8

APPLICATIONS

MATRICES IN ENGINEERING ■ 8.1

This section will show how engineering problems produce symmetric matrices K (often positive definite matrices). The "linear algebra reason" for symmetry and positive definiteness is their form $K = A^{\mathrm{T}}A$ and $K = A^{\mathrm{T}}CA$. The "physical reason" is that the expression $\frac{1}{2}\boldsymbol{u}^{\mathrm{T}}K\boldsymbol{u}$ represents *energy*—and energy is never negative.

Our first examples come from mechanical and civil and aeronautical engineering. K is the ***stiffness matrix***, and $K^{-1}\boldsymbol{f}$ is the structure's response to forces $\boldsymbol{f}$ from outside. The next section turns to electrical engineering—the matrices come from networks and circuits. The exercises involve chemical engineering and I could go on! Economics and management and engineering design come later in this chapter (there the key is optimization).

Here we present equilibrium equations $K\boldsymbol{u} = \boldsymbol{f}$. With motion, $Md^2\boldsymbol{u}/dt^2 + K\boldsymbol{u} = \boldsymbol{f}$ becomes dynamic. Then we use eigenvalues, or finite differences between time steps.

Before explaining the physical examples, may I write down the matrices? The tridiagonal K_0 appears many times in this textbook. Now we will see its applications. These matrices are all symmetric, and the first four are positive definite:

$$K_0 = A_0^{\mathrm{T}}A_0 = \begin{bmatrix} \mathbf{2} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{2} \end{bmatrix} \qquad A_0^{\mathrm{T}}C_0A_0 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3+c_4 \end{bmatrix}$$

$$K_1 = A_1^{\mathrm{T}}A_1 = \begin{bmatrix} \mathbf{2} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} c_1+c_2 & -c_2 & \\ -c_2 & c_2+c_3 & -c_3 \\ & -c_3 & c_3 \end{bmatrix}$$

$$K_{singular} = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & \mathbf{1} \end{bmatrix} \qquad K_{circular} = \begin{bmatrix} 2 & -1 & \mathbf{-1} \\ -1 & 2 & -1 \\ \mathbf{-1} & -1 & 2 \end{bmatrix}$$

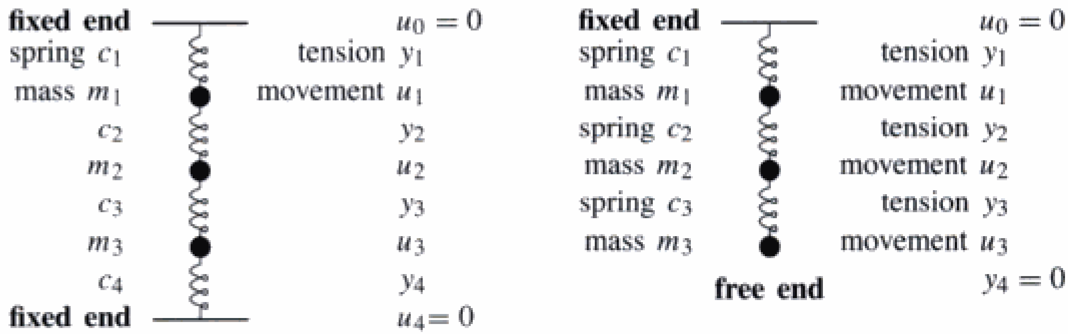


Figure 8.1 Lines of springs and masses with different end conditions: no movement (**fixed-fixed**) and no force at the bottom (**fixed-free**).

The matrices K_0, K_1, $K_{singular}$, and $K_{circular}$ have $C = I$ for simplicity. This means that all the "spring constants" are $c_i = 1$. We included $A_0^{\mathrm{T}} C_0 A_0$ and $A_1^{\mathrm{T}} C_1 A_1$ to show how the spring constants enter the matrix (without changing its positive definiteness). Our first goal is to show where these stiffness matrices come from.

A Line of Springs

Figure 8.1 shows three masses m_1, m_2, m_3 connected by a line of springs. In one case there are four springs, with top and bottom fixed. The fixed-free case has only three springs; the lowest mass hangs freely. The **fixed-fixed** problem will lead to K_0 and $A_0^{\mathrm{T}} C_0 A_0$. The **fixed-free** problem will lead to K_1 and $A_1^{\mathrm{T}} C_1 A_1$. A **free-free** problem, with no support at either end, produces the matrix $K_{singular}$.

We want equations for the mass movements $\boldsymbol{u}$ and the tensions (or compressions) $\boldsymbol{y}$:

$$\begin{aligned} \boldsymbol{u} &= (u_1, u_2, u_3) = \textbf{movements of the masses} \text{ (down or up)} \\ \boldsymbol{y} &= (y_1, y_2, y_3, y_4) \text{ or } (y_1, y_2, y_3) = \textbf{tensions in the springs} \end{aligned}$$

When a mass moves downward, its displacement is positive ($u_i > 0$). For the springs, tension is positive and compression is negative ($y_i < 0$). In tension, the spring is stretched so it pulls the masses inward. Each spring is controlled by its own Hooke's Law $y = c\,e$: (*stretching force*) = (*spring constant*) times (*stretching distance*).

Our job is to link these one-spring equations into a vector equation $K\boldsymbol{u} = \boldsymbol{f}$ for the whole system. The force vector $\boldsymbol{f}$ comes from gravity. The gravitational constant g multiplies each mass to produce $\boldsymbol{f} = (m_1 g, m_2 g, m_3 g)$.

The real problem is to find the stiffness matrix (**fixed-fixed** and **fixed-free**). The best way to create K is in three steps, not one. Instead of connecting the movements $\boldsymbol{u}_i$ directly to the forces $\boldsymbol{f}_i$, it is much better to connect each vector to the next in this list:

$$\begin{aligned} \boldsymbol{u} &= \textbf{\textit{Movements}} \text{ of } n \text{ masses} &&= (u_1, \ldots, u_n) \\ \boldsymbol{e} &= \textbf{\textit{Elongations}} \text{ of } m \text{ springs} &&= (e_1, \ldots, e_m) \\ \boldsymbol{y} &= \textbf{\textit{Internal forces}} \text{ in } m \text{ springs} &&= (y_1, \ldots, y_m) \\ \boldsymbol{f} &= \textbf{\textit{External forces}} \text{ on } n \text{ masses} &&= (f_1, \ldots, f_n) \end{aligned}$$

The framework that connects $\boldsymbol{u}$ to $\boldsymbol{e}$ to $\boldsymbol{y}$ to $\boldsymbol{f}$ looks like this:

$$\begin{array}{ccccc} \boxed{\boldsymbol{u}} & & \boxed{\boldsymbol{f}} & \quad \boldsymbol{e} = A\boldsymbol{u} & \quad A \text{ is } m \text{ by } n \\ A\downarrow & & \uparrow A^{\mathrm{T}} & \quad \boldsymbol{y} = C\boldsymbol{e} & \quad C \text{ is } m \text{ by } m \\ \boxed{\boldsymbol{e}} & \xrightarrow{C} & \boxed{\boldsymbol{y}} & \quad \boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y} & \quad A^{\mathrm{T}} \text{ is } n \text{ by } m \end{array}$$

We will write down the matrices A and C and A^{T} for the two examples, first with fixed ends and then with the lower end free. Forgive the simplicity of these matrices, it is their form that is so important. Especially the appearance of A and A^{T}.

The *elongation* $\boldsymbol{e}$ is the stretching distance—how far the springs are extended. Originally there is no stretching—the system is lying on a table. When it becomes vertical and upright, gravity acts. The masses move down by distances u_1, u_2, u_3. Each spring is stretched or compressed by $e_i = u_i - u_{i-1}$, *the difference in displacements*:

$$\begin{array}{lll} \text{First spring:} & \boldsymbol{e}_1 = \boldsymbol{u}_1 & \text{(the top is fixed so } u_0 = 0) \\ \text{Second spring:} & \boldsymbol{e}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_1 & \\ \text{Third spring:} & \boldsymbol{e}_3 = \boldsymbol{u}_3 - \boldsymbol{u}_2 & \\ \text{Fourth spring:} & \boldsymbol{e}_4 = \qquad - \boldsymbol{u}_3 & \text{(the bottom is fixed so } u_4 = 0) \end{array}$$

If both ends move the same distance, that spring is not stretched: $u_i = u_{i-1}$ and $e_i = 0$. The matrix in those four equations is a 4 by 3 *difference matrix* A, and $\boldsymbol{e} = A\boldsymbol{u}$:

Stretching distances (elongations)
$$\boldsymbol{e} = A\boldsymbol{u} \quad \text{is} \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{1}$$

The next equation $\boldsymbol{y} = C\boldsymbol{e}$ connects spring elongation $\boldsymbol{e}$ with spring tension $\boldsymbol{y}$. *This is Hooke's Law $y_i = c_i e_i$ for each separate spring.* It is the "constitutive law" that depends on the material in the spring. A soft spring has small c, so a moderate force y can produce a large stretching $\boldsymbol{e}$. Hooke's linear law is nearly exact for real springs, before they are overstretched and the material becomes plastic.

Since each spring has its own law, the matrix in $\boldsymbol{y} = C\boldsymbol{e}$ is a diagonal matrix C:

Hooke's Law
$$\begin{array}{rcl} \boldsymbol{y}_1 &=& c_1\boldsymbol{e}_1 \\ \boldsymbol{y}_2 &=& c_2\boldsymbol{e}_2 \\ \boldsymbol{y}_3 &=& c_3\boldsymbol{e}_3 \\ \boldsymbol{y}_4 &=& c_4\boldsymbol{e}_4 \end{array} \quad \text{is} \quad \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & c_4 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = C\boldsymbol{e}. \tag{2}$$

Combining $\boldsymbol{e} = A\boldsymbol{u}$ with $\boldsymbol{y} = C\boldsymbol{e}$, the spring forces are $\boldsymbol{y} = CA\boldsymbol{u}$.

Finally comes the *balance equation*, the most fundamental law of applied mathematics. The internal forces from the springs balance the external forces on the masses. Each mass is pulled or pushed by the spring force y_j above it. From below it feels the spring force y_{j+1} plus f_j from gravity. Thus $y_j = y_{j+1} + f_j$ or $f_j = y_j - y_{j+1}$:

$$\begin{array}{l}\textbf{Force}\\\textbf{balance}\end{array}\quad \begin{aligned} f_1 &= y_1 - y_2 \\ f_2 &= y_2 - y_3 \\ f_3 &= y_3 - y_4 \end{aligned} \quad \text{and} \quad \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}. \tag{3}$$

That matrix is A^{T}. *The equation for balance of forces is* $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$. Nature transposes the rows and columns of the $\boldsymbol{e} - \boldsymbol{u}$ matrix to produce the $\boldsymbol{f} - \boldsymbol{y}$ matrix. This is the beauty of the framework, that A^{T} appears along with A. The three equations combine into $K\boldsymbol{u} = \boldsymbol{f}$, where the ***stiffness matrix*** is $K = A^{\mathrm{T}}CA$:

$$\left\{ \begin{aligned} \boldsymbol{e} &= A\boldsymbol{u} \\ \boldsymbol{y} &= C\boldsymbol{e} \\ \boldsymbol{f} &= A^{\mathrm{T}}\boldsymbol{y} \end{aligned} \right\} \quad \text{combine into} \quad A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f} \quad \text{or} \quad K\boldsymbol{u} = \boldsymbol{f}.$$

In the language of elasticity, $\boldsymbol{e} = A\boldsymbol{u}$ is the **kinematic** equation (for displacement). The force balance $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$ is the **static** equation (for equilibrium). The **constitutive law** is $\boldsymbol{y} = C\boldsymbol{e}$ (from the material). Then $A^{\mathrm{T}}CA$ is n by n = (n by m)(m by m)(m by n).

Finite element programs spend major effort on assembling $K = A^{\mathrm{T}}CA$ from thousands of smaller pieces. We do it for four springs by multiplying A^{T} times CA:

$$\begin{array}{l}\textbf{FIXED}\\\textbf{FIXED}\end{array}\quad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ -c_2 & c_2 & 0 \\ 0 & -c_3 & c_3 \\ 0 & 0 & -c_4 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix}$$

If all springs are identical, with $c_1 = c_2 = c_3 = c_4 = 1$, then $C = I$. The stiffness matrix reduces to $A^{\mathrm{T}}A$. It becomes the special matrix

$$K_0 = A_0^{\mathrm{T}}A_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}. \tag{4}$$

Note the difference between $A^{\mathrm{T}}A$ from engineering and LL^{T} from linear algebra. The matrix A from four springs is 4 by 3. The triangular matrix L from elimination is square. The stiffness matrix K is assembled from $A^{\mathrm{T}}A$, and then broken up into LL^{T}. One step is applied mathematics, the other is computational mathematics. Each K is built from rectangular matrices and factored into square matrices.

May I list some properties of $K = A^{\mathrm{T}}CA$? You know almost all of them:

1. K is **tridiagonal**, because mass 3 is not connected to mass 1.
2. K is **symmetric**, because C is symmetric and A^{T} comes with A.
3. K is **positive definite**, because $c_i > 0$ and A has **independent columns**.
4. K^{-1} is a full matrix in equation (5) with **all positive entries**.

That last property leads to an important fact about $\boldsymbol{u} = K^{-1}\boldsymbol{f}$: *If all forces act downwards* ($f_j > 0$) *then all movements are downwards* ($u_j > 0$). Notice that "positiveness" is different from "positive definiteness". Here K^{-1} is positive (K is not). Both K and K^{-1} are positive definite.

Example 1 Suppose all $c_i = c$ and $m_j = m$. Find the movements $\boldsymbol{u}$ and tensions $\boldsymbol{y}$.

All springs are the same and all masses are the same. But all movements and elongations and tensions will *not* be the same. K^{-1} includes $\frac{1}{c}$ because $A^{\mathrm{T}}CA$ includes c:

$$\boldsymbol{u} = K^{-1}\boldsymbol{f} = \frac{1}{4c}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \end{bmatrix} \tag{5}$$

The displacement u_2, for the mass in the middle, is greater than u_1 and u_3. The units are correct: the force mg divided by force per unit length c gives a length u. Then

$$\boldsymbol{e} = A\boldsymbol{u} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\frac{mg}{c}\begin{bmatrix} \frac{3}{2} \\ 2 \\ \frac{3}{2} \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}.$$

Those elongations add to zero because the ends of the line are fixed. (The sum $u_1 + (u_2 - u_1) + (u_3 - u_2) + (-u_3)$ is certainly zero.) For each spring force y_i we just multiply e_i by c. So y_1, y_2, y_3, y_4 are $\frac{3}{2}mg$, $\frac{1}{2}mg$, $-\frac{1}{2}mg$, $-\frac{3}{2}mg$. The upper two springs are stretched, the lower two springs are compressed.

Notice how $\boldsymbol{u}, \boldsymbol{e}, \boldsymbol{y}$ are computed in that order. We assembled $K = A^{\mathrm{T}}CA$ from rectangular matrices. To find $\boldsymbol{u} = K^{-1}\boldsymbol{f}$, we work with the whole matrix and not its three pieces! The rectangular matrices A and A^{T} do not have (two-sided) inverses.

> **Warning:** *Normally you cannot write* $K^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}$.

The three matrices are mixed together by $A^{\mathrm{T}}CA$, and they cannot easily be untangled. In general, $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ has many solutions. And four equations $A\boldsymbol{u} = \boldsymbol{e}$ would usually have no solution with three unknowns. But $A^{\mathrm{T}}CA$ gives the correct solution to all three equations in the framework. Only when $m = n$ and the matrices are square can we go from $\boldsymbol{y} = (A^{\mathrm{T}})^{-1}\boldsymbol{f}$ to $\boldsymbol{e} = C^{-1}\boldsymbol{y}$ to $\boldsymbol{u} = A^{-1}\boldsymbol{e}$. We will see that now.

Fixed End and Free End

Remove the fourth spring. All matrices become 3 by 3. The pattern does not change! The matrix A loses its fourth row and (of course) A^{T} loses its fourth column. The new stiffness matrix K_1 becomes a product of square matrices:

$$A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 & & \\ & c_2 & \\ & & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

The missing column of A^{T} and row of A multiplied the missing c_4. So the quickest way to find the new $A^{\mathrm{T}}CA$ is to set $c_4 = 0$ in the old one:

FIXED
FREE

$$K_1 = A_1^{\mathrm{T}}C_1A_1 = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}. \qquad (6)$$

If $c_1 = c_2 = c_3 = 1$ and $C = I$, this is the -1, 2, -1 tridiagonal matrix, except the last entry is 1 instead of 2. The spring at the bottom is free.

Example 2 All $c_i = c$ and all $m_j = m$ in the fixed-free hanging line of springs. Then

$$K_1 = c\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad K_1^{-1} = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

The forces mg from gravity are the same. But the movements change from the previous example because the stiffness matrix has changed:

$$\boldsymbol{u} = K_1^{-1}\boldsymbol{f} = \frac{1}{c}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}.$$

Those movements are greater in this fixed-free case. The number 3 appears in u_1 because all three masses are pulling the first spring down. The next mass moves by that 3 plus an additional 2 from the masses below it. The third mass drops even more $(3+2+1=6)$. The elongations $\boldsymbol{e} = A\boldsymbol{u}$ in the springs display those numbers 3, 2, 1:

$$\boldsymbol{e} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \frac{mg}{c}\begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \frac{mg}{c}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Multiplying by c, the forces $\boldsymbol{y}$ in the three springs are $3mg$ and $2mg$ and mg. And the special point of square matrices is that $\boldsymbol{y}$ can be found directly from $\boldsymbol{f}$! The balance equation $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ determines $\boldsymbol{y}$ immediately, because $m = n$ and A^{T} is square. We are allowed to write $(A^{\mathrm{T}}CA)^{-1} = A^{-1}C^{-1}(A^{\mathrm{T}})^{-1}$:

$$\boldsymbol{y} = (A^{\mathrm{T}})^{-1}\boldsymbol{f} \text{ is } \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} mg \\ mg \\ mg \end{bmatrix} = \begin{bmatrix} 3mg \\ 2mg \\ 1mg \end{bmatrix}.$$

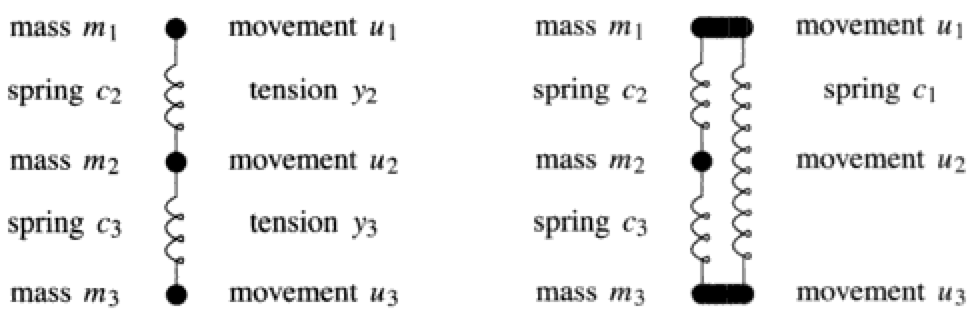


Figure 8.2 **Free-free ends**: A line of springs and a "circle" of springs: *Singular* K's. The masses can move without stretching the springs so $A\boldsymbol{u} = \mathbf{0}$ has nonzero solutions.

