and so does the ratio $|AB|/|B|$.

Property 3 (Linearity) When row 1 of A is multiplied by t , so is row 1 of AB . This multiplies $|AB|$ by t and multiplies the ratio by t —as desired.

If row 1 of A is added to row 1 of A' , then row 1 of AB is added to row 1 of $A'B$. By rule 3, the determinants add. After dividing by $|B|$, the ratios add.

Conclusion This ratio $|AB|/|B|$ has the same three properties that define $|A|$. Therefore it equals $|A|$. This proves the product rule $|AB| = |A||B|$. The case $|B| = 0$ is separate and easy, because AB is singular when B is singular. The rule $|AB| = |A||B|$ becomes $0 = 0$.

10 The transpose A^T has the same determinant as A .

$$\text{Check: } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} \quad \text{since both sides equal } ad - bc.$$

The equation $|A^T| = |A|$ becomes $0 = 0$ when A is singular (we know that A^T is also singular). Otherwise A has the usual factorization $PA = LU$. Transposing both sides gives $A^TP^T = U^TL^T$. The proof of $|A| = |A^T|$ comes by using rule 9 for products:

$$\text{Compare } \det P \det A = \det L \det U \quad \text{with} \quad \det A^T \det P^T = \det U^T \det L^T.$$

First, $\det L = \det L^T = 1$ (both have 1's on the diagonal). Second, $\det U = \det U^T$ (transposing leaves the main diagonal unchanged, and triangular determinants only involve that diagonal). Third, $\det P = \det P^T$ (permutations have $P^T = P^{-1}$, so $|P||P^T| = 1$ by rule 9; thus $|P|$ and $|P^T|$ both equal 1 or both equal -1). Fourth and finally, the comparison proves that $\det A$ equals $\det A^T$.

Important comment Rule 10 practically doubles our list of properties. Every rule for the rows can apply also to the columns (just by transposing, since $|A| = |A^T|$). The determinant changes sign when two columns are exchanged. *A zero column or two*

equal columns will make the determinant zero. If a column is multiplied by t , so is the determinant. The determinant is a linear function of each column separately.

It is time to stop. The list of properties is long enough. Next we find and use an explicit formula for the determinant.

■ REVIEW OF THE KEY IDEAS ■

1. The determinant is defined by $\det I = 1$, sign reversal, and linearity in each row.
2. After elimination $\det A$ is \pm (product of the pivots).
3. The determinant is zero exactly when A is not invertible.
4. Two remarkable properties are $\det AB = (\det A)(\det B)$ and $\det A^T = \det A$.

■ WORKED EXAMPLES ■

5.1 A Apply these operations to A and find the determinants of M_1, M_2, M_3, M_4 :

In M_1 , each a_{ij} is multiplied by $(-1)^{i+j}$. This gives the sign pattern shown below.

In M_2 , rows 1, 2, 3 of A are *subtracted* from rows 2, 3, 1.

In M_3 , rows 1, 2, 3 of A are *added* to rows 2, 3, 1.

The i, j entry of M_4 is (row i of A) \cdot (row j of A).

How are the determinants of M_1, M_2, M_3, M_4 related to the determinant of A ?

$$\begin{bmatrix} a_{11} & -a_{12} & a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ a_{31} & -a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \text{row 1} - \text{row 3} \\ \text{row 2} - \text{row 1} \\ \text{row 3} - \text{row 2} \end{bmatrix} \begin{bmatrix} \text{row 1} + \text{row 3} \\ \text{row 2} + \text{row 1} \\ \text{row 3} + \text{row 2} \end{bmatrix} \begin{bmatrix} \text{row 1} \cdot \text{row 1} \dots \\ \text{row 2} \cdot \text{row 1} \dots \\ \text{row 3} \cdot \text{row 1} \dots \end{bmatrix}$$

Solution The four determinants are $\det A$, 0, $2\det A$, and $(\det A)^2$. Here are reasons:

$$M_1 = \begin{bmatrix} 1 & -1 & \\ & 1 & \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \quad \text{so } \det M_1 = (-1)(\det A)(-1).$$

The matrix M_2 is singular because its rows add to the zero row. Then $\det M_2 = 0$.

The matrix M_3 can be split into *eight matrices* by Rule 3 (linearity in each row):

$$\det M_3 = \left| \begin{array}{c} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array} \right| + \left| \begin{array}{c} \text{row 3} \\ \text{row 2} \\ \text{row 3} \end{array} \right| + \left| \begin{array}{c} \text{row 1} \\ \text{row 1} \\ \text{row 3} \end{array} \right| + \cdots + \left| \begin{array}{c} \text{row 3} \\ \text{row 1} \\ \text{row 2} \end{array} \right|.$$

All but the first and last have repeated rows and zero determinant. The first is A and the last has *two* row exchanges. So $\det M_3 = \det A + \det A$. (Try $A = I$.)

The matrix M_4 is exactly AA^T . Its determinant is $(\det A)(\det A^T) = (\det A)^2$.

5.1 B Find the determinant of A by subtracting row 1 from row 2, then column 3 from column 2, then row or column exchanges to make the matrix lower triangular:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 0 & b & 1 \end{bmatrix} \quad \text{is singular for which } a \text{ and } b?$$

Solution Subtract row 1 from row 2, then column 3 from column 2. *Two* exchanges make the matrix triangular. Then $\det A = (a - 1)(b - 1)$.

$$A \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ a - 1 & 0 & 0 \\ 0 & b - 1 & 1 \end{bmatrix} \xrightarrow{\substack{\text{rows 1} \leftrightarrow 2 \\ \text{columns 2} \leftrightarrow 3}} \begin{bmatrix} a - 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & b - 1 \end{bmatrix}.$$

Note that $a = 1$ gives equal rows in A and $b = 1$ gives equal columns. So not surprising that $(a - 1)$ and $(b - 1)$ are factors of $\det A$.

Problem Set 5.1

Questions 1–12 are about the rules for determinants.

- 1 If a 4 by 4 matrix has $\det A = \frac{1}{2}$, find $\det(2A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.
- 2 If a 3 by 3 matrix has $\det A = -1$, find $\det(\frac{1}{2}A)$ and $\det(-A)$ and $\det(A^2)$ and $\det(A^{-1})$.
- 3 True or false, with a reason if true or a counterexample if false:
 - (a) The determinant of $I + A$ is $1 + \det A$.
 - (b) The determinant of ABC is $|A||B||C|$.
 - (c) The determinant of $4A$ is $4|A|$.
 - (d) The determinant of $AB - BA$ is zero. (Try an example.)

- 4 Which row exchanges show that these “reverse identity matrices” J_3 and J_4 have $|J_3| = -1$ but $|J_4| = +1$?

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = -1 \quad \text{but} \quad \det \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = +1.$$

- 5 For $n = 5, 6, 7$, count the row exchanges to permute the reverse identity J_n to the identity matrix I_n . Propose a rule for every size n and predict whether J_{101} has determinant $+1$ or -1 .
- 6 Show how Rule 6 (determinant = 0 if a row is all zero) comes from Rule 3.
- 7 Find the determinants of rotations and reflections:
- $$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 - 2 \cos^2 \theta & -2 \cos \theta \sin \theta \\ -2 \cos \theta \sin \theta & 1 - 2 \sin^2 \theta \end{bmatrix}.$$
- 8 Prove that every orthogonal matrix ($Q^T Q = I$) has determinant 1 or -1 .
- Use the product rule $|AB| = |A||B|$ and the transpose rule $|Q| = |Q^T|$.
 - Use only the product rule. If $|\det Q| > 1$ then $\det Q^n = (\det Q)^n$ blows up. How do you know this can't happen to Q^n ?
- 9 Do these matrices have determinant 0, 1, 2, or 3?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- 10 If the entries in every row of A add to zero, solve $Ax = \mathbf{0}$ to prove $\det A = 0$. If those entries add to one, show that $\det(A - I) = 0$. Does this mean $\det A = 1$?
- 11 Suppose that $CD = -DC$ and find the flaw in this reasoning: Taking determinants gives $|C||D| = -|D||C|$. Therefore $|C| = 0$ or $|D| = 0$. One or both of the matrices must be singular. (That is not true.)
- 12 The inverse of a 2 by 2 matrix seems to have determinant = 1:

$$\det A^{-1} = \det \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{ad - bc}{ad - bc} = 1.$$

What is wrong with this calculation? What is the correct $\det A^{-1}$?

Questions 13–27 use the rules to compute specific determinants.

- 13 Reduce A to U and find $\det A =$ product of the pivots:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}.$$

- 14 By applying row operations to produce an upper triangular U , compute

$$\det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}.$$

- 15 Use row operations to simplify and compute these determinants:

$$\det \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

- 16 Find the determinants of a rank one matrix and a skew-symmetric matrix:

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}.$$

- 17 A skew-symmetric matrix has $K^T = -K$. Insert a, b, c for 1, 3, 4 in Question 16 and show that $|K| = 0$. Write down a 4 by 4 example with $|K| = 1$.

- 18 Use row operations to show that the 3 by 3 “Vandermonde determinant” is

$$\det \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$$

- 19 Find the determinants of U and U^{-1} and U^2 :

$$U = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

- 20 Suppose you do two row operations at once, going from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} a - Lc & b - Ld \\ c - la & d - lb \end{bmatrix}.$$

Find the second determinant. Does it equal $ad - bc$?

- 21** *Row exchange:* Add row 1 of A to row 2, then subtract row 2 from row 1. Then add row 1 to row 2 and multiply row 1 by -1 to reach B . Which rules show

$$\det B = \begin{vmatrix} c & d \\ a & b \end{vmatrix} \text{ equals } -\det A = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}?$$

Those rules could replace Rule 2 in the definition of the determinant.

- 22** From $ad - bc$, find the determinants of A and A^{-1} and $A - \lambda I$:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix}.$$

Which two numbers λ lead to $\det(A - \lambda I) = 0$? Write down the matrix $A - \lambda I$ for each of those numbers λ —it should not be invertible.

- 23** From $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ find A^2 and A^{-1} and $A - \lambda I$ and their determinants. Which two numbers λ lead to $|A - \lambda I| = 0$?

- 24** Elimination reduces A to U . Then $A = LU$:

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 6 & 8 & 7 \\ -3 & 5 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{bmatrix} = LU.$$

Find the determinants of L , U , A , $U^{-1}L^{-1}$, and $U^{-1}L^{-1}A$.

- 25** If the i, j entry of A is i times j , show that $\det A = 0$. (Exception when $A = [1]$.)

- 26** If the i, j entry of A is $i + j$, show that $\det A = 0$. (Exception when $n = 1$ or 2.)

- 27** Compute the determinants of these matrices by row operations:

$$A = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}.$$

- 28** True or false (give a reason if true or a 2 by 2 example if false):

- (a) If A is not invertible then AB is not invertible.
- (b) The determinant of A is always the product of its pivots.
- (c) The determinant of $A - B$ equals $\det A - \det B$.
- (d) AB and BA have the same determinant.

- 29** What is wrong with this proof that projection matrices have $\det P = 1$?

$$P = A(A^T A)^{-1} A^T \quad \text{so} \quad |P| = |A| \frac{1}{|A^T||A|} |A^T| = 1.$$

- 30** (Calculus question) Show that the partial derivatives of $\ln(\det A)$ give A^{-1} !

$$f(a, b, c, d) = \ln(ad - bc) \quad \text{leads to} \quad \begin{bmatrix} \frac{\partial f}{\partial a} & \frac{\partial f}{\partial c} \\ \frac{\partial f}{\partial b} & \frac{\partial f}{\partial d} \end{bmatrix} = A^{-1}.$$

- 31** (MATLAB) The Hilbert matrix **hilb**(n) has i, j entry equal to $1/(i + j - 1)$. Print the determinants of **hilb**(1), **hilb**(2), ..., **hilb**(10). Hilbert matrices are hard to work with! What are the pivots?
- 32** (MATLAB) What is a typical determinant (experimentally) of **rand**(n) and **randn**(n) for $n = 50, 100, 200, 400$? (And what does “Inf” mean in MATLAB?)
- 33** (MATLAB) Find the largest determinant of a 6 by 6 matrix of 1’s and -1 ’s.
- 34** If you know that $\det A = 6$, what is the determinant of B ?

$$\det A = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} = 6 \qquad \det B = \begin{vmatrix} \text{row 3 + row 2 + row 1} \\ \text{row 2 + row 1} \\ \text{row 1} \end{vmatrix} = ?$$

PERMUTATIONS AND COFACTORS ■ 5.2

A computer finds the determinant from the pivots. This section explains two other ways to do it. There is a “big formula” using all $n!$ permutations. There is a “cofactor formula” using determinants of size $n - 1$. The best example is my favorite 4 by 4 matrix:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } \det A = 5.$$

We can find this determinant in all three ways: *pivots, big formula, cofactors*.

1. The product of the pivots is $2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4}$. Cancellation produces 5.
2. The “big formula” in equation (8) has $4! = 24$ terms. Only five terms are nonzero:

$$\det A = 16 - 4 - 4 - 4 + 1 = 5.$$

The 16 comes from $2 \cdot 2 \cdot 2 \cdot 2$ on the diagonal of A . Where do -4 and $+1$ come from? When you can find those five terms, you have understood formula (8).

3. The numbers 2, -1 , 0, 0 in the first row multiply their cofactors 4, 3, 2, 1 from the other rows. That gives $2 \cdot 4 - 1 \cdot 3 = 5$. Those cofactors are 3 by 3 determinants. They use the rows and columns that are *not* used by the entry in the first row.
Every term in a determinant uses each row and column once!

The Pivot Formula

Elimination leaves the pivots d_1, \dots, d_n on the diagonal of the upper triangular U . If no row exchanges are involved, ***multiply those pivots*** to find the determinant:

$$\det A = (\det L)(\det U) = (1)(d_1 d_2 \cdots d_n). \quad (1)$$

This formula for $\det A$ appeared in the previous section, with the further possibility of row exchanges. The permutation matrix in $PA = LU$ has determinant -1 or $+1$. This factor $\det P = \pm 1$ enters the determinant of A :

$$(\det P)(\det A) = (\det L)(\det U) \quad \text{gives} \quad \det A = \pm(d_1 d_2 \cdots d_n). \quad (2)$$

When A has fewer than n pivots, $\det A = 0$ by Rule 8. The matrix is singular.

Example 1 A row exchange produces pivots 4, 2, 1 and that important minus sign:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad PA = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \det A = -(4)(2)(1) = -8.$$

The odd number of row exchanges (namely one exchange) means that $\det P = -1$.

The next example has no row exchanges. It may be the first matrix we factored into $L U$ (when it was 3 by 3). What is remarkable is that we can go directly to n by n . Pivots give the determinant. We will also see how determinants give the pivots.

Example 2 The first pivots of this tridiagonal matrix A are $2, \frac{3}{2}, \frac{4}{3}$. The next are $\frac{5}{4}$ and $\frac{6}{5}$ and eventually $\frac{n+1}{n}$. Factoring this n by n matrix reveals its determinant:

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -\frac{1}{2} & 1 & & \\ & -\frac{2}{3} & 1 & \\ & & \ddots & \ddots \\ & & & -\frac{n-1}{n} & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & & \\ \frac{3}{2} & -1 & & \\ \frac{4}{3} & -1 & & \\ & \ddots & \ddots & \\ & & & \frac{n+1}{n} \end{bmatrix}$$

The pivots are on the diagonal of U (the last matrix). When 2 and $\frac{3}{2}$ and $\frac{4}{3}$ and $\frac{5}{4}$ are multiplied, the fractions cancel. The determinant of the 4 by 4 matrix is 5. The 3 by 3 determinant is 4. *The n by n determinant is $n + 1$:*

$$\det A = (2) \left(\frac{3}{2}\right) \left(\frac{4}{3}\right) \cdots \left(\frac{n+1}{n}\right) = n + 1.$$

Important point: The first pivots depend only on the *upper left corner* of the original matrix A . This is a rule for all matrices without row exchanges.

The first k pivots come from the k by k matrix A_k in the top left corner of A . ***The determinant of that corner submatrix A_k is $d_1 d_2 \cdots d_k$.***

The 1 by 1 matrix A_1 contains the very first pivot d_1 . This is $\det A_1$. The 2 by 2 matrix in the corner has $\det A_2 = d_1 d_2$. Eventually the n by n determinant uses the product of all n pivots to give $\det A_n$ which is $\det A$.

Elimination deals with the corner matrix A_k while starting on the whole matrix. We assume no row exchanges—then $A = LU$ and $A_k = L_k U_k$. Dividing one determinant by the previous determinant ($\det A_k$ divided by $\det A_{k-1}$) cancels everything but the latest pivot d_k . ***This gives a ratio of determinants formula for the pivots:***

Pivots from determinants	The k th pivot is $d_k = \frac{d_1 d_2 \cdots d_k}{d_1 d_2 \cdots d_{k-1}} = \frac{\det A_k}{\det A_{k-1}}$.	(3)
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In the $-1, 2, -1$ matrices this ratio correctly gives the pivots $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+1}{n}$. The Hilbert matrices in Problem 5.1.31 also build from the upper left corner.

We don't need row exchanges when all these corner submatrices have $\det A_k \neq 0$.

The Big Formula for Determinants

Pivots are good for computing. They concentrate a lot of information—enough to find the determinant. But it is hard to connect them to the original a_{ij} . That part will be

clearer if we go back to rules 1-2-3, linearity and sign reversal and $\det I = 1$. We want to derive a single explicit formula for the determinant, directly from the entries a_{ij} .

The formula has $n!$ terms. Its size grows fast because $n! = 1, 2, 6, 24, 120, \dots$. For $n = 11$ there are forty million terms. For $n = 2$, the two terms are ad and bc . Half the terms have minus signs (as in $-bc$). The other half have plus signs (as in ad). For $n = 3$ there are $3! = (3)(2)(1)$ terms. Here are those six terms:

$$\begin{array}{l} \text{3 by 3} \\ \text{determinant} \end{array} \left| \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right| = \begin{array}{l} +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \end{array} \quad (4)$$

Notice the pattern. Each product like $a_{11}a_{23}a_{32}$ has **one entry from each row**. It also has **one entry from each column**. The column order 1, 3, 2 means that this particular term comes with a minus sign. The column order 3, 1, 2 in $a_{13}a_{21}a_{32}$ has a plus sign. It will be “permutations” that tell us the sign.

The next step ($n = 4$) brings $4! = 24$ terms. There are 24 ways to choose one entry from each row and column. Down the main diagonal, $a_{11}a_{22}a_{33}a_{44}$ with column order 1, 2, 3, 4 always has a plus sign. That is the “identity permutation”.

To derive the big formula I start with $n = 2$. The goal is to reach $ad - bc$ in a systematic way. Break each row into two simpler rows:

$$[a \ b] = [a \ 0] + [0 \ b] \quad \text{and} \quad [c \ d] = [c \ 0] + [0 \ d].$$

Now apply linearity, first in row 1 (with row 2 fixed) and then in row 2 (with row 1 fixed):

$$\begin{aligned} \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| &= \left| \begin{array}{cc} a & 0 \\ c & d \end{array} \right| + \left| \begin{array}{cc} 0 & b \\ c & d \end{array} \right| \\ &= \left| \begin{array}{cc} a & 0 \\ c & 0 \end{array} \right| + \left| \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right| + \left| \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right| + \left| \begin{array}{cc} 0 & b \\ 0 & d \end{array} \right|. \end{aligned} \quad (5)$$

The last line has $2^2 = 4$ determinants. The first and fourth are zero because their rows are dependent—one row is a multiple of the other row. We are left with $2! = 2$ determinants to compute:

$$\left| \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right| + \left| \begin{array}{cc} 0 & b \\ c & 0 \end{array} \right| = ad \left| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right| + bc \left| \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right| = ad - bc.$$

The splitting led to permutation matrices. Their determinants give a plus or minus sign. The 1's are multiplied by numbers that come from A . The permutation tells the column sequence, in this case (1, 2) or (2, 1).

Now try $n = 3$. Each row splits into 3 simpler rows like $[a_{11} \ 0 \ 0]$. Using linearity in each row, $\det A$ splits into $3^3 = 27$ simple determinants. If a column choice is repeated—for example if we also choose $[a_{21} \ 0 \ 0]$ —then the simple determinant is zero. We pay attention only when **the nonzero terms come from different columns**.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{22} & a_{33} \\ & a_{22} & a_{33} \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{23} & a_{31} \\ a_{21} & a_{32} & a_{31} \\ a_{31} & a_{32} & a_{31} \end{vmatrix} + \begin{vmatrix} a_{21} & a_{13} & a_{32} \\ a_{31} & a_{32} & a_{22} \\ a_{32} & a_{22} & a_{13} \end{vmatrix} \\ + \begin{vmatrix} a_{11} & a_{23} & a_{32} \\ & a_{23} & a_{32} \\ & & a_{32} \end{vmatrix} + \begin{vmatrix} a_{12} & a_{13} & a_{33} \\ a_{21} & a_{31} & a_{33} \\ a_{31} & a_{33} & a_{21} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{13} & a_{31} \\ a_{32} & a_{31} & a_{21} \\ a_{31} & a_{21} & a_{12} \end{vmatrix}.$$

There are $3! = 6$ ways to order the columns, so six determinants. The six permutations of $(1, 2, 3)$ include the identity permutation $(1, 2, 3)$ from $P = I$:

$$\text{Column numbers } = (1, 2, 3), (2, 3, 1), (3, 1, 2), (1, 3, 2), (2, 1, 3), (3, 2, 1). \quad (6)$$

The last three are *odd permutations* (one exchange). The first three are *even permutations* (0 or 2 exchanges). When the column sequence is (α, β, ω) , we have chosen the entries $a_{1\alpha}a_{2\beta}a_{3\omega}$ —and the column sequence comes with a plus or minus sign. The determinant of A is now split into six simple terms. Factor out the a_{ij} :

$$\det A = a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} \\ + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix}. \quad (7)$$

The first three (even) permutations have $\det P = +1$, the last three (odd) permutations have $\det P = -1$. We have proved the 3 by 3 formula in a systematic way.

Now you can see the n by n formula. There are $n!$ orderings of the columns. The columns $(1, 2, \dots, n)$ go in each possible order $(\alpha, \beta, \dots, \omega)$. Taking $a_{1\alpha}$ from row 1 and $a_{2\beta}$ from row 2 and eventually $a_{n\omega}$ from row n , the determinant contains the product $a_{1\alpha}a_{2\beta} \cdots a_{n\omega}$ times $+1$ or -1 . Half the column orderings have sign -1 .

The complete determinant of A is the sum of these $n!$ simple determinants, times 1 or -1 . The simple determinants $a_{1\alpha}a_{2\beta} \cdots a_{n\omega}$ choose **one entry from every row and column**:

$$\begin{aligned} \det A &= \text{sum over all } n! \text{ column permutations } P = (\alpha, \beta, \dots, \omega) \\ &= \sum (\det P) a_{1\alpha}a_{2\beta} \cdots a_{n\omega} = \text{BIG FORMULA.} \end{aligned} \quad (8)$$

The 2 by 2 case is $+a_{11}a_{22} - a_{12}a_{21}$ (which is $ad - bc$). Here P is $(1, 2)$ or $(2, 1)$.

The 3 by 3 case has three products “down to the right” (see Problem 30) and three products “down to the left”. Warning: Many people believe they should follow this pattern in the 4 by 4 case. They only take 8 products—but we need 24.

Example 3 (Determinant of U) When U is upper triangular, only one of the $n!$ products can be nonzero. This one term comes from the diagonal: $\det U = +u_{11}u_{22}\cdots u_{nn}$. All other column orderings pick at least one entry below the diagonal, where U has zeros. As soon as we pick a number like $u_{21} = 0$ from below the diagonal, that term in equation (8) is sure to be zero.

Of course $\det I = 1$. The only nonzero term is $+(1)(1)\cdots(1)$ from the diagonal.

Example 4 Suppose Z is the identity matrix except for column 3. Then

$$\text{determinant of } Z = \begin{vmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & d & 1 \end{vmatrix} = c. \quad (9)$$

The term $(1)(1)(c)(1)$ comes from the main diagonal with a plus sign. There are 23 other products (choosing one factor from each row and column) but they are all zero. Reason: If we pick a, b , or d from column 3, that column is used up. Then the only available choice from row 3 is zero.

Here is a different reason for the same answer. If $c = 0$, then Z has a row of zeros and $\det Z = c = 0$ is correct. If c is not zero, use elimination. Subtract multiples of row 3 from the other rows, to knock out a, b, d . That leaves a diagonal matrix and $\det Z = c$.

This example will soon be used for “Cramer’s Rule”. If we move a, b, c, d into the first column of Z , the determinant is $\det Z = a$. (Why?) Changing one column of I leaves Z with an easy determinant, coming from its main diagonal only.

Example 5 Suppose A has 1’s just above and below the main diagonal. Here $n = 4$:

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{have determinant 1.}$$

The only nonzero choice in the first row is column 2. The only nonzero choice in row 4 is column 3. Then rows 2 and 3 must choose columns 1 and 4. In other words P_4 is the only permutation that picks out nonzeros in A_4 . The determinant of P_4 is $+1$ (two exchanges to reach 2, 1, 4, 3). Therefore $\det A_4 = +1$.

Determinant by Cofactors

Formula (8) is a direct definition of the determinant. It gives you everything at once—but you have to digest it. Somehow this sum of $n!$ terms must satisfy rules 1-2-3 (then all the other properties follow). The easiest is $\det I = 1$, already checked. The rule of linearity becomes clear, if you separate out the factor a_{11} or a_{12} or $a_{1\alpha}$ that comes from the first row. With $n = 3$ we separate the determinant into

$$\det A = a_{11} (a_{22}a_{33} - a_{23}a_{32}) + a_{12} (a_{23}a_{31} - a_{21}a_{33}) + a_{13} (a_{21}a_{32} - a_{22}a_{31}). \quad (10)$$

Those three quantities in parentheses are called “*cofactors*”. They are 2 by 2 determinants, coming from matrices in rows 2 and 3. The first row contributes the factors a_{11}, a_{12}, a_{13} . The lower rows contribute the cofactors C_{11}, C_{12}, C_{13} . Certainly the determinant $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ depends linearly on a_{11}, a_{12}, a_{13} —this is rule 3.

The cofactor of a_{11} is $C_{11} = a_{22}a_{33} - a_{23}a_{32}$. You can see it in this splitting:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{21} & a_{22} \\ & a_{31} & a_{32} \end{vmatrix}.$$

We are still choosing ***one entry from each row and column***. Since a_{11} uses up row 1 and column 1, that leaves a 2 by 2 determinant as its cofactor.

As always, we have to watch signs. The 2 by 2 determinant that goes with a_{12} looks like $a_{21}a_{33} - a_{23}a_{31}$. But in the cofactor C_{12} , *its sign is reversed*. Then $a_{12}C_{12}$ is the correct 3 by 3 determinant. The sign pattern for cofactors along the first row is plus-minus-plus-minus. **You cross out row 1 and column j to get a submatrix M_{1j} of size $n - 1$.** Multiply its determinant by $(-1)^{1+j}$ to get the cofactor:

The cofactors along row 1 are $C_{1j} = (-1)^{1+j} \det M_{1j}$.

The cofactor expansion is $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$. (11)

In the big formula (8), the terms that multiply a_{11} combine to give $\det M_{11}$. The sign is $(-1)^{1+1}$, meaning *plus*. Equation (11) is another form of equation (8) and also equation (10), with factors from row 1 multiplying cofactors from the other rows.

Note Whatever is possible for row 1 is possible for row i . The entries a_{ij} in that row also have cofactors C_{ij} . Those are determinants of order $n - 1$, multiplied by $(-1)^{i+j}$. Since a_{ij} accounts for row i and column j , **the submatrix M_{ij} throws out row i and column j** . The display shows a_{43} and M_{43} (with row 4 and column 3 crossed out). The sign $(-1)^{4+3}$ multiplies the determinant of M_{43} to give C_{43} . The sign matrix shows the \pm pattern:

$$A = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ & & & a_{43} \end{bmatrix} \quad \text{signs } (-1)^{i+j} = \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}.$$

5A The determinant is the dot product of any row i of A with its cofactors:

COFACTOR FORMULA $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$. (12)

Each cofactor C_{ij} (order $n - 1$, without row i and column j) includes its correct sign:

$$C_{ij} = (-1)^{i+j} \det M_{ij}.$$

A determinant of order n is a combination of determinants of order $n - 1$. A recursive person would keep going. Each subdeterminant breaks into determinants of order $n - 2$. We could define all determinants via equation (12). This rule goes from order n to $n - 1$ to $n - 2$ and eventually to order 1. Define the 1 by 1 determinant $|a|$ to be the number a . Then the cofactor method is complete.

We preferred to construct $\det A$ from its properties (linearity, sign reversal, and $\det I = 1$). The big formula (8) and the cofactor formulas (10)–(12) follow from those properties. One last formula comes from the rule that $\det A = \det A^T$. We can expand in cofactors, down a column instead of across a row. Down column j the entries are a_{1j} to a_{nj} . The cofactors are C_{1j} to C_{nj} . The determinant is the dot product:

$$\text{Cofactors down column } j: \quad \det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}. \quad (13)$$

Cofactors are most useful when the matrices have many zeros—as in the next examples.

Example 6 The $-1, 2, -1$ matrix has only two nonzeros in its first row. So only two cofactors C_{11} and C_{12} are involved in the determinant. I will highlight C_{12} :

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{vmatrix}. \quad (14)$$

You see 2 times C_{11} first on the right, from crossing out row 1 and column 1. This cofactor has exactly the same $-1, 2, -1$ pattern as the original A —but one size smaller.

To compute the boldface C_{12} , use cofactors down its first column. The only nonzero is at the top. That contributes another -1 (so we are back to minus). Its cofactor is the $-1, 2, -1$ determinant which is 2 by 2, two sizes smaller than the original A .

Summary Equation (14) gives the 4 by 4 determinant D_4 from $2D_3$ minus D_2 . Each D_n (the $-1, 2, -1$ determinant of order n) comes from D_{n-1} and D_{n-2} :

$$D_4 = 2D_3 - D_2 \quad \text{and generally} \quad D_n = 2D_{n-1} - D_{n-2}. \quad (15)$$

Direct calculation gives $D_2 = 3$ and $D_3 = 4$. Therefore $D_4 = 2(4) - 3 = 5$. These determinants 3, 4, 5 fit the formula $D_n = n+1$. That “special tridiagonal answer” also came from the product of pivots in Example 2.

The idea behind cofactors is to reduce the order one step at a time. The determinants $D_n = n+1$ obey the recursion formula $n+1 = 2n - (n-1)$. As they must.

Example 7 This is the same matrix, except the first entry (upper left) is now 1:

$$B_4 = \begin{bmatrix} 1 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

All pivots of this matrix turn out to be 1. So its determinant is 1. How does that come from cofactors? Expanding on row 1, the cofactors all agree with Example 6. Just change $a_{11} = 2$ to $b_{11} = 1$:

$$\det B_4 = D_3 - D_2 \quad \text{instead of} \quad \det A_4 = 2D_3 - D_2.$$

The determinant of B_4 is $4 - 3 = 1$. The determinant of every B_n is $n - (n - 1) = 1$. Problem 13 asks you to use cofactors of the *last* row. You still find $\det B_n = 1$.

■ REVIEW OF THE KEY IDEAS ■

- With no row exchanges, $\det A = (\text{product of the pivots})$. In the upper left corner, $\det A_k = (\text{product of the first } k \text{ pivots})$.
- Every term in the big formula (8) uses each row and column once. Half of the $n!$ terms have plus signs (when $\det P = +1$) and half have minus signs.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the smaller determinant that omits row i and column j (because a_{ij} uses that row and column).
- The determinant is the dot product of any row of A with its row of cofactors. When a row of A has a lot of zeros, we only need a few cofactors.

■ WORKED EXAMPLES ■

5.2 A A *Hessenberg matrix* is a triangular matrix with one extra diagonal. Use cofactors of row 1 to show that the 4 by 4 determinant satisfies Fibonacci's rule $|H_4| = |H_3| + |H_2|$. The same rule will continue for all sizes, $|H_n| = |H_{n-1}| + |H_{n-2}|$. Which Fibonacci number is $|H_n|$?

$$H_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad H_3 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad H_4 = \begin{bmatrix} 2 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution The cofactor C_{11} for H_4 is the determinant $|H_3|$. We also need C_{12} (in boldface):

$$C_{12} = - \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Rows 2 and 3 stayed the same and we used linearity in row 1. The two determinants on the right are $-|H_3|$ and $+|H_2|$. Then the 4 by 4 determinant is

$$|H_4| = 2C_{11} + 1C_{12} = 2|H_3| - |H_3| + |H_2| = |H_3| + |H_2|.$$

The actual numbers are $|H_2| = 3$ and $|H_3| = 5$ (and of course $|H_1| = 2$). Since $|H_n|$ follows Fibonacci's rule $|H_{n-1}| + |H_{n-2}|$, it must be $|H_n| = F_{n+2}$.

5.2 B These questions use the \pm signs (even and odd P 's) in the big formula for $\det A$:

1. If A is the 10 by 10 all-ones matrix, how does the big formula give $\det A = 0$?
2. If you multiply all $n!$ permutations together into a single P , is it odd or even?
3. If you multiply each a_{ij} by the fraction $\frac{i}{j}$, why is $\det A$ unchanged?

Solution In Question 1, with all $a_{ij} = 1$, all the products in the big formula (8) will be 1. Half of them come with a plus sign, and half with minus. So they cancel to leave $\det A = 0$. (Of course the all-ones matrix is singular.)

In Question 2, multiplying $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives an odd permutation. Also for 3 by 3, the three odd permutations multiply (in any order) to give *odd*. But for $n > 3$ the product of all permutations will be *even*. There are $n!/2$ odd permutations and that is an even number as soon as it includes the factor 4.

In Question 3, each a_{ij} is multiplied by i/j . So each product $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ in the big formula is multiplied by all the row numbers $i = 1, 2, \dots, n$ and divided by all the column numbers $j = 1, 2, \dots, n$. (The columns come in some permuted order!) Then each product is unchanged and $\det A$ stays the same.

Another approach to Question 3: We are multiplying the matrix A by the diagonal matrix $D = \text{diag}(1 : n)$ when row i is multiplied by i . And we are postmultiplying by D^{-1} when column j is divided by j . The determinant of DAD^{-1} is $\det A$ by the product rule.

Problem Set 5.2

Problems 1–10 use the big formula with $n!$ terms: $|\mathbf{A}| = \sum \pm a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$.

- 1 Compute the determinants of A, B, C from six terms. Are their rows independent?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 2 Compute the determinants of A , B , C . Are their columns independent?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

- 3 Show that $\det A = 0$, regardless of the five nonzeros marked by x 's:

$$A = \begin{bmatrix} x & x & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}. \quad (\text{What is the rank of } A?)$$

- 4 This problem shows in two ways that $\det A = 0$ (the x 's are any numbers):

$$A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}.$$

- (a) How do you know that the rows are linearly dependent?
 (b) Explain why all 120 terms are zero in the big formula for $\det A$.
- 5 Find two ways to choose nonzeros from four different rows and columns:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 3 & 4 & 5 \\ 5 & 4 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix} \quad (B \text{ has the same zeros as } A).$$

- Is $\det A$ equal to $1 + 1$ or $1 - 1$ or $-1 - 1$? What is $\det B$?
- 6 Place the smallest number of zeros in a 4 by 4 matrix that will guarantee $\det A = 0$. Place as many zeros as possible while still allowing $\det A \neq 0$.
- 7 (a) If $a_{11} = a_{22} = a_{33} = 0$, how many of the six terms in $\det A$ will be zero?
 (b) If $a_{11} = a_{22} = a_{33} = a_{44} = 0$, how many of the 24 products $a_{1j}a_{2k}a_{3l}a_{4m}$ are sure to be zero?
- 8 How many 5 by 5 permutation matrices have $\det P = +1$? Those are even permutations. Find one that needs four exchanges to reach the identity matrix.
- 9 If $\det A$ is not zero, at least one of the $n!$ terms in formula (8) is not zero. Deduce that some ordering of the rows of A leaves no zeros on the diagonal. (Don't use P from elimination; that PA can have zeros on the diagonal.)
- 10 Show that 4 is the largest determinant for a 3 by 3 matrix of 1's and -1's.

- 11** How many permutations of $(1, 2, 3, 4)$ are even and what are they? Extra credit: What are all the possible 4 by 4 determinants of $I + P_{\text{even}}$?

Problems 12–24 use cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$. Remove row i and column j .

- 12** Find all cofactors and put them into a cofactor matrix C . Find $\det B$ by cofactors:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

- 13** Find the cofactor matrix C and multiply A times C^T . Compare AC^T with A^{-1} :

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

- 14** The matrix B_n is the $-1, 2, -1$ matrix A_n except that $b_{11} = 1$ instead of $a_{11} = 2$. Using cofactors of the *last* row of B_4 show that $|B_4| = 2|B_3| - |B_2|$ and find $|B_4|$:

$$B_4 = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \quad B_3 = \begin{bmatrix} 1 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix}.$$

The recursion $|B_n| = 2|B_{n-1}| - |B_{n-2}|$ is satisfied when every $|B_n| = 1$. This recursion is the same as for the A 's. The difference is in the starting values 1, 1, 1 for $n = 1, 2, 3$.

- 15** The n by n determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

- (a) What are these determinants C_1, C_2, C_3, C_4 ?
 (b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2} . Find C_{10} .
- 16** The matrices in Problem 15 have 1's just above and below the main diagonal. Going down the matrix, which order of columns (if any) gives all 1's? Explain why that permutation is *even* for $n = 4, 8, 12, \dots$ and *odd* for $n = 2, 6, 10, \dots$. Then

$$C_n = 0 \text{ (odd } n\text{)} \quad C_n = 1 \text{ (} n = 4, 8, \dots \text{)} \quad C_n = -1 \text{ (} n = 2, 6, \dots \text{)}.$$

- 17 The tridiagonal 1, 1, 1 matrix of order n has determinant E_n :

$$E_1 = |1| \quad E_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \quad E_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad E_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}.$$

- (a) By cofactors show that $E_n = E_{n-1} - E_{n-2}$.
 - (b) Starting from $E_1 = 1$ and $E_2 = 0$ find E_3, E_4, \dots, E_8 .
 - (c) By noticing how these numbers eventually repeat, find E_{100} .
- 18 F_n is the determinant of the 1, 1, -1 tridiagonal matrix of order n :

$$F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2 \quad F_3 = \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 3 \quad F_4 = \begin{vmatrix} 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 \end{vmatrix} \neq 4.$$

Expand in cofactors to show that $F_n = F_{n-1} + F_{n-2}$. These determinants are *Fibonacci numbers* 1, 2, 3, 5, 8, 13, The sequence usually starts 1, 1, 2, 3 (with two 1's) so our F_n is the usual F_{n+1} .