Two Free Ends: K is Singular

The first line of springs in Figure 8.2 is free at *both ends*. This means trouble (the whole line can move). The matrix A is 2 by 3, short and wide. Here is $\boldsymbol{e} = A\boldsymbol{u}$:

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_2 - u_1 \\ u_3 - u_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}. \tag{7}$$

Now there is a nonzero solution to $A\boldsymbol{u} = \mathbf{0}$. ***The masses can move with no stretching of the springs***. The whole line can shift by $\boldsymbol{u} = (1, 1, 1)$ and this leaves $\boldsymbol{e} = (0, 0)$. A has *dependent columns* and the vector $(1, 1, 1)$ is in its nullspace:

$$A\boldsymbol{u} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \textbf{no stretching}. \tag{8}$$

$A\boldsymbol{u} = \mathbf{0}$ certainly leads to $A^{\mathrm{T}}CA\boldsymbol{u} = \mathbf{0}$. So $A^{\mathrm{T}}CA$ is only *positive semidefinite*, without c_1 and c_4. The pivots will be c_2 and c_3 and *no third pivot*:

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_2 & \\ & c_3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{9}$$

Two eigenvalues will be positive but $\boldsymbol{x} = (1, 1, 1)$ is an eigenvector for $\lambda = 0$. We can solve $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}$ only for special vectors $\boldsymbol{f}$. The forces have to add to $f_1 + f_2 + f_3 = 0$, or the whole line of springs (with both ends free) will take off like a rocket.

Circle of Springs

A third spring will complete the circle from mass 3 back to mass 1. This doesn't make K invertible—the new matrix is still singular. That stiffness matrix $K_{circular}$ is not tridiagonal, but it is symmetric (always) and *semidefinite*:

$$A^{\mathrm{T}}_{circular}A_{circular} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \quad (10)$$

The only pivots are 2 and $\frac{3}{2}$. The eigenvalues are 3 and 3 and 0. The determinant is zero. The nullspace still contains $\boldsymbol{x} = (1, 1, 1)$, when all masses move together (nothing is holding them) and the springs are not stretched. This movement vector $(1, 1, 1)$ is in the nullspace of $A_{circular}$ and $K_{circular}$, even after the diagonal matrix C of spring constants is included:

$$(A^{\mathrm{T}}CA)_{circular} = \begin{bmatrix} c_1 + c_2 & -c_2 & -c_1 \\ -c_2 & c_2 + c_3 & -c_3 \\ -c_1 & -c_3 & c_3 + c_1 \end{bmatrix}. \quad (11)$$

Continuous Instead of Discrete

Matrix equations are discrete. Differential equations are continuous. We will see the differential equation that corresponds to the tridiagonal $-1, 2, -1$ matrix $A^{\mathrm{T}}A$. And it is a pleasure to see the boundary conditions that go with K_0 and K_1.

The matrices A and A^{T} correspond to the derivatives d/dx and $-d/dx$! Remember that $\boldsymbol{e} = A\boldsymbol{u}$ took differences $u_i - u_{i-1}$, and $\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y}$ took differences $y_i - y_{i+1}$. Now the springs are infinitesimally short, and those differences become derivatives:

$$\frac{u_i - u_{i-1}}{\Delta x} \quad \text{is like} \quad \frac{du}{dx} \qquad \frac{y_i - y_{i+1}}{\Delta x} \quad \text{is like} \quad -\frac{dy}{dx}$$

The factor Δx didn't appear earlier—we imagined the distance between masses was 1. To approximate a continuous solid bar, we take many more masses (smaller and closer). Let me jump to the three steps A, C, A^{T} in the continuous model, when there is stretching and Hooke's Law and force balance at every point x:

$$e(x) = Au = \frac{du}{dx} \quad y(x) = c(x)e(x) \quad A^{\mathrm{T}}y = -\frac{dy}{dx} = f(x)$$

Combining those equations into $A^{\mathrm{T}}CAu(x) = f(x)$, we have a differential equation not a matrix equation. The line of springs becomes an elastic bar:

Solid Elastic Bar $\quad A^{\mathrm{T}}CAu(x) = f(x) \quad$ is $\quad -\dfrac{d}{dx}\left(c(x)\dfrac{du}{dx}\right) = f(x) \quad$ (12)

$A^{\mathrm{T}}A$ corresponds to a second derivative. A is a "difference matrix" and $A^{\mathrm{T}}A$ is a "second difference matrix". ***The matrix has*** $-1, 2, -1$ ***and the equation has*** $-d^2u/dx^2$:

$$-u_{i+1} + 2u_i - u_{i-1} \text{ is a } \textbf{second difference} \quad -\frac{d^2u}{dx^2} \text{ is a } \textbf{second derivative}.$$

Now we see why this symmetric matrix is a favorite. When we meet a first derivative du/dx, we have three choices (*forward, backward, and centered differences*):

$$\frac{du}{dx} \simeq \frac{u(x+\Delta x) - u(x)}{\Delta x} \quad \text{or} \quad \frac{u(x) - u(x-\Delta x)}{\Delta x} \quad \text{or} \quad \frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x}.$$

When we meet d^2u/dx^2, the natural choice is $u(x+\Delta x) - 2u(x) + u(x-\Delta x)$, divided by $(\Delta x)^2$. *Why reverse these signs to* $-1, 2, -1$? Because the positive definite matrix has $+2$ on the diagonal. First derivatives are *anti*symmetric; the transpose has a minus sign. So second differences are negative definite, and we change to $-d^2u/dx^2$.

We have moved from vectors to functions. Scientific computing moves the other way. It starts with a differential equation like (12). Sometimes there is a formula for the solution $u(x)$, more often not. In reality we *create* the discrete matrix K by approximating the continuous problem. Watch how the boundary conditions on u come in! By missing u_0 we treat it (correctly) as zero:

$$\begin{array}{l}\textbf{FIXED}\\ \textbf{FIXED}\end{array} \quad A\boldsymbol{u} = \frac{1}{\Delta x}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \approx \frac{du}{dx} \quad \text{with} \quad \begin{array}{l}\boldsymbol{u_0 = 0}\\ \boldsymbol{u_4 = 0}\end{array} \qquad (13)$$

Fixing the top end gives the boundary condition $u_0 = 0$. What about the free end, when the bar hangs in the air? Row 4 of A is gone and so is u_4. The boundary condition must come from A^{T}. It is the missing y_4 that we are treating (correctly) as zero:

$$\begin{array}{l}\textbf{FIXED}\\ \textbf{FREE}\end{array} \quad A^{\mathrm{T}}\boldsymbol{y} = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \approx -\frac{dy}{dx} \quad \text{with} \quad \begin{array}{l}\boldsymbol{u_0 = 0}\\ \boldsymbol{y_4 = 0}\end{array} \qquad (14)$$

The boundary condition $y_4 = 0$ *at the free end becomes* $du/dx = 0$, since $\boldsymbol{y} = A\boldsymbol{u}$ corresponds to du/dx. The force balance $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ at that end (in the air) is $0 = 0$. The last row of $K_1\boldsymbol{u} = \boldsymbol{f}$ has entries $-1, 1$ to reflect this condition $du/dx = 0$.

May I summarize this section? I hope this example will help you turn calculus into linear algebra, replacing differential equations by difference equations. If your step Δx is small enough, you will have a totally satisfactory solution.

$$\textbf{The equation is } -\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \textbf{ with } u(0) = 0 \textbf{ and } \left[u(1) \text{ or } \frac{du}{dx}(1)\right] = 0$$

Divide the bar into N pieces of length Δx. Replace du/dx by $A\boldsymbol{u}$ and $-dy/dx$ by $A^{\mathrm{T}}\boldsymbol{y}$. Now A and A^{T} include $1/\Delta x$. The end conditions are $u_0 = 0$ and $[u_N = 0$ or $y_N = 0]$. The three steps $-d/dx$ and $c(x)$ and d/dx correspond to A^{T} and C and A:

$$\boldsymbol{f} = A^{\mathrm{T}}\boldsymbol{y} \text{ and } \boldsymbol{y} = C\boldsymbol{e} \text{ and } \boldsymbol{e} = A\boldsymbol{u} \text{ give } A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}.$$

This is a fundamental example in computational science and engineering. Our book concentrates on Step 3 in that process (linear algebra). Now we have taken Step 2.

1. Model the problem by a differential equation

2. Discretize the differential equation to a difference equation

3. Understand and solve the difference equation (and boundary conditions!)

4. Interpret the solution; visualize it; redesign if needed.

Numerical simulation has become a third branch of science, together with experiment and deduction. Designing the Boeing 777 was much less expensive on a computer than in a wind tunnel. Our discussion still has to move from ordinary to partial differential equations, and from linear to nonlinear. The text *Introduction to Applied Mathematics* (Wellesley-Cambridge Press) develops this whole subject further—see the course page **math.mit.edu/18085**. The principles remain the same, and I hope this book helps you to see the framework behind the computations.

Problem Set 8.1

1 Show that $\det A_0^{\mathrm{T}}C_0A_0 = c_1c_2c_3+c_1c_3c_4+c_1c_2c_4+c_2c_3c_4$. Find also $\det A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example.

2 Invert $A_1^{\mathrm{T}}C_1A_1$ in the fixed-free example by multiplying $A_1^{-1}C_1^{-1}(A_1^{\mathrm{T}})^{-1}$.

3 In the free-free case when $A^{\mathrm{T}}CA$ in equation (9) is singular, add the three equations $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}$ to show that we need $f_1 + f_2 + f_3 = 0$. Find a solution to $A^{\mathrm{T}}CA\boldsymbol{u} = \boldsymbol{f}$ when the forces $\boldsymbol{f} = (-1, 0, 1)$ balance themselves. Find all solutions!

4 Both end conditions for the free-free differential equation are $du/dx = 0$:

$$-\frac{d}{dx}\left(c(x)\frac{du}{dx}\right) = f(x) \quad \text{with} \quad \frac{du}{dx} = 0 \text{ at both ends.}$$

Integrate both sides to show that the force $f(x)$ must balance itself, $\int f(x)\,dx = 0$, or there is no solution. The complete solution is one particular solution $u(x)$ plus any constant. The constant corresponds to $\boldsymbol{u} = (1, 1, 1)$ in the nullspace of $A^{\mathrm{T}}CA$.

5 In the fixed-free problem, the matrix A is square and invertible. We can solve $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ separately from $A\boldsymbol{u} = \boldsymbol{e}$. Do the same for the differential equation:

$$\text{Solve} \quad -\frac{dy}{dx} = f(x) \ \text{ with } \ y(1) = 0. \quad \text{Graph } \ y(x) \ \text{ if } \ f(x) = 1.$$

6 The 3 by 3 matrix $K_1 = A_1^{\mathrm{T}} C_1 A_1$ in equation (6) splits into three "element matrices" $c_1 E_1 + c_2 E_2 + c_3 E_3$. Write down those pieces, one for each c. Show how they come from *column times row* multiplication of $A_1^{\mathrm{T}} C_1 A_1$. This is how finite element stiffness matrices are actually assembled.

7 For five springs and four masses with both ends fixed, what are the matrices A and C and K? With $C = I$ solve $K\boldsymbol{u} = \text{ones}(4)$.

8 Compare the solution $\boldsymbol{u} = (u_1, u_2, u_3, u_4)$ in Problem 7 to the solution of the continuous problem $-u'' = 1$ with $u(0) = 0$ and $u(1) = 0$. The parabola $u(x)$ should correspond at $x = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$ to $\boldsymbol{u}$—is there a $(\Delta x)^2$ factor to account for?

9 Solve the fixed-free problem $-u'' = mg$ with $u(0) = 0$ and $u'(1) = 0$. Compare $u(x)$ at $x = \frac{1}{3}, \frac{2}{3}, \frac{3}{3}$ with the vector $\boldsymbol{u} = (3mg, 5mg, 6mg)$ in Example 2.

10 (MATLAB) Find the displacements $u(1), \ldots, u(100)$ of 100 masses connected by springs all with $c = 1$. Each force is $f(i) = .01$. Print graphs of $\boldsymbol{u}$ with **fixed-fixed** and **fixed-free** ends. Note that $\text{diag}(\text{ones}(n, 1), d)$ is a matrix with n ones along diagonal d. This print command will graph a vector u:

```
plot(u, '+'); xlabel('mass number'); ylabel('movement'); print
```

11 (MATLAB) Chemical engineering has a first derivative du/dx from fluid velocity as well as d^2u/dx^2 from diffusion. Replace du/dx by a forward difference and then by a backward difference, with $\Delta x = \frac{1}{8}$. Graph your numerical solutions of

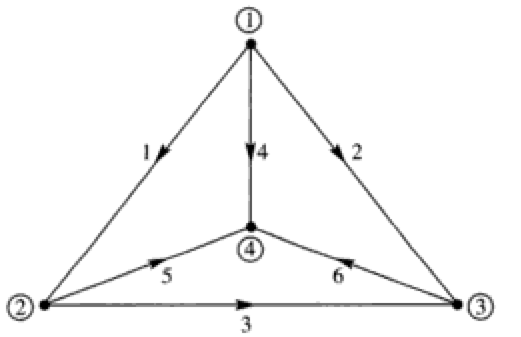
$$-\frac{d^2u}{dx^2} + 10\,\frac{du}{dx} = 1 \ \text{ with } \ u(0) = u(1) = 0.$$

$Ax = \mathbf{0}$. For A and U above, $x = (1, 1, 1, 1)$ is perpendicular to all rows and thus to the whole row space.

This review of the subspaces applies to any matrix A—only the example was special. Now we concentrate on that example. It is the incidence matrix for a particular graph, and we look to the graph for the meaning of every subspace.

Directed Graphs and Incidence Matrices

Figure 8.4 displays a *graph* with $m = 6$ edges and $n = 4$ nodes, so the matrix A is 6 by 4. It tells which nodes are connected by which edges. The entries -1 and $+1$ also tell the direction of each arrow (this is a *directed* graph). The first row of A gives a record of the first edge:



The first edge goes from node 1 to node 2. The first row has -1 in column 1 and $+1$ in column 2.

$$\begin{array}{c} \textbf{node} \\ ①\ ②\ ③\ ④ \end{array}$$

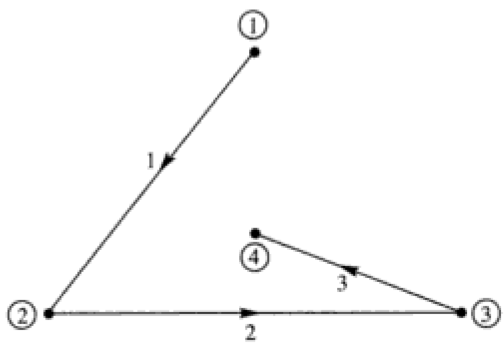
$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \quad \textbf{edge}$$

Figure 8.4a Complete graph with $m = 6$ edges and $n = 4$ nodes.

Row numbers are edge numbers, column numbers are node numbers.

You can write down A immediately by looking at the graph.

The second graph has the same four nodes but only three edges. Its incidence matrix is 3 by 4:



$$\begin{array}{c} \textbf{node} \\ ①\ ②\ ③\ ④ \end{array}$$

$$B = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \quad \textbf{edge}$$

Figure 8.4b Tree with 3 edges and 4 nodes and no loops.

The first graph is *complete*—every pair of nodes is connected by an edge. The second graph is a ***tree***—the graph has ***no closed loops***. Those graphs are the two extremes, with the maximum number of edges $m = \frac{1}{2}n(n-1)$ and the minimum number $m = n - 1$. We are assuming that the graph is connected, and it makes no fundamental difference which way the arrows go. On each edge, flow with the arrow is "positive." Flow in the opposite direction counts as negative. The flow might be a current or a signal or a force—or even oil or gas or water.

The rows of B match the nonzero rows of U—the echelon form found earlier. ***Elimination reduces every graph to a tree***. The loops produce zero rows in U. Look at the loop from edges 1, 2, 3 in the first graph, which leads to a zero row:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Those steps are typical. When two edges share a node, elimination produces the "shortcut edge" without that node. If the graph already has this shortcut edge, elimination gives a row of zeros. When the dust clears we have a tree.

An idea suggests itself: ***Rows are dependent when edges form a loop***. Independent rows come from trees. This is the key to the row space.

For the column space we look at $A\boldsymbol{x}$, which is a vector of differences:

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix}. \tag{1}$$

The unknowns x_1, x_2, x_3, x_4 represent ***potentials*** at the nodes. Then $A\boldsymbol{x}$ gives the ***potential differences*** across the edges. It is these differences that cause flows. We now examine the meaning of each subspace.

1 The ***nullspace*** contains the solutions to $A\boldsymbol{x} = \boldsymbol{0}$. All six potential differences are zero. This means: *All four potentials are equal.* Every $\boldsymbol{x}$ in the nullspace is a constant vector (c, c, c, c). The nullspace of A is a line in $\mathbf{R}^n$—its dimension is $n - r = 1$.

The second incidence matrix B has the same nullspace. It contains $(\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{1})$:

$$B\boldsymbol{x} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We can raise or lower all potentials by the same amount c, without changing the differences. There is an "arbitrary constant" in the potentials. Compare this with the same statement for functions. We can raise or lower $f(x)$ by the same amount C, without changing its derivative. There is an arbitrary constant C in the integral.

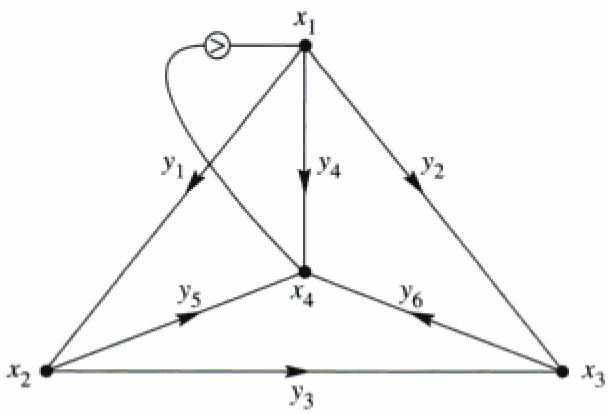


Figure 8.5 The currents in a network with a source S into node 1.

current law $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$, we get $A^{\mathrm{T}}CA\boldsymbol{x} = \mathbf{0}$. This is *almost* the central equation for network flows. The only thing wrong is the zero on the right side! The network needs power from outside—a voltage source or a current source—to make something happen.

Note about signs In circuit theory we change from $A\boldsymbol{x}$ to $-A\boldsymbol{x}$. The flow is from higher potential to lower potential. There is (positive) current from node 1 to node 2 when $x_1 - x_2$ is positive—whereas $A\boldsymbol{x}$ was constructed to yield $x_2 - x_1$. The minus sign in physics and electrical engineering is a plus sign in mechanical engineering and economics. $A\boldsymbol{x}$ versus $-A\boldsymbol{x}$ is a general headache but unavoidable.

Note about applied mathematics Every new application has its own form of Ohm's law. For elastic structures $\boldsymbol{y} = CA\boldsymbol{x}$ is Hooke's law. The stress $\boldsymbol{y}$ is (elasticity C) times (stretching $A\boldsymbol{x}$). For heat conduction, $A\boldsymbol{x}$ is a temperature gradient. For oil flows it is a pressure gradient. There is a similar law for least square regression in statistics. My textbook *Introduction to Applied Mathematics* (Wellesley-Cambridge Press) is practically built on $A^{\mathrm{T}}CA$. This is the key to equilibrium in matrix equations and also in differential equations.

Applied mathematics is more organized than it looks. *I have learned to watch for $A^{\mathrm{T}}CA$.*

We now give an example with a current source. Kirchhoff's law changes from $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$, to balance the source $\boldsymbol{f}$ from outside. *Flow into each node still equals flow out.* Figure 8.5 shows the network with its conductances $c_1, \ldots, c_6$, and it shows the current source going into node 1. The source comes out at node 4 to keep the balance (in = out). The problem is: ***Find the currents $y_1, \ldots, y_6$ on the six edges***.

Example 1 All conductances are $c = 1$, so that $C = I$. A current y_4 travels directly from node 1 to node 4. Other current goes the long way from node 1 to node 2 to node 4 (this is $y_1 = y_5$). Current also goes from node 1 to node 3 to node 4 (this is $y_2 = y_6$). We can find the six currents by using special rules for symmetry, or we can do it right by using $A^{\mathrm{T}}CA$. Since $C = I$, this matrix is $A^{\mathrm{T}}A$:

$$\begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

That last matrix is not invertible! We cannot solve for all potentials because $(1, 1, 1, 1)$ is in the nullspace. One node has to be grounded. Setting $x_4 = 0$ removes the fourth row and column, and this leaves a 3 by 3 invertible matrix. Now we solve $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ for the unknown potentials x_1, x_2, x_3, with source S into node 1:

$$\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S \\ 0 \\ 0 \end{bmatrix} \quad \text{gives} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

Ohm's law $\boldsymbol{y} = -CA\boldsymbol{x}$ yields the six currents. Remember $C = I$ and $x_4 = 0$:

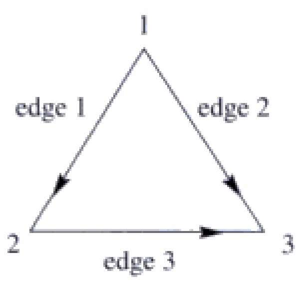
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = - \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} S/2 \\ S/4 \\ S/4 \\ 0 \end{bmatrix} = \begin{bmatrix} S/4 \\ S/4 \\ 0 \\ S/2 \\ S/4 \\ S/4 \end{bmatrix}.$$

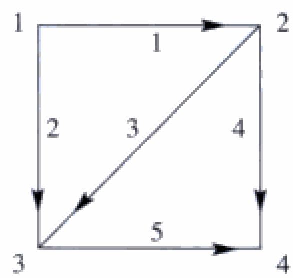
Half the current goes directly on edge 4. That is $y_4 = S/2$. No current crosses from node 2 to node 3. Symmetry indicated $y_3 = 0$ and now the solution proves it.

The same matrix $A^{\mathrm{T}}A$ appears in least squares. Nature distributes the currents to minimize the heat loss. Statistics chooses $\widehat{\boldsymbol{x}}$ to minimize the least squares error.

Problem Set 8.2

Problems 1–7 and 8–14 are about the incidence matrices for these graphs.





1 Write down the 3 by 3 incidence matrix A for the triangle graph. The first row has -1 in column 1 and $+1$ in column 2. What vectors (x_1, x_2, x_3) are in its nullspace? How do you know that $(1, 0, 0)$ is not in its row space?

2 Write down A^{T} for the triangle graph. Find a vector $\boldsymbol{y}$ in its nullspace. The components of $\boldsymbol{y}$ are currents on the edges—how much current is going around the triangle?

3 Eliminate x_1 and x_2 from the third equation to find the echelon matrix U. What tree corresponds to the two nonzero rows of U?

$$\begin{aligned} -x_1 + x_2 &= b_1 \\ -x_1 + x_3 &= b_2 \\ -x_2 + x_3 &= b_3. \end{aligned}$$

4 Choose a vector (b_1, b_2, b_3) for which $A\boldsymbol{x} = \boldsymbol{b}$ can be solved, and another vector $\boldsymbol{b}$ that allows no solution. How are those $\boldsymbol{b}$'s related to $\boldsymbol{y} = (1, -1, 1)$?

5 Choose a vector (f_1, f_2, f_3) for which $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ can be solved, and a vector $\boldsymbol{f}$ that allows no solution. How are those $\boldsymbol{f}$'s related to $\boldsymbol{x} = (1, 1, 1)$? The equation $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$ is Kirchhoff's _____ law.

6 Multiply matrices to find $A^{\mathrm{T}}A$. Choose a vector $\boldsymbol{f}$ for which $A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{f}$ can be solved, and solve for $\boldsymbol{x}$. Put those potentials $\boldsymbol{x}$ and the currents $\boldsymbol{y} = -A\boldsymbol{x}$ and current sources $\boldsymbol{f}$ onto the triangle graph. Conductances are 1 because $C = I$.

7 With conductances $c_1 = 1$ and $c_2 = c_3 = 2$, multiply matrices to find $A^{\mathrm{T}}CA$. For $\boldsymbol{f} = (1, 0, -1)$ find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$. Write the potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the triangle graph, when the current source $\boldsymbol{f}$ goes into node 1 and out from node 3.

8 Write down the 5 by 4 incidence matrix A for the square graph with two loops. Find one solution to $A\boldsymbol{x} = \boldsymbol{0}$ and two solutions to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{0}$.

9 Find two requirements on the b's for the five differences $x_2 - x_1, x_3 - x_1, x_3 - x_2, x_4 - x_2, x_4 - x_3$ to equal b_1, b_2, b_3, b_4, b_5. You have found Kirchhoff's _____ law around the two _____ in the graph.

10 Reduce A to its echelon form U. The three nonzero rows give the incidence matrix for what graph? You found one tree in the square graph—find the other seven trees.

11 Multiply matrices to find $A^{\mathrm{T}}A$ and guess how its entries come from the graph:

(a) The diagonal of $A^{\mathrm{T}}A$ tells how many _____ into each node.

(b) The off-diagonals -1 or 0 tell which pairs of nodes are _____.

12 Why is each statement true about $A^{\mathrm{T}}A$? *Answer for* $A^{\mathrm{T}}A$ *not* A.

(a) Its nullspace contains $(1, 1, 1, 1)$. Its rank is $n - 1$.

(b) It is positive semidefinite but not positive definite.

(c) Its four eigenvalues are real and their signs are _____.

13 With conductances $c_1 = c_2 = 2$ and $c_3 = c_4 = c_5 = 3$, multiply the matrices $A^{\mathrm{T}}CA$. Find a solution to $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f} = (1, 0, 0, -1)$. Write these potentials $\boldsymbol{x}$ and currents $\boldsymbol{y} = -CA\boldsymbol{x}$ on the nodes and edges of the square graph.

14 The matrix $A^{\mathrm{T}}CA$ is not invertible. What vectors $\boldsymbol{x}$ are in its nullspace? Why does $A^{\mathrm{T}}CA\boldsymbol{x} = \boldsymbol{f}$ have a solution if and only if $f_1 + f_2 + f_3 + f_4 = 0$?

15 A connected graph with 7 nodes and 7 edges has how many loops?

16 For the graph with 4 nodes, 6 edges, and 3 loops, add a new node. If you connect it to one old node, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$. If you connect it to two old nodes, Euler's formula becomes $(\quad) - (\quad) + (\quad) = 1$.

17 Suppose A is a 12 by 9 incidence matrix from a connected (but unknown) graph.

(a) How many columns of A are independent?

(b) What condition on $\boldsymbol{f}$ makes it possible to solve $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{f}$?

(c) The diagonal entries of $A^{\mathrm{T}}A$ give the number of edges into each node. What is the sum of those diagonal entries?

18 Why does a complete graph with $n = 6$ nodes have $m = 15$ edges? A tree connecting 6 nodes has _____ edges.

Reason: The components of $\boldsymbol{u}_0$ add to 1 when $\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \boldsymbol{u}_0 = 1$. This is true for each column of A by Property 2. Then by matrix multiplication it is true for $A\boldsymbol{u}_0$:

$$\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} A\boldsymbol{u}_0 = \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \boldsymbol{u}_0 = 1.$$

The same facts apply to $\boldsymbol{u}_2 = A\boldsymbol{u}_1$ and $\boldsymbol{u}_3 = A\boldsymbol{u}_2$. ***Every vector $\boldsymbol{u}_k = A^k\boldsymbol{u}_0$ is nonnegative with components adding to*** 1. These are "***probability vectors***." The limit $\boldsymbol{u}_\infty$ is also a probability vector—but we have to prove that there is a limit! The existence of a steady state will follow from **1** and **2** but not so quickly. We must show that $\lambda = 1$ is an eigenvalue of A, and we must estimate the other eigenvalues.

Example 1 The fraction of rental cars in Denver starts at $\frac{1}{50} = .02$. The fraction outside Denver is .98. Every month those fractions (which add to 1) are multiplied by the Markov matrix A:

$$A = \begin{bmatrix} .80 & .05 \\ .20 & .95 \end{bmatrix} \quad \text{leads to} \quad \boldsymbol{u}_1 = A\boldsymbol{u}_0 = A\begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .065 \\ .935 \end{bmatrix}.$$

That is a single step of a ***Markov chain***. In one month, the fraction of cars in Denver is up to .065. The chain of vectors is $\boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{u}_2, \ldots$, and each step multiplies by A:

$$\boldsymbol{u}_1 = A\boldsymbol{u}_0, \qquad \boldsymbol{u}_2 = A^2\boldsymbol{u}_0, \qquad \ldots \quad \text{produces} \quad \boldsymbol{u}_k = A^k\boldsymbol{u}_0.$$

All these vectors are nonnegative because A is nonnegative. Furthermore $.065 + .935 = 1.000$. Each vector $\boldsymbol{u}_k$ will have its components adding to 1. The vector $\boldsymbol{u}_2 = A\boldsymbol{u}_1$ is (.09875, .90125). The first component has grown from .02 to .065 to nearly .099. Cars are moving toward Denver. What happens in the long run?

This section involves powers of matrices. The understanding of A^k was our first and best application of diagonalization. Where A^k can be complicated, the diagonal matrix Λ^k is simple. The eigenvector matrix S connects them: A^k equals $S\Lambda^k S^{-1}$. The new application to Markov matrices follows up on this idea—to use the eigenvalues (in Λ) and the eigenvectors (in S). We will show that $\boldsymbol{u}_\infty$ is an eigenvector corresponding to $\lambda = 1$.

Since every column of A adds to 1, nothing is lost or gained. We are moving rental cars or populations, and no cars or people suddenly appear (or disappear). The fractions add to 1 and the matrix A keeps them that way. The question is how they are distributed after k time periods—which leads us to A^k.

Solution to Example 1 After k steps the fractions in and out of Denver are the components of $A^k\boldsymbol{u}_0$. To study the powers of A we diagonalize it. The eigenvalues are $\lambda = 1$ and $\lambda = .75$. The first eigenvector, with components adding to 1, is $\boldsymbol{x}_1 = (.2, .8)$:

$$|A - \lambda I| = \begin{vmatrix} .80 - \lambda & .05 \\ .20 & .95 - \lambda \end{vmatrix} = \lambda^2 - 1.75\lambda + .75 = (\lambda - 1)(\lambda - .75)$$

$$A\begin{bmatrix} .2 \\ .8 \end{bmatrix} = 1\begin{bmatrix} .2 \\ .8 \end{bmatrix} \quad \text{and} \quad A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = .75\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Those eigenvectors are $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. They are the columns of S. The starting vector $\boldsymbol{u}_0$ is a combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, in this case with coefficients 1 and .18:

$$\boldsymbol{u}_0 = \begin{bmatrix} .02 \\ .98 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + .18 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Now multiply by A to find $\boldsymbol{u}_1$. The eigenvectors are multiplied by $\lambda_1 = 1$ and $\lambda_2 = .75$:

$$\boldsymbol{u}_1 = 1 \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)(.18) \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Each time we multiply by A, another .75 multiplies the last vector. The eigenvector $\boldsymbol{x}_1$ is unchanged:

$$\boldsymbol{u}_k = A^k \boldsymbol{u}_0 = \begin{bmatrix} .2 \\ .8 \end{bmatrix} + (.75)^k (.18) \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

This equation reveals what happens. ***The eigenvector $\boldsymbol{x}_1$ with $\lambda = 1$ is the steady state*** $\boldsymbol{u}_\infty$. The other eigenvector $\boldsymbol{x}_2$ gradually disappears because $|\lambda| < 1$. The more steps we take, the closer we come to $\boldsymbol{u}_\infty = (.2, .8)$. In the limit, $\frac{2}{10}$ of the cars are in Denver and $\frac{8}{10}$ are outside. This is the pattern for Markov chains:

8A If A is a *positive* Markov matrix (entries $a_{ij} > 0$, each column adds to **1**), then $\lambda = 1$ is larger than any other eigenvalue. The eigenvector $\boldsymbol{x}_1$ is the ***steady state***:

$$u_k = x_1 + c_2(\lambda_2)^k x_2 + \cdots + c_n(\lambda_n)^k x_n \quad \textit{always approaches} \quad u_\infty = x_1.$$

Assume that the components of $\boldsymbol{u}_0$ add to 1. Then this is true of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots$. The key point is that *we approach a multiple of $\boldsymbol{x}_1$ from every starting vector $\boldsymbol{u}_0$*. If all cars start outside Denver, or all start inside, the limit is still $\boldsymbol{u}_\infty = \boldsymbol{x}_1 = (.2, .8)$.

The first point is to see that $\lambda = 1$ is an eigenvalue of A. *Reason*: Every column of $A - I$ adds to $1 - 1 = 0$. The rows of $A - I$ add up to the zero row. Those rows are linearly dependent, so $A - I$ is singular. Its determinant is zero and $\lambda = 1$ is an eigenvalue. Since the trace of A was 1.75, the other eigenvalue had to be $\lambda_2 = .75$.

The second point is that no eigenvalue can have $|\lambda| > 1$. With such an eigenvalue, the powers A^k would grow. But A^k is also a Markov matrix with nonnegative entries adding to 1—and that leaves no room to get large.

A lot of attention is paid to the possibility that another eigenvalue has $|\lambda| = 1$. Suppose the entries of A or any power A^k are all *positive*—zero is not allowed. In this "regular" case $\lambda = 1$ is strictly bigger than any other eigenvalue. When A and its powers have zero entries, another eigenvalue could be as large as $\lambda_1 = 1$.

Example 2 $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has no steady state because $\lambda_2 = -1$.

This matrix sends all cars from inside Denver to outside, and vice versa. The powers A^k alternate between A and I. The second eigenvector $\boldsymbol{x}_2 = (-1, 1)$ is multiplied by $\lambda_2 = -1$ at every step—and does not become smaller. With a regular Markov matrix, the powers A^k approach the rank one matrix that has the steady state $\boldsymbol{x}_1$ in every column.

Example 3 (**"Everybody moves"**) Start with three groups. At each time step, half of group 1 goes to group 2 and the other half goes to group 3. The other groups also *split in half and move*. If the starting populations are p_1, p_2, p_3, then after one step the new populations are

$$\boldsymbol{u}_1 = A\boldsymbol{u}_0 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p_2 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_3 \\ \frac{1}{2}p_1 + \frac{1}{2}p_2 \end{bmatrix}.$$

A is a Markov matrix. Nobody is born or lost. It is true that A contains zeros, which gave trouble in Example 2. But after two steps in this new example, the zeros disappear from A^2:

$$\boldsymbol{u}_2 = A^2\boldsymbol{u}_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}.$$

What is the steady state? The eigenvalues of A are $\lambda_1 = 1$ (because A is Markov) and $\lambda_2 = \lambda_3 = -\frac{1}{2}$. ***The eigenvector*** $\boldsymbol{x}_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ ***for*** $\lambda = 1$ ***will be the steady state.*** When three equal populations split in half and move, the final populations are again equal. When the populations start from $\boldsymbol{u}_0 = (8, 16, 32)$, the Markov chain approaches its steady state:

$$\boldsymbol{u}_0 = \begin{bmatrix} 8 \\ 16 \\ 32 \end{bmatrix} \qquad \boldsymbol{u}_1 = \begin{bmatrix} 24 \\ 20 \\ 12 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 16 \\ 18 \\ 22 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 20 \\ 19 \\ 17 \end{bmatrix}.$$

The step to $\boldsymbol{u}_4$ will split some people in half. This cannot be helped. The total population is $8+16+32 = 56$ (and later the total is still $20+19+17 = 56$). The steady state populations $\boldsymbol{u}_\infty$ are 56 times $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. You can see the three populations approaching, but never reaching, their final limits 56/3.

Linear Algebra in Economics: The Consumption Matrix

A long essay about linear algebra in economics would be out of place here. A short note about one matrix seems reasonable. The ***consumption matrix*** tells how much of each input goes into a unit of output. We have n industries like chemicals, food, and oil. To produce a unit of chemicals may require .2 units of chemicals, .3 units of food, and .4 units of oil. Those numbers go into row 1 of the consumption matrix A:

$$\begin{bmatrix} \text{chemical output} \\ \text{food output} \\ \text{oil output} \end{bmatrix} = \begin{bmatrix} .2 & .3 & .4 \\ .4 & .4 & .1 \\ .5 & .1 & .3 \end{bmatrix} \begin{bmatrix} \text{chemical input} \\ \text{food input} \\ \text{oil input} \end{bmatrix}.$$

Row 2 shows the inputs to produce food—a heavy use of chemicals and food, not so much oil. Row 3 of A shows the inputs consumed to refine a unit of oil. The real consumption matrix for the United States in 1958 contained 83 industries. The models in the 1990's are much larger and more precise. We chose a consumption matrix that has a convenient eigenvector.