- 19 Go back to B_n in Problem 14. It is the same as A_n except for $b_{11} = 1$. So use linearity in the first row, where $[1 \ -1 \ 0]$ equals $[2 \ -1 \ 0]$ minus $[1 \ 0 \ 0]$:

$$|B_n| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & A_{n-1} & 0 \\ 0 & 0 & A_{n-1} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & A_{n-1} & 0 \\ 0 & 0 & A_{n-1} \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ -1 & A_{n-1} & 0 \\ 0 & 0 & A_{n-1} \end{vmatrix}.$$

Linearity gives $|B_n| = |A_n| - |A_{n-1}| = \underline{\hspace{2cm}}$.

- 20 Explain why the 4 by 4 Vandermonde determinant contains x^3 but not x^4 or x^5 :

$$V_4 = \det \begin{bmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & x & x^2 & x^3 \end{bmatrix}.$$

The determinant is zero at $x = \underline{\hspace{2cm}}, \underline{\hspace{2cm}}, \text{ and } \underline{\hspace{2cm}}$. The cofactor of x^3 is $V_3 = (b-a)(c-a)(c-b)$. Then $V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$.

- 21 Find G_2 and G_3 and then by row operations G_4 . Can you predict G_n ?

$$G_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad G_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} \quad G_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

- 22 Compute the determinants S_1, S_2, S_3 of these 1, 3, 1 tridiagonal matrices:

$$S_1 = |3| \quad S_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \quad S_3 = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

Make a Fibonacci guess for S_4 and verify that you are right.

- 23 Cofactors of the 1, 3, 1 matrices in Problem 22 give a recursion $S_n = 3S_{n-1} - S_{n-2}$.

Challenge: Show that S_n is the Fibonacci number F_{2n+2} by proving $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using the Fibonacci's rule $F_k = F_{k-1} + F_{k-2}$.

- 24 Change 3 to 2 in the upper left corner of the matrices in Problem 22. Why does that subtract S_{n-1} from the determinant S_n ? Show that the determinants become the Fibonacci numbers 2, 5, 13 (always F_{2n+1}).

Problems 25–28 are about block matrices and block determinants.

- 25 With 2 by 2 blocks in 4 by 4 matrices, you cannot always use block determinants:

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A||D| \quad \text{but} \quad \begin{vmatrix} A & B \\ C & D \end{vmatrix} \neq |A||D| - |C||B|.$$

- (a) Why is the first statement true? Somehow B doesn't enter.
 - (b) Show by example that equality fails (as shown) when C enters.
 - (c) Show by example that the answer $\det(AD - CB)$ is also wrong.
- 26 With block multiplication, $A = LU$ has $A_k = L_k U_k$ in the top left corner:
- $$A = \begin{bmatrix} A_k & * \\ * & * \end{bmatrix} = \begin{bmatrix} L_k & 0 \\ * & * \end{bmatrix} \begin{bmatrix} U_k & * \\ 0 & * \end{bmatrix}.$$
- (a) Suppose the first three pivots of A are 2, 3, -1 . What are the determinants of L_1, L_2, L_3 (with diagonal 1's) and U_1, U_2, U_3 and A_1, A_2, A_3 ?
 - (b) If A_1, A_2, A_3 have determinants 5, 6, 7 find the three pivots from equation (3).
- 27 Block elimination subtracts CA^{-1} times the first row $[A \ B]$ from the second row $[C \ D]$. This leaves the *Schur complement* $D - CA^{-1}B$ in the corner:

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Take determinants of these block matrices to prove correct rules for square blocks:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{aligned} |A||D - CA^{-1}B| &= |AD - CB| \\ \text{if } A^{-1} \text{ exists} &\quad \text{if } AC = CA. \end{aligned}$$

- 28 If A is m by n and B is n by m , block multiplication gives $\det M = \det AB$:

$$M = \begin{bmatrix} 0 & A \\ -B & I \end{bmatrix} = \begin{bmatrix} AB & A \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}.$$

If A is a single row and B is a single column what is $\det M$? If A is a column and B is a row what is $\det M$? Do a 3 by 3 example of each.

- 29 (A calculus question based on the cofactor expansion) Show that the derivative of $\det A$ with respect to a_{11} is the cofactor C_{11} . The other entries are fixed—we are only changing a_{11} .
- 30 A 3 by 3 determinant has three products “down to the right” and three “down to the left” with minus signs. Compute the six terms in the figure to find D . Then explain without determinants why this matrix is or is not invertible:

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

- 31 For E_4 in Problem 17, five of the $4! = 24$ terms in the big formula (8) are nonzero. Find those five terms to show that $E_4 = -1$.
- 32 For the 4 by 4 tridiagonal matrix (entries $-1, 2, -1$) find the five terms in the big formula that give $\det A = 16 - 4 - 4 - 4 + 1$.
- 33 Find the determinant of this cyclic P by cofactors of row 1 and then the “big formula”. How many exchanges reorder $4, 1, 2, 3$ into $1, 2, 3, 4$? Is $|P^2| = 1$ or -1 ?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

- 34 The $-1, 2, -1$ matrix is $2*\text{eye}(n)-\text{diag}(\text{ones}(n-1, 1), 1)-\text{diag}(\text{ones}(n-1, 1), -1)$. Change $A(1, 1)$ to 1 so $\det A = 1$. Predict the entries of A^{-1} based on $n = 3$ and test the prediction for $n = 4$.
- 35 (MATLAB) The $-1, 2, -1$ matrices have determinant $n+1$. Compute $(n+1)A^{-1}$ for $n = 3$ and 4, and verify your guess for $n = 5$. (Inverses of tridiagonal matrices have the rank one form uv^T above the diagonal.)

- 36 The symmetric Pascal matrices have determinant 1. If I subtract 1 from the n, n entry, why does the determinant become zero? (Use rule 3 or cofactors in row n .)

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix} = 1 \text{ (known)} \quad \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 19 \end{bmatrix} = 0 \text{ (to explain).}$$

CRAMER'S RULE, INVERSES, AND VOLUMES ■ 5.3

This section applies determinants to solve $Ax = b$ and also to invert A . In the entries of A^{-1} , you will see $\det A$ in every denominator—we divide by it. (If $\det A = 0$ then we can't divide and A^{-1} doesn't exist.) Each number in A^{-1} is a determinant divided by another determinant. So is every component of $x = A^{-1}b$.

Cramer's Rule solves $Ax = b$. A neat idea gives the first component x_1 . Replacing the first column of I by x gives a matrix with determinant x_1 . When you multiply by A , the first column becomes Ax which is b . The other columns are copied from A :

$$\left[\begin{array}{c} A \\ \hline \end{array} \right] \left[\begin{array}{ccc} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{array} \right] = B_1. \quad (1)$$

We multiplied a column at a time. Now take determinants. The product rule is:

$$(\det A)(x_1) = \det B_1 \quad \text{or} \quad x_1 = \frac{\det B_1}{\det A}. \quad (2)$$

This is the first component of x in Cramer's Rule! Changing a column of A gives B_1 .

To find x_2 , put the vector x into the second column of the identity matrix:

$$\left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ \hline \end{array} \right] \left[\begin{array}{ccc} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ 0 & x_3 & 1 \end{array} \right] = \left[\begin{array}{ccc} a_1 & b & a_3 \\ \hline \end{array} \right] = B_2. \quad (3)$$

Take determinants to find $(\det A)(x_2) = \det B_2$. This gives x_2 in Cramer's Rule:

5B (CRAMER's RULE) If $\det A$ is not zero, $Ax = b$ has the unique solution

$$x_1 = \frac{\det B_1}{\det A} \quad x_2 = \frac{\det B_2}{\det A} \quad \dots \quad x_n = \frac{\det B_n}{\det A}$$

The matrix B_j has the j th column of A replaced by the vector b .

A MATLAB program for Cramer's rule only needs one line to find B_j and x_j :

$$x(j) = \det([A(:, 1:j-1) \ b \ A(:, j+1:n)]) / \det(A)$$

To solve an n by n system, Cramer's Rule evaluates $n+1$ determinants (of A and the n different B 's). When each one is the sum of $n!$ terms—applying the “big formula” with all permutations—this makes a total of $(n+1)!$ terms. *It would be crazy to solve equations that way.* But we do finally have an explicit formula for the solution \mathbf{x} .

Example 1 Use Cramer's Rule (it needs four determinants) to solve

$$\begin{array}{l} x_1 + x_2 + x_3 = 1 \\ -2x_1 + x_2 = 0 \\ -4x_1 + x_3 = 0 \end{array} \quad \text{with } \det A = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 7.$$

The right side $(1, 0, 0)$ goes into columns 1, 2, 3 to produce the matrices B_1 , B_2 , B_3 :

$$|B_1| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad |B_2| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2 \quad |B_3| = \begin{vmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 0 \end{vmatrix} = 4.$$

Cramer's Rule takes ratios to find the components of \mathbf{x} . Always divide by $\det A$:

$$x_1 = \frac{|B_1|}{|A|} = \frac{1}{7} \quad x_2 = \frac{|B_2|}{|A|} = \frac{2}{7} \quad x_3 = \frac{|B_3|}{|A|} = \frac{4}{7}.$$

I always substitute x_1, x_2, x_3 back into the equations, to check the calculations.

A Formula for A^{-1}

In Example 1, the right side \mathbf{b} was the first column of I . The solution $(\frac{1}{7}, \frac{2}{7}, \frac{4}{7})$ must be the first column of A^{-1} . Then the first column of $AA^{-1} = I$ is correct.

Important point: When $\mathbf{b} = (1, 0, 0)$ replaces a column of A , the determinant is 1 times a *cofactor*. Look back to see how the determinants $|B_j|$ are 2 by 2 cofactors of A :

$$\begin{aligned} |B_1| = 1 \text{ is the cofactor } C_{11} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} && \text{Cross out row 1, column 1} \\ |B_2| = 2 \text{ is the cofactor } C_{12} &= -\begin{vmatrix} -2 & 0 \\ -4 & 1 \end{vmatrix} && \text{Cross out row 1, column 2} \\ |B_3| = 4 \text{ is the cofactor } C_{13} &= \begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} && \text{Cross out row 1, column 3} \end{aligned}$$

Main point: *The numerators in A^{-1} are cofactors. They are divided by $\det A$.*

For the second column of A^{-1} , change \mathbf{b} to $(0, 1, 0)$. The determinants of B_1, B_2, B_3 are cofactors (in bold) from crossing out row 2. The cofactor signs are $(-)(+)(-)$:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & \mathbf{0} & 1 \end{vmatrix} = -1 \quad \begin{vmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 5 \quad \begin{vmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ -4 & \mathbf{0} & 0 \end{vmatrix} = -4.$$

Divide $-1, 5, -4$ by $|\mathbf{A}| = 7$ to get the second column of A^{-1} .

For the third column of A^{-1} , the right side is $\mathbf{b} = (0, 0, 1)$. The determinants of the three B 's become cofactors of the *third row*. Those are $-1, -2, 3$. We always divide by $|A| = 7$. Now we have all columns of A^{-1} :

$$\text{Inverse matrix } A^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 & -1 \\ 2 & 5 & -2 \\ 4 & -4 & 3 \end{bmatrix} = \frac{\text{cofactors of } A}{\text{determinant of } A}.$$

Summary In solving $AA^{-1} = I$, the columns of I lead to the columns of A^{-1} . Then Cramer's Rule using \mathbf{b} = columns of I gives the short formula (4) for A^{-1} . We will include a separate direct proof of this formula below.

5C The i, j entry of A^{-1} is the cofactor C_{ji} (not C_{ij}) divided by $\det A$:

$$\text{FORMULA FOR } A^{-1} \quad (A^{-1})_{ij} = \frac{C_{ji}}{\det A} \quad \text{and} \quad A^{-1} = \frac{C^T}{\det A}. \quad (4)$$

The cofactors C_{ij} go into the “cofactor matrix” C . **Its transpose leads to A^{-1} .**

To compute the i, j entry of A^{-1} , cross out row j and column i of A . Multiply the determinant by $(-1)^{i+j}$ to get the cofactor, and divide by $\det A$.

Example 2 The matrix $A = [a \ b \ c \ d]$ has cofactor matrix $C = [d \ -c \ b \ a]$. Look at A times the transpose of C :

$$AC^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}. \quad (5)$$

The matrix on the right is $\det A$ times I . So divide by $\det A$. Then A times $C^T/\det A$ is I , which reveals A^{-1} :

$$A^{-1} \text{ is } \frac{C^T}{\det A} \text{ which is } \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (6)$$

This 2 by 2 example uses letters. The 3 by 3 example used numbers. Inverting a 4 by 4 matrix would need sixteen cofactors (each one is a 3 by 3 determinant). Elimination is faster—but now we know an explicit formula for A^{-1} .

Direct proof of the formula $A^{-1} = C^T/\det A$ The idea is to multiply A times C^T :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{bmatrix}. \quad (7)$$

Row 1 of A times column 1 of the cofactors yields the first $\det A$ on the right:

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \det A \text{ by the cofactor rule.}$$

Similarly row 2 of A times column 2 of C^T yields $\det A$. The entries a_{2j} are multiplying cofactors C_{2j} as they should, to give the determinant.

How to explain the zeros off the main diagonal in equation (7)? Rows of A are multiplying cofactors from different rows. Row 2 of A times column 1 of C^T gives zero, but why?

$$a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13} = 0. \quad (8)$$

Answer: This is the cofactor rule for a new matrix, when the second row of A is copied into its first row. The new matrix A^* has two equal rows, so $\det A^* = 0$ in equation (8). Notice that A^* has the same cofactors C_{11}, C_{12}, C_{13} as A —because all rows agree after the first row. Thus the remarkable matrix multiplication (7) is correct:

$$AC^T = (\det A)I \quad \text{or} \quad A^{-1} = \frac{C^T}{\det A}.$$

Example 3 A triangular matrix of 1's has determinant 1. Then A^{-1} contains cofactors:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{has inverse} \quad A^{-1} = \frac{C^T}{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Cross out row 1 and column 1 of A to see the 3 by 3 cofactor $C_{11} = 1$. Now cross out row 1 and column 2 for C_{12} . The 3 by 3 submatrix is still triangular with determinant 1. But the cofactor C_{12} is -1 because of the sign $(-1)^{1+2}$. This number -1 goes into the (2, 1) entry of A^{-1} —don't forget to transpose C !

The inverse of a triangular matrix is triangular. Cofactors give a reason why.

Example 4 If all cofactors are nonzero, is A sure to be invertible? *No way.*

Example 5 Here is part of a direct computation of A^{-1} (see Problem 14):

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad |A| = 5 \quad \text{and} \quad C_{12} = -(-2) \quad \text{and} \quad C_{22} = -6 \quad \text{and} \quad A^{-1} = \frac{1}{5} \begin{bmatrix} x & x & x \\ 2 & -6 & x \\ x & x & x \end{bmatrix}.$$

Area of a Triangle

Everybody knows the area of a rectangle—base times height. The area of a triangle is *half* the base times the height. But here is a question that those formulas don't answer. *If we know the corners (x_1, y_1) and (x_2, y_2) and (x_3, y_3) of a triangle, what is the area?*

Using the corners to find the base and height is not a good way. Determinants are much better. There are square roots in the base and height, but they cancel out in the good formula. ***The area of a triangle is half of a 3 by 3 determinant.*** If one corner is at the origin, say $(x_3, y_3) = (0, 0)$, the determinant is only 2 by 2.

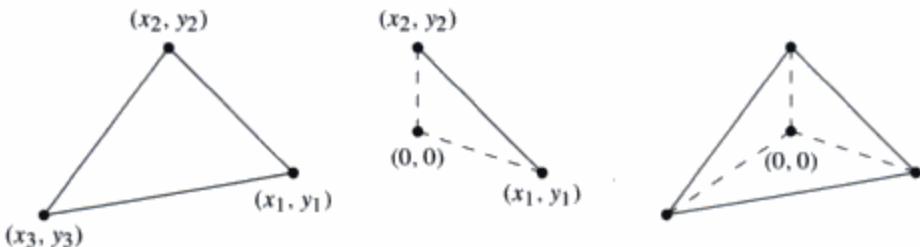


Figure 5.1 General triangle; special triangle from $(0, 0)$; general from three specials.

The triangle with corners (x_1, y_1) and (x_2, y_2) and (x_3, y_3) has area $= \frac{1}{2}$ (determinant):

$$\text{Area of triangle} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \text{ when } (x_3, y_3) = (0, 0).$$

When you set $x_3 = y_3 = 0$ in the 3 by 3 determinant, you get the 2 by 2 determinant. These formulas have no square roots—they are reasonable to memorize. The 3 by 3 determinant breaks into a sum of three 2 by 2's, just as the third triangle in Figure 5.1 breaks into three triangles from $(0, 0)$:

$$\text{Area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}(x_1y_2 - x_2y_1) + \frac{1}{2}(x_2y_3 - x_3y_2) + \frac{1}{2}(x_3y_1 - x_1y_3). \quad (9)$$

This shows the area of the general triangle as the sum of three special areas. If $(0, 0)$ is outside the triangle, two of the special areas can be negative—but the sum is still correct. The real problem is to explain the special area $\frac{1}{2}(x_1y_2 - x_2y_1)$.

Why is this the area of a triangle? We can remove the factor $\frac{1}{2}$ and change to a parallelogram (twice as big, because the parallelogram contains two equal triangles). We now prove that the parallelogram area is the determinant $x_1y_2 - x_2y_1$. This area in Figure 5.2 is 11, and therefore the triangle has area $\frac{11}{2}$.

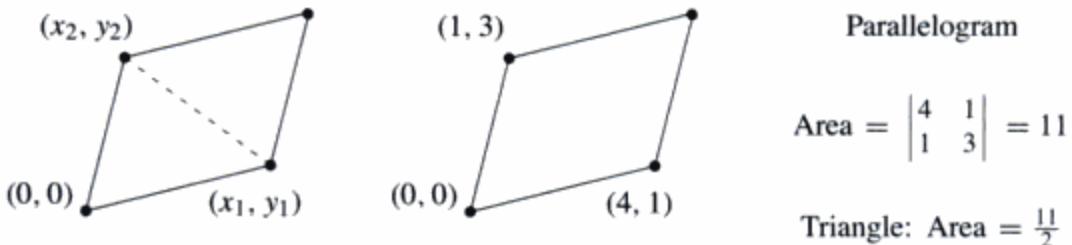


Figure 5.2 A triangle is half of a parallelogram. Area is half of a determinant.

Proof that a parallelogram starting from $(0, 0)$ has area = 2 by 2 determinant.

There are many proofs but this one fits with the book. We show that the area has the same properties 1-2-3 as the determinant. Then area = determinant! Remember that those three rules defined the determinant and led to all its other properties.

- 1 When $A = I$, the parallelogram becomes the unit square. Its area is $\det I = 1$.
- 2 When rows are exchanged, the determinant reverses sign. The absolute value (positive area) stays the same—it is the same parallelogram.
- 3 If row 1 is multiplied by t , Figure 5.3a shows that the area is also multiplied by t . Suppose a new row (x'_1, y'_1) is added to (x_1, y_1) (keeping row 2 fixed). Figure 5.3b shows that the solid parallelogram areas add to the dotted parallelogram area (because the two triangles completed by dotted lines are the same).

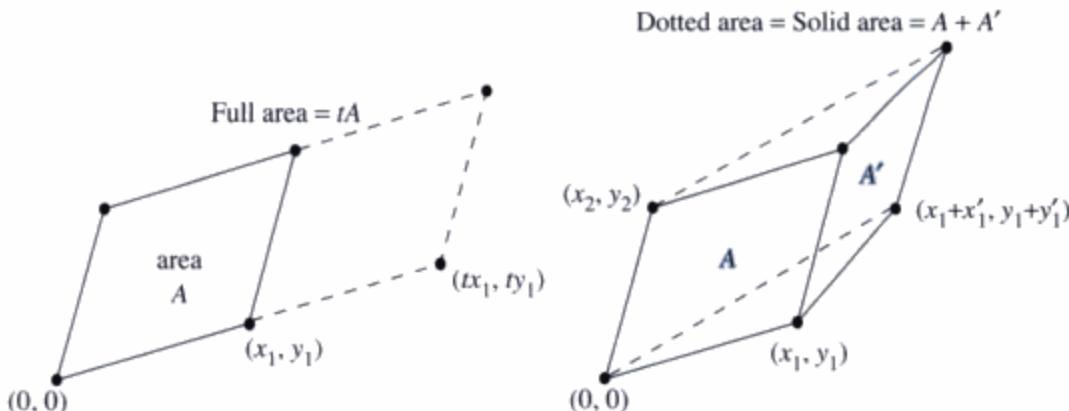


Figure 5.3 Areas obey the rule of linearity (keeping the side (x_2, y_2) constant).

That is an exotic proof, when we could use plane geometry. But the proof has a major attraction—it applies in n dimensions. The n edges going out from the origin are given by the rows of an n by n matrix. This is like the triangle with two edges going out from $(0, 0)$. The box is completed by more edges, just as the parallelogram was completed from a triangle. Figure 5.4 shows a three-dimensional box—whose edges are not at right angles.

The volume of the box in Figure 5.4 equals the absolute value of $\det A$. Our proof checks again that rules 1–3 for determinants are also obeyed by volumes. When an edge is stretched by a factor t , the volume is multiplied by t . When edge 1 is added to edge 1', the new box has edge $1 + 1'$. Its volume is the sum of the two original volumes. This is Figure 5.3b lifted into three dimensions or n dimensions. I would draw the boxes but this paper is only two-dimensional.

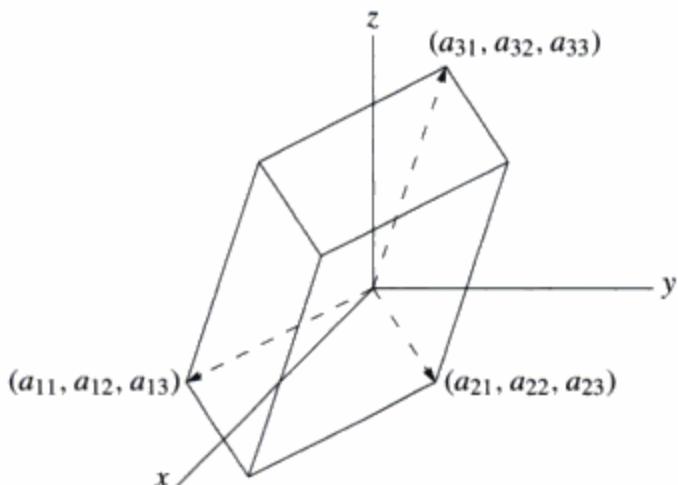


Figure 5.4 Three-dimensional box formed from the three rows of A .

The unit cube has volume = 1, which is $\det I$. This leaves only rule 2 to be checked. Row exchanges or edge exchanges leave the same box and the same absolute volume. The determinant changes sign, to indicate whether the edges are a *right-handed triple* ($\det A > 0$) or a *left-handed triple* ($\det A < 0$). The box volume follows the rules for determinants, so volume of the box (or parallelepiped) = absolute value of the determinant.

Example 6 Suppose a rectangular box (90° angles) has side lengths r, s , and t . Its volume is r times s times t . The diagonal matrix with entries r, s , and t produces those three sides. Then $\det A$ also equals $r s t$.

Example 7 In calculus, the box is infinitesimally small! To integrate over a circle, we might change x and y to r and θ . Those are polar coordinates: $x = r \cos \theta$ and $y = r \sin \theta$. The area of a “polar box” is a determinant J times $dr d\theta$:

$$J = \begin{vmatrix} \partial x / \partial r & \partial x / \partial \theta \\ \partial y / \partial r & \partial y / \partial \theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r.$$

This determinant is the r in the small area $dA = r dr d\theta$. The stretching factor J goes into double integrals just as dx/du goes into an ordinary integral $\int dx = \int (dx/du) du$. For triple integrals the Jacobian matrix J with nine derivatives will be 3 by 3.

The Cross Product

This is an extra (and optional) application, special for three dimensions. Start with vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. These pages are about their *cross product*. Unlike the dot product, which is a number, the cross product is a vector—also in three dimensions. It is written $\mathbf{u} \times \mathbf{v}$ and pronounced “ \mathbf{u} cross \mathbf{v} .” We will quickly give the components of this vector, and also the properties that make it useful in geometry and physics.

This time we bite the bullet, and write down the formula before the properties.

DEFINITION The *cross product* of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is the vector

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2) \mathbf{i} + (u_3 v_1 - u_1 v_3) \mathbf{j} + (u_1 v_2 - u_2 v_1) \mathbf{k}. \quad (10)$$

This vector is perpendicular to \mathbf{u} and \mathbf{v} . The cross product $\mathbf{v} \times \mathbf{u}$ is $-(\mathbf{u} \times \mathbf{v})$.

Comment The 3 by 3 determinant is the easiest way to remember $\mathbf{u} \times \mathbf{v}$. It is not especially legal, because the first row contains vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and the other rows contain numbers. In the determinant, the vector $\mathbf{i} = (1, 0, 0)$ multiplies $u_2 v_3$ and $-u_3 v_2$. The result is $(u_2 v_3 - u_3 v_2, 0, 0)$, which displays the first component of the cross product.

Notice the cyclic pattern of the subscripts: 2 and 3 give component 1, then 3 and 1 give component 2, then 1 and 2 give component 3. This completes the definition of $\mathbf{u} \times \mathbf{v}$. Now we list the properties of the cross product:

Property 1 $\mathbf{v} \times \mathbf{u}$ reverses rows 2 and 3 in the determinant so it equals $-(\mathbf{u} \times \mathbf{v})$.

Property 2 The cross product $\mathbf{u} \times \mathbf{v}$ is perpendicular to \mathbf{u} (and also to \mathbf{v}). The direct proof is to watch terms cancel. Perpendicularity is a zero dot product:

$$\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) = 0. \quad (11)$$

The determinant now has rows \mathbf{u}, \mathbf{u} and \mathbf{v} so it is zero.

Property 3 The cross product of any vector with itself (two equal rows) is $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

When \mathbf{u} and \mathbf{v} are parallel, the cross product is zero. When \mathbf{u} and \mathbf{v} are perpendicular, the dot product is zero. One involves $\sin \theta$ and the other involves $\cos \theta$:

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| |\sin \theta| \quad \text{and} \quad |\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \|\mathbf{v}\| |\cos \theta|. \quad (12)$$

Example 8 Since $\mathbf{u} = (3, 2, 0)$ and $\mathbf{v} = (1, 4, 0)$ are in the xy plane, $\mathbf{u} \times \mathbf{v}$ goes up the z axis:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 10\mathbf{k}. \quad \text{The cross product is } \mathbf{u} \times \mathbf{v} = (0, 0, 10).$$

The length of $\mathbf{u} \times \mathbf{v}$ equals the area of the parallelogram with sides \mathbf{u} and \mathbf{v} . This will be important: In this example the area is 10.

Example 9 The cross product of $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 1, 2)$ is $(1, -1, 0)$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j}.$$

This vector $(1, -1, 0)$ is perpendicular to $(1, 1, 1)$ and $(1, 1, 2)$ as predicted. Area $= \sqrt{2}$.

Example 10 The cross product of $(1, 0, 0)$ and $(0, 1, 0)$ obeys the *right hand rule*. It goes up not down:

$$\begin{matrix} \mathbf{i} \times \mathbf{j} = \mathbf{k} \\ \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right| = \mathbf{k} \end{matrix}$$

Rule $\mathbf{u} \times \mathbf{v}$ points along your right thumb when the fingers curl from \mathbf{u} to \mathbf{v} .

Thus $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. The right hand rule also gives $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. Note the cyclic order. In the opposite order (anti-cyclic) the thumb is reversed and the cross product goes the other way: $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ and $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. You see the three plus signs and three minus signs from a 3 by 3 determinant.

The definition of $\mathbf{u} \times \mathbf{v}$ can be based on vectors instead of their components:

DEFINITION The *cross product* is a vector with length $\|\mathbf{u}\| \|\mathbf{v}\| |\sin \theta|$. Its direction is perpendicular to \mathbf{u} and \mathbf{v} . It points “up” or “down” by the right hand rule.

This definition appeals to physicists, who hate to choose axes and coordinates. They see (u_1, u_2, u_3) as the position of a mass and (F_x, F_y, F_z) as a force acting on it. If \mathbf{F} is parallel to \mathbf{u} , then $\mathbf{u} \times \mathbf{F} = \mathbf{0}$ —there is no turning. The mass is pushed out or pulled in. The cross product $\mathbf{u} \times \mathbf{F}$ is the turning force or *torque*. It points along the turning axis (perpendicular to \mathbf{u} and \mathbf{F}). Its length $\|\mathbf{u}\| \|\mathbf{F}\| \sin \theta$ measures the “moment” that produces turning.

Triple Product = Determinant = Volume

Since $\mathbf{u} \times \mathbf{v}$ is a vector, we can take its dot product with a third vector \mathbf{w} . That produces the *triple product* $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$. It is called a “scalar” triple product, because it is a number. In fact it is a determinant:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \quad (13)$$

We can put \mathbf{w} in the top or bottom row. The two determinants are the same because _____ row exchanges go from one to the other. Notice when this determinant is zero:

$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = 0$ exactly when the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ lie in the *same plane*.

First reason $\mathbf{u} \times \mathbf{v}$ is perpendicular to that plane so its dot product with \mathbf{w} is zero.

Second reason Three vectors in a plane are dependent. The matrix is singular ($\det = 0$).

Third reason Zero volume when the $\mathbf{u}, \mathbf{v}, \mathbf{w}$ box is squashed onto a plane.

It is remarkable that $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ equals the volume of the box with sides $\mathbf{u}, \mathbf{v}, \mathbf{w}$. This 3 by 3 determinant carries tremendous information. Like $ad - bc$ for a 2 by 2 matrix, it separates invertible from singular. Chapter 6 will be looking for singular.

■ REVIEW OF THE KEY IDEAS ■

1. Cramer's Rule solves $A\mathbf{x} = \mathbf{b}$ by ratios like $x_1 = |B_1|/|A| = |\mathbf{b} \mathbf{a}_2 \cdots \mathbf{a}_n|/|A|$.
2. When C is the cofactor matrix for A , the inverse is $A^{-1} = C^T / \det A$.
3. The volume of a box is $|\det A|$, when the box edges are the rows of A .
4. Area and volume are needed to change variables in double and triple integrals.
5. In \mathbb{R}^3 , the cross product $\mathbf{u} \times \mathbf{v}$ is perpendicular to \mathbf{u} and \mathbf{v} .

■ WORKED EXAMPLES ■

5.3 A Use Cramer's Rule with ratios $\det B_j / \det A$ to solve $A\mathbf{x} = \mathbf{b}$. Also find the inverse matrix $A^{-1} = C^T / \det A$. Why is the solution \mathbf{x} in the first part the same as column 3 of A^{-1} ? Which cofactors are involved in computing that column \mathbf{x} ?

$$A\mathbf{x} = \mathbf{b} \quad \text{is} \quad \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Find the volumes of the boxes whose edges are *columns* of A and then *rows* of A^{-1} .

Solution The determinants of the B_j (with right side \mathbf{b} placed in column j) are

$$|B_1| = \begin{vmatrix} 0 & 6 & 2 \\ 0 & 4 & 2 \\ 1 & 9 & 0 \end{vmatrix} = 4 \quad |B_2| = \begin{vmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ 5 & 1 & 0 \end{vmatrix} = -2 \quad |B_3| = \begin{vmatrix} 2 & 6 & 0 \\ 1 & 4 & 0 \\ 5 & 9 & 1 \end{vmatrix} = 2.$$

Those are cofactors C_{31}, C_{32}, C_{33} of row 3. Their dot product with row 3 is $\det A$:

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (5, 9, 0) \cdot (4, -2, 2) = 2.$$

The three ratios $\det B_j / \det A$ give the three components of $\mathbf{x} = (2, -1, 1)$. This \mathbf{x} is the third column of A^{-1} because $\mathbf{b} = (0, 0, 1)$ is the third column of I . The cofactors

along the other rows of A , divided by $\det A = 2$, give the other columns of A^{-1} :

$$A^{-1} = \frac{C^T}{\det A} = \frac{1}{2} \begin{bmatrix} -18 & 18 & 4 \\ 10 & -10 & -2 \\ -11 & 12 & 2 \end{bmatrix}. \text{ Multiply to check } AA^{-1} = I$$

The box from the columns of A has volume = $\det A = 2$ (the same as the box from the rows, since $|A^T| = |A|$). The box from rows of A^{-1} has volume $|A^{-1}| = 1/|A| = \frac{1}{2}$.

5.3 B If A is singular, the equation $AC^T = (\det A)I$ becomes $AC^T = \text{zero matrix}$. Then each column of C^T is in the nullspace of A . Those columns contain cofactors along rows of A . So the cofactors quickly find the nullspace of a 3 by 3 matrix—my apologies that this comes so late!

Solve $Ax = \mathbf{0}$ by $x = \text{cofactors along a row}$, for these singular matrices of rank 2:

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 9 \\ 2 & 2 & 8 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Any nonzero column of C^T will give the desired solution to $Ax = \mathbf{0}$ (with rank 2, A has at least one nonzero cofactor). If A has rank 1 we get $x = \mathbf{0}$ and the idea will not work.

Solution The first matrix has these cofactors along its top row (note the minus sign):

$$\begin{vmatrix} 3 & 9 \\ 2 & 8 \end{vmatrix} = 6 \quad - \begin{vmatrix} 2 & 9 \\ 2 & 8 \end{vmatrix} = 2 \quad \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

Then $x = (6, 2, -2)$ solves $Ax = \mathbf{0}$. The cofactors along the second row are $(-18, -6, 6)$ which is just $-3x$. This is also in the one-dimensional nullspace of A .

The second matrix has zero cofactors along its first row. The nullvector $x = (0, 0, 0)$ is not interesting. The cofactors of row 2 give $x = (1, -1, 0)$ which solves $Ax = \mathbf{0}$.

Every n by n matrix of rank $n-1$ has at least one nonzero cofactor by Problem 3.3.9. But for rank $n-2$, all cofactors are zero. In that case cofactors only find $x = \mathbf{0}$.

Problem Set 5.3

Problems 1–5 are about Cramer's Rule for $x = A^{-1}b$.

1 Solve these linear equations by Cramer's Rule $x_j = \det B_j / \det A$:

$$(a) \begin{array}{l} 2x_1 + 5x_2 = 1 \\ x_1 + 4x_2 = 2 \end{array} \quad (b) \begin{array}{l} 2x_1 + x_2 = 1 \\ x_1 + 2x_2 + x_3 = 0 \\ x_2 + 2x_3 = 0. \end{array}$$

- 2 Use Cramer's Rule to solve for y (only). Call the 3 by 3 determinant D :

$$(a) \begin{array}{l} ax + by = 1 \\ cx + dy = 0 \end{array} \quad (b) \begin{array}{l} ax + by + cz = 1 \\ dx + ey + fz = 0 \\ gx + hy + iz = 0. \end{array}$$

- 3 Cramer's Rule breaks down when $\det A = 0$. Example (a) has no solution while (b) has infinitely many. What are the ratios $x_j = \det B_j / \det A$ in these two cases?

$$(a) \begin{array}{l} 2x_1 + 3x_2 = 1 \\ 4x_1 + 6x_2 = 1 \end{array} \text{(parallel lines)} \quad (b) \begin{array}{l} 2x_1 + 3x_2 = 1 \\ 4x_1 + 6x_2 = 2 \end{array} \text{(same line)}$$

- 4 *Quick proof of Cramer's rule.* The determinant is a linear function of column 1. It is zero if two columns are equal. When $\mathbf{b} = A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ goes into the first column of A , the determinant of this matrix B_1 is

$$|\mathbf{b} \quad \mathbf{a}_2 \quad \mathbf{a}_3| = |x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 \quad \mathbf{a}_2 \quad \mathbf{a}_3| = x_1|\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3| = x_1 \det A.$$

- (a) What formula for x_1 comes from left side = right side?
 (b) What steps lead to the middle equation?
 5 If the right side \mathbf{b} is the first column of A , solve the 3 by 3 system $A\mathbf{x} = \mathbf{b}$. How does each determinant in Cramer's Rule lead to this solution \mathbf{x} ?

Problems 6–16 are about $A^{-1} = C^T / \det A$. Remember to transpose C .

- 6 Find A^{-1} from the cofactor formula $C^T / \det A$. Use symmetry in part (b).

$$(a) A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} \quad (b) A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

- 7 If all the cofactors are zero, how do you know that A has no inverse? If none of the cofactors are zero, is A sure to be invertible?
 8 Find the cofactors of A and multiply AC^T to find $\det A$:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 6 & -3 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \quad \text{and } AC^T = \underline{\hspace{2cm}}.$$

If you change that 4 to 100, why is $\det A$ unchanged?

- 9 Suppose $\det A = 1$ and you know all the cofactors. How can you find A ?
 10 From the formula $AC^T = (\det A)I$ show that $\det C = (\det A)^{n-1}$.
 11 (for professors only) If you know all 16 cofactors of a 4 by 4 invertible matrix A , how would you find A ?

- 12 If all entries of A are integers, and $\det A = 1$ or -1 , prove that all entries of A^{-1} are integers. Give a 2 by 2 example.
- 13 If all entries of A and A^{-1} are integers, prove that $\det A = 1$ or -1 . Hint: What is $\det A$ times $\det A^{-1}$?
- 14 Complete the calculation of A^{-1} by cofactors in Example 5.
- 15 L is lower triangular and S is symmetric. Assume they are invertible:

$$L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} \quad S = \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

- (a) Which three cofactors of L are zero? Then L^{-1} is lower triangular.
- (b) Which three pairs of cofactors of S are equal? Then S^{-1} is symmetric.
- 16 For $n = 5$ the matrix C contains _____ cofactors and each 4 by 4 cofactor contains _____ terms and each term needs _____ multiplications. Compare with $5^3 = 125$ for the Gauss-Jordan computation of A^{-1} in Section 2.4.

Problems 17–27 are about area and volume by determinants.

- 17 (a) Find the area of the parallelogram with edges $v = (3, 2)$ and $w = (1, 4)$.
- (b) Find the area of the triangle with sides v , w , and $v + w$. Draw it.
- (c) Find the area of the triangle with sides v , w , and $w - v$. Draw it.
- 18 A box has edges from $(0, 0, 0)$ to $(3, 1, 1)$ and $(1, 3, 1)$ and $(1, 1, 3)$. Find its volume and also find the area of each parallelogram face using $\|u \times v\|$.
- 19 (a) The corners of a triangle are $(2, 1)$ and $(3, 4)$ and $(0, 5)$. What is the area?
- (b) Add a corner at $(-1, 0)$ to make a lopsided region (four sides). What is the area?
- 20 The parallelogram with sides $(2, 1)$ and $(2, 3)$ has the same area as the parallelogram with sides $(2, 2)$ and $(1, 3)$. Find those areas from 2 by 2 determinants and say why they must be equal. (I can't see why from a picture. Please write to me if you do.)
- 21 The Hadamard matrix H has orthogonal rows. The box is a hypercube!