Now comes the question: Can this economy meet demands y_1, y_2, y_3 for chemicals, food, and oil? To do that, the inputs p_1, p_2, p_3 will have to be higher—because part of $\boldsymbol{p}$ is consumed in producing $\boldsymbol{y}$. The input is $\boldsymbol{p}$ and the consumption is $A\boldsymbol{p}$, which leaves $\boldsymbol{p} - A\boldsymbol{p}$. This net production is what meets the demand $\boldsymbol{y}$:

Problem Find a vector $\boldsymbol{p}$ such that $\boldsymbol{p} - A\boldsymbol{p} = \boldsymbol{y}$ or $(I - A)\boldsymbol{p} = \boldsymbol{y}$ or $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$.

Apparently the linear algebra question is whether $I - A$ is invertible. But there is more to the problem. The demand vector $\boldsymbol{y}$ is nonnegative, and so is A. *The production levels in $\boldsymbol{p} = (I - A)^{-1}\boldsymbol{y}$ must also be nonnegative.* The real question is:

When is $(I - A)^{-1}$ a nonnegative matrix?

This is the test on $(I - A)^{-1}$ for a productive economy, which can meet any positive demand. If A is small compared to I, then $A\boldsymbol{p}$ is small compared to $\boldsymbol{p}$. There is plenty of output. If A is too large, then production consumes more than it yields. In this case the external demand $\boldsymbol{y}$ cannot be met.

"Small" or "large" is decided by the largest eigenvalue λ_1 of A (which is positive):

If $\lambda_1 > 1$ then $(I - A)^{-1}$ has negative entries
If $\lambda_1 = 1$ then $(I - A)^{-1}$ fails to exist
If $\lambda_1 < 1$ then $(I - A)^{-1}$ is nonnegative as desired.

The main point is that last one. The reasoning makes use of a nice formula for $(I - A)^{-1}$, which we give now. The most important infinite series in mathematics is the ***geometric series*** $1 + x + x^2 + \cdots$. This series adds up to $1/(1 - x)$ provided x is between -1 and 1. (When $x = 1$ the series is $1 + 1 + 1 + \cdots = \infty$. When $|x| \geq 1$ the terms x^n don't go to zero and the series cannot converge.) The nice formula for $(I - A)^{-1}$ is the ***geometric series of matrices***:

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots.$$

If you multiply this series by A, you get the same series S except for I. Therefore $S - AS = I$, which is $(I - A)S = I$. The series adds to $S = (I - A)^{-1}$ if it converges. And it converges if $|\lambda_{\max}| < 1$.

5 Every year 2% of young people become old and 3% of old people become dead. (No births.) Find the steady state for

$$\begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_{k+1} = \begin{bmatrix} .98 & .00 & 0 \\ .02 & .97 & 0 \\ .00 & .03 & 1 \end{bmatrix} \begin{bmatrix} \text{young} \\ \text{old} \\ \text{dead} \end{bmatrix}_k .$$

6 The sum of the components of $\boldsymbol{x}$ equals the sum of the components of $A\boldsymbol{x}$. If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\lambda \neq 1$, prove that the components of this non-steady eigenvector $\boldsymbol{x}$ add to zero.

7 Find the eigenvalues and eigenvectors of A. Factor A into $S\Lambda S^{-1}$:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}.$$

This was a MATLAB example in Chapter 1. There A^{16} was computed by squaring four times. What are the factors in $A^{16} = S\Lambda^{16}S^{-1}$?

8 Explain why the powers A^k in Problem 7 approach this matrix A^∞:

$$A^\infty = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Challenge problem: Which Markov matrices produce that steady state (.6, .4)?

9 This permutation matrix is also a Markov matrix:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The steady state eigenvector for $\lambda = 1$ is $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. This is *not* approached when $\boldsymbol{u}_0 = (0, 0, 0, 1)$. What are $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$? What are the four eigenvalues of P, which solve $\lambda^4 = 1$?

10 Prove that the square of a Markov matrix is also a Markov matrix.

11 If $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a Markov matrix, its eigenvalues are 1 and _____. The steady state eigenvector is $\boldsymbol{x}_1 =$ _____.

12 Complete the last row to make A a Markov matrix and find the steady state eigenvector:

$$A = \begin{bmatrix} .7 & .1 & .2 \\ .1 & .6 & .3 \\ \_ & \_ & \_ \end{bmatrix}.$$

When A is a symmetric Markov matrix, why is $\boldsymbol{x}_1 = (1, \ldots, 1)$ its steady state?

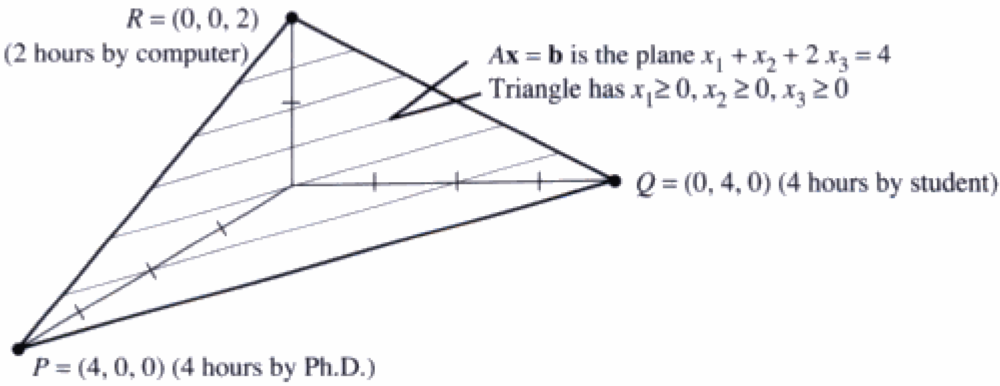


Figure 8.6 The triangle containing nonnegative solutions: $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$. The lowest cost solution $\boldsymbol{x}^*$ is one of the corners $\boldsymbol{P}$, $\boldsymbol{Q}$, or $\boldsymbol{R}$.

The first plane to touch the triangle has minimum cost C. *The point where it touches is the solution* $\boldsymbol{x}^*$. This touching point must be one of the corners $\boldsymbol{P}$ or $\boldsymbol{Q}$ or $\boldsymbol{R}$. A moving plane could not reach the inside of the triangle before it touches a corner! So check the cost $5x_1 + 3x_2 + 8x_3$ at each corner:

$$\boldsymbol{P} = (4, 0, 0) \text{ costs } 20 \qquad \boldsymbol{Q} = (0, 4, 0) \text{ costs } 12 \qquad \boldsymbol{R} = (0, 0, 2) \text{ costs } 16.$$

The winner is $\boldsymbol{Q}$. Then $\boldsymbol{x}^* = (0, 4, 0)$ solves the linear programming problem.

If the cost vector $\boldsymbol{c}$ is changed, the parallel planes are tilted. For small changes, $\boldsymbol{Q}$ is still the winner. For the cost $\boldsymbol{c} \cdot \boldsymbol{x} = 5x_1 + 4x_2 + 7x_3$, the optimum $\boldsymbol{x}^*$ moves to $\boldsymbol{R} = (0, 0, 2)$. The minimum cost is now $7 \cdot 2 = 14$.

Note 1 Some linear programs *maximize profit* instead of minimizing cost. The mathematics is almost the same. The parallel planes start with a large value of C, instead of a small value. They move toward the origin (instead of away), as C gets smaller. *The first touching point is still a corner.*

Note 2 The requirements $A\boldsymbol{x} = \boldsymbol{b}$ and $\boldsymbol{x} \geq 0$ could be impossible to satisfy. The equation $x_1 + x_2 + x_3 = -1$ cannot be solved with $\boldsymbol{x} \geq 0$. The feasible set is empty.

Note 3 It could also happen that the feasible set is *unbounded*. If I change the requirement to $x_1 + x_2 - 2x_3 = 4$, the large positive vector $(100, 100, 98)$ is now a candidate. So is the larger vector $(1000, 1000, 998)$. The plane $A\boldsymbol{x} = \boldsymbol{b}$ is no longer chopped off to a triangle. The two corners $\boldsymbol{P}$ and $\boldsymbol{Q}$ are still candidates for $\boldsymbol{x}^*$, but the third corner has moved to infinity.

Note 4 With an unbounded feasible set, the minimum cost could be $-\infty$ (*minus infinity*). Suppose the cost is $-x_1 - x_2 + x_3$. Then the vector $(100, 100, 98)$ costs $C = -102$. The vector $(1000, 1000, 998)$ costs $C = -1002$. We are being paid to include x_1 and x_2. Instead of paying a cost for those components. In realistic applications this will not happen. But it is theoretically possible that changes in A, $\boldsymbol{b}$, and $\boldsymbol{c}$ can produce unexpected triangles and costs.

Background to Linear Programming

This first problem is made up to fit the previous example. The unknowns x_1, x_2, x_3 represent hours of work by a Ph.D. and a student and a machine. The costs per hour are \$5, \$3, and \$8. (*I apologize for such low pay.*) The number of hours cannot be negative: $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$. The Ph.D. and the student get through one homework problem per hour—*the machine solves two problems in one hour*. In principle they can share out the homework, which has four problems to be solved: $x_1 + x_2 + 2x_3 = 4$.

The problem is to finish the four problems at minimum cost.

If all three are working, the job takes one hour: $x_1 = x_2 = x_3 = 1$. The cost is $5 + 3 + 8 = 16$. But certainly the Ph.D. should be put out of work by the student (who is just as fast and costs less—this problem is getting realistic). When the student works two hours and the machine works one, the cost is $6 + 8$ and all four problems get solved. We are on the edge $\boldsymbol{QR}$ of the triangle because the Ph.D. is unemployed: $x_1 = 0$. But the best point is at a corner—all work by student (at $\boldsymbol{Q}$) or all work by machine (at $\boldsymbol{R}$). In this example the student solves four problems in four hours for \$12—the minimum cost.

With only one equation in $A\boldsymbol{x} = \boldsymbol{b}$, the corner $(0, 4, 0)$ has only one nonzero component. When $A\boldsymbol{x} = \boldsymbol{b}$ has m equations, corners have m nonzeros. As in Chapter 3, $n - m$ free variables are set to zero. We solve $A\boldsymbol{x} = \boldsymbol{b}$ for the m basic variables (pivot variables). But unlike Chapter 3, we don't know which m variables to choose as basic. Our choice must minimize the cost.

The number of possible corners is the number of ways to choose m components out of n. This number "n choose m" is heavily involved in gambling and probability. With $n = 20$ unknowns and $m = 8$ equations (still small numbers), the "feasible set" can have $20!/8!12!$ corners. That number is $(20)(19)\cdots(13) = 5{,}079{,}110{,}400$.

Checking three corners for the minimum cost was fine. Checking five billion corners is not the way to go. The simplex method described below is much faster.

The Dual Problem In linear programming, problems come in pairs. There is a minimum problem and a maximum problem—the original and its "dual." The original problem was specified by a matrix A and two vectors $\boldsymbol{b}$ and $\boldsymbol{c}$. The dual problem has the same input, but A is transposed and $\boldsymbol{b}$ and $\boldsymbol{c}$ are switched. Here is the dual to our example:

> **A cheater offers to solve homework problems by looking up the answers.** The charge is y dollars per problem, or $4y$ altogether. (Note how $\boldsymbol{b} = 4$ has gone into the cost.) The cheater must be as cheap as the Ph.D. or student or machine: $y \leq 5$ and $y \leq 3$ and $2y \leq 8$. (Note how $\boldsymbol{c} = (5, 3, 8)$ has gone into inequality constraints). The cheater maximizes the income $4y$.

Dual Problem $\boxed{\textit{Maximize } b \cdot y \textit{ subject to } A^{\mathrm{T}} y \le c}$.

The maximum occurs when $y = 3$. The income is $4y = 12$. The maximum in the dual problem (\$12) equals the minimum in the original (\$12). This is always true:

Duality Theorem If either problem has a best vector (x^* or y^*) then so does the other. ***The minimum cost $c \cdot x^*$ equals the maximum income $b \cdot y^*$.***

Please note that I personally often look up the answers. It's not cheating.

This book started with a row picture and a column picture. The first "duality theorem" was about rank: The number of independent rows equals the number of independent columns. That theorem, like this one, was easy for small matrices. A proof that minimum cost = maximum income is in our text *Linear Algebra and Its Applications*. Here we establish the easy half of the theorem: ***The cheater's income cannot exceed the honest cost***:

$$\textit{If} \quad Ax = b, x \ge 0, A^{\mathrm{T}} y \le c \quad \textit{then} \quad b^{\mathrm{T}} y = (Ax)^{\mathrm{T}} y = x^{\mathrm{T}}(A^{\mathrm{T}} y) \le x^{\mathrm{T}} c.$$

The full duality theorem says that when $b^{\mathrm{T}} y$ reaches its maximum and $x^{\mathrm{T}} c$ reaches its minimum, they are equal: $b \cdot y^* = c \cdot x^*$.

The Simplex Method

Elimination is the workhorse for linear equations. The simplex method is the workhorse for linear inequalities. We cannot give the simplex method as much space as elimination—but the idea can be briefly described. *The simplex method goes from one corner to a neighboring corner of lower cost.* Eventually (and quite soon in practice) it reaches the corner of minimum cost. This is the solution x^*.

A ***corner*** is a vector $x \ge 0$ that satisfies the m equations $Ax = b$ with at most m positive components. The other $n - m$ components are *zero*. (Those are the free variables. Back substitution gives the basic variables. All variables must be nonnegative or x is a false corner.) For a *neighboring corner*, one zero component becomes positive and one positive component becomes zero.

The simplex method must decide which component "enters" by becoming positive, and which component "leaves" by becoming zero. That exchange is chosen so as to lower the total cost. This is one step of the simplex method.

Here is the overall plan. Look at each zero component at the current corner. If it changes from 0 to 1, the other nonzeros have to adjust to keep $Ax = b$. Find the new x by back substitution and compute the change in the total cost $c \cdot x$. This change what they did from 0 to 2π. They are "*periodic*." The distance between repetitions (the period) is 2π.

Remember: The list is infinite. The Fourier series is an infinite series. Just as we avoided the vector $\boldsymbol{v} = (1, 1, 1, \ldots)$ because its length is infinite, so we avoid a function like $\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots$. (*Note*: This is π times the famous delta function. It is an infinite "spike" above a single point. At $x = 0$ its height $\frac{1}{2}+1+1+\cdots$ is infinite. At all points inside $0 < x < 2\pi$ the series adds in some average way to zero.) The delta function has infinite length, and regretfully it is excluded from our space of functions.

Compute the length of a typical sum $f(x)$:

$$\begin{aligned}(f, f) &= \int_0^{2\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots)^2 \, dx \\ &= \int_0^{2\pi} (a_0^2 + a_1^2 \cos^2 x + b_1^2 \sin^2 x + a_2^2 \cos^2 2x + \cdots) \, dx \\ &= 2\pi a_0^2 + \pi(a_1^2 + b_1^2 + a_2^2 + \cdots). \end{aligned} \tag{6}$$

The step from line 1 to line 2 used orthogonality. All products like $\cos x \cos 2x$ and $\sin x \cos 3x$ integrate to give zero. Line 2 contains what is left—the integrals of each sine and cosine squared. Line 3 evaluates those integrals. Unfortunately the integral of 1^2 is 2π, when all other integrals give π. If we divide by their lengths, our functions become *orthonormal*:

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots \textbf{\textit{ is an orthonormal basis for our function space.}}$$

These are unit vectors. We could combine them with coefficients $A_0, A_1, B_1, A_2, \ldots$ to yield a function $F(x)$. Then the 2π and the π's drop out of the formula for length. Equation 6 becomes *function length = vector length*:

$$\|F\|^2 = (F, F) = A_0^2 + A_1^2 + B_1^2 + A_2^2 + \cdots. \tag{7}$$

Here is the important point, for $f(x)$ as well as $F(x)$. *The function has finite length exactly when the vector of coefficients has finite length.* The integral of $\left(F(x)\right)^2$ matches the sum of coefficients squared. Through Fourier series, we have a perfect match between function space and infinite-dimensional Hilbert space. On one side is the function, on the other side are its Fourier coefficients.

8B The function space contains $f(x)$ exactly when the Hilbert space contains the vector $\boldsymbol{v} = (a_0, a_1, b_1, \ldots)$ of Fourier coefficients. Both $f(x)$ and $\boldsymbol{v}$ have finite length.

Example 3 Suppose $f(x)$ is a "square wave," equal to -1 for negative x and $+1$ for positive x. That looks like a step function, not a wave. But remember that $f(x)$

must repeat after each interval of length 2π. We should have said

$$f(x) = -1 \text{ for } \quad -\pi < x < 0 + 1 \text{ for } \quad 0 < x < \pi.$$

The wave goes back to -1 for $\pi < x < 2\pi$. It is an odd function like the sines, and all its cosine coefficients are zero. We will find its Fourier series, containing only sines:

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]. \tag{8}$$

This square wave has length $\sqrt{2\pi}$, because at every point $\left(f(x)\right)^2$ is $(-1)^2$ or $(+1)^2$:

$$\|f\|^2 = \int_0^{2\pi} \left(f(x)\right)^2 dx = \int_0^{2\pi} 1\, dx = 2\pi.$$

At $x = 0$ the sines are zero and the Fourier series 8 gives zero. This is half way up the jump from -1 to $+1$. The Fourier series is also interesting when $x = \frac{\pi}{2}$. At this point the square wave equals 1, and the sines in equation 8 alternate between $+1$ and -1:

$$1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right). \tag{9}$$

Multiply through by π to find a magical formula $4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots)$ for that famous number.

The Fourier Coefficients

How do we find the a's and b's which multiply the cosines and sines? For a given function $f(x)$, we are asking for its Fourier coefficients:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \cdots.$$

Here is the way to find a_1. ***Multiply both sides by*** $\cos x$. ***Then integrate from*** 0 ***to*** 2π. The key is orthogonality! All integrals on the right side are zero, except the integral of $a_1 \cos^2 x$:

$$\int_0^{2\pi} f(x) \cos x\, dx = \int_0^{2\pi} a_1 \cos^2 x\, dx = \pi a_1. \tag{10}$$

Divide by π and you have a_1. To find any other a_k, multiply the Fourier series by $\cos kx$. Integrate from 0 to 2π. Use orthogonality, so only the integral of $a_k \cos^2 kx$ is left. That integral is πa_k, and divide by π:

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos kx\, dx \quad \text{and similarly} \quad b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin kx\, dx. \tag{11}$$

The exception is a_0. This time we multiply by $\cos 0x = 1$. The integral of 1 is 2π:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot 1\,dx = \text{ average value of } f(x). \tag{12}$$

I used those formulas to find the coefficients in 8 for the square wave. The integral of $f(x)\cos kx$ was zero. The integral of $f(x)\sin kx$ was $4/k$ for odd k.

The point to emphasize is how this infinite-dimensional case is so much like the n-dimensional case. Suppose the nonzero vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ are orthogonal. We want to write the vector $\boldsymbol{b}$ as a combination of those $\boldsymbol{v}$'s:

$$\boldsymbol{b} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_n\boldsymbol{v}_n. \tag{13}$$

Multiply both sides by $\boldsymbol{v}_1^{\mathrm{T}}$. Use orthogonality, so $\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_2 = 0$. Only the c_1 term is left:

$$\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{b} = c_1\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_1 + 0 + \cdots + 0. \quad \text{Therefore } c_1 = \frac{\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_1}. \tag{14}$$

The denominator $\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{v}_1$ is the length squared, like π in equation (11). The numerator $\boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{b}$ is the inner product like $\int f(x)\cos kx\,dx$. ***Coefficients are easy to find when the basis vectors are orthogonal.*** We are just doing one-dimensional projections, to find the components along each basis vector.

The formulas are even better when the vectors are orthonormal. Then we have unit vectors. The denominators $\boldsymbol{v}_k^{\mathrm{T}}\boldsymbol{v}_k$ are all 1. In this orthonormal case,

$$c_1 = \boldsymbol{v}_1^{\mathrm{T}}\boldsymbol{b} \quad \text{and} \quad c_2 = \boldsymbol{v}_2^{\mathrm{T}}\boldsymbol{b} \quad \text{and} \quad c_n = \boldsymbol{v}_n^{\mathrm{T}}\boldsymbol{b}. \tag{15}$$

You know this in another form. The equation for the c's is

$$c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = \boldsymbol{b} \quad \text{or} \quad \begin{bmatrix} & & \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_n \\ & & \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{b}.$$

This is an orthogonal matrix Q! Its inverse is Q^{T}. That gives the c's in (15):

$$Q\boldsymbol{c} = \boldsymbol{b} \quad \text{yields} \quad \boldsymbol{c} = Q^{\mathrm{T}}\boldsymbol{b}. \quad \text{Row by row this is } c_i = \boldsymbol{v}_i^{\mathrm{T}}\boldsymbol{b}.$$

Fourier series is like having a matrix with infinitely many orthogonal columns. Those columns are the basis functions $1, \cos x, \sin x, \dots$. After dividing by their lengths we have an "infinite orthogonal matrix." Its inverse is its transpose. The formulas for the Fourier coefficients are like (15) when we have unit vectors and like (14) when we don't. Orthogonality is what reduces an infinite series to one single term.

Problem Set 8.5

1 Integrate the trig identity $2\cos jx \cos kx = \cos(j+k)x + \cos(j-k)x$ to show that $\cos jx$ is orthogonal to $\cos kx$, provided $j \neq k$. What is the result when $j = k$?

2 Show that the three functions $1, x$, and $x^2 - \frac{1}{3}$ are orthogonal, when the integration is from $x = -1$ to $x = 1$. Write $f(x) = 2x^2$ as a combination of those orthogonal functions.

3 Find a vector $(w_1, w_2, w_3, \ldots)$ that is orthogonal to $\boldsymbol{v} = (1, \frac{1}{2}, \frac{1}{4}, \ldots)$. Compute its length $\|\boldsymbol{w}\|$.

4 The first three *Legendre polynomials* are $1, x$, and $x^2 - \frac{1}{3}$. Choose the number c so that the fourth polynomial $x^3 - cx$ is orthogonal to the first three. The integrals still go from -1 to 1.

5 For the square wave $f(x)$ in Example 3, show that

$$\int_0^{2\pi} f(x)\cos x \, dx = 0 \quad \int_0^{2\pi} f(x)\sin x \, dx = 4 \quad \int_0^{2\pi} f(x)\sin 2x \, dx = 0.$$

Which Fourier coefficients come from those integrals?

6 The square wave has $\|f\|^2 = 2\pi$. This equals what remarkable sum by equation 6?

7 Graph the square wave. Then graph by hand the sum of two sine terms in its series, or graph by machine the sum of two, three, and four terms.

8 Find the lengths of these vectors in Hilbert space:

(a) $\boldsymbol{v} = \left(\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \ldots\right)$

(b) $\boldsymbol{v} = (1, a, a^2, \ldots)$

(c) $f(x) = 1 + \sin x$.

9 Compute the Fourier coefficients a_k and b_k for $f(x)$ defined from 0 to 2π:

(a) $f(x) = 1$ for $0 \le x \le \pi$, $f(x) = 0$ for $\pi < x < 2\pi$

(b) $f(x) = x$.

10 When $f(x)$ has period 2π, why is its integral from $-\pi$ to π the same as from 0 to 2π? If $f(x)$ is an *odd* function, $f(-x) = -f(x)$, show that $\int_0^{2\pi} f(x)\,dx$ is zero.

11 From trig identities find the only two terms in the Fourier series for $f(x)$:

(a) $f(x) = \cos^2 x$

(b) $f(x) = \cos\left(x + \frac{\pi}{3}\right)$

12 The functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$ are a basis for Hilbert space. Write the derivatives of those first five functions as combinations of the same five functions. What is the 5 by 5 "differentiation matrix" for these functions?

13 Write the complete solution to $dy/dx = \cos x$ as a particular solution plus any solution to $dy/dx = 0$.

Important: S is not cI. We keep the 1 in the lower corner. Then $[x, y, 1]$ times S is the correct answer in homogeneous coordinates. The origin stays in position because $[0\ 0\ 1]S = [0\ 0\ 1]$.

If we change that 1 to c, the result is strange. ***The point*** (cx, cy, cz, c) ***is the same as*** $(x, y, z, 1)$. The special property of homogeneous coordinates is that *multiplying by cI does not move the point.* The origin in $\mathbf{R}^3$ has homogeneous coordinates $(0, 0, 0, 1)$ and $(0, 0, 0, c)$ for every nonzero c. This is the idea behind the word "homogeneous."

Scaling can be different in different directions. To fit a full-page picture onto a half-page, scale the y direction by $\frac{1}{2}$. To create a margin, scale the x direction by $\frac{3}{4}$. The graphics matrix is diagonal but not 2 by 2. It is 3 by 3 to rescale a plane and 4 by 4 to rescale a space:

$$\textbf{\textit{Scaling matrices}} \quad S = \begin{bmatrix} \frac{3}{4} & & \\ & \frac{1}{2} & \\ & & 1 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & c_3 & \\ & & & 1 \end{bmatrix}.$$

That last matrix S rescales the x, y, z directions by positive numbers c_1, c_2, c_3. The point at the origin doesn't move, because $\begin{bmatrix}0\ 0\ 0\ 1\end{bmatrix} S = \begin{bmatrix}0\ 0\ 0\ 1\end{bmatrix}$.

Summary The scaling matrix S is the same size as the translation matrix T. They can be multiplied. To translate and then rescale, multiply $\boldsymbol{v}TS$. To rescale and then translate, multiply $\boldsymbol{v}ST$. (Are those different? *Yes.*) The extra column in all these matrices leaves the extra 1 at the end of every vector.

The point (x, y, z) in $\mathbf{R}^3$ has homogeneous coordinates $(x, y, z, 1)$ in $\mathbf{P}^3$. This "projective space" is not the same as $\mathbf{R}^4$. It is still three-dimensional. To achieve such a thing, (cx, cy, cz, c) is the same point as $(x, y, z, 1)$. Those points of $\mathbf{P}^3$ are really lines through the origin in $\mathbf{R}^4$.

Computer graphics uses ***affine*** transformations, *linear plus shift.* An affine transformation T is executed on $\mathbf{P}^3$ by a 4 by 4 matrix with a special fourth column:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix} = \begin{bmatrix} T(1,0,0) & 0 \\ T(0,1,0) & 0 \\ T(0,0,1) & 0 \\ T(0,0,0) & 1 \end{bmatrix}.$$

The usual 3 by 3 matrix tells us three outputs, this tells four. The usual outputs come from the inputs $(1, 0, 0)$ and $(0, 1, 0)$ and $(0, 0, 1)$. When the transformation is linear, three outputs reveal everything. When the transformation is affine, the matrix also contains the output from $(0, 0, 0)$.Then we know the shift.

3. Rotation A rotation in $\mathbf{R}^2$ or $\mathbf{R}^3$ is achieved by an orthogonal matrix Q. The determinant is $+1$. (With determinant -1 we get an extra reflection through a mirror.) Include the extra column when you use homogeneous coordinates!

$$\textbf{\textit{Plane rotation}} \quad Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{becomes} \quad R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix rotates the plane around the origin. ***How would we rotate around a different point*** $(4,5)$? The answer brings out the beauty of homogeneous coordinates. ***Translate*** $(4,5)$ ***to*** $(0,0)$***, then rotate by*** θ***, then translate*** $(0,0)$ ***back to*** $(4,5)$:

$$\boldsymbol{v}\,T_-RT_+ = \begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -5 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}.$$

I won't multiply. The point is to apply the matrices one at a time: $\boldsymbol{v}$ translates to $\boldsymbol{v}T_-$, then rotates to $\boldsymbol{v}T_-R$, and translates back to $\boldsymbol{v}T_-RT_+$. Because each point $\begin{bmatrix} x & y & 1 \end{bmatrix}$ is a row vector, T_- acts first. The center of rotation $(4,5)$—otherwise known as $(4,5,1)$—moves first to $(0,0,1)$. Rotation doesn't change it. Then T_+ moves it back to $(4,5,1)$. All as it should be. The point $(4,6,1)$ moves to $(0,1,1)$, then turns by θ and moves back.

In three dimensions, every rotation Q turns around an axis. The axis doesn't move—it is a line of eigenvectors with $\lambda = 1$. Suppose the axis is in the z direction. The 1 in Q is to leave the z axis alone, the extra 1 in R is to leave the origin alone:

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} & & & 0 \\ & Q & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now suppose the rotation is around the unit vector $\boldsymbol{a} = (a_1, a_2, a_3)$. With this axis $\boldsymbol{a}$, the rotation matrix Q which fits into R has three parts:

$$Q = (\cos\theta)I + (1-\cos\theta)\begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}. \quad (1)$$

The axis doesn't move because $\boldsymbol{a}Q = \boldsymbol{a}$. When $\boldsymbol{a} = (0,0,1)$ is in the z direction, this Q becomes the previous Q—for rotation around the z axis.

The linear transformation Q always goes in the upper left block of R. Below it we see zeros, because rotation leaves the origin in place. When those are not zeros, the transformation is affine and the origin moves.

4. Projection In a linear algebra course, most planes go through the origin. In real life, most don't. A plane through the origin is a vector space. The other planes are affine spaces, sometimes called "flats." An affine space is what comes from translating a vector space.

We want to project three-dimensional vectors onto planes. Start with a plane through the origin, whose unit normal vector is $\boldsymbol{n}$. (We will keep $\boldsymbol{n}$ as a column vector.) The vectors in the plane satisfy $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{v} = 0$. ***The usual projection onto the plane is the matrix*** $\boldsymbol{I - nn^{\mathrm{T}}}$. To project a vector, multiply by this matrix. The vector $\boldsymbol{n}$ is projected to zero, and the in-plane vectors $\boldsymbol{v}$ are projected onto themselves:

$$(I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{n}) = \boldsymbol{0} \quad \text{and} \quad (I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}})\boldsymbol{v} = \boldsymbol{v} - \boldsymbol{n}(\boldsymbol{n}^{\mathrm{T}}\boldsymbol{v}) = \boldsymbol{v}.$$

9

NUMERICAL LINEAR ALGEBRA

GAUSSIAN ELIMINATION IN PRACTICE ■ 9.1

Numerical linear algebra is a struggle for *quick* solutions and also *accurate* solutions. We need efficiency but we have to avoid instability. In Gaussian elimination, the main freedom (always available) is to exchange equations. This section explains when to exchange rows for the sake of speed, and when to do it for the sake of accuracy.

The key to accuracy is to avoid unnecessarily large numbers. Often that requires us to avoid small numbers! A small pivot generally means large multipliers (since we divide by the pivot). Also, a small pivot now means a large pivot later. The product of the pivots is a fixed number (except for its sign). That number is the determinant.

A good plan is to choose the *largest candidate* in each new column as the pivot. This is called "***partial pivoting***." The competitors are in the pivot position and below. We will see why this strategy is built into computer programs.

Other row exchanges are done to save elimination steps. In practice, most large matrices have only a small percentage of nonzero entries. The user probably knows their location. Elimination is generally fastest when the equations are ordered to put those nonzeros close to the diagonal. Then the matrix is as "banded" as possible.

New questions arise for machines with many processors in parallel. Now the problem is communication—to send processors the data they need, when they need it. This is a major research area. The brief comments in this section will try to introduce you to thinking in parallel.

Section 9.2 is about instability that can't be avoided. It is built into the problem, and this sensitivity is measured by the "***condition number***." Then Section 9.3 describes how to solve $A\boldsymbol{x} = \boldsymbol{b}$ by ***iterations***. Instead of direct elimination, the computer solves an easier equation many times. Each answer $\boldsymbol{x}_k$ goes back into the same equation to find the next guess $\boldsymbol{x}_{k+1}$. For good iterations, the $\boldsymbol{x}_k$ converge quickly to $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

Roundoff Error and Partial Pivoting

Up to now, any pivot (nonzero of course) was accepted. In practice a small pivot is dangerous. A catastrophe can occur when numbers of different sizes are added. Computers keep a fixed number of significant digits (say three decimals, for a very weak machine). The sum $10{,}000 + 1$ is rounded off to $10{,}000$. The "1" is completely lost. Watch how that changes the solution to this problem:

$$\begin{aligned} .0001u + v &= 1 \\ -u + v &= 0 \end{aligned} \qquad \text{starts with coefficient matrix} \qquad A = \begin{bmatrix} .0001 & 1 \\ -1 & 1 \end{bmatrix}.$$

If we accept .0001 as the pivot, elimination adds 10,000 times row 1 to row 2. Roundoff leaves

$$10{,}000v = 10{,}000 \qquad \text{instead of} \qquad 10{,}001v = 10{,}000.$$

The computed answer $v = 1$ is near the true $v = .9999$. But then back substitution leads to

$$.0001\,u + 1 = 1 \qquad \text{instead of} \qquad .0001\,u + .9999 = 1.$$

The first equation gives $u = 0$. The correct answer (look at the second equation) is $u = 1.000$. By losing the "1" in the matrix, we have lost the solution. ***The change from* 10,001 *to* 10,000 *has changed the answer from u* = 1 *to u* = 0** (100% error!).

If we exchange rows, even this weak computer finds an answer that is correct to three places:

$$\begin{aligned} -u + v &= 0 \\ .0001u + v &= 1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} -u + v &= 0 \\ v &= 1 \end{aligned} \quad \longrightarrow \quad \begin{aligned} u &= 1 \\ v &= 1. \end{aligned}$$

The original pivots were .0001 and 10,000—badly scaled. After a row exchange the exact pivots are -1 and 1.0001—well scaled. The computed pivots -1 and 1 come close to the exact values. Small pivots bring numerical instability, and the remedy is ***partial pivoting***. The kth pivot is decided when we reach and search column k:

Choose the largest number in row k or below. Exchange its row with row k.

The strategy of *complete pivoting* looks also in later columns for the largest pivot. It exchanges columns as well as rows. This expense is seldom justified, and all major codes use partial pivoting. Multiplying a row or column by a scaling constant can also be worthwhile. *If the first equation above is* $u + 10{,}000v = 10{,}000$ *and we don't rescale, then* 1 *is the pivot but we are in trouble again.*

For positive definite matrices, row exchanges are *not* required. It is safe to accept the pivots as they appear. Small pivots can occur, but the matrix is not improved by row exchanges. When its condition number is high, the problem is in the matrix and not in the order of elimination steps. In this case the output is unavoidably sensitive to the input.

The reader now understands how a computer actually solves $Ax = b$—***by elimination with partial pivoting***. Compared with the theoretical description—***find*** A^{-1} ***and multiply*** $A^{-1}b$—the details took time. But in computer time, elimination is much faster. I believe this algorithm is also the best approach to the algebra of row spaces and nullspaces.

Operation Counts: Full Matrices and Band Matrices

Here is a practical question about cost. *How many separate operations are needed to solve $Ax = b$ by elimination*? This decides how large a problem we can afford.

Look first at A, which changes gradually into U. When a multiple of row 1 is subtracted from row 2, we do n operations. The first is a division by the pivot, to find the multiplier ℓ. For the other $n-1$ entries along the row, the operation is a "multiply-subtract." For convenience, we count this as a single operation. If you regard multiplying by ℓ and subtracting from the existing entry as two separate operations, multiply all our counts by 2.

The matrix A is n by n. The operation count applies to all $n-1$ rows below the first. Thus it requires n times $n-1$ operations, or $n^2 - n$, to produce zeros below the first pivot. *Check*: *All n^2 entries are changed, except the n entries in the first row.*

When elimination is down to k equations, the rows are shorter. We need only $k^2 - k$ operations (instead of $n^2 - n$) to clear out the column below the pivot. This is true for $1 \le k \le n$. The last step requires no operations ($1^2 - 1 = 0$), since the pivot is set and forward elimination is complete. The total count to reach U is the sum of $k^2 - k$ over all values of k from 1 to n:

$$(1^2 + \ldots + n^2) - (1 + \ldots + n) = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{n^3 - n}{3}.$$

Those are known formulas for the sum of the first n numbers and the sum of the first n squares. Substituting $n = 1$ into $n^3 - n$ gives zero. Substituting $n = 100$ gives a million minus a hundred—then divide by 3. (That translates into one second on a workstation.) We will ignore the last term n in comparison with the larger term n^3, to reach our main conclusion:

The operation count for forward elimination (A to U) is $\frac{1}{3}n^3$.

That means $\frac{1}{3}n^3$ multiplications and $\frac{1}{3}n^3$ subtractions. Doubling n increases this cost by eight (because n is cubed). 100 equations are OK, 1000 are expensive, 10000 are impossible. We need a faster computer or a lot of zeros or a new idea.