What is $|H| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix}$ = volume of a hypercube in \mathbf{R}^4 ?

- 22 If the columns of a 4 by 4 matrix have lengths L_1, L_2, L_3, L_4 , what is the largest possible value for the determinant (based on volume)? If all entries are 1 or -1 , what are those lengths and the maximum determinant?
- 23 Show by a picture how a rectangle with area x_1y_2 minus a rectangle with area x_2y_1 produces the same area as our parallelogram.
- 24 When the edge vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are perpendicular, the volume of the box is $\|\mathbf{a}\|$ times $\|\mathbf{b}\|$ times $\|\mathbf{c}\|$. The matrix $A^T A$ is _____. Find $\det A^T A$ and $\det A$.
- 25 The box with edges i and j and $w = 2i + 3j + 4k$ has height _____. What is the volume? What is the matrix with this determinant? What is $i \times j$ and what is its dot product with w ?
- 26 An n -dimensional cube has how many corners? How many edges? How many $(n - 1)$ -dimensional faces? The cube whose edges are the rows of $2I$ has volume _____. A hypercube computer has parallel processors at the corners with connections along the edges.
- 27 The triangle with corners $(0, 0), (1, 0), (0, 1)$ has area $\frac{1}{2}$. The pyramid with four corners $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$ has volume _____. What is the volume of a pyramid in \mathbf{R}^4 with five corners at $(0, 0, 0, 0)$ and the rows of I ?

Problems 28–31 are about areas dA and volumes dV in calculus.

- 28 Polar coordinates satisfy $x = r \cos \theta$ and $y = r \sin \theta$. Polar area $J dr d\theta$ includes J :

$$J = \begin{vmatrix} \partial x / \partial r & \partial x / \partial \theta \\ \partial y / \partial r & \partial y / \partial \theta \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix}.$$

The two columns are orthogonal. Their lengths are _____. Thus $J = _____.$

- 29 Spherical coordinates ρ, ϕ, θ satisfy $x = \rho \sin \phi \cos \theta$ and $y = \rho \sin \phi \sin \theta$ and $z = \rho \cos \phi$. Find the 3 by 3 matrix of partial derivatives: $\partial x / \partial \rho, \partial x / \partial \phi, \partial x / \partial \theta$ in row 1. Simplify its determinant to $J = \rho^2 \sin \phi$. Then dV in a sphere is $\rho^2 \sin \phi d\rho d\phi d\theta$.

- 30 The matrix that connects r, θ to x, y is in Problem 27. Invert that 2 by 2 matrix:

$$J^{-1} = \begin{vmatrix} \partial r / \partial x & \partial r / \partial y \\ \partial \theta / \partial x & \partial \theta / \partial y \end{vmatrix} = \begin{vmatrix} \cos \theta & ? \\ ? & ? \end{vmatrix} = ?$$

It is surprising that $\partial r / \partial x = \partial x / \partial r$ (*Calculus*, Gilbert Strang, p. 501). Multiplying the matrices in 28 and 30 gives the chain rule $\frac{\partial x}{\partial x} = \frac{\partial x}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta} \frac{\partial \theta}{\partial x} = 1$.

- 31 The triangle with corners $(0, 0), (6, 0)$, and $(1, 4)$ has area _____. When you rotate it by $\theta = 60^\circ$ the area is _____. The determinant of the rotation matrix is

$$J = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & ? \\ ? & ? \end{vmatrix} = ?$$

Problems 32–39 are about the triple product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ in three dimensions.

- 32 A box has base area $\|\mathbf{u} \times \mathbf{v}\|$. Its perpendicular height is $\|\mathbf{w}\| \cos \theta$. Base area times height = volume = $\|\mathbf{u} \times \mathbf{v}\| \|\mathbf{w}\| \cos \theta$ which is $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$. Compute base area, height, and volume for $\mathbf{u} = (2, 4, 0)$, $\mathbf{v} = (-1, 3, 0)$, $\mathbf{w} = (1, 2, 2)$.
- 33 The volume of the same box is given more directly by a 3 by 3 determinant. Evaluate that determinant.
- 34 Expand the 3 by 3 determinant in equation (13) in cofactors of its row u_1, u_2, u_3 . This expansion is the dot product of \mathbf{u} with the vector ____.
- 35 Which of the triple products $(\mathbf{u} \times \mathbf{w}) \cdot \mathbf{v}$ and $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$ and $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$ are the same as $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$? Which orders of the rows \mathbf{u} , \mathbf{v} , \mathbf{w} give the correct determinant?
- 36 Let $P = (1, 0, -1)$ and $Q = (1, 1, 1)$ and $R = (2, 2, 1)$. Choose S so that $PQRS$ is a parallelogram and compute its area. Choose T, U, V so that $OPQRSTU$ is a tilted box and compute its volume.
- 37 Suppose (x, y, z) and $(1, 1, 0)$ and $(1, 2, 1)$ lie on a plane through the origin. What determinant is zero? What equation does this give for the plane?
- 38 Suppose (x, y, z) is a linear combination of $(2, 3, 1)$ and $(1, 2, 3)$. What determinant is zero? What equation does this give for the plane of all combinations?
- 39 (a) Explain from volumes why $\det 2A = 2^n \det A$ for n by n matrices.
 (b) For what size matrix is the false statement $\det A + \det A = \det(A + A)$ true?

6

EIGENVALUES AND EIGENVECTORS

INTRODUCTION TO EIGENVALUES ■ 6.1

Linear equations $Ax = b$ come from steady state problems. Eigenvalues have their greatest importance in *dynamic problems*. The solution of $d\mathbf{u}/dt = A\mathbf{u}$ is changing with time—growing or decaying or oscillating. We can't find it by elimination. This chapter enters a new part of linear algebra. All matrices in this chapter are square.

A good model comes from the powers A, A^2, A^3, \dots of a matrix. Suppose you need the hundredth power A^{100} . The starting matrix A becomes unrecognizable after a few steps:

$$\begin{array}{cccc} \left[\begin{array}{cc} .8 & .3 \\ .2 & .7 \end{array} \right] & \left[\begin{array}{cc} .70 & .45 \\ .30 & .55 \end{array} \right] & \left[\begin{array}{cc} .650 & .525 \\ .350 & .475 \end{array} \right] & \cdots \\ A & A^2 & A^3 & \\ & & & \boxed{A^{100}} \end{array}$$

A^{100} was found by using the *eigenvalues* of A , not by multiplying 100 matrices. Those eigenvalues are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A . *Certain exceptional vectors x are in the same direction as Ax . Those are the “eigenvectors”*. Multiply an eigenvector by A , and the vector Ax is a number λ times the original x .

The *basic equation* is $Ax = \lambda x$. The number λ is the “*eigenvalue*”. It tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A . We may find $\lambda = 2$ or $\frac{1}{2}$ or -1 or 1 . The eigenvalue λ could be zero! Then $Ax = 0x$ means that this eigenvector x is in the nullspace.

If A is the identity matrix, every vector has $Ax = x$. All vectors are eigenvectors. The eigenvalue (the number lambda) is $\lambda = 1$. This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. This section teaches

how to compute the x 's and λ 's. It can come early in the course because we only need the determinant of a 2 by 2 matrix.

For the matrix A in our model above, here are eigenvectors x_1 and x_2 . Multiplying those vectors by A gives x_1 and $\frac{1}{2}x_2$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$:

$$x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad Ax_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = x_1 \quad (Ax = x \text{ means that } \lambda_1 = 1)$$

$$x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad Ax_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \frac{1}{2}x_2 \text{ so } \lambda_2 = \frac{1}{2}).$$

If we again multiply x_1 by A , we still get x_1 . Every power of A will give $A^n x_1 = x_1$. Multiplying x_2 by A gave $\frac{1}{2}x_2$, and if we multiply again we get $(\frac{1}{2})^2 x_2$. **When A is squared, the eigenvectors x_1 and x_2 stay the same.** The λ 's are now 1^2 and $(\frac{1}{2})^2$. **The eigenvalues are squared!** This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of A^{100} are the same x_1 and x_2 . The eigenvalues of A^{100} are $1^{100} = 1$ and $(\frac{1}{2})^{100} =$ very small number.

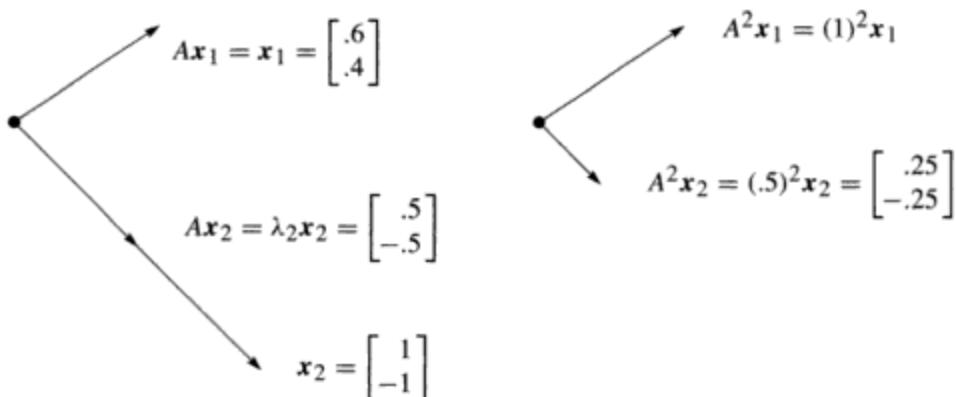


Figure 6.1 The eigenvectors keep their directions. A^2 has eigenvalues 1^2 and $(.5)^2$.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination $x_1 + (.2)x_2$:

$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} \quad \text{is} \quad x_1 + (.2)x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}. \quad (1)$$

Multiplying by A gives the first column of A^2 . Do it separately for x_1 and $(.2)x_2$. Of course $Ax_1 = x_1$. And A multiplies x_2 by its eigenvalue $\frac{1}{2}$:

$$A \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \quad \text{is} \quad x_1 + \frac{1}{2}(.2)x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix}.$$

Each eigenvector is multiplied by its eigenvalue, when we multiply by A . We didn't need these eigenvectors to find A^2 . But it is the good way to do 99 multiplications.

At every step x_1 is unchanged and x_2 is multiplied by $(\frac{1}{2})$, so we have $(\frac{1}{2})^{99}$:

$$A^{99} \begin{bmatrix} .8 \\ .2 \end{bmatrix} \text{ is really } x_1 + (.2)(\frac{1}{2})^{99}x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}.$$

This is the first column of A^{100} . The number we originally wrote as .6000 was not exact. We left out $(.2)(\frac{1}{2})^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector x_1 is a “steady state” that doesn’t change (because $\lambda_1 = 1$). The eigenvector x_2 is a “decaying mode” that virtually disappears (because $\lambda_2 = .5$). The higher the power of A , the closer its columns approach the steady state.

We mention that this particular A is a **Markov matrix**. Its entries are positive and every column adds to 1. Those facts guarantee that the largest eigenvalue is $\lambda = 1$ (as we found). Its eigenvector $x_1 = (.6, .4)$ is the *steady state*—which all columns of A^k will approach. Section 8.3 shows how Markov matrices appear in applications.

For projections we can spot the steady state ($\lambda = 1$) and the nullspace ($\lambda = 0$).

Example 1 The projection matrix $P = \begin{bmatrix} \frac{5}{5} & \frac{5}{5} \\ \frac{5}{5} & \frac{5}{5} \end{bmatrix}$ has eigenvalues 1 and 0.

Its eigenvectors are $x_1 = (1, 1)$ and $x_2 = (1, -1)$. For those vectors, $Px_1 = x_1$ (steady state) and $Px_2 = \mathbf{0}$ (nullspace). This example illustrates three things that we mention now:

1. Each column of P adds to 1, so $\lambda = 1$ is an eigenvalue.
2. P is singular, so $\lambda = 0$ is an eigenvalue.
3. P is symmetric, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

The only possible eigenvalues of a projection matrix are 0 and 1. The eigenvectors for $\lambda = 0$ (which means $Px = 0x$) fill up the nullspace. The eigenvectors for $\lambda = 1$ (which means $Px = x$) fill up the column space. The nullspace is projected to zero. The column space projects onto itself.

An in-between vector like $v = (3, 1)$ partly disappears and partly stays:

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \text{ projects onto } Pv = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The projection keeps the column space part of v and destroys the nullspace part. To emphasize: *Special properties of a matrix lead to special eigenvalues and eigenvectors.* That is a major theme of this chapter (it is captured in a table at the very end).

Projections have $\lambda = 0$ and 1. Permutations have all $|\lambda| = 1$. The next matrix R (a reflection and at the same time a permutation) is also special.

Example 2 The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.

The eigenvector $(1, 1)$ is unchanged by R . The second eigenvector is $(1, -1)$ —its signs are reversed by R . A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P , because $R = 2P - I$:

$$R = 2P - I \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (2)$$

Here is the point. If $Px = \lambda x$ then $2Px = 2\lambda x$. The eigenvalues are doubled when the matrix is doubled. Now subtract $Ix = x$. The result is $(2P - I)x = (2\lambda - 1)x$. **When a matrix is shifted by I , each λ is shifted by 1.** No change in eigenvectors.



Figure 6.2 Projections have eigenvalues 1 and 0. Reflections have $\lambda = 1$ and -1 . A typical x changes direction, but not the eigenvectors x_1 and x_2 .

The eigenvalues are related exactly as the matrices are related:

$$R = 2P - I \quad \text{so the eigenvalues of } R \text{ are} \quad 2(1) - 1 = 1 \text{ and } 2(0) - 1 = -1.$$

The eigenvalues of R^2 are λ^2 . In this case $R^2 = I$. Check $(1)^2 = 1$ and $(-1)^2 = 1$.

The Equation for the Eigenvalues

In small examples we found λ 's and x 's by trial and error. Now we use determinants and linear algebra. *This is the key calculation in the chapter—to solve $Ax = \lambda x$.*

First move λx to the left side. Write the equation $Ax = \lambda x$ as $(A - \lambda I)x = \mathbf{0}$. The matrix $A - \lambda I$ times the eigenvector x is the zero vector. **The eigenvectors make up the nullspace of $A - \lambda I$!** When we know an eigenvalue λ , we find an eigenvector by solving $(A - \lambda I)x = \mathbf{0}$.

Eigenvalues first. If $(A - \lambda I)x = \mathbf{0}$ has a nonzero solution, $A - \lambda I$ is not invertible. **The determinant of $A - \lambda I$ must be zero.** This is how to recognize an eigenvalue λ :

6A The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular:

$$\det(A - \lambda I) = 0. \quad (3)$$

This “characteristic equation” involves only λ , not \mathbf{x} . When A is n by n , $\det(A - \lambda I) = 0$ is an equation of degree n . Then A has n eigenvalues and each λ leads to \mathbf{x} :

For each λ solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ or $A\mathbf{x} = \lambda\mathbf{x}$ to find an eigenvector \mathbf{x} . (4)

Example 3 $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is already singular (zero determinant). Find its λ 's and \mathbf{x} 's.

When A is singular, $\lambda = 0$ is one of the eigenvalues. The equation $A\mathbf{x} = 0\mathbf{x}$ has solutions. They are the eigenvectors for $\lambda = 0$. But here is the way to find *all* λ 's and \mathbf{x} 's! Always subtract λI from A :

Subtract λ from the diagonal to find $A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}$.

Take the determinant “ $ad - bc$ ” of this 2 by 2 matrix. From $1 - \lambda$ times $4 - \lambda$, the “ ad ” part is $\lambda^2 - 5\lambda + 4$. The “ bc ” part, not containing λ , is 2 times 2.

$$\det \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix} = (1 - \lambda)(4 - \lambda) - (2)(2) = \lambda^2 - 5\lambda. \quad (5)$$

Set this determinant $\lambda^2 - 5\lambda$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$:

$$\det(A - \lambda I) = \lambda^2 - 5\lambda = 0 \quad \text{yields the eigenvalues } \lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 5$$

Now find the eigenvectors. Solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$(A - 0I)\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$

$$(A - 5I)\mathbf{x} = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

The matrices $A - 0I$ and $A - 5I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2, -1)$ and $(1, 2)$ are in the nullspaces: $(A - \lambda I)\mathbf{x} = \mathbf{0}$ is $A\mathbf{x} = \lambda\mathbf{x}$.

We need to emphasize: *There is nothing exceptional about $\lambda = 0$.* Like every other number, zero might be an eigenvalue and it might not. If A is singular, it is. The eigenvectors fill the nullspace: $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$. If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to *make it singular*. In the example, the shifted matrix $A - 5I$ was singular and 5 was the other eigenvalue.

Summary To solve the eigenvalue problem for an n by n matrix, follow these steps:

1. **Compute the determinant of $A - \lambda I$.** With λ subtracted along the diagonal, this determinant starts with λ^n or $-\lambda^n$. It is a polynomial in λ of degree n .
2. **Find the roots of this polynomial,** by solving $\det(A - \lambda I) = 0$. The n roots are the n eigenvalues of A . They make $A - \lambda I$ singular.
3. For each eigenvalue λ , **solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ to find an eigenvector \mathbf{x} .**

A note on quick computations, when A is 2 by 2. The determinant of $A - \lambda I$ is a quadratic (starting with λ^2). From factoring or the quadratic formula, we find its two roots (the eigenvalues). Then the eigenvectors come immediately from $A - \lambda I$. This matrix is singular, so both rows are multiples of a vector (a, b) . *The eigenvector is any multiple of $(b, -a)$.* The example had $\lambda = 0$ and $\lambda = 5$:

$$\begin{aligned}\lambda = 0 &: \text{rows of } A - 0I \text{ in the direction } (1, 2); \text{ eigenvector in the direction } (2, -1) \\ \lambda = 5 &: \text{rows of } A - 5I \text{ in the direction } (-4, 2); \text{ eigenvector in the direction } (2, 4).\end{aligned}$$

Previously we wrote that last eigenvector as $(1, 2)$. Both $(1, 2)$ and $(2, 4)$ are correct. There is a whole *line of eigenvectors*—any nonzero multiple of x is as good as x . MATLAB's `eig(A)` divides by the length, to make the eigenvector into a unit vector.

We end with a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A = I$ has equal eigenvalues and plenty of eigenvectors.) Similarly some n by n matrices don't have n independent eigenvectors. Without n eigenvectors, we don't have a basis. We can't write every v as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without n independent eigenvectors.

Good News, Bad News

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the λ 's.* The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A ! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{has } \lambda = 0 \text{ and } \lambda = 2.$$

Good news second: The *product λ_1 times λ_2 and the sum $\lambda_1 + \lambda_2$ can be found quickly from the matrix*. For this A , the product is 0 times 2. That agrees with the determinant (which is 0). The sum of eigenvalues is $0 + 2$. That agrees with the sum down the main diagonal (which is $1 + 1$). These quick checks always work:

6B *The product of the n eigenvalues equals the determinant of A .*

6C *The sum of the n eigenvalues equals the sum of the n diagonal entries of A .* This sum along the main diagonal is called the *trace* of A :

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{trace} = a_{11} + a_{22} + \cdots + a_{nn}. \quad (6)$$

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing λ 's. But when the computation is wrong, they generally tell us so. To compute correct λ 's, go back to $\det(A - \lambda I) = 0$.

The determinant test makes the *product* of the λ 's equal to the *product* of the pivots (assuming no row exchanges). But the sum of the λ 's is not the sum of the pivots—as the example showed. The individual λ 's have almost nothing to do with the individual pivots. In this new part of linear algebra, the key equation is really *nonlinear*: λ multiplies x .

Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

Example 4 *The 90° rotation $Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has no real eigenvectors or eigenvalues.* No vector Qx stays in the same direction as x (except the zero vector which is useless). There cannot be an eigenvector, unless we go to *imaginary numbers*. Which we do.

To see how i can help, look at Q^2 which is $-I$. If Q is rotation through 90° , then Q^2 is rotation through 180° . Its eigenvalues are -1 and -1 . (Certainly $-Ix = -1x$.) Squaring Q is supposed to square its eigenvalues λ , so we must have $\lambda^2 = -1$. *The eigenvalues of the 90° rotation matrix Q are $+i$ and $-i$* , because $i^2 = -1$.

Those λ 's come as usual from $\det(Q - \lambda I) = 0$. This equation gives $\lambda^2 + 1 = 0$. Its roots are $\lambda_1 = i$ and $\lambda_2 = -i$. They add to zero (which is the trace of Q). The product is $(i)(-i) = 1$ (which is the determinant).

We meet the imaginary number i also in the eigenvectors of Q :

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = -i \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

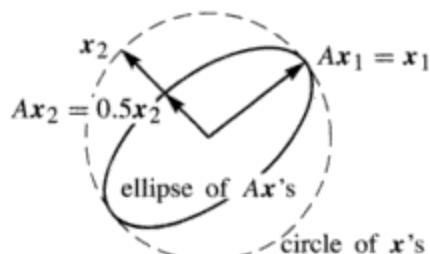
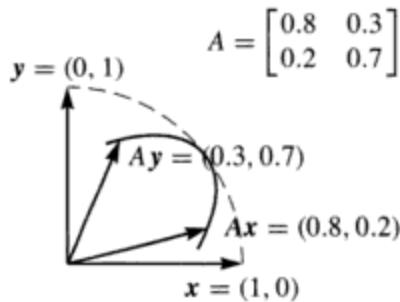
Somehow these complex vectors $x_1 = (1, i)$ and $x_2 = (i, 1)$ keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues. The particular eigenvalues i and $-i$ also illustrate two special properties of Q :

1. Q is an orthogonal matrix so the absolute value of each λ is $|\lambda| = 1$.
2. Q is a skew-symmetric matrix so each λ is pure imaginary.

A symmetric matrix ($A^T = A$) can be compared to a real number. A skew-symmetric matrix ($A^T = -A$) can be compared to an imaginary number. An orthogonal matrix ($A^T A = I$) can be compared to a complex number with $|\lambda| = 1$. For the eigenvalues those are more than analogies—they are theorems to be proved in Section 6.4. The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (in Chapter 10).

Eigshow

There is a MATLAB demo (just type `eigshow`), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $\mathbf{x} = (1, 0)$. *The mouse makes this vector move around the unit circle.* At the same time the screen shows $A\mathbf{x}$, in color and also moving. Possibly $A\mathbf{x}$ is ahead of \mathbf{x} . Possibly $A\mathbf{x}$ is behind \mathbf{x} . *Sometimes $A\mathbf{x}$ is parallel to \mathbf{x} .* At that parallel moment, $A\mathbf{x} = \lambda\mathbf{x}$ (twice in the second figure).



The eigenvalue λ is the length of $A\mathbf{x}$, when the unit eigenvector \mathbf{x} is parallel. The built-in choices for A illustrate three possibilities:

1. There are *no real eigenvectors*. $A\mathbf{x}$ stays *behind or ahead of \mathbf{x}* . This means the eigenvalues and eigenvectors are complex, as they are for the rotation Q .
2. There is only *one* line of eigenvectors (unusual). The moving directions $A\mathbf{x}$ and \mathbf{x} meet but don't cross. This happens for the last 2 by 2 matrix below.
3. There are eigenvectors in *two* independent directions. This is typical! $A\mathbf{x}$ crosses \mathbf{x} at the first eigenvector \mathbf{x}_1 , and it crosses back at the second eigenvector \mathbf{x}_2 .

Suppose A is singular (rank one). Its column space is a line. The vector $A\mathbf{x}$ has to stay on that line while \mathbf{x} circles around. One eigenvector \mathbf{x} is along the line. Another eigenvector appears when $A\mathbf{x}_2 = \mathbf{0}$. Zero is an eigenvalue of a singular matrix.

You can mentally follow \mathbf{x} and $A\mathbf{x}$ for these six matrices. How many eigenvectors and where? When does $A\mathbf{x}$ go clockwise, instead of counterclockwise with \mathbf{x} ?

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

■ REVIEW OF THE KEY IDEAS ■

1. $Ax = \lambda x$ says that x keeps the same direction when multiplied by A .
2. $Ax = \lambda x$ also says that $\det(A - \lambda I) = 0$. This determines n eigenvalues.
3. The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1} , with the same eigenvectors.
4. The sum and product of the λ 's equal the trace (sum of a_{ii}) and determinant.
5. Special matrices like projections P and rotations Q have special eigenvalues !

■ WORKED EXAMPLES ■

6.1 A Find the eigenvalues and eigenvectors of A and A^2 and A^{-1} and $A + 4I$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace $\lambda_1 + \lambda_2$ and the determinant $\lambda_1\lambda_2$ for A and also A^2 .

Solution The eigenvalues of A come from $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into $(\lambda - 1)(\lambda - 3) = 0$ so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. For the trace, the sum 2 + 2 agrees with 1 + 3. The determinant 3 agrees with the product $\lambda_1\lambda_2 = 3$. The eigenvectors come separately by solving $(A - \lambda I)x = \mathbf{0}$ which is $Ax = \lambda x$:

$$\lambda = 1: \quad (A - I)x = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda = 3: \quad (A - 3I)x = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A^2 and A^{-1} and $A + 4I$ keep the *same eigenvectors*. Their eigenvalues are λ^2 , λ^{-1} , $\lambda + 4$:

$$A^2 \text{ has } 1^2 = 1 \text{ and } 3^2 = 9 \quad A^{-1} \text{ has } \frac{1}{1} \text{ and } \frac{1}{3} \quad A + 4I \text{ has } 1 + 4 = 5 \text{ and } 3 + 4 = 7$$

The trace of A^2 is $5 + 5 = 1 + 9 = 10$. The determinant is $25 - 16 = 9$.

Notes for later sections: A has *orthogonal eigenvectors* (Section 6.4 on symmetric matrices). A can be *diagonalized* (Section 6.2). A is *similar* to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.6). A is a *positive definite matrix* (Section 6.5) since $A = A^T$ and the λ 's are positive.

- 3 Compute the eigenvalues and eigenvectors of A and A^{-1} :

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

A^{-1} has the _____ eigenvectors as A . When A has eigenvalues λ_1 and λ_2 , its inverse has eigenvalues _____.

- 4 Compute the eigenvalues and eigenvectors of A and A^2 :

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

A^2 has the same _____ as A . When A has eigenvalues λ_1 and λ_2 , A^2 has eigenvalues _____.

- 5 Find the eigenvalues of A and B and $A + B$:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

Eigenvalues of $A + B$ (are equal to)(are not equal to) eigenvalues of A plus eigenvalues of B .

- 6 Find the eigenvalues of A and B and AB and BA :

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Eigenvalues of AB (are equal to)(are not equal to) eigenvalues of A times eigenvalues of B . Eigenvalues of AB (are equal to)(are not equal to) eigenvalues of BA .

- 7 Elimination produces $A = LU$. The eigenvalues of U are on its diagonal; they are the _____. The eigenvalues of L are on its diagonal; they are all _____. The eigenvalues of A are not the same as _____.

- 8 (a) If you know x is an eigenvector, the way to find λ is to _____.
 (b) If you know λ is an eigenvalue, the way to find x is to _____.

- 9 What do you do to $Ax = \lambda x$, in order to prove (a), (b), and (c)?

- (a) λ^2 is an eigenvalue of A^2 , as in Problem 4.
 (b) λ^{-1} is an eigenvalue of A^{-1} , as in Problem 3.
 (c) $\lambda + 1$ is an eigenvalue of $A + I$, as in Problem 2.

- 10 Find the eigenvalues and eigenvectors for both of these Markov matrices A and A^∞ . Explain why A^{100} is close to A^∞ :

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^\infty = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}.$$

- 11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_1 \neq \lambda_2$: The columns of $A - \lambda_1 I$ are multiples of the eigenvector x_2 . Any idea why this should be?
- 12 Find the eigenvalues and eigenvectors for the projection matrices P and P^{100} :

$$P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If two eigenvectors share the same λ , so do all their linear combinations. Find an eigenvector of P with no zero components.

- 13 From the unit vector $u = (\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6})$ construct the rank one projection matrix $P = uu^T$.
- Show that $Pu = u$. Then u is an eigenvector with $\lambda = 1$.
 - If v is perpendicular to u show that $Pv = \mathbf{0}$. Then $\lambda = 0$.
 - Find three independent eigenvectors of P all with eigenvalue $\lambda = 0$.
- 14 Solve $\det(Q - \lambda I) = 0$ by the quadratic formula to reach $\lambda = \cos \theta \pm i \sin \theta$:

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ rotates the } xy \text{ plane by the angle } \theta.$$

Find the eigenvectors of Q by solving $(Q - \lambda I)x = \mathbf{0}$. Use $i^2 = -1$.

- 15 Every permutation matrix leaves $x = (1, 1, \dots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ 's for these permutations:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 16 Prove that the determinant of A equals the product $\lambda_1 \lambda_2 \cdots \lambda_n$. Start with the polynomial $\det(A - \lambda I)$ separated into its n factors. Then set $\lambda = \underline{\hspace{2cm}}$:

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda) \quad \text{so} \quad \det A = \underline{\hspace{2cm}}.$$

- 17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0.$$

If A has $\lambda_1 = 3$ and $\lambda_2 = 4$ then $\det(A - \lambda I) = \underline{\hspace{2cm}}$. The quadratic formula gives the eigenvalues $\lambda = (a + d + \sqrt{\quad})/2$ and $\lambda = \underline{\hspace{2cm}}$. Their sum is $\underline{\hspace{2cm}}$.

- 28 Find the rank and the four eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

- 29 Subtract I from the previous A . Find the λ 's and then the determinant:

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

When A (all ones) is 5 by 5, the eigenvalues of A and $B = A - I$ are _____ and _____.

- 30 (Review) Find the eigenvalues of A , B , and C :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}.$$

- 31 When $a+b=c+d$ show that $(1, 1)$ is an eigenvector and find both eigenvalues of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- 32 When P exchanges rows 1 and 2 *and* columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and PAP for $\lambda = 11$:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad PAP = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

- 33 Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors u, v, w .

- (a) Give a basis for the nullspace and a basis for the column space.
 - (b) Find a particular solution to $Ax = v + w$. Find all solutions.
 - (c) Show that $Ax = u$ has no solution. (If it did then _____ would be in the column space.)
- 34 Is there a real 2 by 2 matrix (other than I) with $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = 1$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would this give? Construct A .

- 35 Find the eigenvalues of this permutation matrix P . Which vectors are not changed by the permutation? They are eigenvectors for $\lambda = 1$. Can you find two more eigenvectors?

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 36 There are six 3 by 3 permutation matrices P . What numbers can be the *determinants* of P ? What numbers can be *pivots*? What numbers can be the *trace* of P ? What *four numbers* can be eigenvalues of P ?

DIAGONALIZING A MATRIX ■ 6.2

When \mathbf{x} is an eigenvector, multiplication by A is just multiplication by a single number: $A\mathbf{x} = \lambda\mathbf{x}$. All the difficulties of matrices are swept away. Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a *diagonal matrix*, with no off-diagonal interconnections. The 100th power of a diagonal matrix is easy.

The point of this section is very direct. ***The matrix A turns into a diagonal matrix Λ when we use the eigenvectors properly.*** This is the matrix form of our key idea. We start right off with that one essential computation.

6D Diagonalization Suppose the n by n matrix A has n linearly independent eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$. Put them into the columns of an *eigenvector matrix* S . Then $S^{-1}AS$ is the *eigenvalue matrix* Λ :

$$S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \quad (1)$$

The matrix A is “diagonalized.” We use capital lambda for the eigenvalue matrix, because of the small λ ’s (the eigenvalues) on its diagonal.

Proof Multiply A times its eigenvectors, which are the columns of S . The first column of AS is $A\mathbf{x}_1$. That is $\lambda_1\mathbf{x}_1$. Each column of S is multiplied by its eigenvalue:

$$AS = A \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{x}_1 & \cdots & \lambda_n\mathbf{x}_n \end{bmatrix}.$$

The trick is to split this matrix AS into S times Λ :

$$\begin{bmatrix} \lambda_1 \mathbf{x}_1 & \cdots & \lambda_n \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = S\Lambda.$$

Keep those matrices in the right order! Then λ_1 multiplies the first column \mathbf{x}_1 , as shown. The diagonalization is complete, and we can write $AS = S\Lambda$ in two good ways:

$$AS = S\Lambda \quad \text{is} \quad S^{-1}AS = \Lambda \quad \text{or} \quad A = S\Lambda S^{-1}. \quad (2)$$

The matrix S has an inverse, because its columns (the eigenvectors of A) were assumed to be linearly independent. *Without n independent eigenvectors, we can't diagonalize.*

The matrices A and Λ have the same eigenvalues $\lambda_1, \dots, \lambda_n$. The eigenvectors are different. The job of the original eigenvectors was to diagonalize A —those eigenvectors of A went into S . The new eigenvectors, for the diagonal matrix Λ , are just the columns of I . By diagonalizing A and reaching Λ , we can solve differential equations or difference equations or even $Ax = b$.

Example 1 The projection matrix $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ has $\lambda = 1$ and 0. Put the eigenvectors $(1, 1)$ and $(-1, 1)$ into S . Then $S^{-1}PS$ is the eigenvalue matrix Λ :

$$\begin{array}{ccccc} \begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix} & \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} & \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} & = & \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ S^{-1} & P & S & = & \Lambda \end{array}$$

The original projection satisfied $P^2 = P$. The new projection satisfies $\Lambda^2 = \Lambda$. The column space has swung around from $(1, 1)$ to $(1, 0)$. The nullspace has swung around from $(-1, 1)$ to $(0, 1)$. Diagonalization lines up the eigenvectors with the xy axes.

Here are four small remarks about diagonalization, before the applications.

Remark 1 Suppose the numbers $\lambda_1, \dots, \lambda_n$ are all different. Then it is automatic that the eigenvectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ are independent. See **6E** below. Therefore *any matrix that has no repeated eigenvalues can be diagonalized.*

Remark 2 The eigenvector matrix S is not unique. *We can multiply eigenvectors by any nonzero constants.* Suppose we multiply the columns of S by 5 and -1 . Divide the rows of S^{-1} by 5 and -1 to find the new inverse:

$$S_{\text{new}}^{-1}P S_{\text{new}} = \begin{bmatrix} .1 & .1 \\ .5 & -.5 \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \text{same } \Lambda.$$

The extreme case is $A = I$, when every vector is an eigenvector. Any invertible matrix S can be the eigenvector matrix. Then $S^{-1}IS = I$ (which is Λ).

Remark 3 To diagonalize A we *must* use an eigenvector matrix. From $S^{-1}AS = \Lambda$ we know that $AS = S\Lambda$. Suppose the first column of S is \mathbf{x} . Then the first columns of AS and $S\Lambda$ are $A\mathbf{x}$ and $\lambda_1\mathbf{x}$. For those to be equal, \mathbf{x} must be an eigenvector.

The eigenvectors in S come in the same order as the eigenvalues in Λ . To reverse the order in S and Λ , put $(-1, 1)$ before $(1, 1)$:

$$\begin{bmatrix} -.5 & .5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \Lambda \text{ in the new order } 0, 1.$$

Remark 4 (repeated warning for repeated eigenvalues) Some matrices have too few eigenvectors. *Those matrices are not diagonalizable.* Here are two examples:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Their eigenvalues happen to be 0 and 0. Nothing is special about $\lambda = 0$ —it is the repetition of λ that counts. All eigenvectors of the second matrix are multiples of $(1, 0)$!

$$Ax = 0x \quad \text{means} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

There is no second eigenvector, so the unusual matrix A cannot be diagonalized. This matrix is the best example to test any statement about eigenvectors. In many true-false questions, this matrix leads to *false*.

Remember that there is no connection between invertibility and diagonalizability:

- *Invertibility* is concerned with the *eigenvalues* (zero or not).
- *Diagonalizability* is concerned with the *eigenvectors* (too few or enough).

Each eigenvalue has at least one eigenvector! If $(A - \lambda I)x = \mathbf{0}$ leads you to $x = \mathbf{0}$, then λ is *not* an eigenvalue. Look for a mistake in solving $\det(A - \lambda I) = 0$. **The eigenvectors for n different λ 's are independent and A is diagonalizable.**

6E (Independent x from different λ) Eigenvectors x_1, \dots, x_j that correspond to distinct (all different) eigenvalues are linearly independent. An n by n matrix that has n different eigenvalues (no repeated λ 's) must be diagonalizable.

Proof Suppose $c_1x_1 + c_2x_2 = \mathbf{0}$. Multiply by A to find $c_1\lambda_1x_1 + c_2\lambda_2x_2 = \mathbf{0}$. Multiply by λ_2 to find $c_1\lambda_2x_1 + c_2\lambda_2x_2 = \mathbf{0}$. Now subtract one from the other:

$$\text{Subtraction leaves } (\lambda_1 - \lambda_2)c_1x_1 = \mathbf{0}. \quad \text{Therefore } c_1 = 0.$$

Since the λ 's are different and $x_1 \neq \mathbf{0}$, we are forced to the conclusion that $c_1 = 0$. Similarly $c_2 = 0$. No other combination gives $c_1x_1 + c_2x_2 = \mathbf{0}$, so the eigenvectors x_1 and x_2 must be independent.

It seems that β times λ is an eigenvalue. When x is an eigenvector for A and B , this proof is correct. **The mistake is to expect that A and B automatically share the same eigenvector x .** Usually they don't. Eigenvectors of A are not generally eigenvectors of B . A and B can have all eigenvalues $\lambda = 0$ and $\beta = 0$ while 1 is an eigenvalue of AB :

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad \text{then} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For the same reason, the eigenvalues of $A + B$ are generally not $\lambda + \beta$. Here $\lambda + \beta = 0$ while $A + B$ has eigenvalues 1 and -1 . (At least they add to zero.)

The false proof suggests what is true. Suppose x really is an eigenvector for both A and B . Then we do have $ABx = \lambda\beta x$. Sometimes all n eigenvectors are shared, and we *can* multiply eigenvalues. The test $AB = BA$ for shared eigenvectors is important in quantum mechanics—time out to mention this application of linear algebra:

6F Commuting matrices share eigenvectors Suppose A and B can be diagonalized. They share the same eigenvector matrix S if and only if $AB = BA$.

Heisenberg's uncertainty principle In quantum mechanics, the position matrix P and the momentum matrix Q do not commute. In fact $QP - PQ = I$ (these are infinite matrices). Then we cannot have $Px = \mathbf{0}$ at the same time as $Qx = \mathbf{0}$ (unless $x = \mathbf{0}$). If we knew the position exactly, we could not also know the momentum exactly. Problem 32 derives Heisenberg's uncertainty principle from the Schwarz inequality.

Fibonacci Numbers

We present a famous example, where eigenvalues tell how fast the Fibonacci numbers grow. **Every new Fibonacci number is the sum of the two previous F 's:**

The sequence $0, 1, 1, 2, 3, 5, 8, 13, \dots$ **comes from** $F_{k+2} = F_{k+1} + F_k$.

These numbers turn up in a fantastic variety of applications. Plants and trees grow in a spiral pattern, and a pear tree has 8 growths for every 3 turns. For a willow those numbers can be 13 and 5. The champion is a sunflower of Daniel O'Connell, which had 233 seeds in 144 loops. Those are the Fibonacci numbers F_{13} and F_{12} . Our problem is more basic.

Problem: Find the Fibonacci number F_{100} The slow way is to apply the rule $F_{k+2} = F_{k+1} + F_k$ one step at a time. By adding $F_6 = 8$ to $F_7 = 13$ we reach $F_8 = 21$. Eventually we come to F_{100} . Linear algebra gives a better way.

The key is to begin with a matrix equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$. That is a one-step rule for vectors, while Fibonacci gave a two-step rule for scalars. We match them by putting two Fibonacci numbers into a vector:

Example 3 Compute $A^k = S\Lambda^k S^{-1}$ when S and Λ and S^{-1} contain whole numbers:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{has} \quad \lambda_1 = 1 \quad \text{and} \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 2 \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

A is triangular, with 1 and 2 on the diagonal. A^k is also triangular, with 1 and 2^k on the diagonal. Those numbers stay separate in Λ^k . They are combined in A^k :

$$A^k = S\Lambda^k S^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1^k & \\ & 2^k \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2^k - 1 \\ 0 & 2^k \end{bmatrix}.$$

With $k = 1$ we get A . With $k = 0$ we get I . With $k = -1$ we get A^{-1} .

Note The zeroth power of every nonsingular matrix is $A^0 = I$. The product $S\Lambda^0 S^{-1}$ becomes SIS^{-1} which is I . Every λ to the zeroth power is 1. But the rule breaks down when $\lambda = 0$. Then 0^0 is not determined. We don't know A^0 when A is singular.

Nondiagonalizable Matrices (Optional)

Suppose λ is an eigenvalue of A . We discover that fact in two ways:

1. Eigenvectors (geometric) There are nonzero solutions to $A\mathbf{x} = \lambda\mathbf{x}$.
2. Eigenvalues (algebraic) The determinant of $A - \lambda I$ is zero.

The number λ may be a simple eigenvalue or a multiple eigenvalue, and we want to know its **multiplicity**. Most eigenvalues have multiplicity $M = 1$ (simple eigenvalues). Then there is a single line of eigenvectors, and $\det(A - \lambda I)$ does not have a double factor. For exceptional matrices, an eigenvalue can be **repeated**. Then there are two different ways to count its multiplicity:

1. **(Geometric Multiplicity = GM)** Count the independent eigenvectors for λ . This is the dimension of the nullspace of $A - \lambda I$.
2. **(Algebraic Multiplicity = AM)** Count the repetitions of λ among the eigenvalues. Look at the n roots of $\det(A - \lambda I) = 0$.

The following matrix A is the standard example of trouble. Its eigenvalue $\lambda = 0$ is repeated. It is a double eigenvalue ($AM = 2$) with only one eigenvector ($GM = 1$). The geometric multiplicity can be below the algebraic multiplicity—it is never larger:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2.$$

There “should” be two eigenvectors, because $\lambda^2 = 0$ has a double root. The double factor λ^2 makes $AM = 2$. But there is only one eigenvector $\mathbf{x} = (1, 0)$. **This shortage of eigenvectors when $GM < AM$ means that A is not diagonalizable.**

The vector called “repeats” in the Teaching Code **eigval** gives the algebraic multiplicity AM for each eigenvalue. When repeats = [1 1. . . 1] we know that the n eigenvalues are all different. A is certainly diagonalizable in that case. The sum of all components in “repeats” is always n , because the n th degree equation $\det(A - \lambda I) = 0$ always has n roots (counting repetitions).

The diagonal matrix \mathbf{D} in the Teaching Code **eigvec** gives the geometric multiplicity GM for each eigenvalue. This counts the independent eigenvectors. The total number of independent eigenvectors might be less than n . The n by n matrix A is diagonalizable if and only if this total number is n .

We have to emphasize: There is nothing special about $\lambda = 0$. It makes for easy computations, but these three matrices also have the same shortage of eigenvectors. Their repeated eigenvalue is $\lambda = 5$. Traces are 10, determinants are 25:

$$A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}.$$

Those all have $\det(A - \lambda I) = (\lambda - 5)^2$. The algebraic multiplicity is AM = 2. But $A - 5I$ has rank $r = 1$. The geometric multiplicity is GM = 1. There is only one eigenvector for $\lambda = 5$, and these matrices are not diagonalizable.

■ REVIEW OF THE KEY IDEAS ■

1. If A has n independent eigenvectors (they go into the columns of S), then $S^{-1}AS$ is diagonal: $S^{-1}AS = \Lambda$ and $A = S\Lambda S^{-1}$.
2. The powers of A are $A^k = S\Lambda^k S^{-1}$. The eigenvectors in S are unchanged.
3. The eigenvalues of A^k are $(\lambda_1)^k, \dots, (\lambda_n)^k$. The eigenvalues of A^{-1} are $1/\lambda_i$.
4. The solution to $\mathbf{u}_{k+1} = A\mathbf{u}_k$ starting from \mathbf{u}_0 is $\mathbf{u}_k = A^k\mathbf{u}_0 = S\Lambda^k S^{-1}\mathbf{u}_0$:

$$\mathbf{u}_k = c_1(\lambda_1)^k \mathbf{x}_1 + \cdots + c_n(\lambda_n)^k \mathbf{x}_n \quad \text{provided } \mathbf{u}_0 = c_1 \mathbf{x}_1 + \cdots + c_n \mathbf{x}_n.$$

5. A is diagonalizable if every eigenvalue has enough eigenvectors (GM=AM).

■ WORKED EXAMPLES ■

6.2 A The Lucas numbers are like the Fibonacci numbers except they start with $L_1 = 1$ and $L_2 = 3$. Following the rule $L_{k+2} = L_{k+1} + L_k$, the next Lucas numbers are 4, 7, 11, 18. Show that the Lucas number L_{100} is $\lambda_1^{100} + \lambda_2^{100}$.

Solution $\mathbf{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}_k$ is the same as for Fibonacci, because $L_{k+2} = L_{k+1} + L_k$ is the same rule (with different starting values). We can copy equation (5):

$$\text{Let } \mathbf{u}_k = \begin{bmatrix} L_{k+1} \\ L_k \end{bmatrix}. \text{ The rule } \begin{cases} L_{k+2} = L_{k+1} + L_k \\ L_{k+1} = L_k \end{cases} \text{ becomes } \mathbf{u}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{u}_k.$$

The eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ still come from $\lambda^2 = \lambda + 1$:

$$\lambda_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \mathbf{x}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} \qquad \lambda_2 = \frac{1 - \sqrt{5}}{2} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}.$$

Now solve $c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = \mathbf{u}_1 = (3, 1)$. The coefficients are $c_1 = \lambda_1$ and $c_2 = \lambda_2$! Check:

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} \text{trace of } A^2 \\ \text{trace of } A \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \mathbf{u}_1.$$

The solution $\mathbf{u}_{100} = A^{99} \mathbf{u}_1$ tells us the Lucas numbers (L_{101}, L_{100}). The second components of \mathbf{x}_1 and \mathbf{x}_2 are 1, so the second component of \mathbf{u}_{100} is

$$L_{100} = c_1 \lambda_1^{99} + c_2 \lambda_2^{99} = \lambda_1^{100} + \lambda_2^{100}.$$

Every $L_k = \lambda_1^k + \lambda_2^k$ is a whole number (why)? Since λ_2 is very small, L_k must be close to λ_1^k . Lucas starts faster than Fibonacci, and ends up larger by a factor near $\sqrt{5}$.

6.2 B Find all eigenvector matrices S that diagonalize A (rank 1) to give $S^{-1}AS = \Lambda$:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \ 1 \ 1]$$

What is A^n ? Which matrices B commute with A (so that $AB = BA$)?

Solution Since A has rank 1, its nullspace is a two-dimensional plane. Any vector with $x + y + z = 0$ (components adding to zero) solves $A\mathbf{x} = \mathbf{0}$. So $\lambda = 0$ is an eigenvalue with multiplicity 2. There are two independent eigenvectors ($GM = 2$). The other eigenvalue must be $\lambda = 3$ because the trace of A is $1 + 1 + 1 = 3$. Check these λ 's:

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 1 & 1 \\ 1 & 1 - \lambda & 1 \\ 1 & 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^3 + 2 - 3(1 - \lambda) = -\lambda^3 + 3\lambda^2.$$

Then $\lambda^2(3 - \lambda) = 0$ and the eigenvalues are $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 3$. The eigenvectors for $\lambda = 3$ are multiples of $\mathbf{x}_3 = (1, 1, 1)$. The eigenvectors for $\lambda_1 = \lambda_2 = 0$ are *any two independent vectors in the plane $x + y + z = 0$* . These are the columns of all possible eigenvector matrices S :

$$S = \begin{bmatrix} x & X & c \\ y & Y & c \\ -x - y & -X - Y & c \end{bmatrix} \quad \text{and} \quad S^{-1}AS = \Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

where $c \neq 0$ and $xY \neq yX$. The powers A^n come quickly by multiplication:

$$A^2 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3A \quad \text{and} \quad A^n = 3^{n-1}A.$$

To find matrices B that commute with A , look at AB and BA . The 1's in A produce the column sums C_1, C_2, C_3 and the row sums R_1, R_2, R_3 of B :

$$AB = \text{column sums} = \begin{bmatrix} C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \quad BA = \text{row sums} = \begin{bmatrix} R_1 & R_1 & R_1 \\ R_2 & R_2 & R_2 \\ R_3 & R_3 & R_3 \end{bmatrix}$$

If $AB = BA$, *all six column and row sums of B must be the same*. One possible B is A itself, since $AA = AA$. B is any linear combination of permutation matrices!

This is a 5-dimensional space (Problem 3.5.39) of matrices that commute with A . All B 's share the eigenvector $(1, 1, 1)$. Their other eigenvectors are in the plane $x + y + z = 0$. Three degrees of freedom in the λ 's and two in the unit eigenvectors.

Problem Set 6.2

Questions 1–8 are about the eigenvalue and eigenvector matrices.

- 1 Factor these two matrices into $A = S\Lambda S^{-1}$:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

- 2 If $A = S\Lambda S^{-1}$ then $A^3 = (\)(\)()$ and $A^{-1} = (\)()()$.

- 3 If A has $\lambda_1 = 2$ with eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda_2 = 5$ with $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, use SAS^{-1} to find A . No other matrix has the same λ 's and \mathbf{x} 's.

- 4 Suppose $A = S\Lambda S^{-1}$. What is the eigenvalue matrix for $A + 2I$? What is the eigenvector matrix? Check that $A + 2I = (\)()^{-1}$.

- 31 The eigenvalues of A are 1 and 9, the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = S\sqrt{\Lambda}S^{-1}$. Why is there no real matrix square root of B ?

- 32 (**Heisenberg's Uncertainty Principle**) $AB - BA = I$ can happen for infinite matrices with $A = A^T$ and $B = -B^T$. Then

$$\mathbf{x}^T \mathbf{x} = \mathbf{x}^T A B \mathbf{x} - \mathbf{x}^T B A \mathbf{x} \leq 2 \|A\mathbf{x}\| \|B\mathbf{x}\|.$$

Explain that last step by using the Schwarz inequality. Then the inequality says that $\|A\mathbf{x}\|/\|\mathbf{x}\|$ times $\|B\mathbf{x}\|/\|\mathbf{x}\|$ is at least $\frac{1}{2}$. It is impossible to get the position error and momentum error both very small.

- 33 If A and B have the same λ 's with the same independent eigenvectors, their factorizations into _____ are the same. So $A = B$.
- 34 Suppose the same S diagonalizes both A and B , so that $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$. Prove that $AB = BA$.
- 35 Substitute $A = S\Lambda S^{-1}$ into the product $(A - \lambda_1 I)(A - \lambda_2 I) \cdots (A - \lambda_n I)$ and explain why this produces the zero matrix. We are substituting the matrix A for the number λ in the polynomial $p(\lambda) = \det(A - \lambda I)$. The **Cayley-Hamilton Theorem** says that this product is always $p(A) = \text{zero matrix}$, even if A is not diagonalizable.
- 36 Test the Cayley-Hamilton Theorem on Fibonacci's matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.
- 37 If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ then $\det(A - \lambda I)$ is $(\lambda - a)(\lambda - d)$. Check the Cayley-Hamilton statement that $(A - aI)(A - dI) = \text{zero matrix}$.
- 38 (a) When do the eigenvectors for $\lambda = 0$ span the nullspace $N(A)$?
 (b) When do all the eigenvectors for $\lambda \neq 0$ span the column space $C(A)$?
- 39 Find the eigenvalues and eigenvectors and the k th power of A . Worked Example 2.4 C described A as the “adjacency matrix” for this 3-node graph. The i, j entry of A^k counts the k -step paths from i to j —what is the 2, 2 entry of A^4 and which 4-step paths along edges of the graph begin and end at node 2?

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- 40** If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $AB = BA$, show that $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is also a diagonal matrix. B has the same eigen _____ as A but different eigen _____. These diagonal matrices B form a two-dimensional subspace of matrix space. $AB - BA = 0$ gives four equations for the unknowns a, b, c, d —find the rank of the 4 by 4 matrix.
- 41** If A is 5 by 5, then $AB - BA =$ zero matrix gives 25 equations for the 25 entries in B . How do you know that the 25 by 25 matrix is singular (and there is always a nonzero solution B)?
- 42** The powers A^k approach zero if all $|\lambda_i| < 1$ and they blow up if any $|\lambda_i| > 1$. Peter Lax gives these striking examples in his book *Linear Algebra*:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ -5 & -3 \end{bmatrix} \quad C = \begin{bmatrix} 5 & 7 \\ -3 & -4 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 6.9 \\ -3 & -4 \end{bmatrix}$$

$$\|A^{1024}\| > 10^{700} \quad B^{1024} = I \quad C^{1024} = -C \quad \|D^{1024}\| < 10^{-78}$$

Find the eigenvalues $\lambda = e^{i\theta}$ of B and C to show $B^4 = I$ and $C^3 = -I$.

APPLICATIONS TO DIFFERENTIAL EQUATIONS ■ 6.3

Eigenvalues and eigenvectors and $A = S\Lambda S^{-1}$ are perfect for matrix powers A^k . They are also perfect for differential equations. This section is mostly linear algebra, but to read it you need one fact from calculus: *The derivative of $e^{\lambda t}$ is $\lambda e^{\lambda t}$.* It helps to know what e is, but I am not even sure that is essential. The whole point of the section is this: **To convert differential equations into linear algebra.**

The ordinary scalar equation $du/dt = u$ is solved by $u = e^t$. The equation $du/dt = 4u$ is solved by $u = e^{4t}$. Constant coefficient equations have exponential solutions!

$$\frac{du}{dt} = \lambda u \quad \text{has the solutions } u(t) = Ce^{\lambda t}. \quad (1)$$

The number C turns up on both sides of $du/dt = \lambda u$. At $t = 0$ the solution $Ce^{\lambda t}$ reduces to C (because $e^0 = 1$). By choosing $C = u(0)$, **the solution that starts from $u(0)$ at $t = 0$ is $u(0)e^{\lambda t}$.**

We just solved a 1 by 1 problem. Linear algebra moves to n by n . The unknown is a vector \mathbf{u} (now boldface). It starts from the initial vector $\mathbf{u}(0)$, which is given. The n equations contain a square matrix A :

Problem

Solve $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ starting from the vector $\mathbf{u}(0)$ at $t = 0$. (2)

This system of differential equations is *linear*. If $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are solutions, so is $C\mathbf{u}(t) + D\mathbf{v}(t)$. We will need n constants like C and D to match the n components of $\mathbf{u}(0)$. Our first job is to find n “pure exponential solutions” to the equation $d\mathbf{u}/dt = A\mathbf{u}$.

Notice that A is a *constant* matrix. In other linear equations, A changes as t changes. In nonlinear equations, A changes as \mathbf{u} changes. We don’t have either of these difficulties. Equation (2) is “linear with constant coefficients”. Those and only those are the differential equations that we will convert directly to linear algebra. The main point will be: **Solve linear constant coefficient equations by exponentials $e^{\lambda t}\mathbf{x}$.**

Solution of $d\mathbf{u}/dt = A\mathbf{u}$

Our pure exponential solution will be $e^{\lambda t}$ times a fixed vector \mathbf{x} . You may guess that λ is an eigenvalue of A , and \mathbf{x} is the eigenvector. Substitute $\mathbf{u}(t) = e^{\lambda t}\mathbf{x}$ into the equation $d\mathbf{u}/dt = A\mathbf{u}$ to prove you are right (the factor $e^{\lambda t}$ will cancel):

$A\mathbf{u} = Ae^{\lambda t}\mathbf{x}$ agrees with $\frac{d\mathbf{u}}{dt} = \lambda e^{\lambda t}\mathbf{x}$ provided $A\mathbf{x} = \lambda\mathbf{x}$. (3)

All components of this special solution $\mathbf{u} = e^{\lambda t} \mathbf{x}$ share the same $e^{\lambda t}$. The solution grows when $\lambda > 0$. It decays when $\lambda < 0$. In general λ can be a complex number. Then the real part of λ decides growth or decay, while the imaginary part gives oscillation like a sine wave.

Example 1 Solve $\frac{d\mathbf{u}}{dt} = A\mathbf{u} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{u}$ starting from $\mathbf{u}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

This is a vector equation for \mathbf{u} . It contains two scalar equations for the components y and z . They are “coupled together” because the matrix is not diagonal:

Equation $\frac{d\mathbf{u}}{dt} = A\mathbf{u}$ $\frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$ means that $\frac{dy}{dt} = z$ and $\frac{dz}{dt} = y$.

The idea of eigenvectors is to combine those equations in a way that gets back to 1 by 1 problems. The combinations $y + z$ and $y - z$ will do it:

$$\frac{d}{dt}(y + z) = z + y \quad \text{and} \quad \frac{d}{dt}(y - z) = -(y - z).$$

The combination $y + z$ grows like e^t , because it has $\lambda = 1$. The combination $y - z$ decays like e^{-t} , because it has $\lambda = -1$. Here is the point: We don't have to juggle the original equations $d\mathbf{u}/dt = A\mathbf{u}$, looking for these special combinations. The eigenvectors and eigenvalues do it for us.

This matrix A has eigenvalues 1 and -1 . Here are two eigenvectors:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \mathbf{x}_1 \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\mathbf{x}_2.$$

The pure exponential solutions \mathbf{u}_1 and \mathbf{u}_2 take the form $e^{\lambda t} \mathbf{x}$ with $\lambda = 1$ and -1 :

$$u_1(t) = e^{\lambda_1 t} \mathbf{x}_1 = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad u_2(t) = e^{\lambda_2 t} \mathbf{x}_2 = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (4)$$

Notice: These \mathbf{u} 's are eigenvectors. They satisfy $A\mathbf{u}_1 = \mathbf{u}_1$ and $A\mathbf{u}_2 = -\mathbf{u}_2$, just like \mathbf{x}_1 and \mathbf{x}_2 . The factors e^t and e^{-t} change with time. Those factors give $du_1/dt = \mathbf{u}_1 = A\mathbf{u}_1$ and $du_2/dt = -\mathbf{u}_2 = A\mathbf{u}_2$. We have two solutions to $d\mathbf{u}/dt = A\mathbf{u}$. To find all other solutions, multiply those special solutions by any C and D and add:

$$\text{General solution } \mathbf{u}(t) = Ce^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}. \quad (5)$$

With these constants C and D , we can match any starting vector $\mathbf{u}(0)$. Set $t = 0$ and $e^0 = 1$. The problem asked for $\mathbf{u}(0) = (4, 2)$:

$$C \begin{bmatrix} 1 \\ 1 \end{bmatrix} + D \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{yields} \quad C = 3 \quad \text{and} \quad D = 1.$$

With $C = 3$ and $D = 1$ in the solution (5), the initial value problem is solved.

We summarize the steps. The same three steps that solved $\mathbf{u}_{k+1} = A\mathbf{u}_k$ now solve $d\mathbf{u}/dt = A\mathbf{u}$. The powers A^k led to λ^k . The differential equation leads to $e^{\lambda t}$:

- Find the eigenvalues λ_i and n independent eigenvectors \mathbf{x}_i of A .
- Write $\mathbf{u}(0)$ as a combination $c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n$ of the eigenvectors.
- Multiply each eigenvector \mathbf{x}_i by $e^{\lambda_i t}$. Then $\mathbf{u}(t)$ is the combination

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t} \mathbf{x}_1 + \cdots + c_n e^{\lambda_n t} \mathbf{x}_n. \quad (6)$$

Example 2 Solve $d\mathbf{u}/dt = A\mathbf{u}$ knowing the eigenvalues $\lambda = 1, 2, 3$ of A :

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{u} \quad \text{starting from } \mathbf{u}(0) = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}.$$

Step 1 The eigenvectors are $\mathbf{x}_1 = (1, 0, 0)$ and $\mathbf{x}_2 = (1, 1, 0)$ and $\mathbf{x}_3 = (1, 1, 1)$.

Step 2 The vector $\mathbf{u}(0) = (6, 5, 4)$ is $\mathbf{x}_1 + \mathbf{x}_2 + 4\mathbf{x}_3$. Thus $(c_1, c_2, c_3) = (1, 1, 4)$.

Step 3 The pure exponential solutions are $e^t \mathbf{x}_1$ and $e^{2t} \mathbf{x}_2$ and $e^{3t} \mathbf{x}_3$.

Solution: The combination that starts from $\mathbf{u}(0)$ is $\mathbf{u}(t) = e^t \mathbf{x}_1 + e^{2t} \mathbf{x}_2 + 4e^{3t} \mathbf{x}_3$. The coefficients 1, 1, 4 came from solving the linear equation $c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + c_3 \mathbf{x}_3 = \mathbf{u}(0)$:

$$\begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{which is } S\mathbf{c} = \mathbf{u}(0). \quad (7)$$

You now have the basic idea—how to solve $d\mathbf{u}/dt = A\mathbf{u}$. The rest of this section goes further. We solve equations that contain *second* derivatives, because they arise so often in applications. We also decide whether $\mathbf{u}(t)$ approaches zero or blows up or just oscillates. At the end comes the *matrix exponential* e^{At} . Then $e^{At}\mathbf{u}(0)$ solves the equation $d\mathbf{u}/dt = A\mathbf{u}$ in the same way that $A^k\mathbf{u}_0$ solves the equation $\mathbf{u}_{k+1} = A\mathbf{u}_k$.

All these steps use the λ 's and \mathbf{x} 's. With extra time, this section makes a strong connection to the whole topic of differential equations. It solves the constant coefficient problems that turn into linear algebra. Use this section to clarify these simplest but most important differential equations—whose solution is completely based on $e^{\lambda t}$.

Second Order Equations

The most important equation in mechanics is $my'' + by' + ky = 0$. The first term is the mass m times the acceleration $a = y''$. This term ma balances the force F (*Newton's Law!*). The force includes the damping $-by'$ and the elastic restoring force $-ky$, proportional to distance moved. This is a second-order equation because it contains the second derivative $y'' = d^2y/dt^2$. It is still linear with constant coefficients m, b, k .

In a differential equations course, the method of solution is to substitute $y = e^{\lambda t}$. Each derivative brings down a factor λ . We want $y = e^{\lambda t}$ to solve the equation:

$$m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = (m\lambda^2 + b\lambda + k)e^{\lambda t} = 0. \quad (8)$$

Everything depends on $m\lambda^2 + b\lambda + k = 0$. This equation for λ has two roots λ_1 and λ_2 . Then the equation for y has two pure solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. Their combinations $c_1 y_1 + c_2 y_2$ give the complete solution.

In a linear algebra course we expect matrices and eigenvalues. Therefore we turn the scalar equation (with y'') into a vector equation (first derivative only!). Suppose $m = 1$. The unknown vector \mathbf{u} has components y and y' . The equation is $d\mathbf{u}/dt = A\mathbf{u}$:

$$\begin{aligned} \frac{dy}{dt} &= y' \\ \frac{dy'}{dt} &= -ky - by' \end{aligned} \quad \text{converts to} \quad \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}. \quad (9)$$

The first equation $dy/dt = y'$ is trivial (but true). The second equation connects y'' to y' and y . Together the equations connect \mathbf{u}' to \mathbf{u} . So we solve by eigenvalues:

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -k & -b - \lambda \end{bmatrix} \quad \text{has determinant} \quad \lambda^2 + b\lambda + k = 0.$$

The equation for the λ 's is the same! It is still $\lambda^2 + b\lambda + k = 0$, since $m = 1$. The roots λ_1 and λ_2 are now *eigenvalues of A*. The eigenvectors and the complete solution are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \quad \mathbf{u}(t) = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

In the first component of $\mathbf{u}(t)$, you see $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ —the same solution as before.

It can't be anything else. In the second component of $\mathbf{u}(t)$ you see dy/dt . The vector problem is completely consistent with the scalar problem.

Note 1 Real engineering and real physics deal with systems (not just a single mass at one point). The unknown \mathbf{y} is a vector. The coefficient of \mathbf{y}'' is a *mass matrix M*, not a number m . The coefficient of \mathbf{y} is a *stiffness matrix K*, not a number k . The coefficient of \mathbf{y}' is a damping matrix which might be zero.

The equation $M\mathbf{y}'' + K\mathbf{y} = \mathbf{f}$ is a major part of computational mechanics. It is controlled by the eigenvalues of $M^{-1}K$ in $K\mathbf{x} = \lambda M\mathbf{x}$.

Note 2 In linear algebra the serious danger is a shortage of eigenvectors. Our eigenvectors $(1, \lambda_1)$ and $(1, \lambda_2)$ are the same if $\lambda_1 = \lambda_2$. Then we can't diagonalize A . In this case we don't yet have a complete solution to $d\mathbf{u}/dt = A\mathbf{u}$.

In differential equations the danger is also a repeated λ . After $y = e^{\lambda t}$, a second solution has to be found. It turns out to be $y = te^{\lambda t}$.

This “impure” solution (with the extra t) will soon appear also in e^{At} .

Example 3 Solve $y'' + 4y' + 3y = 0$ by substituting $e^{\lambda t}$ and also by linear algebra.

Solution Substituting $y = e^{\lambda t}$ yields $(\lambda^2 + 4\lambda + 3)e^{\lambda t} = 0$. The quadratic factors into $\lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0$. Therefore $\lambda_1 = -1$ and $\lambda_2 = -3$. The pure solutions are $y_1 = e^{-t}$ and $y_2 = e^{-3t}$. The complete solution $c_1 y_1 + c_2 y_2$ approaches zero.

To use linear algebra we set $\mathbf{u} = (y, y')$. This leads to a vector equation $\mathbf{u}' = A\mathbf{u}$:

$$\begin{aligned} dy/dt &= y' \\ dy'/dt &= -3y - 4y' \end{aligned} \text{ converts to } \frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \mathbf{u}.$$

This A is called a “companion matrix” and we find its eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -3 & -4 - \lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0.$$

The λ 's are still -1 and -3 and the solution is the same. With constant coefficients and pure exponentials, calculus goes back to algebra.

Stability of 2 by 2 Matrices

For the solution of $d\mathbf{u}/dt = A\mathbf{u}$, there is a fundamental question. *Does the solution approach $\mathbf{u} = \mathbf{0}$ as $t \rightarrow \infty$?* Is the problem *stable*? Example 3 was certainly stable, because both pure solutions e^{-t} and e^{-3t} approach zero. Stability depends on the eigenvalues -1 and -3 , and the eigenvalues depend on A .

The complete solution $\mathbf{u}(t)$ is built from pure solutions $e^{\lambda t}\mathbf{x}$. If the eigenvalue λ is real, we know exactly when $e^{\lambda t}$ approaches zero: *The number λ must be negative.* If the eigenvalue is a complex number $\lambda = r + is$, *the real part r must be negative.* When $e^{\lambda t}$ splits into $e^{rt}e^{ist}$, the factor e^{ist} has absolute value fixed at 1:

$$e^{ist} = \cos st + i \sin st \quad \text{has} \quad |e^{ist}|^2 = \cos^2 st + \sin^2 st = 1.$$

The factor e^{rt} controls growth ($r > 0$ is instability) or decay ($r < 0$ is stability).

The question is: **Which matrices have negative eigenvalues?** More accurately, when are the **real parts** of the λ 's all negative? 2 by 2 matrices allow a clear answer.

6G Stability The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is *stable* and $\mathbf{u}(t) \rightarrow \mathbf{0}$ when the eigenvalues have negative real parts. The matrix A must pass two tests:

The trace $T = a + d$ must be negative.

The determinant $D = ad - bc$ must be positive.

Reason If the λ 's are real and negative, their sum is negative. This is the trace T . Their product is positive. This is the determinant D . The argument also goes in the reverse direction. If $D = \lambda_1\lambda_2$ is positive, then λ_1 and λ_2 have the same sign. If $T = \lambda_1 + \lambda_2$ is negative, that sign will be negative. We can test T and D .

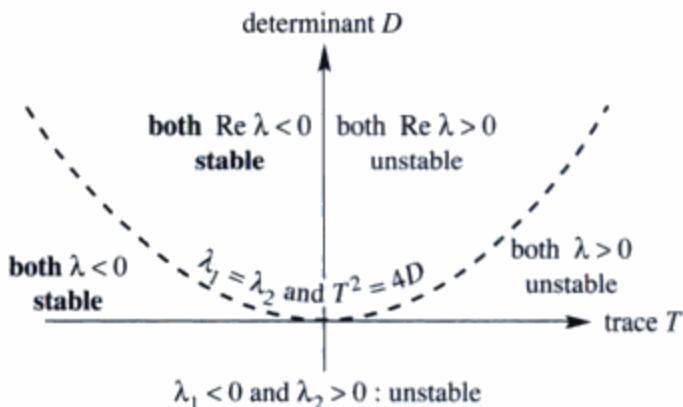


Figure 6.3 A 2 by 2 matrix is stable ($\mathbf{u}(t) \rightarrow \mathbf{0}$) when $T < 0$ and $D > 0$.

If the λ 's are complex numbers, they must have the form $r + is$ and $r - is$. Otherwise T and D will not be real. The determinant D is automatically positive, since $(r + is)(r - is) = r^2 + s^2$. The trace T is $r + is + r - is = 2r$. So a negative trace means that the real part r is negative and the matrix is stable. Q.E.D.

Figure 6.3 shows the parabola $T^2 = 4D$ which separates real from complex eigenvalues. Solving $\lambda^2 - T\lambda + D = 0$ leads to $\sqrt{T^2 - 4D}$. This is real below the parabola and imaginary above it. The stable region is the *upper left quarter* of the figure—where the trace T is negative and the determinant D is positive.

Example 4 Which of these matrices is stable?

$$A_1 = \begin{bmatrix} 0 & -1 \\ -2 & -3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 4 & 5 \\ -6 & -7 \end{bmatrix} \quad A_3 = \begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix}.$$

The Exponential of a Matrix

We return briefly to write the solution $\mathbf{u}(t)$ in a new form $e^{At}\mathbf{u}(0)$. This gives a perfect parallel with $A^k\mathbf{u}_0$ in the previous section. First we have to say what e^{At} means.

The matrix e^{At} has a matrix in the exponent. To define e^{At} , copy e^x . The direct definition of e^x is by the infinite series $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$. When you substitute the matrix At for x , this series defines the matrix e^{At} :

Matrix exponential

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \dots \quad (10)$$

Its t derivative is

$$A + A^2t + \frac{1}{2}A^3t^2 + \dots = Ae^{At}.$$

The number that divides $(At)^n$ is “ n factorial.” This is $n! = (1)(2)\cdots(n-1)(n)$. The factorials after 1, 2, 6 are $4! = 24$ and $5! = 120$. They grow quickly. The series

$$I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots = \begin{bmatrix} 1 - \frac{1}{2}t^2 + \cdots & t - \frac{1}{6}t^3 + \cdots \\ -t + \frac{1}{6}t^3 - \cdots & 1 - \frac{1}{2}t^2 + \cdots \end{bmatrix}.$$

On the left side is the series for e^{At} . The top row of the matrix shows the series for $\cos t$ and $\sin t$. We have found e^{At} directly:

$$e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}. \quad (15)$$

At $t = 0$ this gives $e^0 = I$. Most important, the derivative of e^{At} is Ae^{At} :

$$\frac{d}{dt} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

A is a skew-symmetric matrix ($A^T = -A$). Its exponential e^{At} is an orthogonal matrix. The eigenvalues of A are i and $-i$. The eigenvalues of e^{At} are e^{it} and e^{-it} . This illustrates two general rules:

- 1 *The eigenvalues of e^{At} are $e^{\lambda t}$.*
- 2 *When A is skew-symmetric, e^{At} is orthogonal.*

Example 6 Solve $\frac{du}{dt} = Au = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} u$ starting from $u(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ at $t = 0$.

Solution The eigenvalues 1 and 2 are on the diagonal of A (since A is triangular). The eigenvectors are $x_1 = (1, 0)$ and $x_2 = (1, 1)$:

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Step 2 writes $u(0)$ as a combination $x_1 + x_2$ of these eigenvectors. This is $Sc = u(0)$. In this case $c_1 = c_2 = 1$. Then $u(t)$ is the same combination of pure exponential solutions:

$$u(t) = e^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

That is the clearest solution $e^{\lambda_1 t} x_1 + e^{\lambda_2 t} x_2$. In matrix form, the eigenvectors go into S :

$$u(t) = Se^{At}S^{-1}u(0) \text{ is } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & \\ & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} u(0) = \begin{bmatrix} e^t & e^{2t} + e^t \\ 0 & e^{2t} \end{bmatrix} u(0).$$

That last matrix is e^{At} . It's not bad to see what a matrix exponential looks like (this is a particularly nice one). The situation is the same as for $Ax = b$ and inverses. We don't really need A^{-1} to find x , and we don't need e^{At} to solve $du/dt = Au$. But as quick formulas for the answers, $A^{-1}b$ and $e^{At}u(0)$ are unbeatable.

positive viewpoint, for *positive definite matrices*.) Matrices like A and $-A$ are the key to all the engineering applications in Section 8.1.

6.3 B Solve the four equations $da/dt = 0$, $db/dt = a$, $dc/dt = 2b$, $dz/dt = 3c$ in that order starting from $\mathbf{u}(0) = (a(0), b(0), c(0), z(0))$. Note the matrix in $\mathbf{u}' = A\mathbf{u}$ and solve the same equations by the matrix exponential in $\mathbf{u}(t) = e^{At}\mathbf{u}(0)$:

$$\frac{d}{dt} \begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ z \end{bmatrix} \quad \text{is} \quad \frac{d\mathbf{u}}{dt} = A\mathbf{u}.$$

First find A^2 , A^3 , A^4 and then $e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3$. Why does the series stop there? Check that e^A is Pascal's triangular matrix at $t = 1$, and the derivative of e^{At} at $t = 0$ is A . Finally, verify $(e^A)(e^A) = (e^{2A})$. Why is this true for any A ?

Solution Integrate $da/dt = 0$, then $db/dt = a$, then $dc/dt = 2b$ and $dz/dt = 3c$:

$$\begin{aligned} a(t) &= a(0) \\ b(t) &= ta(0) + b(0) \\ c(t) &= t^2a(0) + 2tb(0) + c(0) \\ z(t) &= t^3a(0) + 3t^2b(0) + 3tc(0) + z(0) \end{aligned} \quad \text{which must match } e^{At}\mathbf{u}(0).$$

The powers of A are all zero after A^3 . So the series for e^{At} stops after four terms:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix}$$

The diagonals move down at each step and disappear for A^4 . (There must be a diagonal-diagonal rule to go with the row-column and column-row rules for multiplying matrices.) The matrix exponential is the same one that multiplied $(a(0), b(0), c(0), z(0))$ above:

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{6} = \begin{bmatrix} 1 & & & \\ t & 1 & & \\ t^2 & 2t & 1 & \\ t^3 & 3t^2 & 3t & 1 \end{bmatrix}$$

At $t = 1$, e^A is *Pascal's triangular matrix* P_L . The derivative of e^{At} at $t = 0$ is A :

$$\lim_{t \rightarrow 0} \frac{e^{At} - I}{t} = \lim_{t \rightarrow 0} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ t & 2 & 0 & 0 \\ t^2 & 3t & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = A.$$

A is the **matrix logarithm** of Pascal's e^A ! The inverse of Pascal is e^{-A} with two negative diagonals. The square of e^A is always e^{2A} (and also $e^{As}e^{At} = e^{A(s+t)}$) for many reasons:

Solving with e^A from $t = 0$ to 1 and then 1 to 2 agrees with e^{2A} from 0 to 2. The squared series $(I + A + \frac{A^2}{2} + \dots)^2$ agrees with $I + 2A + \frac{(2A)^2}{2} + \dots = e^{2A}$. If A can be diagonalized (this A can't!) then $(Se^A S^{-1})(Se^A S^{-1}) = Se^{2A} S^{-1}$.

Problem Set 6.3

- 1** Find λ 's and x 's so that $\mathbf{u} = e^{\lambda t} \mathbf{x}$ solves

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \mathbf{u}.$$

What combination $\mathbf{u} = c_1 e^{\lambda_1 t} \mathbf{x}_1 + c_2 e^{\lambda_2 t} \mathbf{x}_2$ starts from $\mathbf{u}(0) = (5, -2)$?

- 2** Solve Problem 1 for $\mathbf{u} = (y, z)$ by back substitution:

First solve $\frac{dz}{dt} = z$ starting from $z(0) = -2$.

Then solve $\frac{dy}{dt} = 4y + 3z$ starting from $y(0) = 5$.

The solution for y will be a combination of e^{4t} and e^t .

- 3** Find A to change the scalar equation $y'' = 5y' + 4y$ into a vector equation for $\mathbf{u} = (y, y')$:

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\mathbf{u}.$$

What are the eigenvalues of A ? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 5y' + 4y$.

- 4** The rabbit and wolf populations show fast growth of rabbits (from $6r$) but loss to wolves (from $-2w$):

$$\frac{dr}{dt} = 6r - 2w \quad \text{and} \quad \frac{dw}{dt} = 2r + w.$$

Find the eigenvalues and eigenvectors. If $r(0) = w(0) = 30$ what are the populations at time t ? After a long time, is the ratio of rabbits to wolves 1 to 2 or is it 2 to 1?

- 5** A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \quad \text{and} \quad \frac{dw}{dt} = v - w.$$

Show that the total $v + w$ is constant (40 people). Find the matrix in $d\mathbf{u}/dt = A\mathbf{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$?