On the right side of the equations, the steps go much faster. We operate on single numbers, not whole rows. ***Each right side needs exactly n^2 operations.*** Remember that we solve two triangular systems, $Lc = b$ forward and $Ux = c$ backward. In back substitution, the last unknown needs only division by the last pivot. The equation above

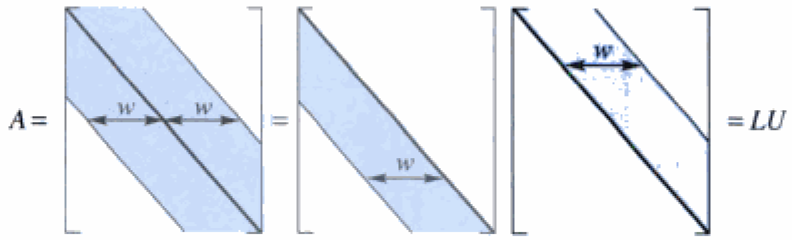


Figure 9.1 $A = LU$ for a band matrix. Good zeros in A stay zero in L and U.

it needs two operations—substituting x_n and dividing by *its* pivot. The kth step needs k operations, and the total for back substitution is

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \approx \tfrac{1}{2}n^2 \quad \text{operations.}$$

The forward part is similar. *The n^2 total exactly equals the count for multiplying $A^{-1}b$!* This leaves Gaussian elimination with two big advantages over $A^{-1}b$:

1 **Elimination requires $\frac{1}{3}n^3$ operations compared to n^3 for A^{-1}.**

2 **If A is *banded* so are L and U. But A^{-1} is full of nonzeros.**

Band Matrices

These counts are improved when A has "*good zeros*." A good zero is an entry that remains zero in L and U. The most important good zeros are at the beginning of a row. No elimination steps are required (the multipliers are zero). So we also find those same good zeros in L. That is especially clear for this *tridiagonal matrix* A:

$$\begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}.$$

Rows 3 and 4 of A begin with zeros. No multiplier is needed, so L has the same zeros. Also rows 1 and 2 *end* with zeros. When a multiple of row 1 is subtracted from row 2, no calculation is required beyond the second column. The rows are short. They stay short! Figure 9.1 shows how a band matrix A has band factors L and U.

These zeros lead to a complete change in the operation count, for "half-bandwidth" w:

$$\textbf{\textit{A band matrix has } } a_{ij} = 0 \textbf{\textit{ when }} |i - j| > w.$$

Thus $w = 1$ for a diagonal matrix and $w = 2$ for a tridiagonal matrix. The length of the pivot row is at most w. There are no more than $w - 1$ nonzeros below any pivot.

Each stage of elimination is complete after $w(w-1)$ operations, and *the band structure survives*. There are n columns to clear out. Therefore:

Forward elimination on a band matrix needs less than $w^2 n$ operations.

For a band matrix, the count is proportional to n instead of n^3. It is also proportional to w^2. A full matrix has $w = n$ and we are back to n^3. For a closer count, remember that the bandwidth drops below w in the lower right corner (not enough space). The exact count to find L and U is

$$\frac{w(w-1)(3n-2w+1)}{3} \quad \text{for a band matrix}$$

$$\frac{n(n-1)(n+1)}{3} = \frac{n^3-n}{3} \quad \text{when} \quad w = n.$$

On the right side, to find $\boldsymbol{x}$ from $\boldsymbol{b}$, the cost is about $2wn$ (compared to the usual n^2). *Main point*: *For a band matrix the operation counts are* ***proportional to*** n. This is extremely fast. A tridiagonal matrix of order 10,000 is very cheap, provided we don't compute A^{-1}. That inverse matrix has no zeros at all:

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{has} \quad A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We are actually worse off knowing A^{-1} than knowing L and U. Multiplication by A^{-1} needs the full n^2 steps. Solving $L\boldsymbol{c} = \boldsymbol{b}$ and $U\boldsymbol{x} = \boldsymbol{c}$ needs only $2wn$. Here that means $4n$. A band structure is very common in practice, when the matrix reflects connections between near neighbors. We see $a_{13} = 0$ and $a_{14} = 0$ because 1 is not a neighbor of 3 and 4.

We close with two more operation counts:

1 A^{-1} costs n^3 steps. **2** QR costs $\frac{2}{3}n^3$ steps.

1 Start with $AA^{-1} = I$. The jth column of A^{-1} solves $A\boldsymbol{x}_j = j$th column of I. Normally each of those n right sides needs n^2 operations, making n^3 in all. The left side costs $\frac{1}{3}n^3$ as usual. (This is a one-time cost! L and U are not repeated for each new right side.) This count gives $\frac{4}{3}n^3$, but we can get down to n^3.

The special saving for the jth column of I comes from its first $j-1$ zeros. No work is required on the right side until elimination reaches row j. The forward cost is $\frac{1}{2}(n-j)^2$ instead of $\frac{1}{2}n^2$. Summing over j, the total for forward elimination on the

n right sides is $\frac{1}{6}n^3$. Then the final count of multiplications for A^{-1} (with an equal number of subtractions) is n^3 if we actually want the inverse matrix:

$$\frac{n^3}{3}\ (L \text{ and } U) + \frac{n^3}{6}\ (\text{forward}) \ + n\left(\frac{n^2}{2}\right)\ (\text{back substitutions}) \ = n^3. \qquad (1)$$

2 The Gram-Schmidt process works with columns instead of rows—that is not so important to the count. The key difference from elimination is that *the multiplier is decided by a dot product*. So it takes n operations to find the multiplier, where elimination just divides by the pivot. Then there are n "multiply-subtract" operations to remove from column 2 its projection along column 1. (See Section 4.4 and Problem 4.4.28 for the sequence of projections.) The cost for Gram-Schmidt is $2n$ where for elimination it is n. This factor 2 is the price of orthogonality. We are changing a dot product to zero instead of changing an entry to zero.

Caution To judge a numerical algorithm, it is **not enough** to count the operations. Beyond "flop counting" is a study of stability and the flow of data. Van Loan emphasizes the three levels of linear algebra: linear combinations $c\boldsymbol{u}+\boldsymbol{v}$ (level 1), matrix-vector $A\boldsymbol{u}+\boldsymbol{v}$ (level 2), and matrix-matrix $AB+C$ (level 3). For parallel computing, level 3 is best. AB uses $2n^3$ flops (additions and multiplications) and only $2n^2$ data—a good ratio of work to communication overhead. Solving $UX = B$ for matrices is better than $U\boldsymbol{x} = \boldsymbol{b}$ for vectors. Gauss-Jordan partly wins after all!

Plane Rotations

There are two ways to reach the important factorization $A = QR$. One way works to find Q, the other way works to find R. Gram-Schmidt chose the first way, and the columns of A were orthogonalized to go into Q. (Then R was an afterthought. It was upper triangular because of the order of Gram-Schmidt steps.) Now we look at a method that starts with A and aims directly at R.

Elimination gives $A = LU$, orthogonalization gives $A = QR$. What is the difference, when R and U are both upper triangular? For elimination L is a product of E's—with 1's on the diagonal and the multiplier ℓ_{ij} below. *QR uses orthogonal matrices.* The E's are not allowed. We don't want a triangular L, we want an orthogonal Q.

There are two simple orthogonal matrices to take the place of the E's. The ***reflection matrices*** $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ are named after Householder. The ***plane rotation matrices*** are named after Givens. The matrix that rotates the xy plane by θ, and leaves the z direction alone, is Q_{21}:

$$\textbf{Givens Rotation} \qquad Q_{21} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5 For back substitution with a band matrix (width w), show that the number of multiplications to solve $U\boldsymbol{x} = \boldsymbol{c}$ is approximately wn.

6 If you know L and U and Q and R, is it faster to solve $LU\boldsymbol{x} = \boldsymbol{b}$ or $QR\boldsymbol{x} = \boldsymbol{b}$?

7 Show that the number of multiplications to invert an upper triangular n by n matrix is about $\frac{1}{6}n^3$. Use back substitution on the columns of I, upward from 1's.

8 Choosing the largest available pivot in each column (partial pivoting), factor each A into $PA = LU$:

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

9 Put 1's on the three central diagonals of a 4 by 4 tridiagonal matrix. Find the cofactors of the six zero entries. Those entries are nonzero in A^{-1}.

10 (Suggested by C. Van Loan.) Find the LU factorization of $A = \left[\begin{smallmatrix} \varepsilon & 1 \\ 1 & 1 \end{smallmatrix}\right]$. On your computer solve by elimination when $\varepsilon = 10^{-3}, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15}$:

$$\begin{bmatrix} \varepsilon & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon \\ 2 \end{bmatrix}.$$

The true $\boldsymbol{x}$ is $(1, 1)$. Make a table to show the error for each ε. Exchange the two equations and solve again—the errors should almost disappear.

11 Choose $\sin\theta$ and $\cos\theta$ to triangularize A, and find R:

$$Q_{21}A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} = R.$$

12 Choose $\sin\theta$ and $\cos\theta$ to make $Q_{21}AQ_{21}^{-1}$ triangular (same A). What are the eigenvalues?

13 When A is multiplied by Q_{ij}, which of the n^2 entries of A are changed? When $Q_{ij}A$ is multiplied on the right by Q_{ij}^{-1}, which entries are changed now?

14 How many multiplications and how many additions are used to compute $Q_{ij}A$? (A careful organization of the whole sequence of rotations gives $\frac{2}{3}n^3$ multiplications and $\frac{2}{3}n^3$ additions—the same as for QR by reflectors and twice as many as for LU.)

15 (**Turning a robot hand**) The robot produces any 3 by 3 rotation A from plane rotations around the x, y, z axes. Then $Q_{32}Q_{31}Q_{21}A = R$, where A is orthogonal so R is I! The three robot turns are in $A = Q_{21}^{-1}Q_{31}^{-1}Q_{32}^{-1}$. The three angles

Example 2 The norm of a diagonal matrix is its largest entry (using absolute values):

$$\text{The norm of} \quad A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \text{is} \quad \|A\| = 3.$$

The ratio is $\|Ax\| = \sqrt{2^2x_1^2 + 3^2x_2^2}$ divided by $\|x\| = \sqrt{x_1^2 + x_2^2}$. That is a maximum when $x_1 = 0$ and $x_2 = 1$. This vector $x = (0, 1)$ is an eigenvector with $Ax = (0, 3)$. The eigenvalue is 3. This is the largest eigenvalue of A and it equals the norm.

For a positive definite symmetric matrix the norm is $\|A\| = \lambda_{\max}$.

Choose x to be the eigenvector with maximum eigenvalue: $Ax = \lambda_{\max}x$. Then $\|Ax\|/\|x\|$ equals $\lambda_{\max}$. The point is that no other vector x can make the ratio larger. The matrix is $A = Q\Lambda Q^{\mathrm{T}}$, and the orthogonal matrices Q and Q^{T} leave lengths unchanged. So the ratio to maximize is really $\|\Lambda x\|/\|x\|$. The norm $\lambda_{\max}$ is the largest eigenvalue in the diagonal matrix Λ.

Symmetric matrices Suppose A is symmetric but not positive definite—some eigenvalues of A are negative or zero. Then the norm $\|A\|$ is the largest of $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$. We take absolute values of the λ's, because the norm is only concerned with length. For an eigenvector we have $\|Ax\| = \|\lambda x\|$, which is $|\lambda|$ times $\|x\|$. Dividing by $\|x\|$ leaves $|\lambda|$. The x that gives the maximum ratio is the eigenvector for the maximum $|\lambda|$.

Unsymmetric matrices If A is not symmetric, its eigenvalues may not measure its true size. The norm can be large when the eigenvalues are small. *Thus the norm is generally larger than* $|\lambda|_{\max}$. A very unsymmetric example has $\lambda_1 = \lambda_2 = 0$ but its norm is not zero:

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \qquad \text{has norm} \qquad \|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = 2.$$

The vector $x = (0, 1)$ gives $Ax = (2, 0)$. The ratio of lengths is $2/1$. This is the maximum ratio $\|A\|$, even though x is not an eigenvector.

It is the symmetric matrix $A^{\mathrm{T}}A$, not the unsymmetric A, that has $x = (0, 1)$ as its eigenvector. The norm is really decided by *the largest eigenvalue of* $A^{\mathrm{T}}A$, as we now prove.

9A ***The norm of*** A (symmetric or not) ***is the square root of*** $\lambda_{\max}(A^{\mathrm{T}}A)$***:***

$$\|A\|^2 = \max_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \max_{x \neq 0} \frac{x^{\mathrm{T}}A^{\mathrm{T}}Ax}{x^{\mathrm{T}}x} = \lambda_{\max}(A^{\mathrm{T}}A). \tag{4}$$

Proof Choose $\boldsymbol{x}$ to be the eigenvector of $A^{\mathrm{T}}A$ corresponding to its largest eigenvalue $\lambda_{\max}$. The ratio in equation (1) is then $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}(\lambda_{\max})\boldsymbol{x}$ divided by $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$. For this particular $\boldsymbol{x}$, the ratio equals $\lambda_{\max}$.

No other $\boldsymbol{x}$ can give a larger ratio. The symmetric matrix $A^{\mathrm{T}}A$ has orthonormal eigenvectors $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n$. Every $\boldsymbol{x}$ is a combination of those vectors. Try this combination in the ratio and remember that $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$:

$$\frac{\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}} = \frac{(c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n)^{\mathrm{T}}(c_1\lambda_1\boldsymbol{q}_1 + \cdots + c_n\lambda_n\boldsymbol{q}_n)}{(c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n)^{\mathrm{T}}(c_1\boldsymbol{q}_1 + \cdots + c_n\boldsymbol{q}_n)} = \frac{c_1^2\lambda_1 + \cdots + c_n^2\lambda_n}{c_1^2 + \cdots + c_n^2}. \quad (5)$$

That last ratio cannot be larger than $\lambda_{\max}$. The maximum ratio is when all c's are zero, except the one that multiplies $\lambda_{\max}$.

Note 1 The ratio in (5) is known as the *Rayleigh quotient* for the matrix $A^{\mathrm{T}}A$. The maximum is the largest eigenvalue $\lambda_{\max}(A^{\mathrm{T}}A)$. The minimum is $\lambda_{\min}(A^{\mathrm{T}}A)$. If you substitute any vector $\boldsymbol{x}$ into the Rayleigh quotient $\boldsymbol{x}^{\mathrm{T}}A^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$, you are guaranteed to get a number between $\lambda_{\min}$ and $\lambda_{\max}$.

Note 2 The norm $\|A\|$ equals the largest *singular value* $\sigma_{\max}$ of A. The singular values $\sigma_1, \ldots, \sigma_r$ are the square roots of the positive eigenvalues of $A^{\mathrm{T}}A$. So certainly $\sigma_{\max} = (\lambda_{\max})^{1/2}$. This is the norm of A.

Note 3 Check that the unsymmetric example in equation (3) has $\lambda_{\max}(A^{\mathrm{T}}A) = 4$:

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \text{ leads to } A^{\mathrm{T}}A = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} \text{ with } \lambda_{\max} = 4. \text{ So the norm is } \|A\| = \sqrt{4}.$$

The Condition Number of A

Section 9.1 showed that roundoff error can be serious. Some systems are sensitive, others are not so sensitive. The sensitivity to error is measured by the ***condition number***. This is the first chapter in the book which intentionally introduces errors. We want to estimate how much they change $\boldsymbol{x}$.

The original equation is $A\boldsymbol{x} = \boldsymbol{b}$. Suppose the right side is changed to $\boldsymbol{b} + \Delta\boldsymbol{b}$ because of roundoff or measurement error. The solution is then changed to $\boldsymbol{x}+\Delta\boldsymbol{x}$. Our goal is to estimate the change $\Delta\boldsymbol{x}$ in the solution from the change $\Delta\boldsymbol{b}$ in the equation. Subtraction gives the *error equation* $A(\Delta\boldsymbol{x}) = \Delta\boldsymbol{b}$:

$$\text{Subtract } A\boldsymbol{x} = \boldsymbol{b} \text{ from } A(\boldsymbol{x} + \Delta\boldsymbol{x}) = \boldsymbol{b} + \Delta\boldsymbol{b} \quad \text{to find} \quad \boldsymbol{A(\Delta x) = \Delta b}. \quad (6)$$

The error is $\Delta\boldsymbol{x} = A^{-1}\Delta\boldsymbol{b}$. It is large when A^{-1} is large (then A is nearly singular). The error $\Delta\boldsymbol{x}$ is especially large when $\Delta\boldsymbol{b}$ points in the worst direction—which is amplified most by A^{-1}. The worst error has $\|\Delta\boldsymbol{x}\| = \|A^{-1}\|\,\|\Delta\boldsymbol{b}\|$. That is the largest possible output error $\Delta\boldsymbol{x}$.

Example 3 When A is symmetric, $c = \|A\|\,\|A^{-1}\|$ comes from the eigenvalues:

$$A = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \text{ has norm } 6. \qquad A^{-1} = \begin{bmatrix} \frac{1}{6} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \text{ has norm } \tfrac{1}{2}.$$

This A is symmetric positive definite. Its norm is $\lambda_{\max} = 6$. The norm of A^{-1} is $1/\lambda_{\min} = \frac{1}{2}$. Multiplying those norms gives the *condition number*:

$$c = \frac{\lambda_{\max}}{\lambda_{\min}} = \frac{6}{2} = 3.$$

Example 4 Keep the same A, with eigenvalues 6 and 2. To make $\boldsymbol{x}$ small, choose $\boldsymbol{b}$ along the first eigenvector $(1, 0)$. To make $\Delta\boldsymbol{x}$ large, choose $\Delta\boldsymbol{b}$ along the second eigenvector $(0, 1)$. Then $\boldsymbol{x} = \frac{1}{6}\boldsymbol{b}$ and $\Delta\boldsymbol{x} = \frac{1}{2}\boldsymbol{b}$. The ratio $\|\Delta\boldsymbol{x}\|/\|\boldsymbol{x}\|$ is exactly $c = 3$ times the ratio $\|\Delta\boldsymbol{b}\|/\|\boldsymbol{b}\|$.

This shows that the worst error allowed by the condition number can actually happen. Here is a useful rule of thumb, experimentally verified for Gaussian elimination: *The computer can lose* $\log c$ *decimal places to roundoff error.*

Problem Set 9.2

1 Find the norms $\lambda_{\max}$ and condition numbers $\lambda_{\max}/\lambda_{\min}$ of these positive definite matrices:

$$\begin{bmatrix} .5 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.$$

2 Find the norms and condition numbers from the square roots of $\lambda_{\max}(A^{\mathrm{T}}A)$ and $\lambda_{\min}(A^{\mathrm{T}}A)$:

$$\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

3 Explain these two inequalities from the definitions of $\|A\|$ and $\|B\|$:

$$\|AB\boldsymbol{x}\| \le \|A\|\,\|B\boldsymbol{x}\| \le \|A\|\,\|B\|\,\|\boldsymbol{x}\|.$$

From the ratio that gives $\|AB\|$, deduce that $\|AB\| \le \|A\|\,\|B\|$. This is the key to using matrix norms.

4 Use $\|AB\| \le \|A\|\,\|B\|$ to prove that the condition number of any matrix A is at least 1.

5 Why is I the only symmetric positive definite matrix that has $\lambda_{\max} = \lambda_{\min} = 1$? Then the only matrices with $\|A\| = 1$ and $\|A^{-1}\| = 1$ must have $A^{\mathrm{T}}A = I$. They are _____ matrices.

6 Orthogonal matrices have norm $\|Q\| = 1$. If $A = QR$ show that $\|A\| \le \|R\|$ and also $\|R\| \le \|A\|$. Then $\|A\| = \|R\|$. Find an example of $A = LU$ with $\|A\| < \|L\|\|U\|$.

7 (a) Which famous inequality gives $\|(A + B)\boldsymbol{x}\| \le \|A\boldsymbol{x}\| + \|B\boldsymbol{x}\|$ for every $\boldsymbol{x}$?

(b) Why does the definition (4) of matrix norms lead to $\|A+B\| \le \|A\|+\|B\|$?

8 Show that if λ is any eigenvalue of A, then $|\lambda| \le \|A\|$. Start from $A\boldsymbol{x} = \lambda\boldsymbol{x}$.

9 The "*spectral radius*" $\rho(A) = |\lambda_{\max}|$ is the largest absolute value of the eigenvalues. Show with 2 by 2 examples that $\rho(A+B) \le \rho(A)+\rho(B)$ and $\rho(AB) \le \rho(A)\rho(B)$ can both be *false*. The spectral radius is not acceptable as a norm.

10 (a) Explain why A and A^{-1} have the same condition number.

(b) Explain why A and A^{T} have the same norm.

11 Estimate the condition number of the ill-conditioned matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$.

12 Why is the determinant of A no good as a norm? Why is it no good as a condition number?

13 (Suggested by C. Moler and C. Van Loan.) Compute $\boldsymbol{b} - A\boldsymbol{y}$ and $\boldsymbol{b} - A\boldsymbol{z}$ when

$$\boldsymbol{b} = \begin{bmatrix} .217 \\ .254 \end{bmatrix} \quad A = \begin{bmatrix} .780 & .563 \\ .913 & .659 \end{bmatrix} \quad \boldsymbol{y} = \begin{bmatrix} .341 \\ -.087 \end{bmatrix} \quad \boldsymbol{z} = \begin{bmatrix} .999 \\ -1.0 \end{bmatrix}.$$

Is $\boldsymbol{y}$ closer than $\boldsymbol{z}$ to solving $A\boldsymbol{x} = \boldsymbol{b}$? Answer in two ways: Compare the *residual* $\boldsymbol{b} - A\boldsymbol{y}$ to $\boldsymbol{b} - A\boldsymbol{z}$. Then compare $\boldsymbol{y}$ and $\boldsymbol{z}$ to the true $\boldsymbol{x} = (1, -1)$. Both answers can be right. Sometimes we want a small residual, sometimes a small $\Delta\boldsymbol{x}$.

14 (a) Compute the determinant of A in Problem 13. Compute A^{-1}.

(b) If possible compute $\|A\|$ and $\|A^{-1}\|$ and show that $c > 10^6$.

Problems 15–19 are about vector norms other than the usual $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}\cdot\boldsymbol{x}}$.

15 The "l^1 norm" and the "l^∞ norm" of $\boldsymbol{x} = (x_1, \ldots, x_n)$ are

$$\|\boldsymbol{x}\|_1 = |x_1| + \cdots + |x_n| \quad \text{and} \quad \|\boldsymbol{x}\|_\infty = \max_{1\le i\le n} |x_i|.$$

Compute the norms $\|\boldsymbol{x}\|$ and $\|\boldsymbol{x}\|_1$ and $\|\boldsymbol{x}\|_\infty$ of these two vectors in $\mathbf{R}^5$:

$$\boldsymbol{x} = (1, 1, 1, 1, 1) \qquad \boldsymbol{x} = (.1, .7, .3, .4, .5).$$

16 Prove that $\|\boldsymbol{x}\|_\infty \le \|\boldsymbol{x}\| \le \|\boldsymbol{x}\|_1$. Show from the Schwarz inequality that the ratios $\|\boldsymbol{x}\|/\|\boldsymbol{x}\|_\infty$ and $\|\boldsymbol{x}\|_1/\|\boldsymbol{x}\|$ are never larger than $\sqrt{n}$. Which vector $(x_1, \ldots, x_n)$ gives ratios equal to $\sqrt{n}$?

17 All vector norms must satisfy the *triangle inequality*. Prove that

$$\|\boldsymbol{x}+\boldsymbol{y}\|_\infty \le \|\boldsymbol{x}\|_\infty + \|\boldsymbol{y}\|_\infty \qquad \text{and} \qquad \|\boldsymbol{x}+\boldsymbol{y}\|_1 \le \|\boldsymbol{x}\|_1 + \|\boldsymbol{y}\|_1.$$

18 Vector norms must also satisfy $\|c\boldsymbol{x}\| = |c|\,\|\boldsymbol{x}\|$. The norm must be positive except when $\boldsymbol{x} = \boldsymbol{0}$. Which of these are norms for (x_1, x_2)?

$$\|\boldsymbol{x}\|_A = |x_1| + 2|x_2| \qquad \|\boldsymbol{x}\|_B = \min |x_i|$$

$$\|\boldsymbol{x}\|_C = \|\boldsymbol{x}\| + \|\boldsymbol{x}\|_\infty \qquad \|\boldsymbol{x}\|_D = \|A\boldsymbol{x}\| \quad \text{(answer depends on } A\text{)}.$$

ITERATIVE METHODS FOR LINEAR ALGEBRA ■ 9.3

Up to now, our approach to $Ax = b$ has been "direct." We accepted A as it came. We attacked it with Gaussian elimination. This section is about ***iterative methods***, which replace A by a simpler matrix S. The difference $T = S - A$ is moved over to the right side of the equation. The problem becomes easier to solve, with S instead of A. But there is a price—*the simpler system has to be solved over and over.*

An iterative method is easy to invent. Just split A into $S - T$. Then $Ax = b$ is the same as

$$Sx = Tx + b. \tag{1}$$

The novelty is to solve (1) iteratively. Each guess x_k leads to the next x_{k+1}:

$$\boldsymbol{Sx_{k+1} = Tx_k + b.} \tag{2}$$

Start with any x_0. Then solve $Sx_1 = Tx_0 + b$. Continue to the second iteration $Sx_2 = Tx_1 + b$. A hundred iterations are very common—maybe more. Stop when (and if!) the new vector x_{k+1} is sufficiently close to x_k—or when the residual $Ax_k - b$ is near zero. We can choose the stopping test. Our hope is to get near the true solution, more quickly than by elimination. When the sequence x_k converges, its limit $x = x_\infty$ does solve equation (1). The proof is to let $k \to \infty$ in equation (2).

The two goals of the splitting $A = S - T$ are ***speed per step*** and ***fast convergence of the*** x_k. The speed of each step depends on S and the speed of convergence depends on $S^{-1}T$:

1 Equation (2) should be easy to solve for x_{k+1}. The "***preconditioner***" S could be diagonal or triangular. When its LU factorization is known, each iteration step is fast.

2 The difference $x - x_k$ (this is the error e_k) should go quickly to zero. Subtracting equation (2) from (1) cancels b, and it leaves the ***error equation***:

$$Se_{k+1} = Te_k \text{ which means } e_{k+1} = S^{-1}Te_k. \tag{3}$$

At every step the error is multiplied by $S^{-1}T$. If $S^{-1}T$ is small, its powers go quickly to zero. But what is "small"?

The extreme splitting is $S = A$ and $T = 0$. Then the first step of the iteration is the original $Ax = b$. Convergence is perfect and $S^{-1}T$ is zero. But the cost of that step is what we wanted to avoid. The choice of S is a battle between speed per step (a simple S) and fast convergence (S close to A). Here are some popular choices:

J $S =$ diagonal part of A (the iteration is called *Jacobi's method*)

GS $S =$ lower triangular part of A (*Gauss-Seidel method*)

SOR $S =$ combination of Jacobi and Gauss-Seidel (*successive overrelaxation*)

ILU $S =$ approximate L times approximate U (*incomplete LU method*).

Our first question is pure linear algebra: ***When do the x_k's converge to x?*** The answer uncovers the number $|\lambda|_{\max}$ that controls convergence. In examples of **J** and **GS** and **SOR**, we will compute this "*spectral radius*" $|\lambda|_{\max}$. It is the largest eigenvalue of the iteration matrix $S^{-1}T$.

The Spectral Radius Controls Convergence

Equation (3) is $\boldsymbol{e}_{k+1} = S^{-1}T\boldsymbol{e}_k$. Every iteration step multiplies the error by the same matrix $B = S^{-1}T$. The error after k steps is $\boldsymbol{e}_k = B^k\boldsymbol{e}_0$. ***The error approaches zero if the powers of $B = S^{-1}T$ approach zero.*** It is beautiful to see how the eigenvalues of B—the largest eigenvalue in particular—control the matrix powers B^k.

> **9C Convergence** The powers B^k approach zero if and only if every eigenvalue of B satisfies $|\lambda| < 1$. ***The rate of convergence is controlled by the spectral radius*** $|\lambda|_{\max}$.

The test for convergence is $|\lambda|_{\max} < 1$. Real eigenvalues must lie between -1 and 1. Complex eigenvalues $\lambda = a+ib$ must lie inside the unit circle in the complex plane. In that case the absolute value $|\lambda|$ is the square root of a^2+b^2—Chapter 10 will discuss complex numbers. In every case the spectral radius is the largest distance from the origin 0 to the eigenvalues $\lambda_1, \ldots, \lambda_n$. Those are eigenvalues of the iteration matrix $B = S^{-1}T$.

To see why $|\lambda|_{\max} < 1$ is necessary, suppose the starting error $\boldsymbol{e}_0$ happens to be an eigenvector of B. After one step the error is $B\boldsymbol{e}_0 = \lambda\boldsymbol{e}_0$. After k steps the error is $B^k\boldsymbol{e}_0 = \lambda^k\boldsymbol{e}_0$. If we start with an eigenvector, we continue with that eigenvector—and it grows or decays with the powers λ^k. *This factor λ^k goes to zero when $|\lambda| < 1$.* Since this condition is required of every eigenvalue, we need $|\lambda|_{\max} < 1$.

To see why $|\lambda|_{\max} < 1$ is sufficient for the error to approach zero, suppose $\boldsymbol{e}_0$ is a combination of eigenvectors:

$$\boldsymbol{e}_0 = c_1\boldsymbol{x}_1 + \cdots + c_n\boldsymbol{x}_n \quad \text{leads to} \quad \boldsymbol{e}_k = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n. \qquad (4)$$

This is the point of eigenvectors! They grow independently, each one controlled by its eigenvalue. When we multiply by B, the eigenvector $\boldsymbol{x}_i$ is multiplied by λ_i. If all $|\lambda_i| < 1$ then equation (4) ensures that e_k goes to zero.

Example 1 $B = \begin{bmatrix} .6 & .5 \\ .6 & .5 \end{bmatrix}$ has $\lambda_{\max} = 1.1$ $\quad B' = \begin{bmatrix} .6 & 1.1 \\ 0 & .5 \end{bmatrix}$ has $\lambda_{\max} = .6$ B^2 is 1.1 times B. Then B^3 is $(1.1)^2$ times B. The powers of B blow up. Contrast with the powers of B'. The matrix $(B')^k$ has $(.6)^k$ and $(.5)^k$ on its diagonal. The off-diagonal entries also involve $(.6)^k$, which sets the speed of convergence.

Note There is a technical difficulty when B does not have n independent eigenvectors. (To produce this effect in B', change .5 to .6.) The starting error $\boldsymbol{e}_0$ may not be a combination of eigenvectors—there are too few for a basis. Then diagonalization is impossible and equation (4) is not correct. We turn to the *Jordan form*:

$$B = SJS^{-1} \quad \text{and} \quad B^k = SJ^kS^{-1}. \tag{5}$$

Section 6.6 shows how J and J^k are made of "blocks" with one repeated eigenvalue:

$$\text{The powers of a 2 by 2 block are} \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}.$$

If $|\lambda| < 1$ then these powers approach zero. The extra factor k from a double eigenvalue is overwhelmed by the decreasing factor λ^{k-1}. This applies to all Jordan blocks. A larger block has $k^2\lambda^{k-2}$ in J^k, which also approaches zero when $|\lambda| < 1$.

If all $|\lambda| < 1$ then $J^k \to 0$. This proves **9C:** ***Convergence requires*** $\boldsymbol{|\lambda|_{\max} < 1}$.

Jacobi versus Seidel

We now solve a specific 2 by 2 problem. The theory of iteration says that the key number is the spectral radius of $B = S^{-1}T$. Watch for that number $|\lambda|_{\max}$. It is also written $\rho(B)$—the Greek letter "rho" stands for the spectral radius:

$$\begin{aligned} 2u - v &= 4 \\ -u + 2v &= -2 \end{aligned} \quad \text{has the solution} \quad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \tag{6}$$

The first splitting is ***Jacobi's method***. Keep the *diagonal terms* on the left side (this is S). Move the off-diagonal part of A to the right side (this is T). Then iterate:

$$\begin{aligned} 2u_{k+1} &= v_k + 4 \\ 2v_{k+1} &= u_k - 2. \end{aligned}$$

Start the iteration from $u_0 = v_0 = 0$. The first step goes to $u_1 = 2, v_1 = -1$. Keep going:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1/4 \end{bmatrix} \quad \begin{bmatrix} 15/8 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1/16 \end{bmatrix} \quad \text{approaches} \quad \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

This shows convergence. At steps 1, 3, 5 the second component is -1, $-1/4$, $-1/16$. The error is multiplied by $\frac{1}{4}$ every two steps. So is the error in the first component. The values 0, 3/2, 15/8 have errors 2, $\frac{1}{2}$, $\frac{1}{8}$. Those also drop by 4 in each two steps. *The error equation is* $S\boldsymbol{e}_{k+1} = T\boldsymbol{e}_k$:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \boldsymbol{e}_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{e}_k \quad \text{or} \quad \boldsymbol{e}_{k+1} = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \boldsymbol{e}_k. \tag{7}$$

Rewrite $Ax = b$ as $\omega Ax = \omega b$. The matrix S in **SOR** has the diagonal of the original A, but below the diagonal we use ωA. The matrix T on the right side is $S - \omega A$:

$$\begin{aligned} 2u_{k+1} \qquad\qquad &= (2-2\omega)u_k + \omega v_k + 4\omega \\ -\omega u_{k+1} + 2v_{k+1} &= \qquad\qquad (2-2\omega)v_k - 2\omega. \end{aligned} \tag{9}$$

This looks more complicated to us, but the computer goes as fast as ever. Each new u_{k+1} from the first equation is used immediately to find v_{k+1} in the second equation. This is like Gauss-Seidel, with an adjustable number ω. The key matrix is always $S^{-1}T$:

$$S^{-1}T = \begin{bmatrix} 1-\omega & \frac{1}{2}\omega \\ \frac{1}{2}\omega(1-\omega) & 1-\omega+\frac{1}{4}\omega^2 \end{bmatrix}. \tag{10}$$

The determinant is $(1-\omega)^2$. At the best ω, both eigenvalues turn out to equal $7-4\sqrt{3}$, which is close to $(\frac{1}{4})^2$. Therefore **SOR** is twice as fast as Gauss-Seidel in this example. In other examples **SOR** can converge ten or a hundred times as fast.

I will put on record the most valuable test matrix of order n. It is our favorite $-1, 2, -1$ tridiagonal matrix. The diagonal is $2I$. Below and above are -1's. Our example had $n = 2$, which leads to $\cos\frac{\pi}{3} = \frac{1}{2}$ as the Jacobi eigenvalue. (We found that $\frac{1}{2}$ above.) Notice especially that this eigenvalue is squared for Gauss-Seidel:

9D The splittings of the $-1, 2, -1$ matrix of order n yield these eigenvalues of B:

Jacobi ($S = 0, 2, 0$ matrix): $\quad S^{-1}T$ has $|\lambda|_{\max} = \boxed{\cos\frac{\pi}{n+1}}$

Gauss-Seidel ($S = -1, 2, 0$ matrix): $\quad S^{-1}T$ has $|\lambda|_{\max} = \boxed{\left(\cos\frac{\pi}{n+1}\right)^2}$

SOR (with the best ω): $\quad S^{-1}T$ has $|\lambda|_{\max} = \boxed{\left(\cos\frac{\pi}{n+1}\right)^2 \Big/ \left(1+\sin\frac{\pi}{n+1}\right)^2}$.

Let me be clear: For the $-1, 2, -1$ matrix you should not use any of these iterations! Elimination is very fast (exact LU). Iterations are intended for large sparse matrices—when a high percentage of the zero entries are "not good." The not good zeros are inside the band, which is wide. They become nonzero in the exact L and U, which is why elimination becomes expensive.

We mention one more splitting. It is associated with the words "***incomplete*** LU." The idea is to set the small nonzeros in L and U *back to zero*. This leaves triangular matrices L_0 and U_0 which are again sparse. That allows fast computations.

The splitting has $S = L_0U_0$ on the left side. Each step is quick:

$$L_0U_0x_{k+1} = (A - L_0U_0)x_k + b.$$

On the right side we do sparse matrix-vector multiplications. Don't multiply L_0 times U_0—those are matrices. Multiply x_k by U_0 and then multiply that vector by L_0. On the left side we do forward and back substitutions. If L_0U_0 is close to A, then $|\lambda|_{\max}$ is small. A few iterations will give a close answer.

The difficulty with all four of these splittings is that a single large eigenvalue in $S^{-1}T$ would spoil everything. There is a safer iteration—the ***conjugate gradient method***—which avoids this difficulty. Combined with a good preconditioner S (from the splitting $A = S - T$), this produces one of the most popular and powerful algorithms in numerical linear algebra.[1]

Iterative Methods for Eigenvalues

We move from $A\boldsymbol{x} = \boldsymbol{b}$ to $A\boldsymbol{x} = \lambda\boldsymbol{x}$. Iterations are an option for linear equations. They are a necessity for eigenvalue problems. The eigenvalues of an n by n matrix are the roots of an nth degree polynomial. The determinant of $A - \lambda I$ starts with $(-\lambda)^n$. This book must not leave the impression that eigenvalues should be computed from this polynomial. The determinant of $A - \lambda I$ is a very poor approach—except when n is small.

For $n > 4$ there is no formula to solve $\det(A - \lambda I) = 0$. Worse than that, the λ's can be very unstable and sensitive. It is much better to work with A itself, gradually making it diagonal or triangular. (Then the eigenvalues appear on the diagonal.) Good computer codes are available in the LAPACK library—individual routines are free on **www.netlib.org**. This library combines the earlier LINPACK and EISPACK, with improvements. It is a collection of Fortran 77 programs for linear algebra on high-performance computers. (The email message **send index from lapack** brings information.) For your computer and mine, the same efficiency is achieved by high quality matrix packages like MATLAB.