- 6 Reverse the diffusion of people in Problem 5 to $d\mathbf{u}/dt = -A\mathbf{u}$:

$$\frac{dv}{dt} = v - w \quad \text{and} \quad \frac{dw}{dt} = w - v.$$

The total $v + w$ still remains constant. How are the λ 's changed now that A is changed to $-A$? But show that $v(t)$ grows to infinity from $v(0) = 30$.

- 7 The solution to $y'' = 0$ is a straight line $y = C + Dt$. Convert to a matrix equation:

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \text{ has the solution } \begin{bmatrix} y \\ y' \end{bmatrix} = e^{At} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}.$$

This matrix A cannot be diagonalized. Find A^2 and compute $e^{At} = I + At + \frac{1}{2}A^2t^2 + \dots$. Multiply your e^{At} times $(y(0), y'(0))$ to check the straight line $y(t) = y(0) + y'(0)t$.

- 8 Substitute $y = e^{\lambda t}$ into $y'' = 6y' - 9y$ to show that $\lambda = 3$ is a repeated root. This is trouble; we need a second solution after e^{3t} . The matrix equation is

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}.$$

Show that this matrix has $\lambda = 3, 3$ and only one line of eigenvectors. *Trouble here too.* Show that the second solution is $y = te^{3t}$.

- 9 Figure out how to write $my'' + by' + ky = 0$ as a vector equation $M\mathbf{u}' = A\mathbf{u}$.

- 10 The matrix in this question is skew-symmetric ($A^T = -A$):

$$\frac{d\mathbf{u}}{dt} = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix} \mathbf{u} \quad \text{or} \quad \begin{aligned} u'_1 &= cu_2 - bu_3 \\ u'_2 &= au_3 - cu_1 \\ u'_3 &= bu_1 - au_2. \end{aligned}$$

- (a) The derivative of $\|\mathbf{u}(t)\|^2 = u_1^2 + u_2^2 + u_3^2$ is $2u_1u'_1 + 2u_2u'_2 + 2u_3u'_3$. Substitute u'_1, u'_2, u'_3 to get zero. Then $\|\mathbf{u}(t)\|^2$ stays equal to $\|\mathbf{u}(0)\|^2$.

- (b) When A is skew-symmetric, $Q = e^{At}$ is orthogonal. Prove $Q^T = e^{-At}$ from the series for $Q = e^{At}$. Then $Q^T Q = I$.

- 11 (a) Write $(1, 0)$ as a combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ of these two eigenvectors of A :

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = i \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = -i \begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

- (b) The solution to $d\mathbf{u}/dt = A\mathbf{u}$ starting from $(1, 0)$ is $c_1 e^{it}\mathbf{x}_1 + c_2 e^{-it}\mathbf{x}_2$. Substitute $e^{it} = \cos t + i \sin t$ and $e^{-it} = \cos t - i \sin t$ to find $\mathbf{u}(t)$.

- 12 (a) Write down two familiar functions that solve the equation $d^2y/dt^2 = -y$. Which one starts with $y(0) = 1$ and $y'(0) = 0$?
 (b) This second-order equation $y'' = -y$ produces a vector equation $\mathbf{u}' = A\mathbf{u}$:

$$\mathbf{u} = \begin{bmatrix} y \\ y' \end{bmatrix} \quad \frac{d\mathbf{u}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\mathbf{u}.$$

Put $y(t)$ from part (a) into $\mathbf{u}(t) = (y, y')$. This solves Problem 11 again.

- 13 A particular solution to $d\mathbf{u}/dt = A\mathbf{u} - \mathbf{b}$ is $\mathbf{u}_p = A^{-1}\mathbf{b}$, if A is invertible. The solutions to $d\mathbf{u}/dt = A\mathbf{u}$ give \mathbf{u}_n . Find the complete solution $\mathbf{u}_p + \mathbf{u}_n$ to

$$(a) \quad \frac{d\mathbf{u}}{dt} = 2\mathbf{u} - 8 \quad (b) \quad \frac{d\mathbf{u}}{dt} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \mathbf{u} - \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$

- 14 If c is not an eigenvalue of A , substitute $\mathbf{u} = e^{ct}\mathbf{v}$ and find \mathbf{v} to solve $d\mathbf{u}/dt = A\mathbf{u} - e^{ct}\mathbf{b}$. This $\mathbf{u} = e^{ct}\mathbf{v}$ is a particular solution. How does it break down when c is an eigenvalue?

- 15 Find a matrix A to illustrate each of the unstable regions in Figure 6.4:

- (a) $\lambda_1 < 0$ and $\lambda_2 > 0$
- (b) $\lambda_1 > 0$ and $\lambda_2 > 0$
- (c) Complex λ 's with real part $a > 0$.

Questions 16–25 are about the matrix exponential e^{At} .

- 16 Write five terms of the infinite series for e^{At} . Take the t derivative of each term. Show that you have four terms of Ae^{At} . Conclusion: $e^{At}\mathbf{u}_0$ solves $\mathbf{u}' = A\mathbf{u}$.
- 17 The matrix $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ has $B^2 = 0$. Find e^{Bt} from a (short) infinite series. Check that the derivative of e^{Bt} is Be^{Bt} .
- 18 Starting from $\mathbf{u}(0)$ the solution at time T is $e^{AT}\mathbf{u}(0)$. Go an additional time t to reach $e^{At}(e^{AT}\mathbf{u}(0))$. This solution at time $t+T$ can also be written as _____. Conclusion: e^{At} times e^{AT} equals _____.
 19 Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ in the form SAS^{-1} . Find e^{At} from $Se^{At}S^{-1}$.
- 20 If $A^2 = A$ show that the infinite series produces $e^{At} = I + (e^t - 1)A$. For $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ in Problem 19 this gives $e^{At} = _____$.
- 21 Generally e^Ae^B is different from e^Be^A . They are both different from e^{A+B} . Check this using Problems 19–20 and 17:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

- 22 Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as SAS^{-1} . Multiply $Se^{\Lambda t}S^{-1}$ to find the matrix exponential e^{At} . Check e^{At} when $t = 0$.
- 23 Put $A = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ into the infinite series to find e^{At} . First compute A^2 :

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 3t \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} & \\ & \end{bmatrix} + \cdots = \begin{bmatrix} e^t & \\ 0 & \end{bmatrix}.$$

- 24 Give two reasons why the matrix exponential e^{At} is never singular:
- Write down its inverse.
 - Write down its eigenvalues. If $Ax = \lambda x$ then $e^{At}x = \underline{\hspace{2cm}} x$.
- 25 Find a solution $x(t), y(t)$ that gets large as $t \rightarrow \infty$. To avoid this instability a scientist exchanged the two equations:

$$\begin{array}{ll} dx/dt = 0x - 4y & \text{becomes} \\ dy/dt = -2x + 2y & \end{array} \quad \begin{array}{l} dy/dt = -2x + 2y \\ dx/dt = 0x - 4y. \end{array}$$

Now the matrix $\begin{bmatrix} -2 & 2 \\ 0 & -4 \end{bmatrix}$ is stable. It has negative eigenvalues. Comment on this.

SYMMETRIC MATRICES ■ 6.4

For projection onto a line, the eigenvalues are 1 and 0. Eigenvectors are *on* the line (where $Px = x$) and *perpendicular* to the line (where $Px = \mathbf{0}$). Now we open up to all other *symmetric matrices*. It is no exaggeration to say that these are the most important matrices the world will ever see—in the theory of linear algebra and also in the applications. We come immediately to the key questions about symmetry. Not only the questions, but also the answers.

What is special about $Ax = \lambda x$ when A is symmetric? We are looking for special properties of the eigenvalues λ and the eigenvectors x when $A = A^T$.

The diagonalization $A = SAS^{-1}$ will reflect the symmetry of A . We get some hint by transposing to $A^T = (S^{-1})^T \Lambda S^T$. Those are the same since $A = A^T$. Possibly S^{-1} in the first form equals S^T in the second form. Then $S^T S = I$. That makes each eigenvector in S orthogonal to the other eigenvectors. The key facts get first place in the Table at the end of this chapter, and here they are:

1. A symmetric matrix has only *real eigenvalues*.
2. The *eigenvectors* can be chosen *orthonormal*.

Those orthonormal eigenvectors go into the columns of S . There are n of them (independent because they are orthonormal). Every symmetric matrix can be diagonalized.

Its eigenvector matrix S becomes an orthogonal matrix Q . Orthogonal matrices have $Q^{-1} = Q^T$ —what we suspected about S is true. To remember it we write $S = Q$, when we choose orthonormal eigenvectors.

Why do we use the word “choose”? Because the eigenvectors do not *have* to be unit vectors. Their lengths are at our disposal. We will choose unit vectors—eigenvectors of length one, which are orthonormal and not just orthogonal. Then $A = S\Lambda S^{-1}$ is in its special and particular form $Q\Lambda Q^T$ for symmetric matrices:

6H (Spectral Theorem) Every symmetric matrix has the factorization $A = Q\Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in Q :

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T \quad \text{with} \quad Q^{-1} = Q^T.$$

It is easy to see that $Q\Lambda Q^T$ is symmetric. Take its transpose. You get $(Q^T)^T \Lambda^T Q^T$, which is $Q\Lambda Q^T$ again. The harder part is to prove that every symmetric matrix has real λ 's and orthonormal x 's. This is the “*spectral theorem*” in mathematics and the “*principal axis theorem*” in geometry and physics. We have to prove it! No choice. I will approach the proof in three steps:

1. By an example (which only proves that $A = Q\Lambda Q^T$ might be true)
2. By calculating the 2 by 2 case (which convinces most fair-minded people)
3. By a proof when no eigenvalues are repeated (leaving only real diehards).

The diehards are worried about repeated eigenvalues. Are there still n orthonormal eigenvectors? *Yes, there are.* They go into the columns of S (which becomes Q). The last page before the problems outlines this fourth and final step.

We now take steps 1 and 2. In a sense they are optional. The 2 by 2 case is mostly for fun, since it is included in the final n by n case.

Example 1 Find the λ 's and x 's when $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}$.

Solution The equation $\det(A - \lambda I) = 0$ is $\lambda^2 - 5\lambda = 0$. The eigenvalues are 0 and 5 (*both real*). We can see them directly: $\lambda = 0$ is an eigenvalue because A is singular, and $\lambda = 5$ is the other eigenvalue so that $0 + 5$ agrees with $1 + 4$. This is the *trace* down the diagonal of A .

Two eigenvectors are $(2, -1)$ and $(1, 2)$ —orthogonal but not yet orthonormal. The eigenvector for $\lambda = 0$ is in the *nullspace* of A . The eigenvector for $\lambda = 5$ is in the *column space*. We ask ourselves, why are the nullspace and column space perpendicular? The Fundamental Theorem says that the nullspace is perpendicular to the *row space*—not the column space. But our matrix is *symmetric*! Its row and column spaces are the same. Its eigenvectors $(2, -1)$ and $(1, 2)$ must be (and are) perpendicular.

These eigenvectors have length $\sqrt{5}$. Divide them by $\sqrt{5}$ to get unit vectors. Put those into the columns of S (which is Q). Then $Q^{-1}AQ$ is Λ and $Q^{-1} = Q^T$:

$$Q^{-1}AQ = \frac{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \frac{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}}{\sqrt{5}} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = \Lambda.$$

Now comes the calculation for *any* 2 by 2 symmetric matrix $[a \ b ; b \ c]$. First, **real eigenvalues**. Second, **perpendicular eigenvectors**. The λ 's come from

$$\det \begin{bmatrix} a - \lambda & b \\ b & c - \lambda \end{bmatrix} = \lambda^2 - (a + c)\lambda + (ac - b^2) = 0. \quad (1)$$

The test for real roots of $A\lambda^2 + B\lambda + C = 0$ is based on $B^2 - 4AC$. *This must not be negative*, or its square root in the quadratic formula would be imaginary. Our equation has $A = 1$ and $B = -(a + c)$ and $C = ac - b^2$. Look at $B^2 - 4AC$:

Real eigenvalues: $(a + c)^2 - 4(ac - b^2)$ must not be negative.

Rewrite that as $a^2 + 2ac + c^2 - 4ac + 4b^2$. Rewrite again as $(a - c)^2 + 4b^2$. Those squares are not negative! So the roots λ_1 and λ_2 (the eigenvalues) are certainly real.

Perpendicular eigenvectors: Compute \mathbf{x}_1 and \mathbf{x}_2 and their dot product:

$$(A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} a - \lambda_1 & b \\ b & c - \lambda_1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \end{bmatrix} = \mathbf{0} \quad \text{so} \quad \mathbf{x}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \quad \begin{array}{l} \text{from} \\ \text{first} \\ \text{row} \end{array}$$

$$(A - \lambda_2 I)\mathbf{x}_2 = \begin{bmatrix} a - \lambda_2 & b \\ b & c - \lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_2 \end{bmatrix} = \mathbf{0} \quad \text{so} \quad \mathbf{x}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix} \quad \begin{array}{l} \text{from} \\ \text{second} \\ \text{row} \end{array}$$

The dot product of \mathbf{x}_1 and \mathbf{x}_2 proves that these eigenvectors are perpendicular:

$$\mathbf{x}_1 \cdot \mathbf{x}_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = b(\lambda_1 + \lambda_2 - a - c) = 0. \quad (2)$$

This is zero because $\lambda_1 + \lambda_2$ equals the trace $a + c$. Thus $\mathbf{x}_1 \cdot \mathbf{x}_2 = 0$. Eagle eyes might notice the special case $a = c, b = 0$ when $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{0}$. This case has repeated eigenvalues, as in $A = I$. It still has perpendicular eigenvectors $(1, 0)$ and $(0, 1)$.

Now comes the general n by n case, with real λ 's and perpendicular eigenvectors.

6I Real Eigenvalues

The eigenvalues of a real symmetric matrix are real.

Proof Suppose that $A\mathbf{x} = \lambda\mathbf{x}$. Until we know otherwise, λ might be a complex number $a + ib$ (a and b real). Its complex conjugate is $\bar{\lambda} = a - ib$. Similarly the components of \mathbf{x} may be complex numbers, and switching the signs of their imaginary

parts gives \bar{x} . The good thing is that $\bar{\lambda}$ times \bar{x} is always the conjugate of λ times x . So take conjugates of $Ax = \lambda x$, remembering that A is real:

$$Ax = \lambda x \text{ leads to } A\bar{x} = \bar{\lambda}\bar{x}. \quad \text{Transpose to } \bar{x}^T A = \bar{x}^T \bar{\lambda}. \quad (3)$$

Now take the dot product of the first equation with \bar{x} and the last equation with x :

$$\bar{x}^T A x = \bar{x}^T \lambda x \quad \text{and also} \quad \bar{x}^T A x = \bar{x}^T \bar{\lambda} x. \quad (4)$$

The left sides are the same so the right sides are equal. One equation has λ , the other has $\bar{\lambda}$. They multiply $\bar{x}^T x$ which is not zero—it is the squared length of the eigenvector. Therefore λ must equal $\bar{\lambda}$, and $a + ib$ equals $a - ib$. The imaginary part is $b = 0$. Q.E.D.

The eigenvectors come from solving the real equation $(A - \lambda I)x = 0$. So the x 's are also real. The important fact is that they are perpendicular.

6] Orthogonal Eigenvectors Eigenvectors of a real symmetric matrix (when they correspond to different λ 's) are always perpendicular.

A has real eigenvalues and n real orthogonal eigenvectors if and only if $A = A^T$.

Proof Suppose $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$ and $A = A^T$. Take dot products of the first equation with y and the second with x :

$$(\lambda_1 x)^T y = (Ax)^T y = x^T A^T y = x^T A y = x^T \lambda_2 y. \quad (5)$$

The left side is $x^T \lambda_1 y$, the right side is $x^T \lambda_2 y$. Since $\lambda_1 \neq \lambda_2$, this proves that $x^T y = 0$. The eigenvector x (for λ_1) is perpendicular to the eigenvector y (for λ_2).

Example 2 Find the λ 's and x 's for this symmetric matrix with trace zero:

$$A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \quad \text{has} \quad \det(A - \lambda I) = \begin{vmatrix} -3 - \lambda & 4 \\ 4 & 3 - \lambda \end{vmatrix} = \lambda^2 - 25.$$

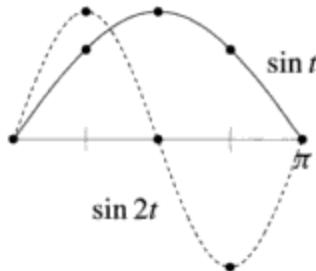
The roots of $\lambda^2 - 25 = 0$ are $\lambda_1 = 5$ and $\lambda_2 = -5$ (both real). The eigenvectors $x_1 = (1, 2)$ and $x_2 = (-2, 1)$ are perpendicular. To make them into unit vectors, divide by their lengths $\sqrt{5}$. The new x_1 and x_2 are the columns of Q , and Q^{-1} equals Q^T :

$$A = Q \Lambda Q^T = \frac{\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}}{\sqrt{5}} \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix} \frac{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}{\sqrt{5}}.$$

This example shows the main goal of this section—to diagonalize symmetric matrices A by orthogonal eigenvector matrices $S = Q$:

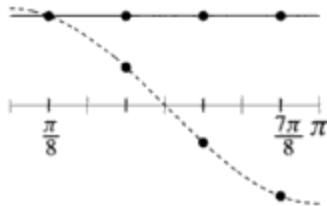
Solution The eigenvalues of A_3 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$. Their sum is 6 (the trace of A_3) and their product is 4 (the determinant). The eigenvector matrix S gives the “Discrete Sine Transform” and the graph shows how the components of the first two eigenvectors fall onto sine curves. Please draw the third eigenvector onto a third sine curve!

$$S = \begin{bmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{bmatrix}$$



The eigenvalues of B_4 are $\lambda = 2 - \sqrt{2}$ and 2 and $2 + \sqrt{2}$ and 0 (the same as for A_3 , plus the zero eigenvalue). The trace is still 6, but the determinant is now zero. The eigenvector matrix C gives the 4-point “Discrete Cosine Transform” and the graph shows how the first two eigenvectors fall onto cosine curves. (Please plot the third eigenvector!) These eigenvectors match cosines at the *halfway points* $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \sqrt{2}-1 & -1 & 1-\sqrt{2} \\ 1 & 1-\sqrt{2} & -1 & \sqrt{2}-1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$



Both S and C have orthogonal columns (these are the eigenvectors of the symmetric A_3 and B_4). When we multiply an input signal by S or C , we split that signal into pure frequencies—like separating a musical chord into pure notes. This Discrete Fourier Transform is the most useful and insightful transform in all of signal processing. We are seeing the sines and cosines (DST and DCT) that go into the DFT. Of course these beautiful patterns continue for larger matrices. Here is a MATLAB code to create B_8 and its eigenvector matrix C_8 and plot the first four eigenvectors onto cosine curves:

```
n = 8; e = ones(n - 1, 1); B = 2 * eye(n) - diag(e, -1) - diag(e, 1); B(1, 1) = 1; B(n, n) = 1; [C, L] = eig(B); plot(C(:, 1:4), '-o')
```

Problem Set 6.4

- 1 Write A as $M + N$, symmetric matrix plus skew-symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 4 & 3 & 0 \\ 8 & 6 & 5 \end{bmatrix} = M + N \quad (M^T = M, N^T = -N).$$

For any square matrix, $M = \frac{A+A^T}{2}$ and $N = \frac{A-A^T}{2}$ add up to A .

- 2 If C is symmetric prove that A^TCA is also symmetric. (Transpose it.) When A is 6 by 3, what are the shapes of C and A^TCA ?
- 3 Find the eigenvalues and the unit eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

- 4 Find an orthogonal matrix Q that diagonalizes $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$.
- 5 Find an orthogonal matrix Q that diagonalizes this symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

- 6 Find *all* orthogonal matrices that diagonalize $A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.
- 7 (a) Find a symmetric matrix $\begin{bmatrix} 1 & b \\ b & 1 \end{bmatrix}$ that has a negative eigenvalue.
 (b) How do you know it must have a negative pivot?
 (c) How do you know it can't have two negative eigenvalues?
- 8 If $A^3 = 0$ then the eigenvalues of A must be _____. Give an example that has $A \neq 0$. But if A is symmetric, diagonalize it to prove that A must be zero.
- 9 If $\lambda = a + ib$ is an eigenvalue of a real matrix A , then its conjugate $\bar{\lambda} = a - ib$ is also an eigenvalue. (If $Ax = \lambda x$ then also $A\bar{x} = \bar{\lambda}\bar{x}$.) Prove that every real 3 by 3 matrix has a real eigenvalue.
- 10 Here is a quick “proof” that the eigenvalues of all real matrices are real:

$$Ax = \lambda x \text{ gives } x^T Ax = \lambda x^T x \text{ so } \lambda = \frac{x^T Ax}{x^T x} \text{ is real.}$$

Find the flaw in this reasoning—a hidden assumption that is not justified.

- 11 Write A and B in the form $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T$ of the spectral theorem $Q \Lambda Q^T$:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \quad (\text{keep } \|x_1\| = \|x_2\| = 1).$$

- 12 Every 2 by 2 symmetric matrix is $\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T = \lambda_1 P_1 + \lambda_2 P_2$. Explain $P_1 + P_2 = x_1 x_1^T + x_2 x_2^T = I$ from columns times rows, $P_1 P_2 = x_1 x_1^T + x_2 x_2^T = 0$ and from row times column.

- 13 What are the eigenvalues of $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$? Create a 3 by 3 skew-symmetric matrix ($A^T = -A$) and verify that its eigenvalues are all imaginary.
- 14 This matrix M is skew symmetric and also _____. Then its eigenvalues are all pure imaginary and they have $|\lambda| = 1$. ($\|M\mathbf{x}\| = \|\mathbf{x}\|$ for every \mathbf{x} so $\|\lambda\mathbf{x}\| = \|\mathbf{x}\|$ for eigenvectors.) Find all four eigenvalues of

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix}.$$

- 15 Show that A (symmetric but complex) does not have two independent eigenvectors:

$$A = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \text{ is not diagonalizable; } \det(A - \lambda I) = \lambda^2.$$

$A^T = A$ is not such a special property for complex matrices. The good property is $\overline{A^T} = A$ (Section 10.2). Then all λ 's are real and eigenvectors are orthogonal.

- 16 Even if A is rectangular, the block matrix $B = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ is symmetric:

$$B\mathbf{x} = \lambda\mathbf{x} \quad \text{is} \quad \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{y} \\ \mathbf{z} \end{bmatrix} \quad \text{which is} \quad \begin{array}{l} A\mathbf{z} = \lambda\mathbf{y} \\ A^T\mathbf{y} = \lambda\mathbf{z}. \end{array}$$

- (a) Show that $-\lambda$ is also an eigenvalue, with the eigenvector $(\mathbf{y}, -\mathbf{z})$.
- (b) Show that $A^TA\mathbf{z} = \lambda^2\mathbf{z}$, so that λ^2 is an eigenvalue of A^TA .
- (c) If $A = I$ (2 by 2) find all four eigenvalues and eigenvectors of B .
- 17 If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in Problem 16, find all three eigenvalues and eigenvectors of B .
- 18 *Another proof that eigenvectors are perpendicular when $A = A^T$.* Suppose $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = 0\mathbf{y}$ and $\lambda \neq 0$. Then \mathbf{y} is in the nullspace and \mathbf{x} is in the column space. They are perpendicular because _____. Go carefully—why are these subspaces orthogonal? If the second eigenvalue is a nonzero number β , apply this argument to $A - \beta I$. The eigenvalue moves to zero and the eigenvectors stay the same—so they are perpendicular.
- 19 Find the eigenvector matrix S for this matrix B . Show that it doesn't collapse at $d = 1$, even though $\lambda = 1$ is repeated. Are the eigenvectors perpendicular?

$$B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix} \quad \text{has} \quad \lambda = -1, 1, d.$$

- 20 From the trace and the determinant find the eigenvalues of

$$A = \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}.$$

Compare the signs of the λ 's with the signs of the pivots.

- 21 True or false. Give a reason or a counterexample.

- (a) A matrix with real eigenvalues and eigenvectors is symmetric.
 - (b) A matrix with real eigenvalues and orthogonal eigenvectors is symmetric.
 - (c) The inverse of a symmetric matrix is symmetric.
 - (d) The eigenvector matrix S of a symmetric matrix is symmetric.
- 22 A *normal matrix* has $A^T A = AA^T$; it has orthogonal eigenvectors. Why is every skew-symmetric matrix normal? Why is every orthogonal matrix normal? When is $\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$ normal?
- 23 (A paradox for instructors) If $AA^T = A^T A$ then A and A^T share the same eigenvectors (true). A and A^T always share the same eigenvalues. Find the flaw in this conclusion: They must have the same S and Λ . Therefore A equals A^T .
- 24 (Recommended) Which of these classes of matrices do A and B belong to: Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B : LU , QR , $S\Lambda S^{-1}$, $Q\Lambda Q^T$?

- 25 What number b in $\begin{bmatrix} 2 & b \\ 1 & 0 \end{bmatrix}$ makes $A = Q\Lambda Q^T$ possible? What number makes $A = S\Lambda S^{-1}$ impossible? What number makes A^{-1} impossible?
- 26 Find all 2 by 2 matrices that are orthogonal and also symmetric. Which two numbers can be eigenvalues?
- 27 This A is nearly symmetric. But its eigenvectors are far from orthogonal:

$$A = \begin{bmatrix} 1 & 10^{-15} \\ 0 & 1 + 10^{-15} \end{bmatrix} \text{ has eigenvectors } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} ? \\ ? \end{bmatrix}$$

What is the angle between the eigenvectors?

- 28 (MATLAB) Take two symmetric matrices with different eigenvectors, say $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 1 \\ 1 & 0 \end{bmatrix}$. Graph the eigenvalues $\lambda_1(A + tB)$ and $\lambda_2(A + tB)$ for $-8 < t < 8$. Peter Lax says on page 113 of *Linear Algebra* that λ_1 and λ_2 appear to be on a collision course at certain values of t . "Yet at the last minute they turn aside." How close do they come?

6M The eigenvalues of $A = A^T$ are positive if and only if the pivots are positive:

$$a > 0 \quad \text{and} \quad \frac{ac - b^2}{a} > 0.$$

The point is to recognize that ratio of positive numbers as the *second pivot* of A :

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \xrightarrow{\text{The first pivot is } a} \begin{bmatrix} a & b \\ 0 & c - \frac{b}{a}b \end{bmatrix} \quad \text{The second pivot is } c - \frac{b^2}{a} = \frac{ac - b^2}{a}$$

The multiplier is b/a

This connects two big parts of linear algebra. **Positive eigenvalues mean positive pivots and vice versa.** We gave a proof for all symmetric matrices in the last section (**Theorem 6K**). So the pivots give a quick test for $\lambda > 0$. They are a lot faster to compute than the eigenvalues. It is very satisfying to see pivots and determinants and eigenvalues come together in this course.

Example 1 This matrix has $a = 1$ (positive). But $ac - b^2 = 3 - 2^2$ is negative:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \text{ has a negative determinant and pivot. So a negative eigenvalue.}$$

The pivots are 1 and -1 . The eigenvalues also multiply to give -1 . One eigenvalue is negative (we don't want its formula, which has a square root, just its sign).

Here is a different way to look at symmetric matrices with positive eigenvalues. From $Ax = \lambda x$, multiply by x^T to get $x^T Ax = \lambda x^T x$. The right side is a positive λ times a positive $x^T x = \|x\|^2$. So $x^T Ax$ is positive for any eigenvector.

The new idea is that **this number $x^T Ax$ is positive for all nonzero vectors x** , not just the eigenvectors. Matrices with this property $x^T Ax > 0$ are **positive definite matrices**. We will prove that exactly these matrices have positive eigenvalues and pivots.

Definition The matrix A is **positive definite** if $x^T Ax > 0$ for every nonzero vector:

$$x^T Ax = [x \ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2 > 0.$$

$x^T Ax$ is a number (1 by 1 matrix). The four entries a, b, b, c give the four parts of $x^T Ax$. From a and c come the pure squares ax^2 and cy^2 . From b and b off the diagonal come the cross terms bxy and byx (the same). Adding those four parts gives $x^T Ax$:

$$f(x, y) = x^T Ax = ax^2 + 2bxy + cy^2 \quad \text{is "second degree."}$$

The rest of this book has been linear (mostly Ax). Now the degree has gone from 1 to 2. The *second* derivatives of $ax^2 + 2bxy + cy^2$ are constant. Those second derivatives are $2a, 2b, 2b, 2c$. They go into the ***second derivative matrix*** $2A$:

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2ax + 2by & \text{and} & \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 2b & 2c \end{bmatrix} = 2A.\end{aligned}$$

This is the 2 by 2 version of what everybody knows for 1 by 1. There the function is ax^2 , its slope is $2ax$, and its second derivative is $2a$. Now the function is $x^T Ax$, its first derivatives are in the vector $2Ax$, and its second derivatives are in the matrix $2A$. Third derivatives are all zero.

First Application: Test for a Minimum

Where does calculus use second derivatives? When f'' is positive, the curve bends up from its tangent line. The point $x = 0$ is a minimum point of $y = x^2$. It is a maximum point of $y = -x^2$. To decide minimum versus maximum for a one-variable function $f(x)$, calculus looks at its second derivative.

For a two-variable function $f(x, y)$, the *matrix of second derivatives* holds the key. One number is not enough to decide minimum versus maximum (versus saddle point). ***The function $f = x^T Ax$ has a minimum at $x = y = 0$ if and only if A is positive definite.*** The statement “ A is a positive definite matrix” is the 2 by 2 version of “ a is a positive number”.

Example 2 This matrix A is positive definite. We test by pivots or determinants:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} \quad \text{has positive pivots and determinants (1 and 3).}$$

More directly, $x^T Ax = x^2 + 4xy + 7y^2$ is positive because it is a **sum of squares**:

$$\text{Rewrite } x^2 + 4xy + 7y^2 \text{ as } (x + 2y)^2 + 3y^2.$$

The pivots 1 and 3 multiply those squares. This is no accident! By the algebra of “completing the square,” this always happens. So when the pivots are positive, the quadratic function $f(x, y) = x^T Ax$ is guaranteed to be positive: a sum of squares.

Comparing our examples $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$, the only difference is that change from 3 to 7. The borderline is $a_{22} = 4$. Above 4, the matrix is positive definite. At $a_{22} = 4$, the borderline matrix is only **semidefinite**. Then (> 0) changes to (≥ 0):

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \text{ is singular. It has eigenvalues 5 and 0.}$$

This matrix has $a > 0$ but $ac - b^2 = 0$. Not quite positive definite!

We will summarize this section so far. We have four ways to recognize a positive definite matrix. Right now it is only 2 by 2.

6N When a 2 by 2 symmetric matrix has one of these four properties, it has them all:

1. Both of the eigenvalues are positive.
2. The 1 by 1 and 2 by 2 determinants are positive: $a > 0$ and $ac - b^2 > 0$.
3. The pivots are positive: $a > 0$ and $(ac - b^2)/a > 0$.
4. The function $x^T A x = ax^2 + 2bxy + cy^2$ is positive except at $(0, 0)$.

When A has one (therefore all) of these four properties, it is a **positive definite matrix**.

Note We deal only with symmetric matrices. The cross derivative $\partial^2 f / \partial x \partial y$ always equals $\partial^2 f / \partial y \partial x$. For $f(x, y, z)$ the nine second derivatives fill a symmetric 3 by 3 matrix. It is positive definite when the three pivots (and the three eigenvalues, and the three determinants) are positive. **When the first derivatives $\partial f / \partial x$ and $\partial f / \partial y$ are zero and the second derivative matrix is positive definite, we have found a local minimum.**

Example 3 Is $f(x, y) = x^2 + 8xy + 3y^2$ everywhere positive—except at $(0, 0)$?

Solution The second derivatives are $f_{xx} = 2$ and $f_{xy} = f_{yx} = 8$ and $f_{yy} = 6$, all positive. But the test is not positive definiteness. We look for positive *definiteness*. The answer is *no*, this function is not always positive. By trial and error we locate a point $x = 1$, $y = -1$ where $f(1, -1) = 1 - 8 + 3 = -4$. Better to do linear algebra, and apply the exact tests to the matrix that produced $f(x, y)$:

$$x^2 + 8xy + 3y^2 = [x \ y] \begin{bmatrix} 1 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The matrix has $ac - b^2 = 3 - 16$. The pivots are 1 and -13 . The eigenvalues are _____ (we don't need them). The matrix is not positive definite.

Note how $8xy$ comes from $a_{12} = 4$ above the diagonal and $a_{21} = 4$ symmetrically below. That matrix multiplication in $x^T A x$ makes the function appear.

Main point The sign of b is not the essential thing. The cross derivative $\partial^2 f / \partial x \partial y$ can be positive or negative—the test uses b^2 . The size of b , compared to a and c , decides whether A is positive definite and the function $f(x, y)$ has a minimum.

Example 4 For which numbers c is $x^2 + 8xy + cy^2$ always positive (or zero)?

Solution The matrix is $A = \begin{bmatrix} 1 & 4 \\ 4 & c \end{bmatrix}$. Again $a = 1$ passes the first test. The second test has $ac - b^2 = c - 16$. For a positive definite matrix we need $c > 16$.

The “semidefinite” borderline is $c = 16$. At that point $\begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}$ has $\lambda = 17$ and 0, determinants 1 and 0, pivots 1 and _____. The function $x^2 + 8xy + 16y^2$ is $(x + 4y)^2$. Its graph does not go below zero, but it stays at zero all along the line $x + 4y = 0$. This is close to positive definite, but each test just misses: $x^T Ax$ equals zero for the vector $x = (4, -1)$. So A is only semidefinite.

Example 5 When A is positive definite, write $f(x, y)$ as a sum of two squares.

Solution This is called “completing the square.” The part $ax^2 + 2bxy$ is correct in the first square $a(x + \frac{b}{a}y)^2$. But that ends with a final $a(\frac{b}{a}y)^2$. To stay even, this added amount b^2y^2/a has to be subtracted off from cy^2 at the end:

$$\text{Completing the Square} \quad ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \left(\frac{ac - b^2}{a}\right)y^2. \quad (1)$$

After that gentle touch of algebra, the situation is clearer. The two squares (never negative) are multiplied by numbers that could be positive or negative. *Those numbers a and $(ac - b^2)/a$ are the pivots!* So positive pivots give a sum of squares and a positive definite matrix. Think back to the pivots and multipliers in LDL^T :

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b/a & 1 \end{bmatrix} \begin{bmatrix} a & (ac - b^2)/a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}. \quad (2)$$

To complete the square, we started with a and b . *Elimination does exactly the same.* It starts with the first column. Inside $(x + \frac{b}{a}y)^2$ are the numbers 1 and $\frac{b}{a}$ from L .

Every positive definite symmetric matrix factors into $A = LDL^T$ with positive pivots in D . The “Cholesky factorization” is $A = (L\sqrt{D})(L\sqrt{D})^T$.

Important to compare $A = LDL^T$ with $A = Q\Lambda Q^T$. One is based on pivots (in D). The other is based on eigenvalues (in Λ). *Please* do not think that pivots equal eigenvalues. Their signs are the same, but the numbers are entirely different.

Positive Definite Matrices: n by n

For a 2 by 2 matrix, the “positive definite test” uses *eigenvalues* or *determinants* or *pivots*. All those numbers must be positive. We hope and expect that the same three tests carry over to n by n symmetric matrices. They do.

6O When a symmetric matrix has one of these four properties, it has them all:

1. All n eigenvalues are positive.
2. All n upper left determinants are positive.
3. All n pivots are positive.
4. $x^T A x$ is positive except at $x = 0$. The matrix A is *positive definite*.

The “upper left determinants” are 1 by 1, 2 by 2, . . . , n by n . The last one is the determinant of the complete matrix A . This remarkable theorem ties together the whole linear algebra course—at least for symmetric matrices. We believe that two examples are more helpful than a proof (we nearly have a proof already). Then we give two applications.

Example 6 Test these matrices A and A^* for positive definiteness:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}.$$

Solution This A is an old friend (or enemy). Its pivots are 2 and $\frac{3}{2}$ and $\frac{4}{3}$, all positive. Its upper left determinants are 2 and 3 and 4, all positive. Its eigenvalues are $2 - \sqrt{2}$ and $2 + \sqrt{2}$, all positive. That completes tests 1, 2, and 3.

We can write $x^T A x$ as a sum of three squares (since $n = 3$). The pivots $2, \frac{3}{2}, \frac{4}{3}$ appear outside the squares. The multipliers $-\frac{1}{2}$ and $-\frac{2}{3}$ in L are inside the squares:

$$\begin{aligned} x^T A x &= 2(x_1^2 - x_1 x_2 + x_2^2 - x_2 x_3 + x_3^2) \\ &= 2(x_1 - \frac{1}{2}x_2)^2 + \frac{3}{2}(x_2 - \frac{2}{3}x_3)^2 + \frac{4}{3}(x_3)^2 > 0. \quad \text{This is positive.} \end{aligned}$$

Go to the second matrix A^* . The determinant test is easiest. The 1 by 1 determinant is 2, the 2 by 2 determinant is 3. The 3 by 3 determinant comes from the whole A^* :

$$\det A^* = 4 + 2b - 2b^2 = (1+b)(4-2b) \quad \text{must be positive.}$$

At $b = -1$ and $b = 2$ we get $\det A^* = 0$. In those cases A^* is positive semidefinite (no inverse, zero eigenvalue, $x^T A^* x \geq 0$). Between $b = -1$ and $b = 2$ the matrix is positive definite. The corner entry $b = 0$ in the first matrix A was safely between.

Second Application: The Ellipse $ax^2 + 2bxy + cy^2 = 1$

Think of a tilted ellipse centered at $(0, 0)$, as in Figure 6.4a. Turn it to line up with the coordinate axes. That is Figure 6.4b. These two pictures show the geometry behind the factorization $A = Q \Lambda Q^{-1} = Q \Lambda Q^T$:

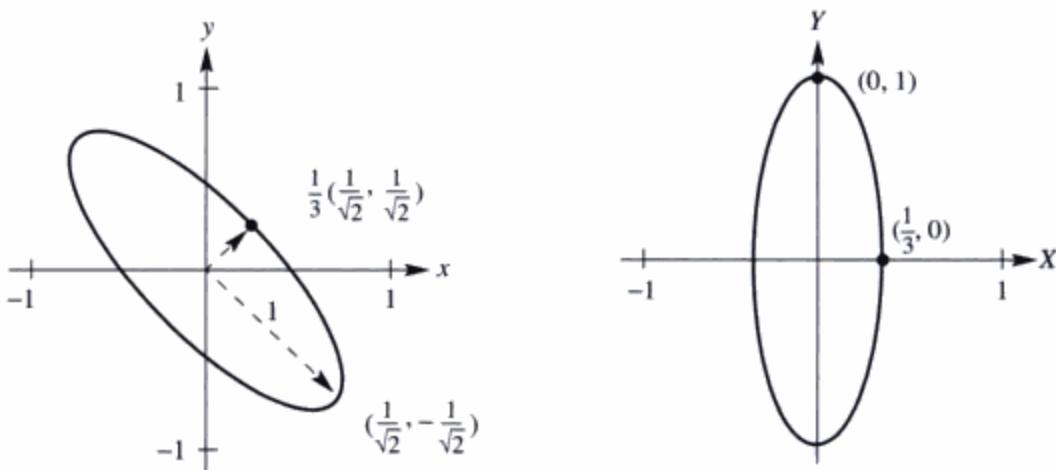


Figure 6.4 The tilted ellipse $5x^2 + 8xy + 5y^2 = 1$. Lined up it is $9X^2 + Y^2 = 1$.

1. The tilted ellipse is associated with A . Its equation is $\mathbf{x}^T A \mathbf{x} = 1$.
2. The lined-up ellipse is associated with Λ . Its equation is $\mathbf{X}^T \Lambda \mathbf{X} = 1$.
3. The rotation matrix from \mathbf{x} to \mathbf{X} that lines up the ellipse is Q .

Example 7 Find the axes of this tilted ellipse $5x^2 + 8xy + 5y^2 = 1$.

Solution Start with the positive definite matrix that matches this equation:

$$\text{The function is } [\mathbf{x}] \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} [\mathbf{x}]^T = 1. \quad \text{The matrix is } A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}.$$

The eigenvalues of A are $\lambda_1 = 9$ and $\lambda_2 = 1$. The eigenvectors are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. To make them unit vectors, divide by $\sqrt{2}$. Then $A = Q\Lambda Q^T$ is

$$\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Now multiply by $[\mathbf{x}]$ on the left and $[\mathbf{x}]^T$ on the right to get back the function $\mathbf{x}^T A \mathbf{x}$:

$$5x^2 + 8xy + 5y^2 = 9 \left(\frac{x+y}{\sqrt{2}} \right)^2 + 1 \left(\frac{x-y}{\sqrt{2}} \right)^2. \quad (3)$$

The function is again a sum of two squares. But this is different from completing the square. The coefficients are not the pivots 5 and 9/5 from D , they are the eigenvalues 9 and 1 from Λ . Inside *these* squares are the eigenvectors $(1, 1)/\sqrt{2}$ and $(1, -1)/\sqrt{2}$.

■ WORKED EXAMPLES ■

6.5 A The great factorizations of a symmetric matrix are $A = LDL^T$ from pivots and multipliers, and $A = Q\Lambda Q^T$ from eigenvalues and eigenvectors. Show that $x^T Ax > 0$ for all nonzero x exactly when the pivots and eigenvalues are positive. Try these n by n tests on pascal(6) and ones(6) and hilb(6) and other matrices in MATLAB's gallery.

Solution To prove $x^T Ax > 0$ put parentheses into $x^T LDL^T x$ and $x^T Q\Lambda Q^T x$:

$$x^T Ax = (L^T x)^T D (L^T x) \quad \text{and} \quad x^T Ax = (Q^T x)^T \Lambda (Q^T x).$$

If x is nonzero, then $y = L^T x$ and $z = Q^T x$ are nonzero (those matrices are invertible). So $x^T Ax = y^T Dy = z^T \Lambda z$ becomes a sum of squares and A is positive definite:

$$\begin{aligned} x^T Ax &= y^T Dy = d_1 y_1^2 + \cdots + d_n y_n^2 > 0 \\ x^T Ax &= z^T \Lambda z = \lambda_1 z_1^2 + \cdots + \lambda_n z_n^2 > 0 \end{aligned}$$

Honesty makes me add one little comment to this fast and beautiful proof. A zero in the pivot position would force a row exchange and a permutation matrix P . So the factorization might be $PAP^T = LDL^T$ (we exchange columns with P^T to maintain symmetry). Now the fast proof applies to $A = (P^{-1}L)D(P^{-1}L)^T$ with no problem.

MATLAB has a gallery of unusual matrices (type help gallery) and here are four: pascal(6) is positive definite because all its pivots are 1 (Worked Example 2.6 A).

ones(6) is positive *semidefinite* because its eigenvalues are 0, 0, 0, 0, 0, 6.

hilb(6) is positive definite even though eig(hilb(6)) shows two eigenvalues very near zero. In fact $x^T \text{hilb}(6) x = \int_0^1 (x_1 + x_2 s + \cdots + x_6 s^5)^2 ds > 0$.

rand(6)+rand(6)' can be positive definite or not (experiments give only 1 in 10000):
 $n = 20000; p = 0;$ for $k = 1:n, A = \text{rand}(6); p = p + \text{all}(\text{eig}(A + A') > 0); \text{end},$
 p/n

6.5 B Find conditions on the blocks $A = A^T$ and $C = C^T$ and B of this matrix M :

When is the symmetric block matrix $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ positive definite?

Solution Test M for positive pivots, starting in the upper left corner. The first pivots of M are the pivots of A ! **First condition** The block A must be positive definite.

Multiply the first row of M by $B^T A^{-1}$ and subtract from the second row to get a block of zeros. The *Schur complement* $S = C - B^T A^{-1} B$ appears in the corner:

$$\begin{bmatrix} I & 0 \\ -B^T A^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & C - B^T A^{-1} B \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & S \end{bmatrix}$$

The last pivots of M are pivots of S ! **Second condition** S must be positive definite.

The two conditions are exactly like $a > 0$ and $c > b^2/a$, except they apply to blocks.

Problem Set 6.5

Problems 1–13 are about tests for positive definiteness.

- 1 Which of A_1, A_2, A_3, A_4 has two positive eigenvalues? Use the test, don't compute the λ 's. Find an x so that $x^T A_1 x < 0$.

$$A_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix} \quad A_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

- 2 For which numbers b and c are these matrices positive definite?

$$A = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}.$$

With the pivots in D and multiplier in L , factor each A into LDL^T .

- 3 What is the quadratic $f = ax^2 + 2bxy + cy^2$ for each of these matrices? Complete the square to write f as a sum of one or two squares $d_1(\)^2 + d_2(\)^2$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}.$$

- 4 Show that $f(x, y) = x^2 + 4xy + 3y^2$ does not have a minimum at $(0, 0)$ even though it has positive coefficients. Write f as a difference of squares and find a point (x, y) where f is negative.

- 5 The function $f(x, y) = 2xy$ certainly has a saddle point and not a minimum at $(0, 0)$. What symmetric matrix A produces this f ? What are its eigenvalues?

- 6 (Important) If A has independent columns then $A^T A$ is square and symmetric and invertible (Section 4.2). Rewrite $x^T A^T A x$ to show why it is positive except when $x = \mathbf{0}$. Then $A^T A$ is more than invertible, it is positive definite.

- 7 Test to see if $A^T A$ is positive definite in each case:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

- 8 The function $f(x, y) = 3(x+2y)^2 + 4y^2$ is positive except at $(0, 0)$. What is the matrix in $f = [x \ y] A [x \ y]^T$? Check that the pivots of A are 3 and 4.

- 9 Find the 3 by 3 matrix A and its pivots, rank, eigenvalues, and determinant:

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4(x_1 - x_2 + 2x_3)^2.$$

SIMILAR MATRICES ■ 6.6

The key step in this chapter was to diagonalize a matrix. That was done by S —the eigenvector matrix. The diagonal matrix $S^{-1}AS$ is Λ —the eigenvalue matrix. But diagonalization was not possible for every A . Some matrices resisted and we had to leave them alone. They had too few eigenvectors to produce S . In this new section, S remains the best choice when we can find it, *but we allow any invertible matrix M* .

Starting from A we go to $M^{-1}AM$. This new matrix may happen to be diagonal—more likely not. It still shares important properties of A . No matter which M we choose, *the eigenvalues stay the same*. The matrices A and $M^{-1}AM$ are called “similar”. A typical matrix A is similar to a whole family of other matrices because there are so many choices of M .

DEFINITION Let M be any invertible matrix. Then $B = M^{-1}AM$ is *similar* to A .

If $B = M^{-1}AM$ then immediately $A = MBM^{-1}$. That means: If B is similar to A then A is similar to B . The matrix in this reverse direction is M^{-1} —just as good as M .

A diagonalizable matrix is similar to Λ . In that special case M is S . We have $A = SAS^{-1}$ and $\Lambda = S^{-1}AS$. They certainly have the same eigenvalues! This section is opening up to other similar matrices $B = M^{-1}AM$.

The combination $M^{-1}AM$ appears when we change variables in a differential equation. Start with an equation for u and set $u = Mv$:

$$\frac{du}{dt} = Au \quad \text{becomes} \quad M \frac{dv}{dt} = AMv \quad \text{which is} \quad \frac{dv}{dt} = M^{-1}AMv.$$

The original coefficient matrix was A , the new one at the right is $M^{-1}AM$. Changing variables leads to a similar matrix. When $M = S$ the new system is diagonal—the maximum in simplicity. But other choices of M also make the new system easier to solve. Since we can always go back to u , similar matrices have to give the same growth or decay. More precisely, *the eigenvalues of A and B are the same*.

6Q (No change in λ 's) Similar matrices A and $M^{-1}AM$ have the same eigenvalues. If x is an eigenvector of A then $M^{-1}x$ is an eigenvector of $B = M^{-1}AM$.

The proof is quick, since $B = M^{-1}AM$ gives $A = MBM^{-1}$. Suppose $Ax = \lambda x$:

$$MBM^{-1}x = \lambda x \quad \text{means that} \quad BM^{-1}x = \lambda M^{-1}x.$$

The eigenvalue of B is the same λ . The eigenvector is now $M^{-1}x$.

The following example finds three matrices that are similar to one projection matrix.

Example 1

The projection $A = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ is similar to $\Lambda = S^{-1}AS = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Now choose $M = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$: the similar matrix $M^{-1}AM$ is $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Also choose $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$: the similar matrix $M^{-1}AM$ is $\begin{bmatrix} .5 & -.5 \\ -.5 & .5 \end{bmatrix}$.

These matrices $M^{-1}AM$ all have the same eigenvalues 1 and 0. **Every 2 by 2 matrix with those eigenvalues is similar to A.** The eigenvectors change with M .

The eigenvalues in that example are 1 and 0, *not repeated*. This makes life easy. Repeated eigenvalues are harder. The next example has eigenvalues 0 and 0. The zero matrix shares those eigenvalues, but it is in a family by itself: $M^{-1}0M = 0$.

The following matrix A is similar to every nonzero matrix with eigenvalues 0 and 0.

Example 2

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to every matrix $B = \begin{bmatrix} cd & d^2 \\ -c^2 & -cd \end{bmatrix}$ except $B = 0$.

These matrices B all have zero determinant (like A). They all have rank one (like A). Their trace is $cd - cd = 0$. Their eigenvalues are 0 and 0 (like A). I chose any $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 1$, and $B = M^{-1}AM$.

These matrices B can't be diagonalized. In fact A is as close to diagonal as possible. It is the "**Jordan form**" for the family of matrices B . This is the outstanding member (my class says "Godfather") of the family. The Jordan form $J = A$ is as near as we can come to diagonalizing these matrices, when there is only one eigenvector.

Chapter 7 will explain another approach to similar matrices. Instead of changing variables by $\mathbf{u} = M\mathbf{v}$, we "**change the basis**". In this approach, similar matrices will represent the same transformation of n -dimensional space. When we choose a basis for \mathbf{R}^n , we get a matrix. The standard basis vectors ($M = I$) lead to $I^{-1}AI$ which is A . Other bases lead to similar matrices $B = M^{-1}AM$.

In this "similarity transformation" from A to B , some things change and some don't. Here is a table to show connections between similar matrices A and B :

<u>Not changed</u>	<u>Changed</u>
Eigenvalues	Eigenvectors
Trace and determinant	Nullspace
Rank	Column space
Number of independent eigenvectors	Row space
Jordan form	Left nullspace
	Singular values

The eigenvalues don't change for similar matrices; the eigenvectors do. The trace is the sum of the λ 's (unchanged). The determinant is the product of the same λ 's.¹ The nullspace consists of the eigenvectors for $\lambda = 0$ (if any), so it can change. Its dimension $n - r$ does not change! The *number* of eigenvectors stays the same for each λ , while the vectors themselves are multiplied by M^{-1} .

The *singular values* depend on $A^T A$, which definitely changes. They come in the next section. The table suggests good exercises in linear algebra. But the last entry in the unchanged column—the *Jordan form*—is more than an exercise. We lead up to it with one more example of similar matrices.

Example 3 This Jordan matrix J has triple eigenvalue 5, 5, 5. Its only eigenvectors are multiples of $(1, 0, 0)$! Algebraic multiplicity 3, geometric multiplicity 1:

$$\text{If } J = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{then } J - 5I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{has rank 2.}$$

Every similar matrix $B = M^{-1}JM$ has the same triple eigenvalue 5, 5, 5. Also $B - 5I$ must have the same rank 2. Its nullspace has dimension $3 - 2 = 1$. So each similar matrix B also has only one independent eigenvector.

The transpose matrix J^T has the same eigenvalues 5, 5, 5, and $J^T - 5I$ has the same rank 2. *Jordan's theory says that J^T is similar to J .* The matrix that produces the similarity happens to be the *reverse identity* M :

$$J^T = M^{-1}JM \quad \text{is} \quad \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

All blank entries are zero. An eigenvector of J^T is $M^{-1}(1, 0, 0) = (0, 0, 1)$. There is one line of eigenvectors $(x_1, 0, 0)$ for J and another line $(0, 0, x_3)$ for J^T .

The key fact is that this matrix J is similar to *every* matrix A with eigenvalues 5, 5, 5 and *one line* of eigenvectors. There is an M with $M^{-1}AM = J$.

Example 4 Since J is as close to diagonal as we can get, the equation $d\mathbf{u}/dt = J\mathbf{u}$ cannot be simplified by changing variables. We must solve it as it stands:

$$\frac{d\mathbf{u}}{dt} = J\mathbf{u} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is} \quad \begin{aligned} dx/dt &= 5x + y \\ dy/dt &= 5y + z \\ dz/dt &= 5z. \end{aligned}$$

¹The determinant is unchanged because $\det B = (\det M^{-1})(\det A)(\det M) = \det A$.

The system is triangular. We think naturally of back substitution. Solve the last equation and work upwards. Main point: *All solutions contain e^{5t}* :

$$\begin{aligned}\frac{dz}{dt} = 5z &\quad \text{yields } z = z(0)e^{5t} \\ \frac{dy}{dt} = 5y + z &\quad \text{yields } y = (y(0) + tz(0))e^{5t} \\ \frac{dx}{dt} = 5x + y &\quad \text{yields } x = (x(0) + ty(0) + \frac{1}{2}t^2z(0))e^{5t}.\end{aligned}$$

The two missing eigenvectors are responsible for the te^{5t} and t^2e^{5t} terms in y and z . The factors t and t^2 enter because $\lambda = 5$ is a triple eigenvalue with one eigenvector.

The Jordan Form

For every A , we want to choose M so that $M^{-1}AM$ is as *nearly diagonal as possible*. When A has a full set of n eigenvectors, they go into the columns of M . Then $M = S$. The matrix $S^{-1}AS$ is diagonal, period. This matrix is the Jordan form of A —when A can be diagonalized. In the general case, eigenvectors are missing and Λ can't be reached.

Suppose A has s independent eigenvectors. Then it is similar to a matrix with s blocks. Each block is like J in Example 3. *The eigenvalue is on the diagonal and the diagonal above it contains 1's*. This block accounts for one eigenvector of A . When there are n eigenvectors and n blocks, they are all 1 by 1. In that case J is Λ .

6R (Jordan form) If A has s independent eigenvectors, it is similar to a matrix J that has s Jordan blocks on its diagonal: There is a matrix M such that

$$M^{-1}AM = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{bmatrix} = J. \quad (1)$$

Each block in J has one eigenvalue λ_i , one eigenvector, and 1's above the diagonal:

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}. \quad (2)$$

A is similar to B if they share the same Jordan form J—not otherwise.

This is the big theorem about matrix similarity. In every family of similar matrices, we are picking one outstanding member called J . It is nearly diagonal (or if possible completely diagonal). For that J , we can solve $du/dt = Ju$ as in Example 4.

We can take powers J^k as in Problems 9–10. Every other matrix in the family has the form $A = M J M^{-1}$. The connection through M solves $d\mathbf{u}/dt = A\mathbf{u}$.

The point you must see is that $M J M^{-1} M J M^{-1} = M J^2 M^{-1}$. That cancellation of $M^{-1}M$ in the middle has been used through this chapter (when M was S). We found A^{100} from $S \Lambda^{100} S^{-1}$ —by diagonalizing the matrix. Now we can't quite diagonalize A . So we use $M J^{100} M^{-1}$ instead.

Jordan's Theorem **6R** is proved in my textbook *Linear Algebra and Its Applications*, published by Brooks-Cole. Please refer to that book (or more advanced books) for the proof. The reasoning is rather intricate and in actual computations the Jordan form is not at all popular—its calculation is not stable. A slight change in A will separate the repeated eigenvalues and remove the off-diagonal 1's—switching to a diagonal Λ . Proved or not, you have caught the central idea of similarity—to make A as simple as possible while preserving its essential properties.

■ REVIEW OF THE KEY IDEAS ■

1. B is similar to A if $B = M^{-1}AM$, for some invertible matrix M .
2. Similar matrices have the same eigenvalues. Eigenvectors are multiplied by M^{-1} .
3. If A has n independent eigenvectors then A is similar to Λ (take $M = S$).
4. Every matrix is similar to a Jordan matrix J (which has Λ as its diagonal part). J has a block for each eigenvector, and 1's for missing eigenvectors.

■ WORKED EXAMPLES ■

6.6 A The 4 by 4 triangular Pascal matrix P_L and its inverse (alternating diagonals) are

$$P_L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \quad \text{and} \quad P_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}.$$

Check that P_L and P_L^{-1} have the same eigenvalues. Find a diagonal matrix D with alternating signs that gives $P_L^{-1} = D^{-1}P_LD$, so P_L is similar to P_L^{-1} . Show that P_LD with columns of alternating signs is its own inverse. P_LD is `pascal(4, 1)` in MATLAB.

Since P_L and P_L^{-1} are similar they have the same Jordan form J . Find J by checking the number of independent eigenvectors of P_L with $\lambda = 1$.

Solution The triangular matrices P_L and P_L^{-1} both have $\lambda = 1, 1, 1, 1$ on their main diagonals. Choose D with alternating 1 and -1 on its diagonal. D equals D^{-1} :

$$D^{-1}P_L D = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix} = P_L^{-1}.$$

Check: Changing signs in rows 1 and 3 of P_L , and columns 1 and 3, produces the four negative entries in P_L^{-1} . We are multiplying row i by $(-1)^i$ and column j by $(-1)^j$, which gives the alternating diagonals. Then $P_L D = \text{pascal}(n, 1)$ has *columns with alternating signs* and equals its own inverse!

$$(P_L D)(P_L D) = P_L D^{-1} P_L D = P_L P_L^{-1} = I.$$

P_L has only one line of eigenvectors $x = (0, 0, 0, x_4)$, with $\lambda = 1$. The rank of $P_L - I$ is certainly 3. So its Jordan form J has only *one block* (also with $\lambda = 1$):

$$P_L \text{ and also } P_L^{-1} \text{ are somehow similar to Jordan's } J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6.6 B If A is similar to A^{-1} , explain why its eigenvalues come in reciprocal pairs $\lambda = a$ and $\lambda = 1/a$. The 3 by 3 Pascal matrix P_S has paired eigenvalues $4 + \sqrt{15}, 4 - \sqrt{15}, 1$. Use $P_L^{-1} = D^{-1}P_L D$ and the symmetric factorization $P_S = P_L P_L^T$ in Worked Example 2.6 A to prove that P_S is similar to P_S^{-1} .

Solution When A has nonzero eigenvalues $\lambda_1, \dots, \lambda_n$, its inverse has eigenvalues $\lambda_1^{-1}, \dots, \lambda_n^{-1}$. Reason: Multiply $Ax = \lambda x$ by A^{-1} and λ^{-1} to get $A^{-1}x = \lambda^{-1}x$.

If A and A^{-1} are similar they have the same set of eigenvalues. So an even number of λ 's must pair off in the form a and $1/a$. The product $(4 + \sqrt{15})(4 - \sqrt{15}) = 16 - 15 = 1$ shows that $4 + \sqrt{15}, 4 - \sqrt{15}, 1$ do pair off properly.

The symmetric Pascal matrices have paired eigenvalues because P_S is similar to P_S^{-1} . To prove this similarity, using $D = D^{-1} = D^T$, start from $P_S = P_L P_L^T$:

$$P_S^{-1} = (P_L^T)^{-1}(P_L^{-1}) = (D^{-1}P_L D)^T(D^{-1}P_L D) = D^{-1}P_L^T P_L D = (P_L D)^{-1}(P_L P_L^T)(P_L D).$$

This is $P_S^{-1} = M^{-1}P_S M$ (similar matrices!) for the matrix $M = P_L D$.

The eigenvalues of larger matrices P_S don't have nice formulas. But `eig(pascal(n))` will confirm that those eigenvalues come in reciprocal pairs a and $1/a$. The Jordan form of P_S is the diagonal Λ , because symmetric matrices always have a complete set of eigenvectors.

Questions 10–14 are about the Jordan form.

- 10** By direct multiplication, find J^2 and J^3 when

$$J = \begin{bmatrix} c & 1 \\ 0 & c \end{bmatrix}.$$

Guess the form of J^k . Set $k = 0$ to find J^0 . Set $k = -1$ to find J^{-1} .

- 11** The text solved $d\mathbf{u}/dt = J\mathbf{u}$ for a 3 by 3 Jordan block J . Add a fourth equation $dw/dt = 5w + x$. Follow the pattern of solutions for z, y, x to find w .
- 12** These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block sizes don't match and they are *not similar*:

$$J = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \text{and} \quad K = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

For any matrix M , compare JM with MK . If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible.

- 13** Prove that A^T is always similar to A (we knew the λ 's are the same):
1. For one Jordan block J_i : Find M_i so that $M_i^{-1}J_iM_i = J_i^T$ (see Example 3).
 2. For any J with blocks J_i : Build M_0 from blocks so that $M_0^{-1}JM_0 = J^T$.
 3. For any $A = MJM^{-1}$: Show that A^T is similar to J^T and so to J and to A .
- 14** Find two more matrices similar to J in Example 3.
- 15** Prove that $\det(A - \lambda I) = \det(M^{-1}AM - \lambda I)$. (You could write $I = M^{-1}M$ and factor out $\det M^{-1}$ and $\det M$.) This says that A and $M^{-1}AM$ have the same characteristic polynomial. So their roots are the same eigenvalues.
- 16** Which pairs are similar? Choose a, b, c, d to prove that the other pairs aren't:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} b & a \\ d & c \end{bmatrix}, \begin{bmatrix} c & d \\ a & b \end{bmatrix}, \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

- 17** True or false, with a good reason:

- (a) An invertible matrix can't be similar to a singular matrix.
- (b) A symmetric matrix can't be similar to a nonsymmetric matrix.
- (c) A can't be similar to $-A$ unless $A = 0$.
- (d) A can't be similar to $A + I$.

- 18** If B is invertible prove that AB has the same eigenvalues as BA .
- 19** If A is 6 by 4 and B is 4 by 6, AB and BA have different sizes. But still
- $$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix} = G.$$
- (a) What sizes are the blocks of G ? They are the same in each matrix.
 (b) This equation is $M^{-1}FM = G$, so F and G have the same 10 eigenvalues. F has the eigenvalues of AB plus 4 zeros; G has the eigenvalues of BA plus 6 zeros. AB has the same eigenvalues as BA plus _____ zeros.
- 20** Why are these statements all true?
- (a) If A is similar to B then A^2 is similar to B^2 .
 (b) A^2 and B^2 can be similar when A and B are not similar (try $\lambda = 0, 0$).
 (c) $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}$.
 (d) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is not similar to $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$.
 (e) If we exchange rows 1 and 2 of A , and then exchange columns 1 and 2, the eigenvalues stay the same.
- 21** If J is the 5 by 5 Jordan block with $\lambda = 0$, find J^2 and count its eigenvectors and find its Jordan form (two blocks).

SINGULAR VALUE DECOMPOSITION (SVD) ■ 6.7

The Singular Value Decomposition is a highlight of linear algebra. A is any m by n matrix, square or rectangular. We will diagonalize it, but not by $S^{-1}AS$. Its row space is r -dimensional (inside \mathbf{R}^n). Its column space is also r -dimensional (inside \mathbf{R}^m). We are going to choose special *orthonormal bases* $\mathbf{v}_1, \dots, \mathbf{v}_r$ for the row space and $\mathbf{u}_1, \dots, \mathbf{u}_r$ for the column space. For those bases, we want each $A\mathbf{v}_i$ to be in the direction of \mathbf{u}_i . In matrix form, *these equations* $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$ *become* $AV = U\Sigma$ or $A = U\Sigma V^T$. This is the SVD.

Image Compression

Unusually, I am going to stop the theory and describe applications. This is the century of data, and often that data is stored in a matrix. A *digital image is really a matrix of pixel values*. Each little picture element or “pixel” has a gray scale number between black and white (it has three numbers for a color picture). The picture might have $512 = 2^9$ pixels in each row and $256 = 2^8$ pixels down each column. We have a 256 by 512 pixel matrix with 2^{17} entries! To store one picture, the computer has no problem. But if you go in for a CT scan or Magnetic Resonance, you produce an image at every cross section—a ton of data. If the pictures are frames in a movie, 30 frames a second means 108,000 images per hour. Compression is especially needed for high definition digital TV, or the equipment could not keep up in real time.

What is compression? We want to replace those 2^{17} matrix entries by a smaller number, *without losing picture quality*. A simple way would be to use larger pixels—replace groups of four pixels by their average value. This is 4 : 1 compression. But if we carry it further, like 16 : 1, our image becomes “blocky”. We want to replace the mn entries by a smaller number, in a way that the human visual system won’t notice.

Compression is a billion dollar problem and everyone has ideas. Later in this book I will describe Fourier transforms (used in **jpeg**) and wavelets (now in **JPEG2000**). Here we try an SVD approach: *Replace the 256 by 512 pixel matrix by a matrix of rank one: a column times a row*. If this is successful, the storage requirement for a column and row becomes $256 + 512$ (plus instead of times!). The compression ratio $(256)(512)/(256 + 512)$ is better than 170 : 1. This is more than we hope for. We may actually use five matrices of rank one (so a matrix approximation of rank 5). The compression is still 34 : 1 and the crucial question is the picture quality.

Where does the SVD come in? *The best rank one approximation to A is the matrix $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T$.* It uses the largest singular value σ_1 and its left and right singular vectors \mathbf{u}_1 and \mathbf{v}_1 . The best rank 5 approximation includes $\sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \dots + \sigma_5 \mathbf{u}_5 \mathbf{v}_5^T$. If we can compute those \mathbf{u} ’s and \mathbf{v} ’s quickly (a big “if” since you will see them as eigenvectors for $A^T A$ and $A A^T$) then this SVD algorithm is competitive.

I will mention a different matrix, one that a library needs to compress. The rows correspond to key words. The columns correspond to titles in the library. The entry in this *word-title matrix* is $a_{ij} = 1$ if word i is in title j (otherwise $a_{ij} = 0$). We might normalize the columns to be unit vectors, so that long titles don’t get an advantage.

Searching the Web

I will end with an application of the SVD to web search engines. When you type a search word, you get a list of related web sites in order of importance. (Regrettably, typing “SVD” produced 13 non-mathematical SVD’s before the real one. “Cofactors” was even worse but “cofactor” had one good entry. “Four subspaces” did much better.) The HITS algorithm that we describe is one way to produce that ranked list. It begins with about 200 sites found from an index of key words, and after that we look only at links between pages. HITS is link-based not content-based.

Start with the 200 sites and all sites that link to them and all sites they link to. That is our list, to be put in order. Importance can be measured in two ways:

1. The site is an **authority**: *links come from many sites*. Especially from hubs.
2. The site is a **hub**: *it links to many sites in the list*. Especially to authorities.

We want numbers x_1, \dots, x_N to rank the authorities and y_1, \dots, y_N to rank the hubs. Start with a simple count: x_i^0 and y_i^0 count the links into and out of site i .

Here is the point: *A good authority has links from important sites* (like hubs). Links from universities count more heavily than links from friends. *A good hub is linked to important sites* (like authorities). A link to amazon.com means more than a link to wellesleycambridge.com. The rankings x^0 and y^0 from counting links are updated to x^1 and y^1 by taking account of *good* links (measuring their quality by x^0 and y^0):

$$\text{Authority values } x_i^1 = \sum_{j \text{ links to } i} y_j^0 \quad \text{Hub values } y_i^1 = \sum_{i \text{ links to } j} x_j^0 \quad (12)$$

In matrix language those are $x^1 = A^T y^0$ and $y^1 = Ax^0$. The matrix A contains 1’s and 0’s, with $a_{ij} = 1$ when i links to j . In the language of graphs, A is an “adjacency matrix” for the World Wide Web. It is pretty large.

The algorithm doesn’t stop there. The new x^1 and y^1 give better rankings, but not the best. Take another step like (12) to x^2 and y^2 . Notice how $A^T A$ and AA^T appear:

$$x^2 = A^T y^1 = A^T A x^0 \quad \text{and} \quad y^2 = A^T x^1 = A A^T y^0. \quad (13)$$

In two steps we are multiplying x^0 by $A^T A$ and y^0 by AA^T . In twenty steps we are multiplying by $(A^T A)^{10}$ and $(AA^T)^{10}$. When we take these powers, the largest eigenvalue σ_1^2 begins to dominate. And the vectors x and y gradually line up with the leading eigenvectors v_1 and u_1 of $A^T A$ and AA^T . We are computing the top terms in the SVD iteratively, by the *power method* that is further discussed in Section 9.3. It is wonderful that linear algebra helps to understand the Web.

Google actually creates rankings by a random walk that follows web links. The more often this random walk goes to a site, the higher the ranking. The frequency of visits gives the leading eigenvector ($\lambda = 1$) of the normalized adjacency matrix for the

With orthonormal rows in V^T , the σ 's in Σ come in: $A = \sigma_1 \mathbf{c}_1 \mathbf{r}_1 + \cdots + \sigma_n \mathbf{c}_n \mathbf{r}_n$. With diagonal 1's in L and U , the pivots d_i come in: $A = LDU = d_1 \mathbf{c}_1 \mathbf{r}_1 + \cdots + d_n \mathbf{c}_n \mathbf{r}_n$. With the diagonal of R placed in H , QR becomes $QHR = h_1 \mathbf{c}_1 \mathbf{r}_1 + \cdots + h_n \mathbf{c}_n \mathbf{r}_n$. These numbers h_i have no standard name and I propose “heights”. Each h_i tells the height of column i above the base from the first $i - 1$ columns. The volume of the full n -dimensional box comes from $A = U\Sigma V^T = LDU = QHR$:

$$|\det A| = |\text{product of } \sigma\text{'s}| = |\text{product of } d\text{'s}| = |\text{product of } h\text{'s}|.$$

Problem Set 6.7

Problems 1–3 compute the SVD of a square singular matrix A .

- 1 Compute $A^T A$ and its eigenvalues $\sigma_1^2, 0$ and unit eigenvectors $\mathbf{v}_1, \mathbf{v}_2$:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}.$$

- 2 (a) Compute AA^T and its eigenvalues $\sigma_1^2, 0$ and unit eigenvectors $\mathbf{u}_1, \mathbf{u}_2$.
 (b) Verify from Problem 1 that $A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1$. Find all entries in the SVD:

$$\begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^T.$$

- 3 Write down orthonormal bases for the four fundamental subspaces of this A .

Problems 4–7 ask for the SVD of matrices of rank 2.

- 4 (a) Find the eigenvalues and unit eigenvectors of $A^T A$ and AA^T for the Fibonacci matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

- (b) Construct the singular value decomposition of A .
 5 Show that the vectors in Problem 4 satisfy $A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1$ and $A\mathbf{v}_2 = \sigma_2 \mathbf{u}_2$.
 6 Use the SVD part of the MATLAB demo **eigshow** to find the same vectors \mathbf{v}_1 and \mathbf{v}_2 graphically.
 7 Compute $A^T A$ and AA^T and their eigenvalues and unit eigenvectors for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Multiply the three matrices $U\Sigma V^T$ to recover A .

Problems 8–15 bring out the underlying ideas of the SVD.

- 8 Suppose $\mathbf{u}_1, \dots, \mathbf{u}_n$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ are orthonormal bases for \mathbf{R}^n . Construct the matrix A that transforms each \mathbf{v}_j into \mathbf{u}_j to give $A\mathbf{v}_1 = \mathbf{u}_1, \dots, A\mathbf{v}_n = \mathbf{u}_n$.

- 9 Construct the matrix with rank one that has $A\mathbf{v} = 12\mathbf{u}$ for $\mathbf{v} = \frac{1}{2}(1, 1, 1, 1)$ and $\mathbf{u} = \frac{1}{3}(2, 2, 1)$. Its only singular value is $\sigma_1 = \underline{\hspace{2cm}}$.
- 10 Suppose A has orthogonal columns $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ of lengths $\sigma_1, \sigma_2, \dots, \sigma_n$. What are U , Σ , and V in the SVD?
- 11 Explain how the SVD expresses the matrix A as the sum of r rank one matrices:

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T.$$

- 12 Suppose A is a 2 by 2 symmetric matrix with unit eigenvectors \mathbf{u}_1 and \mathbf{u}_2 . If its eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$, what are the matrices U , Σ , V^T in its SVD?
- 13 If $A = QR$ with an orthonormal matrix Q , then the SVD of A is almost the same as the SVD of R . Which of the three matrices in the SVD is changed because of Q ?
- 14 Suppose A is invertible (with $\sigma_1 > \sigma_2 > 0$). Change A by as small a matrix as possible to produce a singular matrix A_0 . Hint: U and V do not change:

$$A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^T$$

- 15 (a) If A changes to $4A$, what is the change in the SVD?
 (b) What is the SVD for A^T and for A^{-1} ?
- 16 Why doesn't the SVD for $A + I$ just use $\Sigma + I$?
- 17 (MATLAB) Run a random walk starting from web site $x(1) = 1$ and record the visits to each site. From the site $x(k-1) = 1, 2, 3$, or 4 the code chooses $x(k)$ with probabilities given by column $x(k-1)$ of A . At the end \mathbf{p} gives the fraction of time at each site from a histogram (and $A\mathbf{p} \approx \mathbf{p}$ —please check this steady state eigenvector):

$$A = [0 .1 .2 .7; .05 0 .15 .8; .15 .25 0 .6; .1 .3 .6 0]' =$$

Markov matrix

```
n = 1000; x = zeros(1,n); x(1) = 1;
for k = 2 : n x(k) = min(find(rand < cumsum(A(:,x(k-1))))); end
p = hist(x, 1 : 4)/n
```

How are the properties of a matrix reflected in its eigenvalues and eigenvectors? This question is fundamental throughout Chapter 6. A table that organizes the key facts may be helpful. For each class of matrices, here are the special properties of the eigenvalues λ_i and eigenvectors \mathbf{x}_i .

7

LINEAR TRANSFORMATIONS

THE IDEA OF A LINEAR TRANSFORMATION ■ 7.1

When a matrix A multiplies a vector v , it “transforms” v into another vector Av . *In goes v , out comes Av .* This transformation follows the same idea as a function. In goes a number x , out comes $f(x)$. For one vector v or one number x , we multiply by the matrix or we evaluate the function. The deeper goal is to see all v 's at once. We are transforming the whole space when we multiply every v by A .

Start again with a matrix A . It transforms v to Av . It transforms w to Aw . Then we know what happens to $u = v + w$. There is no doubt about Au , it has to equal $Av + Aw$. Matrix multiplication $T(v) = Av$ gives a *linear transformation*:

DEFINITION A transformation T assigns an output $T(v)$ to each input vector v . The transformation is *linear* if it meets these requirements for all v and w :

$$(a) \quad T(v + w) = T(v) + T(w) \qquad (b) \quad T(cv) = cT(v) \quad \text{for all } c.$$

If the input is $v = \mathbf{0}$, the output must be $T(v) = \mathbf{0}$. We combine (a) and (b) into one:

Linearity: $T(cv + dw)$ *must equal* $cT(v) + dT(w)$.

Again I test matrix multiplication: $A(cv + dw) = cAv + dAw$ is *true*.

A linear transformation is highly restricted. Suppose T adds u_0 to every vector. Then $T(v) = v + u_0$ and $T(w) = w + u_0$. This isn't good, or at least *it isn't linear*. Applying T to $v + w$ produces $v + w + u_0$. That is not the same as $T(v) + T(w)$:

$$v + w + u_0 \quad \text{is different from} \quad T(v) + T(w) = v + u_0 + w + u_0.$$

The exception is when $u_0 = \mathbf{0}$. The transformation reduces to $T(v) = v$. This is the *identity transformation* (nothing moves, as in multiplication by I). That is certainly linear. In this case the input space \mathbf{V} is the same as the output space \mathbf{W} .

The linear-plus-shift transformation $T(\mathbf{v}) = A\mathbf{v} + \mathbf{u}_0$ is called “*affine*.” Straight lines stay straight although T is not linear. Computer graphics works with affine transformations. The shift to computer graphics is in Section 8.6.

Example 1 Choose a fixed vector $\mathbf{a} = (1, 3, 4)$, and let $T(\mathbf{v})$ be the dot product $\mathbf{a} \cdot \mathbf{v}$:

The input is $\mathbf{v} = (v_1, v_2, v_3)$. The output is $T(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v} = v_1 + 3v_2 + 4v_3$.

This is linear. The inputs \mathbf{v} come from three-dimensional space, so $\mathbf{V} = \mathbf{R}^3$. The outputs are just numbers, so the output space is $\mathbf{W} = \mathbf{R}^1$. We are multiplying by the row matrix $A = [1 \ 3 \ 4]$. Then $T(\mathbf{v}) = A\mathbf{v}$.

You will get good at recognizing which transformations are linear. If the output involves squares or products or lengths, v_1^2 or v_1v_2 or $\|\mathbf{v}\|$, then T is not linear.

Example 2 The length $T(\mathbf{v}) = \|\mathbf{v}\|$ is not linear. Requirement (a) for linearity would be $\|\mathbf{v} + \mathbf{w}\| = \|\mathbf{v}\| + \|\mathbf{w}\|$. Requirement (b) would be $\|c\mathbf{v}\| = c\|\mathbf{v}\|$. Both are false!

Not (a): The sides of a triangle satisfy an *inequality* $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Not (b): The length $\|-\mathbf{v}\|$ is not $-\|\mathbf{v}\|$. For negative c , we fail.

Example 3 (Important) T is the transformation that rotates every vector by 30° . The domain is the xy plane (where the input vector \mathbf{v} is). The range is also the xy plane (where the rotated vector $T(\mathbf{v})$ is). We described T without mentioning a matrix: just rotate the plane by 30° .

Is rotation linear? Yes it is. We can rotate two vectors and add the results. The sum of rotations $T(\mathbf{v}) + T(\mathbf{w})$ is the same as the rotation $T(\mathbf{v} + \mathbf{w})$ of the sum. The whole plane is turning together, in this linear transformation.

Note Transformations have a language of their own. Where there is no matrix, we can't talk about a column space. But the idea can be rescued and used. The column space consisted of all outputs $A\mathbf{v}$. The nullspace consisted of all inputs for which $A\mathbf{v} = \mathbf{0}$. Translate those into “range” and “kernel”:

Range of T = set of all outputs $T(\mathbf{v})$: corresponds to column space

Kernel of T = set of all inputs for which $T(\mathbf{v}) = \mathbf{0}$: corresponds to nullspace.

The range is in the output space \mathbf{W} . The kernel is in the input space \mathbf{V} . When T is multiplication by a matrix, $T(\mathbf{v}) = A\mathbf{v}$, you can translate to column space and nullspace.

For an m by n matrix, the nullspace is a subspace of $\mathbf{V} = \mathbf{R}^n$. The column space is a subspace of _____. The range might or might not be the whole output space \mathbf{W} .

Examples of Transformations (mostly linear)

Example 4 Project every 3-dimensional vector down onto the xy plane. The range is that plane, which contains every $T(\mathbf{v})$. The kernel is the z axis (which projects down to zero). This projection is linear.

Example 5 Project every 3-dimensional vector onto the horizontal plane $z = 1$. The vector $\mathbf{v} = (x, y, z)$ is transformed to $T(\mathbf{v}) = (x, y, 1)$. This transformation is not linear. Why not? It doesn't even transform $\mathbf{v} = \mathbf{0}$ into $T(\mathbf{v}) = \mathbf{0}$.

Multiply every 3-dimensional vector by a 3 by 3 matrix A . This is definitely a linear transformation!

$$T(v + w) = A(v + w) \quad \text{which does equal} \quad Av + Aw = T(v) + T(w).$$

Example 6 Suppose A is an *invertible matrix*. The kernel of T is the zero vector; the range \mathbf{W} equals the domain \mathbf{V} . Another linear transformation is multiplication by A^{-1} . This is the *inverse transformation* T^{-1} , which brings every vector $T(v)$ back to v :

$$T^{-1}(T(v)) = v \quad \text{matches the matrix multiplication} \quad A^{-1}(Av) = v.$$

We are reaching an unavoidable question. *Are all linear transformations produced by matrices?* Each m by n matrix does produce a linear transformation from $\mathbf{V} = \mathbf{R}^n$ to $\mathbf{W} = \mathbf{R}^m$. Our question is the converse. When a linear T is described as a “rotation” or “projection” or “...”, is there always a matrix hiding behind T ?

The answer is *yes*. This is an approach to linear algebra that doesn’t start with matrices. The next section shows that we still end up with matrices.

Linear Transformations of the Plane

It is more interesting to *see* a transformation than to define it. When a 2 by 2 matrix A multiplies all vectors in \mathbf{R}^2 , we can watch how it acts. Start with a “house” that has eleven endpoints. Those eleven vectors v are transformed into eleven vectors Av . Straight lines between v ’s become straight lines between the transformed vectors Av . (The transformation from house to house is linear!) Applying A to a standard house produces a new house—possibly stretched or rotated or otherwise unlivable.

This part of the book is visual, not theoretical. We will show six houses and the matrices that produce them. The columns of H are the eleven circled points of the first house. (H is 2 by 12, so **plot2d** will connect the 11th circle to the first.) The 11 points in the house matrix H are multiplied by A to produce the other houses. The houses on the cover of the book were produced this way (before Christine Curtis turned them into a quilt for Professor Curtis). H is in the Teaching Code **house**.

$$H = \begin{bmatrix} -6 & -6 & -7 & 0 & 7 & 6 & 6 & -3 & -3 & 0 & 0 & -6 \\ -7 & 2 & 1 & 8 & 1 & 2 & -7 & -7 & -2 & -2 & -7 & -7 \end{bmatrix}.$$

■ REVIEW OF THE KEY IDEAS ■

1. A transformation T takes each v in the input space to $T(v)$ in the output space.
2. Linearity requires that $T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n)$.
3. The transformation $T(v) = Av + v_0$ is linear only if $v_0 = \mathbf{0}$!
4. The quilt on the book cover shows $T(\text{house}) = AH$ for nine matrices A .

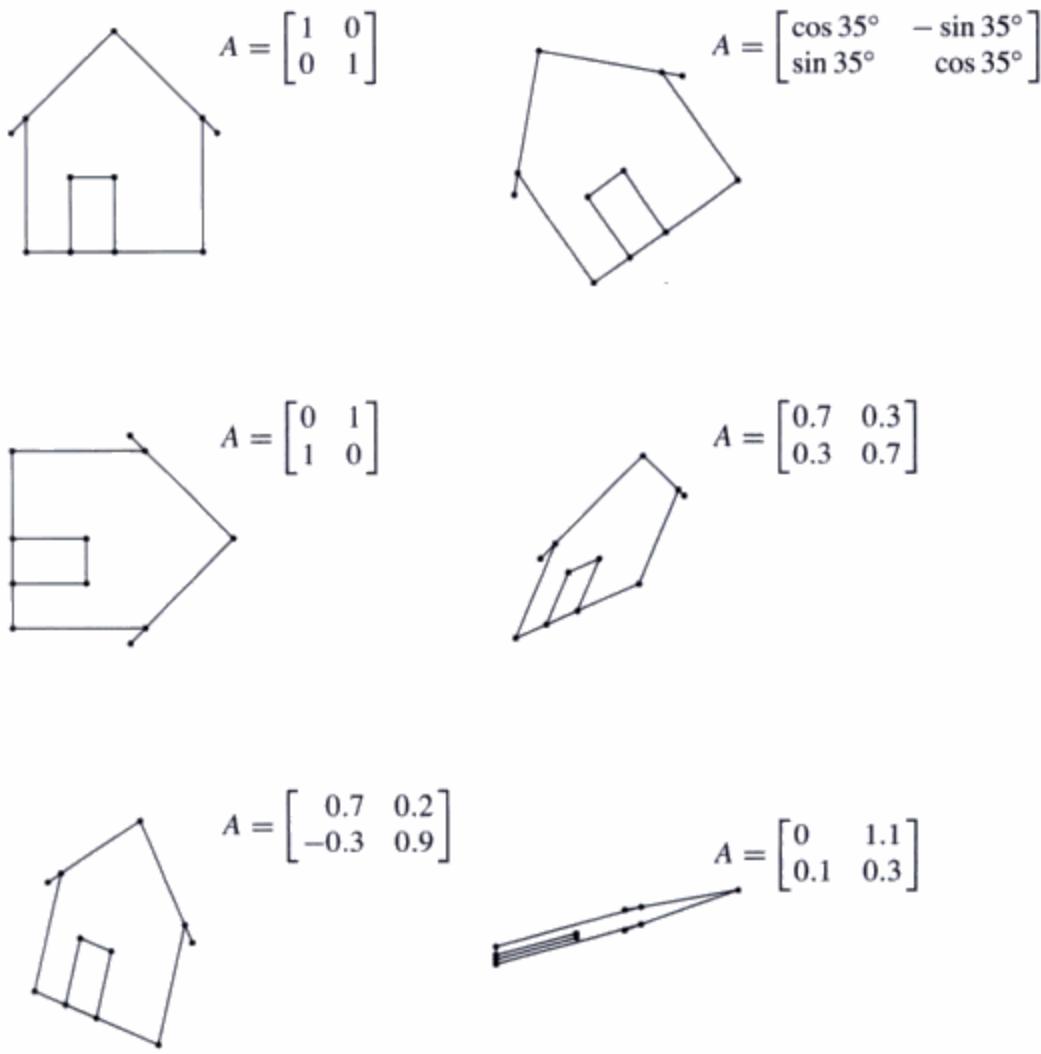


Figure 7.1 Linear transformations of a house drawn by `plot2d(A * H)`.

■ WORKED EXAMPLES ■

7.1 A The matrix $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ gives a *shearing transformation* $T(x, y) = (x, 3x + y)$. Draw the xy plane and show what happens to $(1, 0)$ and $(2, 0)$ on the x axis. What happens to the points on the vertical lines $x = 0$ and $x = a$? If the inputs fill the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$, draw the outputs (the transformed square).

Solution The points $(1, 0)$ and $(2, 0)$ on the x axis transform by T to $(1, 3)$ and $(2, 6)$. The horizontal x axis transforms to the straight line with slope 3 (going through $(0, 0)$ of course). The points on the y axis are *not moved* because $T(0, y) = (0, y)$. The y axis is the line of eigenvectors of T with $\lambda = 1$.

The vertical line $x = a$ is moved up by $3a$, since $3a$ is added to the y component. This is the “shearing”. Vertical lines slide higher and higher as you go from left to right.

The unit square has one side on the y axis (unchanged). The opposite side from $(1, 0)$ to $(1, 1)$ moves upward, to go from $(1, 3)$ to $(1, 4)$. The transformed square has a lower side from $(0, 0)$ to $(1, 3)$ and a parallel upper side from $(0, 1)$ to $(1, 4)$. It is a parallelogram. Multiplication by any A transforms squares to parallelograms!

7.1 B A nonlinear transformation T is invertible if every b in the output space comes from exactly one x in the input space: $T(x) = b$ always has exactly one solution. Which of these transformations (on real numbers x) is invertible and what is T^{-1} ? None are linear, not even T_3 . When you solve $T(x) = b$, you are inverting T :

$$T_1(x) = x^2 \quad T_2(x) = x^3 \quad T_3(x) = x + 9 \quad T_4(x) = e^x \quad T_5(x) = \frac{1}{x} \text{ for nonzero } x\text{'s}$$

Solution T_1 is not invertible because $x^2 = 1$ has two solutions (and $x^2 = -1$ has no solution). T_4 is not invertible because $e^x = -1$ has no solution. (If the output space changes to positive b 's then the inverse of $e^x = b$ is $x = \ln b$.) Notice that $T_5^2 = \text{identity}$. But $T_3^2(x) = x + 18$. What are $T_2^2(x)$ and T_4^2 ?

T_2, T_3, T_5 are invertible. The solutions to $x^3 = b$ and $x + 9 = b$ and $\frac{1}{x} = b$ are unique:

$$x = T_2^{-1}(b) = b^{1/3} \quad x = T_3^{-1}(b) = b - 9 \quad x = T_5^{-1}(b) = \frac{1}{b}$$

Problem Set 7.1

- 1 A linear transformation must leave the zero vector fixed: $T(\mathbf{0}) = \mathbf{0}$. Prove this from $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ by choosing $\mathbf{w} = \underline{\hspace{2cm}}$. Prove it also from requirement (b) by choosing $c = \underline{\hspace{2cm}}$.
- 2 Requirement (b) gives $T(c\mathbf{v}) = cT(\mathbf{v})$ and also $T(d\mathbf{w}) = dT(\mathbf{w})$. Then by addition, requirement (a) gives $T(\underline{\hspace{2cm}}) = (\underline{\hspace{2cm}})$. What is $T(c\mathbf{v} + d\mathbf{w} + e\mathbf{u})$?
- 3 Which of these transformations is not linear? The input is $\mathbf{v} = (v_1, v_2)$:
 - (a) $T(\mathbf{v}) = (v_2, v_1)$
 - (b) $T(\mathbf{v}) = (v_1, v_1)$
 - (c) $T(\mathbf{v}) = (0, v_1)$
 - (d) $T(\mathbf{v}) = (0, 1)$.

- 4** If S and T are linear transformations, is $S(T(\mathbf{v}))$ linear or quadratic?
- (Special case) If $S(\mathbf{v}) = \mathbf{v}$ and $T(\mathbf{v}) = \mathbf{v}$, then $S(T(\mathbf{v})) = \mathbf{v}$ or \mathbf{v}^2 ?
 - (General case) $S(\mathbf{w}_1 + \mathbf{w}_2) = S(\mathbf{w}_1) + S(\mathbf{w}_2)$ and $T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ combine into
- $$S(T(\mathbf{v}_1 + \mathbf{v}_2)) = S(\text{_____}) = \text{_____} + \text{_____}.$$
- 5** Suppose $T(\mathbf{v}) = \mathbf{v}$ except that $T(0, v_2) = (0, 0)$. Show that this transformation satisfies $T(c\mathbf{v}) = cT(\mathbf{v})$ but not $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$.
- 6** Which of these transformations satisfy $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ and which satisfy $T(c\mathbf{v}) = cT(\mathbf{v})$?
- $T(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|$
 - $T(\mathbf{v}) = v_1 + v_2 + v_3$
 - $T(\mathbf{v}) = (v_1, 2v_2, 3v_3)$
 - $T(\mathbf{v}) = \text{largest component of } \mathbf{v}$.
- 7** For these transformations of $\mathbf{V} = \mathbf{R}^2$ to $\mathbf{W} = \mathbf{R}^2$, find $T(T(\mathbf{v}))$. Is this transformation T^2 linear?
- $T(\mathbf{v}) = -\mathbf{v}$
 - $T(\mathbf{v}) = \mathbf{v} + (1, 1)$
 - $T(\mathbf{v}) = 90^\circ \text{ rotation} = (-v_2, v_1)$
 - $T(\mathbf{v}) = \text{projection} = (\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2})$.
- 8** Find the range and kernel (like the column space and nullspace) of T :
- $T(v_1, v_2) = (v_2, v_1)$
 - $T(v_1, v_2, v_3) = (v_1, v_2)$
 - $T(v_1, v_2) = (0, 0)$
 - $T(v_1, v_2) = (v_1, v_1)$.
- 9** The “cyclic” transformation T is defined by $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. What is $T(T(\mathbf{v}))$? What is $T^3(\mathbf{v})$? What is $T^{100}(\mathbf{v})$? Apply T three times and 100 times to \mathbf{v} .
- 10** A linear transformation from \mathbf{V} to \mathbf{W} has an *inverse* from \mathbf{W} to \mathbf{V} when the range is all of \mathbf{W} and the kernel contains only $\mathbf{v} = \mathbf{0}$. Why are these transformations not invertible?
- $T(v_1, v_2) = (v_2, v_2)$ $\mathbf{W} = \mathbf{R}^2$
 - $T(v_1, v_2) = (v_1, v_2, v_1 + v_2)$ $\mathbf{W} = \mathbf{R}^3$
 - $T(v_1, v_2) = v_1$ $\mathbf{W} = \mathbf{R}^1$
- 11** If $T(\mathbf{v}) = A\mathbf{v}$ and A is m by n , then T is “multiplication by A .”
- What are the input and output spaces \mathbf{V} and \mathbf{W} ?
 - Why is range of T = column space of A ?
 - Why is kernel of T = nullspace of A ?

- 22 Draw a picture of T (house) for these matrices:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} .7 & .7 \\ .3 & .3 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

- 23 What are the conditions on $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to ensure that T (house) will

- (a) sit straight up?
- (b) expand the house by 3 in all directions?
- (c) rotate the house with no change in its shape?

- 24 What are the conditions on $\det A = ad - bc$ to ensure that T (house) will

- (a) be squashed onto a line?
- (b) keep its endpoints in clockwise order (not reflected)?
- (c) have the same area as the original house?

If one side of the house stays in place, how do you know that $A = I$?

- 25 Describe T (house) when $T(\mathbf{v}) = -\mathbf{v} + (1, 0)$. This T is “affine.”

- 26 Change the house matrix H to add a chimney.

- 27 This MATLAB program creates a vector of 50 angles called theta, and then draws the unit circle and T (circle) = ellipse. You can change A .

```
A = [2 1; 1 2]
theta = [0:2*pi/50:2*pi];
circle = [cos(theta); sin(theta)];
ellipse = A * circle;
axis([-4 4 -4 4]); axis('square')
plot(circle(1,:), circle(2,:)), ellipse(1,:), ellipse(2,:))
```

- 28 Add two eyes and a smile to the circle in Problem 27. (If one eye is dark and the other is light, you can tell when the face is reflected across the y axis.) Multiply by matrices A to get new faces.

- 29 The standard house is drawn by **plot2d(H)**. Circles from \mathbf{o} and lines from $-:$

```
x = H(1,:)'; y = H(2,:)';
axis([-10 10 -10 10]), axis('square')
plot(x, y, '-o', x, y, '-');
```

Test **plot2d(A' * H)** and **plot2d(A' * A * H)** with the matrices in Figure 7.1.

- 30 Without a computer describe the houses $A * H$ for these matrices A :

$$\begin{bmatrix} 1 & 0 \\ 0 & .1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

- 31 What matrices give the houses on the front cover? The second is $A = I$.

THE MATRIX OF A LINEAR TRANSFORMATION ■ 7.2

The next pages assign a matrix to every linear transformation. For ordinary column vectors, the input \mathbf{v} is in $\mathbf{V} = \mathbf{R}^n$ and the output $T(\mathbf{v})$ is in $\mathbf{W} = \mathbf{R}^m$. The matrix for this transformation T will be m by n .

The standard basis vectors for \mathbf{R}^n and \mathbf{R}^m lead to a standard matrix for T . Then $T(\mathbf{v}) = A\mathbf{v}$ in the normal way. But these spaces also have other bases, so the same T is represented by other matrices. A main theme of linear algebra is to choose the bases that give the best matrix.

When \mathbf{V} and \mathbf{W} are not \mathbf{R}^n and \mathbf{R}^m , they still have bases. Each choice of basis leads to a matrix for T . When the input basis is different from the output basis, the matrix for $T(\mathbf{v}) = \mathbf{v}$ will not be the identity I . It will be the “change of basis matrix.”

Key idea of this section

When we know $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$ for the basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, linearity produces $T(\mathbf{v})$ for every other vector \mathbf{v} .

Reason Every input \mathbf{v} is a unique combination $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$ of the basis vectors. Since T is a linear transformation (here is the moment for linearity), the output $T(\mathbf{v})$ must be the same combination of the known outputs $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$:

$$\begin{aligned} &\text{Suppose } \mathbf{v} = c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n. \\ &\text{Then linearity requires } T(\mathbf{v}) = c_1T(\mathbf{v}_1) + \dots + c_nT(\mathbf{v}_n). \end{aligned} \tag{1}$$

The rule of linearity extends from $c\mathbf{v} + d\mathbf{w}$ to all combinations $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$. Our first example gives the outputs $T(\mathbf{v})$ for the standard basis vectors $(1, 0)$ and $(0, 1)$.

Example 1 Suppose T transforms $\mathbf{v}_1 = (1, 0)$ to $T(\mathbf{v}_1) = (2, 3, 4)$. Suppose the second basis vector $\mathbf{v}_2 = (0, 1)$ goes to $T(\mathbf{v}_2) = (5, 5, 5)$. If T is linear from \mathbf{R}^2 to \mathbf{R}^3 then its “standard matrix” is 3 by 2. Those outputs go into its columns:

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix}. \quad T(\mathbf{v}_1 + \mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2) \quad \text{is} \quad \begin{bmatrix} 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}.$$

Example 2 The derivatives of the functions $1, x, x^2, x^3$ are $0, 1, 2x, 3x^2$. Those are four facts about the transformation T that “*takes the derivative*.” The inputs and outputs are functions! Now add the crucial fact that T is linear:

$$T(\mathbf{v}) = \frac{d\mathbf{v}}{dx} \quad \text{obeys the linearity rule} \quad \frac{d}{dx}(c\mathbf{v} + d\mathbf{w}) = c\frac{d\mathbf{v}}{dx} + d\frac{d\mathbf{w}}{dx}.$$

It is exactly this linearity that you use to find all other derivatives. From the derivative of each separate power $1, x, x^2, x^3$ (those are the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$) you find the derivative of any polynomial like $4 + x + x^2 + x^3$:

$$\frac{d}{dx}(4 + x + x^2 + x^3) = 1 + 2x + 3x^2 \quad (\text{because of linearity!})$$

This example applies T (the derivative d/dx) to the input $4\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$. Here the input space \mathbf{V} contains all combinations of $1, x, x^2, x^3$. I call them vectors, you might call them functions. Those four vectors are a basis for the space \mathbf{V} of cubic polynomials (degree ≤ 3).

For the nullspace of A , we solved $A\mathbf{v} = \mathbf{0}$. For the kernel of the derivative T , we solve $d\mathbf{v}/dx = \mathbf{0}$. The solution is $\mathbf{v} = \text{constant}$. The nullspace of T is one-dimensional, containing all constant functions like $\mathbf{v}_1 = 1$ (the first basis function).

To find the range (or column space), look at all outputs from $T(\mathbf{v}) = d\mathbf{v}/dx$. The inputs are cubic polynomials $a + bx + cx^2 + dx^3$, so the outputs are *quadratic polynomials* (degree ≤ 2). For the output space \mathbf{W} we have a choice. If $\mathbf{W} = \text{cubics}$, then the range of T (the quadratics) is a subspace. If $\mathbf{W} = \text{quadratics}$, then the range is all of \mathbf{W} .

That second choice emphasizes the difference between the domain or input space ($\mathbf{V} = \text{cubics}$) and the image or output space ($\mathbf{W} = \text{quadratics}$). \mathbf{V} has dimension $n = 4$ and \mathbf{W} has dimension $m = 3$. The matrix for T in equation (2) will be 3 by 4.

The range of T is a three-dimensional subspace. The matrix will have rank $r = 3$. The kernel is one-dimensional. The sum $3+1=4$ is the dimension of the input space. This was $r+(n-r)=n$ in the Fundamental Theorem of Linear Algebra. Always **(dimension of range) + (dimension of kernel) = dimension of \mathbf{V}** .

Example 3 The *integral* is the inverse of the derivative. That is the Fundamental Theorem of Calculus. We see it now in linear algebra. The transformation T^{-1} that “takes the integral from 0 to x ” is linear! Apply T^{-1} to $1, x, x^2$, which are $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$:

$$\int_0^x 1 \, dx = x, \quad \int_0^x x \, dx = \frac{1}{2}x^2, \quad \int_0^x x^2 \, dx = \frac{1}{3}x^3.$$

By linearity, the integral of $\mathbf{w} = B + Cx + Dx^2$ is $T^{-1}(\mathbf{w}) = Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$. The integral of a quadratic is a cubic. The input space of T^{-1} is the quadratics, the output space is the cubics. **Integration takes \mathbf{W} back to \mathbf{V}** . Its matrix will be 4 by 3.

Range of T^{-1} The outputs $Bx + \frac{1}{2}Cx^2 + \frac{1}{3}Dx^3$ are cubics with no constant term.

Kernel of T^{-1} The output is zero only if $B = C = D = 0$. The nullspace is \mathbf{Z} , the zero vector. Now $3+0=3$ is the dimension of the input space \mathbf{W} for T^{-1} .

Matrices for the Derivative and Integral

We will show how the matrices A and A^{-1} copy the derivative T and the integral T^{-1} . This is an excellent example from calculus. Then comes the general rule—how to represent any linear transformation T by a matrix A .

Construction of the Matrix

Now we construct a matrix for any linear transformation. Suppose T transforms the space \mathbf{V} (n -dimensional) to the space \mathbf{W} (m -dimensional). We choose a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for \mathbf{V} and a basis $\mathbf{w}_1, \dots, \mathbf{w}_m$ for \mathbf{W} . The matrix A will be m by n . To find its first column, apply T to the first basis vector \mathbf{v}_1 :

$T(\mathbf{v}_1)$ is a combination $a_{11}\mathbf{w}_1 + \dots + a_{m1}\mathbf{w}_m$ of the output basis for \mathbf{W} .

These numbers a_{11}, \dots, a_{m1} go into the first column of A . Transforming \mathbf{v}_1 to $T(\mathbf{v}_1)$ matches multiplying $(1, 0, \dots, 0)$ by A . It yields that first column of the matrix. When T is the derivative and the first basis vector is 1, its derivative is $T(\mathbf{v}_1) = \mathbf{0}$. So for the derivative, the first column of A was all zero.

For the integral, the first basis function is again 1 and its integral is x . This is 1 times the second basis function. So the first column of A^{-1} was $(0, 1, 0, 0)$.

7A Each linear transformation T from \mathbf{V} to \mathbf{W} is represented by a matrix A (after the bases are chosen for \mathbf{V} and \mathbf{W}). The j th column of A is found by applying T to the j th basis vector \mathbf{v}_j :

$$T(\mathbf{v}_j) = \text{combination of basis vectors of } \mathbf{W} = a_{1j}\mathbf{w}_1 + \dots + a_{mj}\mathbf{w}_m. \quad (5)$$

These numbers a_{1j}, \dots, a_{mj} go into column j of A . *The matrix is constructed to get the basis vectors right. Then linearity gets all other vectors right.* Every \mathbf{v} is a combination $c_1\mathbf{v}_1 + \dots + c_n\mathbf{v}_n$, and $T(\mathbf{v})$ is a combination of the \mathbf{w} 's. When A multiplies the coefficient vector $\mathbf{c} = (c_1, \dots, c_n)$ in the \mathbf{v} combination, $A\mathbf{c}$ produces the coefficients in the $T(\mathbf{v})$ combination. This is because matrix multiplication (combining columns) is linear like T .

A tells what T does. Every linear transformation can be converted to a matrix. This matrix depends on the bases.

Example 4 If the bases change, T is the same but the matrix A is different.

Suppose we reorder the basis to $x, x^2, x^3, 1$ for the cubics in \mathbf{V} . Keep the original basis $1, x, x^2$ for the quadratics in \mathbf{W} . Now apply T to the first basis vector \mathbf{v}_1 . The derivative of x is 1. This is the first basis vector of \mathbf{W} . So the first column of A looks different:

$$A_{\text{new}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \begin{array}{l} \text{matrix for the derivative } T \\ \text{when the bases change to} \\ x, x^2, x^3, 1 \text{ and } 1, x, x^2. \end{array}$$

When we reorder the basis of \mathbf{V} , we reorder the columns of A . The input basis vector \mathbf{v}_j is responsible for column j . The output basis vector \mathbf{w}_i is responsible for row i . Soon the changes in the bases will be more than permutations.

Now take the standard basis $(1, 0)$ and $(0, 1)$. Figure 7.2b shows how $(1, 0)$ projects to $(\frac{1}{2}, \frac{1}{2})$. That gives the first column of A . The other basis vector $(0, 1)$ also projects to $(\frac{1}{2}, \frac{1}{2})$. So the standard matrix for this projection is A :

$$\text{Projection } A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad \text{for the same } T \text{ and the standard basis.}$$

Both A 's are projection matrices. If you square A it doesn't change. Projecting twice is the same as projecting once: $T^2 = T$ so $A^2 = A$. Notice what is hidden in that statement: *The matrix for T^2 is A^2 .*

We have come to something important—the real reason for the way matrices are multiplied. *At last we discover why!* Two transformations S and T are represented by two matrices B and A . When we apply T to the output from S , we get the “composition” TS . When we apply A after B , we get the matrix product AB . Matrix multiplication gives the correct matrix AB to represent TS .

The transformation S is from a space \mathbf{U} to \mathbf{V} . Its matrix B uses a basis $\mathbf{u}_1, \dots, \mathbf{u}_p$ for \mathbf{U} and a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for \mathbf{V} . The matrix is n by p . The transformation T is from \mathbf{V} to \mathbf{W} as before. *Its matrix A must use the same basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ for \mathbf{V}* —this is the output space for S and the input space for T . Then AB matches TS :

7B Multiplication The linear transformation TS starts with any vector \mathbf{u} in \mathbf{U} , goes to $S(\mathbf{u})$ in \mathbf{V} and then to $T(S(\mathbf{u}))$ in \mathbf{W} . The matrix AB starts with any \mathbf{x} in \mathbb{R}^p , goes to $B\mathbf{x}$ in \mathbb{R}^n and then to $AB\mathbf{x}$ in \mathbb{R}^m . The matrix AB correctly represents TS :

$$TS : \mathbf{U} \rightarrow \mathbf{V} \rightarrow \mathbf{W} \quad AB : (m \text{ by } n)(n \text{ by } p) = (m \text{ by } p).$$

The input is $\mathbf{u} = x_1\mathbf{u}_1 + \cdots + x_p\mathbf{u}_p$. The output $T(S(\mathbf{u}))$ matches the output $AB\mathbf{x}$. Product of transformations matches product of matrices. The most important cases are when the spaces $\mathbf{U}, \mathbf{V}, \mathbf{W}$ are the same and their bases are the same. With $m = n = p$ we have square matrices.

Example 7 S rotates the plane by θ and T also rotates by θ . Then TS rotates by 2θ . This transformation T^2 corresponds to the rotation matrix A^2 through 2θ :

$$T = S \quad A = B \quad A^2 = \text{rotation by } 2\theta = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}.$$

By matching (transformation)² with (matrix)², we pick up the formulas for $\cos 2\theta$ and $\sin 2\theta$. Multiply A times A :

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos^2 \theta - \sin^2 \theta & -2 \sin \theta \cos \theta \\ 2 \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix}.$$

Comparing with the display above, $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ and $\sin 2\theta = 2 \sin \theta \cos \theta$. Trigonometry comes from linear algebra.

Example 8 S rotates by θ and T rotates by $-\theta$. Then $TS = I$ and $AB = I$.

In this case $T(S(\mathbf{u}))$ is \mathbf{u} . We rotate forward and back. For the matrices to match, $AB\mathbf{x}$ must be \mathbf{x} . The two matrices are inverses. Check this by putting $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ into A :

$$AB = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix}.$$

By the famous identity for $\cos^2\theta + \sin^2\theta$, this is I .

Earlier T took the derivative and S took the integral. Then TS is the identity but not ST . Therefore AB is the identity matrix but not BA :

$$AB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = I \quad \text{but} \quad BA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Identity Transformation and Change of Basis

We need the matrix for the special and boring transformation $T(\mathbf{v}) = \mathbf{v}$. This identity transformation does nothing to \mathbf{v} . The matrix also does nothing, *provided* the output basis is the same as the input basis. The output $T(\mathbf{v}_1)$ is \mathbf{v}_1 . When the bases are the same, this is \mathbf{w}_1 . So the first column of A is $(1, 0, \dots, 0)$.

When each output $T(\mathbf{v}_j) = \mathbf{v}_j$ is the same as \mathbf{w}_j , the matrix is just I .

This seems reasonable: The identity transformation is represented by the identity matrix. But suppose the bases are different. Then $T(\mathbf{v}_1) = \mathbf{v}_1$ is a combination of the \mathbf{w} 's. That combination $m_{11}\mathbf{w}_1 + \dots + m_{n1}\mathbf{w}_n$ tells us the first column of the matrix M . We will use M (instead of A) for a matrix that represents the identity transformation.

When the outputs $T(\mathbf{v}_j) = \mathbf{v}_j$ are combinations $\sum_{i=1}^n m_{ij}\mathbf{w}_i$, the “change of basis matrix” is M .

The basis is changing but the vectors themselves are not changing: $T(\mathbf{v}) = \mathbf{v}$. When the input has one basis and the output has another basis, the matrix is not I .

Example 9 The input basis is $\mathbf{v}_1 = (3, 7)$ and $\mathbf{v}_2 = (2, 5)$. The output basis is $\mathbf{w}_1 = (1, 0)$ and $\mathbf{w}_2 = (0, 1)$. Then the matrix M is easy to compute:

The matrix for $T(\mathbf{v}) = \mathbf{v}$ is $M = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$.

Reason The first input is $\mathbf{v}_1 = (3, 7)$. The output is also $(3, 7)$ but we express it as $3\mathbf{w}_1 + 7\mathbf{w}_2$. Then the first column of M contains 3 and 7.

This seems too simple to be important. It becomes trickier when the change of basis goes the other way. We get the inverse of the previous matrix M :

Example 10 The input basis is $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$. The outputs are just $T(\mathbf{v}) = \mathbf{v}$. But the output basis is $\mathbf{w}_1 = (3, 7)$ and $\mathbf{w}_2 = (2, 5)$.

$$\text{The matrix for } T(\mathbf{v}) = \mathbf{v} \text{ is } \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}.$$

Reason The first input is $\mathbf{v}_1 = (1, 0)$. The output is also \mathbf{v}_1 but we express it as $5\mathbf{w}_1 - 7\mathbf{w}_2$. Check that $5(3, 7) - 7(2, 5)$ does produce $(1, 0)$. We are combining the columns of the previous M to get the columns of I . The matrix to do that is M^{-1} :

$$[\mathbf{w}_1 \quad \mathbf{w}_2] \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} = [\mathbf{v}_1 \quad \mathbf{v}_2] \text{ is } MM^{-1} = I.$$

A mathematician would say that MM^{-1} corresponds to the product of two identity transformations. We start and end with the same basis $(1, 0)$ and $(0, 1)$. Matrix multiplication must give I . So the two change of basis matrices are inverses.

Warning One mistake about M is very easy to make. Example 9 changes from the basis of \mathbf{v} 's to the standard columns of I . But matrix multiplication goes the other way. When you multiply M times the columns of I , you get the \mathbf{v} 's. It seems backward but it is really OK.

One thing is sure. Multiplying A times $(1, 0, \dots, 0)$ gives column 1 of the matrix. The novelty of this section is that $(1, 0, \dots, 0)$ stands for the first vector \mathbf{v}_1 , written in the basis of \mathbf{v} 's. Then column 1 of the matrix is that same vector \mathbf{v}_1 , written in the standard basis. This is when we keep $T = I$ and change the basis for \mathbf{V} .

In the rest of the book we keep the standard basis and T is multiplication by A .

■ REVIEW OF THE KEY IDEAS ■

- If we know $T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)$ for a basis, linearity determines all other $T(\mathbf{v})$.
- $$\left\{ \begin{array}{ll} \text{Linear transformation } T & \text{Matrix } A \text{ (m by n)} \\ \text{Input basis } \mathbf{v}_1, \dots, \mathbf{v}_n & \text{represents } T \\ \text{Output basis } \mathbf{w}_1, \dots, \mathbf{w}_m & \text{in these bases} \end{array} \right\}$$
- The derivative and integral matrices are one-sided inverses: $d(\text{constant})/dx = 0$:

$$(\text{Derivative}) (\text{Integral}) = I = \text{Fundamental Theorem of Calculus !}$$
- The change of basis matrix M represents $T(\mathbf{v}) = \mathbf{v}$. Its columns are the coefficients of the output basis expressed in the input basis: $\mathbf{w}_j = m_{1j}\mathbf{v}_1 + \dots + m_{nj}\mathbf{v}_n$.

■ WORKED EXAMPLES ■

7.2 A Using the standard basis, find the 4 by 4 matrix P that represents a *cyclic permutation* from $\mathbf{x} = (x_1, x_2, x_3, x_4)$ to $T(\mathbf{x}) = (x_4, x_1, x_2, x_3)$. Find the matrix for T^2 . What is the triple shift $T^3(\mathbf{x})$ and why is $T^3 = T^{-1}$? Find two real independent eigenvectors of P . What are all the eigenvalues of P ?

Solution The first vector $(1, 0, 0, 0)$ in the standard basis transforms to $(0, 1, 0, 0)$ which is the second basis vector. So the first column of P is $(0, 1, 0, 0)$. The other three columns come from transforming the other three standard basis vectors:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{Then } P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ copies } T.$$

Since we used the standard basis, T is *ordinary multiplication by P* . The matrix for T^2 is a “double cyclic shift” P^2 and it produces (x_3, x_4, x_1, x_2) .

The triple shift T^3 will transform $\mathbf{x} = (x_1, x_2, x_3, x_4)$ to $T^3(\mathbf{x}) = (x_2, x_3, x_4, x_1)$. If we apply T once more we are back to the original \mathbf{x} , so $T^4 = \text{identity transformation}$. For matrices this is $P^4 = I$. This means that $T^3T = \text{identity}$ and $T^3 = T^{-1}$.

Two real eigenvectors of P are $(1, 1, 1, 1)$ with eigenvalue $\lambda = 1$ and $(1, -1, 1, -1)$ with eigenvalue $\lambda = -1$. The shift leaves $(1, 1, 1, 1)$ unchanged and it reverses signs in $(1, -1, 1, -1)$. The other two eigenvalues are $\lambda_3 = i$ and $\lambda_4 = -i$. The determinant of P is $\lambda_1\lambda_2\lambda_3\lambda_4 = -1$ as in Problem 5.2 which used cofactors of the first row.

Notice that the eigenvalues $1, -1, i, -i$ add to zero (the trace of P). They are the four roots of 1, since $\det(P - \lambda I) = \lambda^4 - 1$. They are equally spaced around the unit circle in the complex plane. I think P must be a 90° rotation times a reflection in \mathbf{R}^4 .

7.2 B The space of 2 by 2 matrices has these four “vectors” as a basis:

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

T is the linear transformation that *transposes* every 2 by 2 matrix. What is the matrix A that represents T in this basis (output basis = input basis)? What is the inverse matrix A^{-1} ? What is the transformation T^{-1} that inverts the transpose operation?

Also, T_2 multiplies each matrix by $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. What 4 by 4 matrix A_2 represents T_2 ?

Solution Transposing those four “basis matrices” permutes them to $\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_2, \mathbf{u}_4$!

$$\begin{aligned} T(\mathbf{u}_1) &= \mathbf{u}_1 \\ T(\mathbf{u}_2) &= \mathbf{u}_3 \\ T(\mathbf{u}_3) &= \mathbf{u}_2 \quad \text{gives the four columns of } A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \\ T(\mathbf{u}_4) &= \mathbf{u}_4 \end{aligned}$$

The inverse matrix A^{-1} is the same as A . The inverse transformation T^{-1} is the same as T . If we transpose and transpose again, the final output equals the original input.

To find the matrix A_2 , multiply the basis matrices $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$ by M :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\mathbf{u}_1 + c\mathbf{u}_3$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = a\mathbf{u}_2 + c\mathbf{u}_4$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix} = b\mathbf{u}_1 + d\mathbf{u}_3$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = b\mathbf{u}_2 + d\mathbf{u}_4$$

gives the columns of $A = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$.

This A is the “Kronecker product” or “tensor product” of M with I , written $M \otimes I$.

Problem Set 7.2

Questions 1–4 extend the first derivative example to higher derivatives.

- 1 The transformation S takes the *second derivative*. Keep $1, x, x^2, x^3$ as the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ and also as $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$. Write $S\mathbf{v}_1, S\mathbf{v}_2, S\mathbf{v}_3, S\mathbf{v}_4$ in terms of the \mathbf{w} 's. Find the 4 by 4 matrix B for S .
- 2 What functions have $\mathbf{v}'' = \mathbf{0}$? They are in the kernel of the second derivative S . What vectors are in the nullspace of its matrix B in Problem 1?
- 3 B is not the square of the 4 by 3 first derivative matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Add a zero row to A , so that output space = input space. Then compare A^2 with B . Conclusion: For $B = A^2$ we also want output basis = _____ basis. Then $m = n$.

- 4 (a) The product TS produces the *third derivative*. Add zero rows to make 4 by 4 matrices, then compute AB .
- (b) The matrix B^2 corresponds to $S^2 =$ *fourth derivative*. Why is this entirely zero?

Questions 5–10 are about a particular T and its matrix A .

- 5 With bases $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$, suppose $T(\mathbf{v}_1) = \mathbf{w}_2$ and $T(\mathbf{v}_2) = T(\mathbf{v}_3) = \mathbf{w}_1 + \mathbf{w}_3$. T is a linear transformation. Find the matrix A .
- 6 (a) What is the output from T in Question 5 when the input is $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$?
- (b) Multiply A times the vector $(1, 1, 1)$.

- 7 Since $T(\mathbf{v}_2) = T(\mathbf{v}_3)$, the solutions to $T(\mathbf{v}) = \mathbf{0}$ are $\mathbf{v} = \underline{\hspace{2cm}}$. What vectors are in the nullspace of A ? Find all solutions to $T(\mathbf{v}) = \mathbf{w}_2$.
- 8 Find a vector that is not in the column space of A . Find a combination of \mathbf{w} 's that is not in the range of T .
- 9 You don't have enough information to determine T^2 . Why not? Why is its matrix not necessarily A^2 ?
- 10 Find the rank of A . This is not the dimension of the output space \mathbf{W} . It is the dimension of the $\underline{\hspace{2cm}}$ of T .

Questions 11–14 are about invertible linear transformations.

- 11 Suppose $T(\mathbf{v}_1) = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ and $T(\mathbf{v}_2) = \mathbf{w}_2 + \mathbf{w}_3$ and $T(\mathbf{v}_3) = \mathbf{w}_3$. Find the matrix for T using these basis vectors. What input vector \mathbf{v} gives $T(\mathbf{v}) = \mathbf{w}_1$?
- 12 Invert the matrix A in Problem 11. Also invert the transformation T —what are $T^{-1}(\mathbf{w}_1)$ and $T^{-1}(\mathbf{w}_2)$ and $T^{-1}(\mathbf{w}_3)$? Find all \mathbf{v} 's that solve $T(\mathbf{v}) = \mathbf{0}$.
- 13 Which of these are true and why is the other one ridiculous?
- (a) $T^{-1}T = I$ (b) $T^{-1}(T(\mathbf{v}_1)) = \mathbf{v}_1$ (c) $T^{-1}(T(\mathbf{w}_1)) = \mathbf{w}_1$.
- 14 Suppose the spaces \mathbf{V} and \mathbf{W} have the same basis $\mathbf{v}_1, \mathbf{v}_2$.
- (a) Describe a transformation T (not I) that is its own inverse.
 (b) Describe a transformation T (not I) that equals T^2 .
 (c) Why can't the same T be used for both (a) and (b)?

Questions 15–20 are about changing the basis.

- 15 (a) What matrix transforms $(1, 0)$ into $(2, 5)$ and transforms $(0, 1)$ to $(1, 3)$?
 (b) What matrix transforms $(2, 5)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?
 (c) Why does no matrix transform $(2, 6)$ to $(1, 0)$ and $(1, 3)$ to $(0, 1)$?
- 16 (a) What matrix M transforms $(1, 0)$ and $(0, 1)$ to (r, t) and (s, u) ?
 (b) What matrix N transforms (a, c) and (b, d) to $(1, 0)$ and $(0, 1)$?
 (c) What condition on a, b, c, d will make part (b) impossible?
- 17 (a) How do M and N in Problem 16 yield the matrix that transforms (a, c) to (r, t) and (b, d) to (s, u) ?
 (b) What matrix transforms $(2, 5)$ to $(1, 1)$ and $(1, 3)$ to $(0, 2)$?
- 18 If you keep the same basis vectors but put them in a different order, the change of basis matrix M is a $\underline{\hspace{2cm}}$ matrix. If you keep the basis vectors in order but change their lengths, M is a $\underline{\hspace{2cm}}$ matrix.

- 19** The matrix that rotates the axis vectors $(1, 0)$ and $(0, 1)$ through an angle θ is Q . What are the coordinates (a, b) of the original $(1, 0)$ using the new (rotated) axes? This can be tricky. Draw a figure or solve this equation for a and b :

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + b \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}.$$

- 20** The matrix that transforms $(1, 0)$ and $(0, 1)$ to $(1, 4)$ and $(1, 5)$ is $M = \underline{\hspace{2cm}}$. The combination $a(1, 4) + b(1, 5)$ that equals $(1, 0)$ has $(a, b) = (\underline{\hspace{1cm}}, \underline{\hspace{1cm}})$. How are those new coordinates of $(1, 0)$ related to M or M^{-1} ?

Questions 21–24 are about the space of quadratic polynomials $A + Bx + Cx^2$.

- 21** The parabola $w_1 = \frac{1}{2}(x^2 + x)$ equals one at $x = 1$ and zero at $x = 0$ and $x = -1$. Find the parabolas w_2, w_3 , and $y(x)$:
- w_2 equals one at $x = 0$ and zero at $x = 1$ and $x = -1$.
 - w_3 equals one at $x = -1$ and zero at $x = 0$ and $x = 1$.
 - $y(x)$ equals 4 at $x = 1$ and 5 at $x = 0$ and 6 at $x = -1$. Use w_1, w_2, w_3 .
- 22** One basis for second-degree polynomials is $v_1 = 1$ and $v_2 = x$ and $v_3 = x^2$. Another basis is w_1, w_2, w_3 from Problem 21. Find two change of basis matrices, from the w 's to the v 's and from the v 's to the w 's.
- 23** What are the three equations for A, B, C if the parabola $Y = A + Bx + Cx^2$ equals 4 at $x = a$ and 5 at $x = b$ and 6 at $x = c$? Find the determinant of the 3 by 3 matrix. For which numbers a, b, c will it be impossible to find this parabola Y ?
- 24** Under what condition on the numbers m_1, m_2, \dots, m_9 do these three parabolas give a basis for the space of all parabolas?

$$v_1(x) = m_1 + m_2x + m_3x^2 \text{ and } v_2(x) = m_4 + m_5x + m_6x^2 \text{ and} \\ v_3(x) = m_7 + m_8x + m_9x^2.$$

- 25** The Gram-Schmidt process changes a basis a_1, a_2, a_3 to an orthonormal basis q_1, q_2, q_3 . These are columns in $A = QR$. Show that R is the change of basis matrix from the a 's to the q 's (a_2 is what combination of q 's when $A = QR$?).
- 26** Elimination changes the rows of A to the rows of U with $A = LU$. Row 2 of A is what combination of the rows of U ? Writing $A^T = U^T L^T$ to work with columns, the change of basis matrix is $M = L^T$. (We have bases provided the matrices are $\underline{\hspace{2cm}}$.)
- 27** Suppose v_1, v_2, v_3 are eigenvectors for T . This means $T(v_i) = \lambda_i v_i$ for $i = 1, 2, 3$. What is the matrix for T when the input and output bases are the v 's?

- 28** Every invertible linear transformation can have I as its matrix. Choose any input basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ and for output basis choose \mathbf{w}_i to be $T(\mathbf{v}_i)$. Why must T be invertible?

Questions 29–32 review some basic linear transformations.

- 29** Using $\mathbf{v}_1 = \mathbf{w}_1$ and $\mathbf{v}_2 = \mathbf{w}_2$ find the standard matrix for these T 's:
- $T(\mathbf{v}_1) = \mathbf{0}$ and $T(\mathbf{v}_2) = 3\mathbf{v}_1$
 - $T(\mathbf{v}_1) = \mathbf{v}_1$ and $T(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1$.
- 30** Suppose T is reflection across the x axis and S is reflection across the y axis. The domain \mathbf{V} is the xy plane. If $\mathbf{v} = (x, y)$ what is $S(T(\mathbf{v}))$? Find a simpler description of the product ST .
- 31** Suppose T is reflection across the 45° line, and S is reflection across the y axis. If $\mathbf{v} = (2, 1)$ then $T(\mathbf{v}) = (1, 2)$. Find $S(T(\mathbf{v}))$ and $T(S(\mathbf{v}))$. This shows that generally $ST \neq TS$.
- 32** Show that the product ST of two reflections is a rotation. Multiply these reflection matrices to find the rotation angle:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}.$$

- 33** True or false: If we know $T(\mathbf{v})$ for n different nonzero vectors in \mathbf{R}^n , then we know $T(\mathbf{v})$ for every vector in \mathbf{R}^n .

CHANGE OF BASIS ■ 7.3

This section returns to one of the fundamental ideas of linear algebra—*a basis for \mathbf{R}^n* . We don't intend to change that idea, but we do intend to change the basis. It often happens (and we will give examples) that one basis is especially suitable for a specific problem. By changing to that basis, the vectors and the matrices reveal the information we want. The whole idea of a *transform* (this book explains the Fourier transform and wavelet transform) is exactly a change of basis.

Remember what it means for the vectors w_1, \dots, w_n to be a basis for \mathbf{R}^n :

1. The w 's are linearly independent.
2. The $n \times n$ matrix W with these columns is invertible.
3. Every vector v in \mathbf{R}^n can be written in exactly one way as a combination of the w 's:

$$v = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n. \quad (1)$$

Here is the key point: Those coefficients c_1, \dots, c_n completely describe the vector v , *after we have decided on the basis*. Originally, a column vector v just has the components v_1, \dots, v_n . In the new basis of w 's, the same vector is described by the different set of numbers c_1, \dots, c_n . It takes n numbers to describe each vector and it also requires a choice of basis. The n numbers are the coordinates of v in that basis:

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and also} \quad v = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad (2)$$

standard basis basis of w 's

A basis is a set of axes for \mathbf{R}^n . The coordinates c_1, \dots, c_n tell how far to go along each axis. The axes are at right angles when the w 's are orthogonal.

Small point: What is the “standard basis”? Those basis vectors are simply the columns of the n by n identity matrix I . These columns e_1, \dots, e_n are the “default basis.” When I write down the vector $v = (2, 4, 5)$ in \mathbf{R}^3 , I am intending and you are expecting the standard basis (the usual xyz axes, where the coordinates are 2, 4, 5):

$$v = 2e_1 + 4e_2 + 5e_3 = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}.$$

The new question is: *What are the coordinates c_1, c_2, c_3 in the new basis w_1, w_2, w_3 ?* As usual we put the basis vectors into the columns of a matrix. This is the *basis matrix* W . Then the fundamental equation $v = c_1 w_1 + \dots + c_n w_n$ has the matrix form $v = Wc$. From this we immediately know $c = W^{-1}v$.

7C The coordinates $c = (c_1, \dots, c_n)$ of v in the basis w_1, \dots, w_n are given by $c = W^{-1}v$. The change of basis matrix W^{-1} is the inverse of the basis matrix W .

The standard basis has $W = I$. The coordinates in that default basis e_1, \dots, e_n are the usual components v_1, \dots, v_n . Our first new example is the wavelet basis for \mathbf{R}^4 .

Example 1 (Wavelet basis) Wavelets are little waves. They have different lengths and they are localized at different places. The first basis vector is not actually a wavelet, it is the very useful flat vector of all ones. The others are “Haar wavelets”:

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{w}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{w}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}. \quad (3)$$

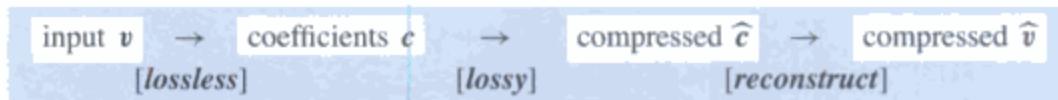
Those vectors are *orthogonal*, which is good. You see how \mathbf{w}_3 is localized in the first half and \mathbf{w}_4 is localized in the second half. Their coefficients c_3 and c_4 tell us about details in the first half and last half of \mathbf{v} . The ultimate in localization is the standard basis.

Why do want to change the basis? I think of v_1, v_2, v_3, v_4 as the intensities of a signal. It could be an audio signal, like music on a CD. It could be a medical signal, like an electrocardiogram. Of course $n = 4$ is very short, and $n = 10,000$ is more realistic. We may need to *compress* that long signal, by keeping only the largest 5% of the coefficients. This is 20 : 1 compression and (to give only one of its applications) it makes modern video conferencing possible.

If we keep only 5% of the *standard* basis coefficients, we lose 95% of the signal. In image processing, most of the image disappears. In audio, 95% of the tape goes blank. But if we choose a better basis of \mathbf{w} 's, 5% of the basis vectors can come very close to the original signal. In image processing and audio coding, you can't see or hear the difference. We don't need the other 95%!

One good basis vector is a flat $(1, 1, 1, 1)$. That part alone can represent the constant background of our image. A short wave like $(0, 0, 1, -1)$ or in higher dimensions $(0, 0, 0, 0, 0, 0, 1, -1)$ represents a detail at the end of the signal.

The three steps of *transform* and *compression* and *inverse transform* are



In linear algebra, where everything is perfect, we omit the compression step. The output $\widehat{\mathbf{v}}$ is exactly the same as the input \mathbf{v} . The transform gives $\mathbf{c} = W^{-1}\mathbf{v}$ and the reconstruction brings back $\mathbf{v} = W\mathbf{c}$. In true signal processing, where nothing is perfect but everything is fast, the transform (lossless) and the compression (which only loses unnecessary information) are absolutely the keys to success. Then $\widehat{\mathbf{v}} = W\widehat{\mathbf{c}}$.

I will show those steps for a typical vector like $v = (6, 4, 5, 1)$. Its wavelet coefficients are 4, 1, 1, 2. This means that v can be reconstructed from $c = (4, 1, 1, 2)$ using w_1, w_2, w_3, w_4 . In matrix form the reconstruction is $v = Wc$:

$$\begin{bmatrix} 6 \\ 4 \\ 5 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \\ 2 \end{bmatrix}. \quad (4)$$

Those coefficients $c = (4, 1, 1, 2)$ are $W^{-1}v$. Inverting this basis matrix W is easy because the w 's in its columns are orthogonal. But they are not unit vectors. So the inverse is the transpose divided by the lengths squared, $W^{-1} = (W^T W)^{-1} W^T$:

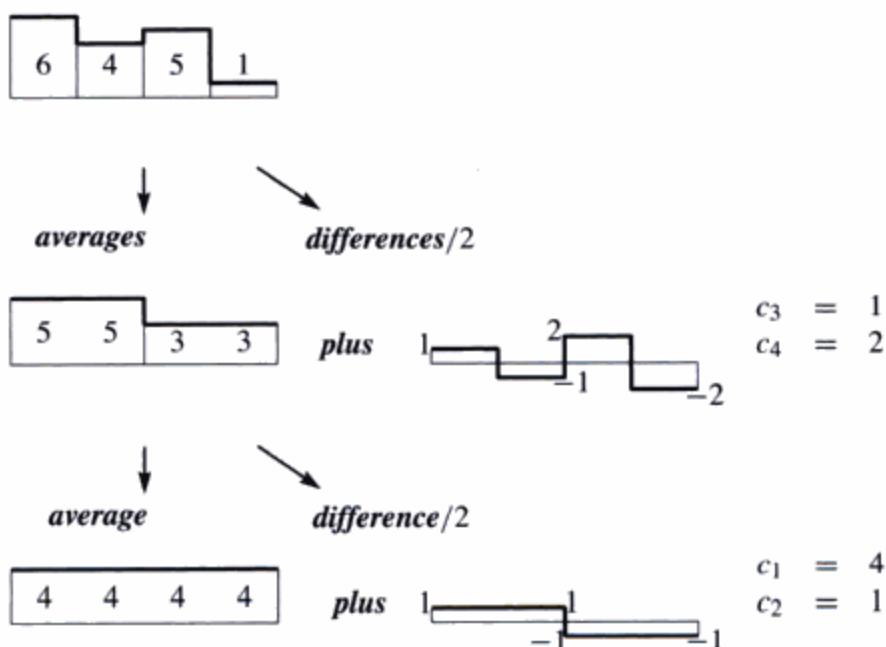
$$W^{-1} = \begin{bmatrix} \frac{1}{4} & & & \\ & \frac{1}{4} & & \\ & & \frac{1}{2} & \\ & & & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

From the 1's in the first row of $c = W^{-1}v$, notice that c_1 is the average of v_1, v_2, v_3, v_4 :

$$c_1 = \frac{6 + 4 + 5 + 1}{4} = 4.$$

Example 2 (Same wavelet basis by recursion) I can't resist showing you a faster way to find the c 's. The special point of the wavelet basis is that you can pick off the details in c_3 and c_4 , before the coarse details in c_2 and the overall average in c_1 . A picture will explain this "multiscale" method, which is in Chapter 1 of my textbook with Nguyen on *Wavelets and Filter Banks*:

Split $v = (6, 4, 5, 1)$ into averages and waves at small scale and then large scale:



the columns in W . In other words $W^{-1} = W^T$. That is the specially important case of an *orthogonal matrix*.

Other bases are not orthonormal. The axes don't have to be perpendicular. The basis matrix W can be invertible without having orthogonal columns.

When the inverse matrices are in the opposite order $WW^{-1} = I$, we learn something new. The columns are w_j , the rows are u_i^T , and each product is a rank one matrix. *Multiply columns times rows:*

$$WW^{-1} = \begin{bmatrix} & \\ w_1 & \cdots & w_n \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix} = w_1 u_1^T + \cdots + w_n u_n^T = I.$$

WW^{-1} is the order that we constantly use to change the basis. The coefficients are in $c = W^{-1}v$. So W^{-1} is the first (with the u_i^T in its rows). Then we reconstruct v from Wc . Use the u 's and w 's to state the basic facts that $c = W^{-1}v$ and $v = Wc = WW^{-1}v$:

The coefficients are $c_i = u_i^T v$ and the vector is $v = \sum_1^n w_i (u_i^T v)$. (5)

The analysis step takes dot products of v with the dual basis to find the c 's. The synthesis step adds up the pieces $c_i w_i$ to reconstruct the vector v .

■ REVIEW OF THE KEY IDEAS ■

1. The new basis vectors w_j are the columns of an invertible matrix W .
2. The coefficients of v in this new basis are $c = W^{-1}v$ (the analysis step).
3. The vector v is reconstructed as $Wc = c_1 w_1 + \cdots + c_n w_n$ (the synthesis step).
4. Compression would simplify c to \hat{c} and we reconstruct $\hat{v} = \hat{c}_1 w_1 + \cdots + \hat{c}_n w_n$.
5. The rows of W^{-1} are the dual basis vectors u_i and $c_i = u_i^T v$. Then $u_i^T w_j = \delta_{ij}$.

■ WORKED EXAMPLES ■

7.3 A Match $a_0 + a_1x + a_2x^2$ with $b_0 + b_1(x+1) + b_2(x+1)^2$, to find the 3 by 3 matrix M_1 that connects these coefficients by $a = M_1b$. M_1 will be familiar to Pascal!

The matrix to reverse that change must be M_1^{-1} , and $b = M_1^{-1}a$. This shifts the center of the series back, so $a_0 + a_1(x-1) + a_2(x-1)^2$ equals $b_0 + b_1x + b_2x^2$. Match

- 4 The wavelet basis matrix W factors into simpler matrices W_1 and W_2 :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $W^{-1} = W_2^{-1}W_1^{-1}$ allows \mathbf{c} to be computed in two steps. The first splitting in Example 2 shows $W_1^{-1}\mathbf{v}$. Then the second splitting applies W_2^{-1} . Find those inverse matrices W_1^{-1} and W_2^{-1} directly from W_1 and W_2 . Apply them to $\mathbf{v} = (6, 4, 5, 1)$.

- 5 The 4 by 4 *Hadamard matrix* is like the wavelet matrix but entirely +1 and -1:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Find H^{-1} and write $\mathbf{v} = (7, 5, 3, 1)$ as a combination of the columns of H .

- 6 Suppose we have two bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_n$ for \mathbf{R}^n . If a vector has coefficients b_i in one basis and c_i in the other basis, what is the change of basis matrix in $\mathbf{b} = M\mathbf{c}$? Start from

$$b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n = V\mathbf{b} = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n = W\mathbf{c}.$$

Your answer represents $T(\mathbf{v}) = \mathbf{v}$ with input basis of \mathbf{v} 's and output basis of \mathbf{w} 's. Because of different bases, the matrix is not I .

- 7 The dual basis vectors $\mathbf{w}_1^*, \dots, \mathbf{w}_n^*$ are the columns of $W^* = (W^{-1})^T$. Show that the original basis $\mathbf{w}_1, \dots, \mathbf{w}_n$ is “the dual of the dual.” In other words, show that the \mathbf{w} 's are the rows of $(W^*)^{-1}$. **Hint:** Transpose the equation $WW^{-1} = I$.

DIAGONALIZATION AND THE PSEUDOINVERSE ■ 7.4

This short section combines the ideas from Section 7.2 (matrix of a linear transformation) and Section 7.3 (change of basis). The combination produces a needed result: *the change of matrix due to change of basis*. The matrix depends on the input basis and output basis. We want to produce a better matrix than A , by choosing a better basis than the standard basis.

By reversing the input and output bases, we will find the pseudoinverse A^+ . It sends \mathbf{R}^m back to \mathbf{R}^n , column space back to row space.

The truth is that all our great factorizations of A can be regarded as a change of basis. But this is a short section, so we concentrate on the two outstanding examples. In both cases the good matrix is *diagonal*. It is either Λ or Σ :

1. $S^{-1}AS = \Lambda$ when the input and output bases are eigenvectors of A .
2. $U^{-1}AV = \Sigma$ when the input and output bases are eigenvectors of $A^T A$ and AA^T .

You see immediately the difference between Λ and Σ . In Λ the bases are the same. The matrix A must be square. And some square matrices cannot be diagonalized by any S , because they don't have n independent eigenvectors.

In Σ the input and output bases are different. The matrix A can be rectangular. The bases are *orthonormal* because $A^T A$ and AA^T are symmetric. Then $U^{-1} = U^T$ and $V^{-1} = V^T$. Every matrix A is allowed, and can be diagonalized. This is the Singular Value Decomposition (**SVD**) of Section 6.7.

I will just note that the Gram-Schmidt factorization $A = QR$ chooses only *one* new basis. That is the orthogonal output basis given by Q . The input uses the standard basis given by I . We don't reach a diagonal Σ , but we do reach a triangular R . The output basis matrix appears on the left and the input basis appears on the right, in $A = QRI$.

We start with input basis equal to output basis. That will produce S and S^{-1} .

Similar Matrices: A and $S^{-1}AS$ and $W^{-1}AW$

We begin with a square matrix and one basis. The input space \mathbf{V} is \mathbf{R}^n and the output space \mathbf{W} is also \mathbf{R}^n . The standard basis vectors are the columns of I . The matrix is n by n , and we call it A . The linear transformation T is “multiplication by A ”.

Most of this book has been about one fundamental problem—*to make the matrix simple*. We made it triangular in Chapter 2 (by elimination) and Chapter 4 (by Gram-Schmidt). We made it diagonal in Chapter 6 (by eigenvectors). Now that change from A to Λ comes from a *change of basis*.

Here are the main facts in advance. When you change the basis for \mathbf{V} , the matrix changes from A to AM . Because \mathbf{V} is the input space, the matrix M goes on the right (to come first). When you change the basis for \mathbf{W} , the new matrix is $M^{-1}A$. We are working with the output space so M^{-1} is on the left (to come last). *If you change both bases in the same way, the new matrix is $M^{-1}AM$.* The good basis vectors are the eigenvectors of A , in the columns of $M = S$. The matrix becomes $S^{-1}AS = \Lambda$.

Polar Decomposition

Every complex number has the polar form $re^{i\theta}$. A nonnegative number r multiplies a number on the unit circle. (Remember that $|e^{i\theta}| = |\cos \theta + i \sin \theta| = 1$.) Thinking of these numbers as 1 by 1 matrices, $r \geq 0$ corresponds to a *positive semidefinite matrix* (call it H) and $e^{i\theta}$ corresponds to an *orthogonal matrix* Q . The **polar decomposition** extends this $re^{i\theta}$ factorization to matrices.

7G Every real square matrix can be factored into $A = QH$, where Q is *orthogonal* and H is *symmetric positive semidefinite*. If A is invertible then H is positive definite.

For the proof we just insert $V^T V = I$ into the middle of the SVD:

$$A = U\Sigma V^T = (UV^T)(V\Sigma V^T) = (Q)(H). \quad (5)$$

The first factor UV^T is Q . The product of orthogonal matrices is orthogonal. The second factor $V\Sigma V^T$ is H . It is positive semidefinite because its eigenvalues are in Σ . If A is invertible then Σ and H are also invertible. **H is the symmetric positive definite square root of $A^T A$.** Equation (5) says that $H^2 = V\Sigma^2 V^T = A^T A$.

There is also a polar decomposition $A = KQ$ in the reverse order. Q is the same but now $K = U\Sigma U^T$. This is the symmetric positive definite square root of AA^T .

Example 3 Find the polar decomposition $A = QH$ from its SVD in Section 6.7:

$$A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} \\ 2\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = U\Sigma V^T.$$

Solution The orthogonal part is $Q = UV^T$. The positive definite part is $H = V\Sigma V^T$. This is also $H = Q^{-1}A$ which is $Q^T A$:

$$\begin{aligned} Q &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ H &= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/\sqrt{2} \end{bmatrix}. \end{aligned}$$

In mechanics, the polar decomposition separates the *rotation* (in Q) from the *stretching* (in H). The eigenvalues of H are the singular values of A . They give the stretching factors. The eigenvectors of H are the eigenvectors of $A^T A$. They give the stretching directions (the principal axes). Then Q rotates the axes.

The polar decomposition just splits the key equation $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$ into two steps. The “ H ” part multiplies \mathbf{v}_i by σ_i . The “ Q ” part swings \mathbf{v}_i around into \mathbf{u}_i .

The Pseudoinverse

By choosing good bases, A multiplies \mathbf{v}_i in the row space to give $\sigma_i \mathbf{u}_i$ in the column space. A^{-1} must do the opposite! If $A\mathbf{v} = \sigma \mathbf{u}$ then $A^{-1}\mathbf{u} = \mathbf{v}/\sigma$. The singular values of A^{-1} are $1/\sigma$, just as the eigenvalues of A^{-1} are $1/\lambda$. The bases are reversed. The \mathbf{u} 's are in the row space of A^{-1} , the \mathbf{v} 's are in the column space.

Until this moment we would have added "if A^{-1} exists." Now we don't. A matrix that multiplies \mathbf{u}_i to produce \mathbf{v}_i/σ_i does exist. It is the pseudoinverse A^+ :

$$\begin{array}{l} \text{Pseudoinverse} \\ A^+ = V\Sigma^+U^T \end{array} = \left[\begin{array}{c} \mathbf{v}_1 \cdots \mathbf{v}_r \cdots \mathbf{v}_n \\ \hline n \text{ by } n \end{array} \right] \left[\begin{array}{ccc} \sigma_1^{-1} & & \\ & \ddots & \\ & & \sigma_r^{-1} \end{array} \right] \left[\begin{array}{c} \mathbf{u}_1 \cdots \mathbf{u}_r \cdots \mathbf{u}_m \\ \hline m \text{ by } m \end{array} \right]^T$$

The *pseudoinverse* A^+ is an n by m matrix. If A^{-1} exists (we said it again), then A^+ is the same as A^{-1} . In that case $m = n = r$ and we are inverting $U\Sigma V^T$ to get $V\Sigma^{-1}U^T$. The new symbol A^+ is needed when $r < m$ or $r < n$. Then A has no two-sided inverse, but it has a *pseudoinverse* A^+ with that same rank r :

$$A^+\mathbf{u}_i = \frac{1}{\sigma_i} \mathbf{v}_i \quad \text{for } i \leq r \quad \text{and} \quad A^+\mathbf{u}_i = \mathbf{0} \quad \text{for } i > r.$$

The vectors $\mathbf{u}_1, \dots, \mathbf{u}_r$ in the column space of A go back to the row space. The other vectors $\mathbf{u}_{r+1}, \dots, \mathbf{u}_m$ are in the left nullspace, and A^+ sends them to zero. When we know what happens to each basis vector \mathbf{u}_i , we know A^+ .

Notice the pseudoinverse Σ^+ of the diagonal matrix Σ . Each σ is replaced by σ^{-1} . The product $\Sigma^+\Sigma$ is as near to the identity as we can get. We get r 1's. We can't do anything about the zero rows and columns! This example has $\sigma_1 = 2$ and $\sigma_2 = 3$:

$$\Sigma^+\Sigma = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7H The pseudoinverse A^+ is the n by m matrix with these two properties:

$$\begin{aligned} AA^+ &= \text{projection matrix onto the column space of } A \\ A^+A &= \text{projection matrix onto the row space of } A \end{aligned}$$

Example 4 Find the pseudoinverse of $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. This matrix is not invertible.

The rank is 1. The only singular value is $\sqrt{10}$. That is inverted to $1/\sqrt{10}$ in Σ^+ :

$$A^+ = V\Sigma^+U^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{10} & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}.$$

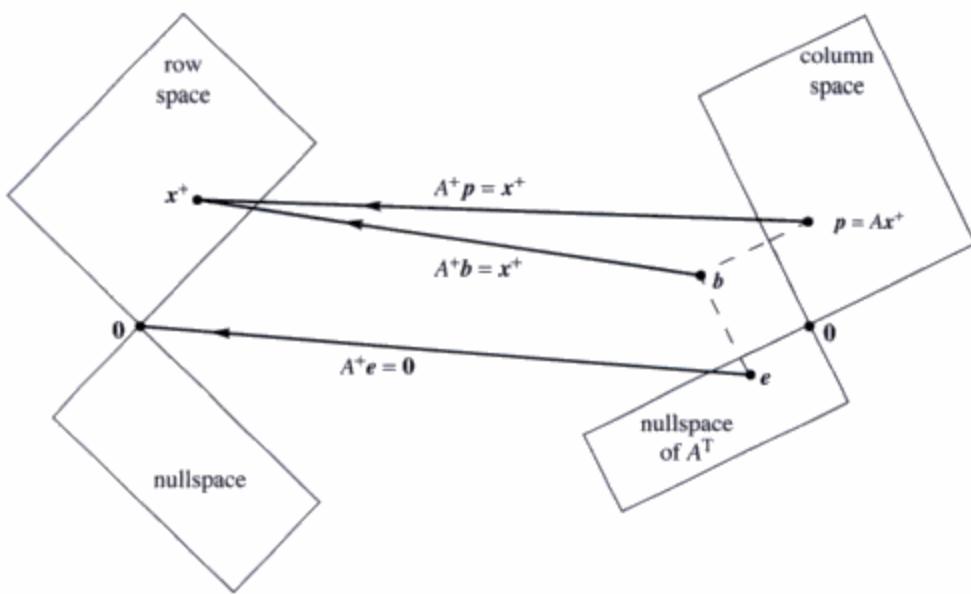


Figure 7.4 A is invertible from row space to column space. A^+ inverts it.

A^+ also has rank 1. Its column space is the row space of A . When A takes $(1, 1)$ in the row space to $(4, 2)$ in the column space, A^+ does the reverse. Every rank one matrix is a column times a row. With unit vectors \mathbf{u} and \mathbf{v} , that is $A = \sigma \mathbf{u} \mathbf{v}^T$. Then the best inverse of a rank one matrix is $A^+ = \mathbf{v} \mathbf{u}^T / \sigma$.

The product AA^+ is $\mathbf{u}\mathbf{u}^T$, the projection onto the line through \mathbf{u} . The product A^+A is $\mathbf{v}\mathbf{v}^T$, the projection onto the line through \mathbf{v} . **For all matrices, AA^+ and A^+A are the projections onto the column space and row space.**

The shortest least squares solution to $Ax = b$ is $x^+ = A^+b$. Any other vector that solves $A^TA\hat{x} = A^Tb$ is longer than x^+ (Problem 18).

■ REVIEW OF THE KEY IDEAS ■

1. Diagonalization $S^{-1}AS = \Lambda$ is the same as a change to the eigenvector basis.
2. The SVD chooses an input basis of \mathbf{v} 's and an output basis of \mathbf{u} 's. Those orthonormal bases diagonalize A . This is $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$, and $A = U\Sigma V^T$.
3. Polar decomposition factors A into QH , rotation times stretching.
4. The pseudoinverse $A^+ = V\Sigma^+U^T$ transforms the column space of A back to its row space. A^+A is the identity on the row space (and zero on the nullspace).

■ WORKED EXAMPLES ■

7.4 A Start with an m by n matrix A . If its rank is n (full column rank) then it has a *left inverse* $C = (A^T A)^{-1} A^T$. This matrix C gives $CA = I$. Explain why the pseudoinverse is $A^+ = C$ in this case. If A has rank m (full row rank) then it has a *right inverse* B with $B = A^T (A A^T)^{-1}$ and $AB = I$. Explain why $A^+ = B$ in this case.

Find B and C if possible and find A^+ for all three matrices:

$$A_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Solution If A has rank n (independent columns) then $A^T A$ is invertible—this is a key point of Section 4.2. Certainly $C = (A^T A)^{-1} A^T$ multiplies A to give $CA = I$. In the opposite order, $AC = A(A^T A)^{-1} A^T$ is the projection matrix (Section 4.2 again) onto the column space. So C meets the requirements **7H** to be A^+ .

If A has rank m (full row rank) then $A A^T$ is invertible. Certainly A multiplies $B = A^T (A A^T)^{-1}$ to give $AB = I$. In the opposite order, $BA = A^T (A A^T)^{-1} A$ is the projection matrix onto the row space. So B is the pseudoinverse A^+ .

The example A_1 has full column rank (for C) and A_2 has full row rank (for B):

$$A_1^+ = (A_1^T A_1)^{-1} A_1^T = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 & 2 \end{bmatrix} \quad A_2^+ = A_2^T (A_2 A_2^T)^{-1} = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Notice $A_1^+ A_1 = [1]$ and $A_2 A_2^+ = [1]$. But A_3 has no left or right inverse. Its pseudoinverse is $A_3^+ = \sigma_1^{-1} \mathbf{v}_1 \mathbf{u}_1^T = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}/4$.

Problem Set 7.4

Problems 1–6 compute and use the SVD of a particular matrix (not invertible).

- 1 Compute $A^T A$ and its eigenvalues and unit eigenvectors \mathbf{v}_1 and \mathbf{v}_2 :

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

What is the only singular value σ_1 ? The rank of A is $r = 1$.

- 2 (a) Compute $A A^T$ and its eigenvalues and unit eigenvectors \mathbf{u}_1 and \mathbf{u}_2 .
 (b) Verify from Problem 1 that $A \mathbf{v}_1 = \sigma_1 \mathbf{u}_1$. Put numbers into the SVD:

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}^T.$$

- 14 What is the only 2 by 3 matrix that has no pivots and no singular values? What is Σ for that matrix? A^+ is the zero matrix, but what shape?
- 15 If $\det A = 0$ how do you know that $\det A^+ = 0$?
- 16 When are the factors in $U\Sigma V^T$ the same as in $Q\Lambda Q^T$? The eigenvalues λ_i must be positive, to equal the σ_i . Then A must be _____ and positive _____.

Problems 17–20 bring out the main properties of A^+ and $x^+ = A^+b$.

- 17 Suppose all matrices have rank one. The vector b is (b_1, b_2) .
- $$A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} .2 & .1 \\ .2 & .1 \end{bmatrix} \quad AA^T = \begin{bmatrix} .8 & .4 \\ .4 & .2 \end{bmatrix} \quad A^TA = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$$
- (a) The equation $A^TA\hat{x} = A^Tb$ has many solutions because A^TA is _____.
- (b) Verify that $x^+ = A^+b = (.2b_1 + .1b_2, .2b_1 + .1b_2)$ does solve $A^TAx^+ = A^Tb$.
- (c) AA^+ projects onto the column space of A . Therefore _____ projects onto the nullspace of A^T . Then $A^T(AA^+ - I)b = \mathbf{0}$. This gives $A^TAx^+ = A^Tb$ and \hat{x} can be x^+ .
- 18 The vector x^+ is the shortest possible solution to $A^TA\hat{x} = A^Tb$. Reason: The difference $\hat{x} - x^+$ is in the nullspace of A^TA . This is also the nullspace of A . Explain how it follows that

$$\|\hat{x}\|^2 = \|x^+\|^2 + \|\hat{x} - x^+\|^2.$$

Any other solution \hat{x} has greater length than x^+ .

- 19 Every b in \mathbf{R}^m is $p + e$. This is the column space part plus the left nullspace part. Every x in \mathbf{R}^n is $x_r + x_n$ = (row space part) + (nullspace part). Then

$$AA^+p = \text{_____} \quad AA^+e = \text{_____} \quad A^+Ax_r = \text{_____} \quad A^+Ax_n = \text{_____}$$

- 20 Find A^+ and A^+A and AA^+ for the 2 by 1 matrix whose SVD is

$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} .6 & -.8 \\ .8 & .6 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} [1].$$

- 21 A general 2 by 2 matrix A is determined by four numbers. If triangular, it is determined by three. If diagonal, by two. If a rotation, by one. An eigenvector, by one. Check that the total count is four for each factorization of A :

$$LU \quad LDU \quad QR \quad U\Sigma V^T \quad SAS^{-1}.$$

- 22 Following Problem 21, check that LDL^T and $Q\Lambda Q^T$ are determined by *three* numbers. This is correct because the matrix A is _____.

- 23 From $A = U\Sigma V^T$ and $A^+ = V\Sigma^+U^T$ explain these splittings into rank 1:

$$A = \sum_1^r \sigma_i \mathbf{u}_i \mathbf{v}_i^T \quad A^+ = \sum_1^r \frac{\mathbf{v}_i \mathbf{u}_i^T}{\sigma_i} \quad A^+ A = \sum_1^r \mathbf{v}_i \mathbf{v}_i^T \quad AA^+ = \sum_1^r \mathbf{u}_i \mathbf{u}_i^T$$

- 24 This problem looks for all matrices A with a given column space in \mathbf{R}^m and a given row space in \mathbf{R}^n . Suppose $\hat{\mathbf{u}}_1, \dots, \hat{\mathbf{u}}_r$ and $\hat{\mathbf{v}}_1, \dots, \hat{\mathbf{v}}_r$ are bases for those two spaces. Make them columns of \hat{U} and \hat{V} . Use $A = U\Sigma V^T$ to show that A has the form $\hat{U}M\hat{V}^T$ for an r by r invertible matrix M .
- 25 A pair of singular vectors \mathbf{v} and \mathbf{u} will satisfy $A\mathbf{v} = \sigma\mathbf{u}$ and $A^T\mathbf{u} = \sigma\mathbf{v}$. This means that the double vector $\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix}$ is an eigenvector of what symmetric matrix? With what eigenvalue?