We will briefly discuss the power method and the QR method for computing eigenvalues. It makes no sense to give full details of the codes.

1 Power methods and inverse power methods. Start with any vector $\boldsymbol{u}_0$. Multiply by A to find $\boldsymbol{u}_1$. Multiply by A again to find $\boldsymbol{u}_2$. If $\boldsymbol{u}_0$ is a combination of the eigenvectors, then A multiplies each eigenvector $\boldsymbol{x}_i$ by λ_i. After k steps we have $(\lambda_i)^k$:

$$\boldsymbol{u}_k = A^k\boldsymbol{u}_0 = c_1(\lambda_1)^k\boldsymbol{x}_1 + \cdots + c_n(\lambda_n)^k\boldsymbol{x}_n. \tag{11}$$

As the power method continues, ***the largest eigenvalue begins to dominate***. The vectors $\boldsymbol{u}_k$ point toward that dominant eigenvector. We saw this for Markov matrices in Chapter 8:

$$A = \begin{bmatrix} .9 & .3 \\ .1 & .7 \end{bmatrix} \quad \text{has} \quad \lambda_{\max} = 1 \quad \text{with eigenvector} \quad \begin{bmatrix} .75 \\ .25 \end{bmatrix}.$$

Start with $\boldsymbol{u}_0$ and multiply at every step by A:

$$\boldsymbol{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{u}_1 = \begin{bmatrix} .9 \\ .1 \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} .84 \\ .16 \end{bmatrix} \quad \text{is approaching} \quad \boldsymbol{u}_\infty = \begin{bmatrix} .75 \\ .25 \end{bmatrix}.$$

[1] Conjugate gradients are described in the author's book *Introduction to Applied Mathematics* and in greater detail by Golub-Van Loan and by Trefethen-Bau.

The other idea is to obtain off-diagonal zeros before the QR method starts. Change A to the similar matrix $L^{-1}AL$ (no change in the eigenvalues):

$$L^{-1}AL = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 5 \\ 1 & 6 & 7 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 \\ 1 & 9 & 5 \\ 0 & 4 & 2 \end{bmatrix}.$$

L^{-1} subtracted row 2 from row 3 to produce the zero in column 1. Then L added column 3 to column 2 and left the zero alone. If I try for another zero (too ambitious), I will fail. Subtracting row 1 from row 2 produces a zero. But now L adds column 2 to column 1 and destroys it.

We must leave those nonzeros 1 and 4 along one subdiagonal. This is a "*Hessenberg matrix*", which is reachable in a fixed number of steps. The zeros in the lower left corner will stay zero through the QR method. The operation count for each QR factorization drops from $O(n^3)$ to $O(n^2)$.

Golub and Van Loan give this example of one shifted QR step on a Hessenberg matrix A. The shift is $cI = 7I$:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & .001 & 7 \end{bmatrix} \quad \text{leads to} \quad A_1 = \begin{bmatrix} -.54 & 1.69 & 0.835 \\ .31 & 6.53 & -6.656 \\ 0 & .00002 & 7.012 \end{bmatrix}.$$

Factoring $A - 7I$ into QR produced $A_1 = RQ + 7I$. Notice the very small number .00002. The diagonal entry 7.012 is almost an exact eigenvalue of A_1, and therefore of A. Another QR step with shift by $7.012I$ would give terrific accuracy.

Problem Set 9.3

Problems 1–12 are about iterative methods for $Ax = b$.

1 Change $Ax = b$ to $x = (I - A)x + b$. What are S and T for this splitting? What matrix $S^{-1}T$ controls the convergence of $x_{k+1} = (I - A)x_k + b$?

2 If λ is an eigenvalue of A, then _____ is an eigenvalue of $B = I - A$. The real eigenvalues of B have absolute value less than 1 if the real eigenvalues of A lie between _____ and _____.

3 Show why the iteration $x_{k+1} = (I - A)x_k + b$ does not converge for $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

4 Why is the norm of B^k never larger than $\|B\|^k$? Then $\|B\| < 1$ guarantees that the powers B^k approach zero (convergence). No surprise since $|\lambda|_{\max}$ is below $\|B\|$.

5 If A is singular then all splittings $A = S - T$ must fail. From $Ax = 0$ show that $S^{-1}Tx = x$. So this matrix $B = S^{-1}T$ has $\lambda = 1$ and fails.

6 Change the 2's to 3's and find the eigenvalues of $S^{-1}T$ for Jacobi's method:

$$S\boldsymbol{x}_{k+1} = T\boldsymbol{x}_k + \boldsymbol{b} \quad \text{is} \quad \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{b}.$$

7 Find the eigenvalues of $S^{-1}T$ for the Gauss-Seidel method applied to Problem 6:

$$\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix} \boldsymbol{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{b}.$$

Does $|\lambda|_{\max}$ for Gauss-Seidel equal $|\lambda|^2_{\max}$ for Jacobi?

8 For any 2 by 2 matrix $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ show that $|\lambda|_{\max}$ equals $|bc/ad|$ for Gauss-Seidel and $|bc/ad|^{1/2}$ for Jacobi. We need $ad \neq 0$ for the matrix S to be invertible.

9 The best ω produces two equal eigenvalues for $S^{-1}T$ in the **SOR** method. Those eigenvalues are $\omega - 1$ because the determinant is $(\omega - 1)^2$. Set the trace in equation (10) equal to $(\omega - 1) + (\omega - 1)$ and find this optimal ω.

10 Write a computer code (MATLAB or other) for the Gauss-Seidel method. You can define S and T from A, or set up the iteration loop directly from the entries a_{ij}. Test it on the -1, 2, -1 matrices A of order 10, 20, 50 with $\boldsymbol{b} = (1, 0, \ldots, 0)$.

11 The Gauss-Seidel iteration at component i is

$$x_i^{\text{new}} = x_i^{\text{old}} + \frac{1}{a_{ii}} \Big(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{\text{new}} - \sum_{j=i}^{n} a_{ij} x_j^{\text{old}} \Big).$$

If every $x_i^{\text{new}} = x_i^{\text{old}}$ how does this show that the solution $\boldsymbol{x}$ is correct? How does the formula change for Jacobi's method? For **SOR** insert ω outside the parentheses.

12 The **SOR** splitting matrix S is the same as for Gauss-Seidel except that the diagonal is divided by ω. Write a program for **SOR** on an n by n matrix. Apply it with $\omega = 1$, 1.4, 1.8, 2.2 when A is the -1, 2, -1 matrix of order $n = 10$.

13 Divide equation (11) by λ_1^k and explain why $|\lambda_2/\lambda_1|$ controls the convergence of the power method. Construct a matrix A for which this method *does not converge*.

14 The Markov matrix $A = \left[\begin{smallmatrix} .9 & .3 \\ .1 & .7 \end{smallmatrix}\right]$ has $\lambda = 1$ and $.6$, and the power method $\boldsymbol{u}_k = A^k \boldsymbol{u}_0$ converges to $\left[\begin{smallmatrix} .75 \\ .25 \end{smallmatrix}\right]$. Find the eigenvectors of A^{-1}. What does the inverse power method $\boldsymbol{u}_{-k} = A^{-k}\boldsymbol{u}_0$ converge to (after you multiply by $.6^k$)?

15 Show that the n by n matrix with diagonals -1, 2, -1 has the eigenvector $\boldsymbol{x}_1 = (\sin \frac{\pi}{n+1}, \sin \frac{2\pi}{n+1}, \ldots, \sin \frac{n\pi}{n+1})$. Find the eigenvalue λ_1 by multiplying $A\boldsymbol{x}_1$. Note: For the other eigenvectors and eigenvalues of this matrix, change π to $j\pi$ in $\boldsymbol{x}_1$ and λ_1.

25 What bound on $|\lambda|_{\max}$ does Problem 24 give for these matrices? What are the three Gershgorin circles that contain all the eigenvalues?

$$A = \begin{bmatrix} .3 & .3 & .2 \\ .3 & .2 & .4 \\ .2 & .4 & .1 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}.$$

26 These matrices are diagonally dominant because each $a_{ii} > r_i =$ absolute sum along the rest of row i. From the Gershgorin circles containing all λ's, show that diagonally dominant matrices are invertible.

$$A = \begin{bmatrix} 1 & .3 & .4 \\ .3 & 1 & .5 \\ .4 & .5 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 5 \end{bmatrix}.$$

The key point for large matrices is that matrix-vector multiplication is much faster than matrix-matrix multiplication. A crucial construction starts with a vector $\boldsymbol{b}$ and computes $A\boldsymbol{b}, A^2\boldsymbol{b}, \ldots$ (but never A^2!). The first N vectors span the ***Nth Krylov subspace***. They are the columns of the ***Krylov matrix*** K_N:

$$K_N = \begin{bmatrix} \boldsymbol{b} & A\boldsymbol{b} & A^2\boldsymbol{b} & \cdots & A^{N-1}\boldsymbol{b} \end{bmatrix}.$$

Here in "pseudocode" are two of the most important algorithms in numerical linear algebra:

Arnoldi Iteration	**Conjugate Gradient Iteration for Positive Definite** A	
$\boldsymbol{q}_1 = \boldsymbol{b}/\\|\boldsymbol{b}\\|$	$\boldsymbol{x}_0 = 0, \boldsymbol{r}_0 = \boldsymbol{b}, \boldsymbol{p}_0 = \boldsymbol{r}_0$	
for $n = 1$ **to** $N-1$	**for** $n = 1$ **to** N	
$\boldsymbol{v} = A\boldsymbol{q}_n$	$\alpha_n = (\boldsymbol{r}_{n-1}^{\mathrm{T}}\boldsymbol{r}_{n-1})/(\boldsymbol{p}_{n-1}^{\mathrm{T}}A\boldsymbol{p}_{n-1})$	step length $\boldsymbol{x}_{n-1}$ to $\boldsymbol{x}_n$
for $j = 1$ **to** n	$\boldsymbol{x}_n = \boldsymbol{x}_{n-1} + \alpha_n \boldsymbol{p}_{n-1}$	approximate solution
$h_{jn} = \boldsymbol{q}_j^{\mathrm{T}}\boldsymbol{v}$	$\boldsymbol{r}_n = \boldsymbol{r}_{n-1} - \alpha_n A\boldsymbol{p}_{n-1}$	new residual $\boldsymbol{b} - A\boldsymbol{x}_n$
$\boldsymbol{v} = \boldsymbol{v} - h_{jn}\boldsymbol{q}_j$	$\beta_n = (\boldsymbol{r}_n^{\mathrm{T}}\boldsymbol{r}_n)/(\boldsymbol{r}_{n-1}^{\mathrm{T}}\boldsymbol{r}_{n-1})$	improvement this step
$h_{n+1,n} = \\|\boldsymbol{v}\\|$	$\boldsymbol{p}_n = \boldsymbol{r}_n + \beta_n \boldsymbol{p}_{n-1}$	next search direction
$\boldsymbol{q}_{n+1} = \boldsymbol{v}/h_{n+1,n}$	% *Notice: only* 1 *matrix-vector multiplication* $A\boldsymbol{q}$ *and* $A\boldsymbol{p}$	

27 In Arnoldi show that $\boldsymbol{q}_2$ is orthogonal to $\boldsymbol{q}_1$. The Arnoldi method is Gram-Schmidt orthogonalization applied to the Krylov matrix: $K_N = Q_N R_N$. The eigenvalues of $Q_N^{\mathrm{T}} A Q_N$ are often very close to those of A even for $N \ll n$. The *Lanczos iteration* is Arnoldi for symmetric matrices (all coded in ARPACK).

28 In Conjugate Gradients show that $\boldsymbol{r}_1$ is orthogonal to $\boldsymbol{r}_0$ (orthogonal residuals) and $\boldsymbol{p}_1^{\mathrm{T}} A \boldsymbol{p}_0 = 0$ (search directions are A-orthogonal). The iteration solves $A\boldsymbol{x} = \boldsymbol{b}$ by minimizing the error $\boldsymbol{e}^{\mathrm{T}} A \boldsymbol{e}$ in the Krylov subspace. It is a fantastic algorithm.

10

COMPLEX VECTORS AND MATRICES

COMPLEX NUMBERS ■ 10.1

A complete theory of linear algebra must include complex numbers. Even when the matrix is real, the eigenvalues and eigenvectors are often complex. Example: A 2 by 2 rotation matrix has no real eigenvectors. Every vector turns by θ—the direction is changed. But there are complex eigenvectors $(1, i)$ and $(1, -i)$. The eigenvalues are also complex numbers $e^{i\theta}$ and $e^{-i\theta}$. If we insist on staying with real numbers, the theory of eigenvalues will be left in midair.

The second reason for allowing complex numbers goes beyond λ and $\boldsymbol{x}$ to the matrix A. ***The matrix itself may be complex***. We will devote a whole section to the most important example—*the Fourier matrix*. Engineering and science and music and economics all use Fourier series. In reality the series is finite, not infinite. Computing the coefficients in $c_1e^{ix} + c_2e^{i2x} + \cdots + c_ne^{inx}$ is a linear algebra problem.

This section gives the main facts about complex numbers. It is a review for some students and a reference for everyone. The underlying fact is that $i^2 = -1$. Everything comes from that. We will get as far as the amazing formula $e^{2\pi i} = 1$.

Adding and Multiplying Complex Numbers

Start with the imaginary number i. Everybody knows that $x^2 = -1$ has no real solution. When you square a real number, the answer is never negative. So the world has agreed on a solution called i. (Except that electrical engineers call it j.) Imaginary numbers follow the normal rules of addition and multiplication, with one difference. *Whenever i^2 appears it is replaced by -1.*

10A ***A complex number*** (say $3+2i$) ***is the sum of a real number*** (3) ***and a pure imaginary number*** ($2i$). Addition keeps the real and imaginary parts separate. Multiplication uses $i^2 = -1$:

$$\textbf{Add:}\quad (3+2i)+(3+2i)=6+4i$$
$$\textbf{Multiply:}\quad (3+2i)(1-i)=3+2i-3i-2i^2=5-i.$$

If I add $3+2i$ to $1-i$, the answer is $4+i$. The real numbers $3+1$ stay separate from the imaginary numbers $2i-i$. We are adding the vectors $(3,2)$ and $(1,-1)$.

The number $(1-i)^2$ is $1-i$ times $1-i$. The rules give the surprising answer $-2i$:

$$(1-i)(1-i)=1-i-i+i^2=-2i.$$

In the complex plane, $1-i$ is at an angle of $-45°$. When we square it to get $-2i$, the angle doubles to $-90°$. If we square again, the answer is $(-2i)^2=-4$. The $-90°$ angle has become $-180°$, which is the direction of a negative real number.

A real number is just a complex number $z=a+bi$, with zero imaginary part: $b=0$. A pure imaginary number has $a=0$:

The ***real part*** is $\quad a=\text{Re }(a+bi)$. The ***imaginary part*** is $\quad b=\text{Im }(a+bi)$.

The Complex Plane

Complex numbers correspond to points in a plane. Real numbers go along the x axis. Pure imaginary numbers are on the y axis. ***The complex number 3+2i is at the point with coordinates* (3, 2)**. The number zero, which is $0+0i$, is at the origin.

Adding and subtracting complex numbers is like adding and subtracting vectors in the plane. The real component stays separate from the imaginary component. The vectors go head-to-tail as usual. The complex plane $\mathbf{C}^1$ is like the ordinary two-dimensional plane $\mathbf{R}^2$, except that we multiply complex numbers and we didn't multiply vectors.

Now comes an important idea. ***The complex conjugate of 3+2i is 3−2i.*** The complex conjugate of $z=1-i$ is $\bar{z}=1+i$. In general the conjugate of $z=a+bi$ is $\bar{z}=a-bi$. (Notice the "*bar*" on the number to indicate the conjugate.) The imaginary parts of z and "z bar" have opposite signs. In the complex plane, $\bar{z}$ is the image of z on the other side of the real axis.

Two useful facts. ***When we multiply conjugates*** $\bar{z}_1$ ***and*** $\bar{z}_2$***, we get the conjugate of*** z_1z_2. When we add $\bar{z}_1$ and $\bar{z}_2$, we get the conjugate of z_1+z_2:

$$\bar{z}_1+\bar{z}_2=(3-2i)+(1+i)=4-i=\text{ conjugate of } z_1+z_2.$$
$$\bar{z}_1\times\bar{z}_2=(3-2i)\times(1+i)=5+i=\text{ conjugate of } z_1\times z_2.$$

Adding and multiplying is exactly what linear algebra needs. By taking conjugates of $Ax=\lambda x$, when A is real, we have another eigenvalue $\bar{\lambda}$ and its eigenvector $\bar{x}$:

$$\textit{If } Ax=\lambda x \textit{ and } A \textit{ is real then } A\bar{x}=\bar{\lambda}\bar{x}. \tag{1}$$

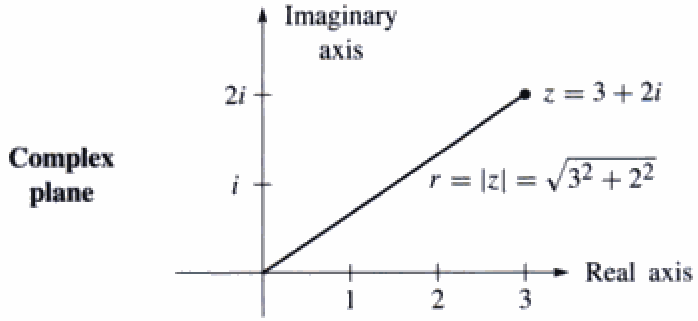


Figure 10.1 $z = a + bi$ corresponds to the point (a, b) and the vector $\begin{bmatrix} a \\ b \end{bmatrix}$.

Something special happens when $z = 3 + 2i$ combines with its own complex conjugate $\bar{z} = 3 - 2i$. The result from adding $z + \bar{z}$ or multiplying $z\bar{z}$ is always real:

$$(3 + 2i) + (3 - 2i) = 6 \quad \textbf{(real)}$$
$$(3 + 2i) \times (3 - 2i) = 9 + 6i - 6i - 4i^2 = 13 \quad \textbf{(real)}.$$

The sum of $z = a + bi$ and its conjugate $\bar{z} = a - bi$ is the real number $2a$. The product of z times $\bar{z}$ is the real number $a^2 + b^2$:

$$\boldsymbol{z\bar{z} = (a + bi)(a - bi) = a^2 + b^2.} \tag{2}$$

The next step with complex numbers is division. The best idea is to multiply the denominator by its conjugate to produce $a^2 + b^2$ which is real:

$$\frac{1}{a + ib} = \frac{1}{a + ib}\frac{a - ib}{a - ib} = \frac{a - ib}{a^2 + b^2} \qquad \frac{1}{3 + 2i} = \frac{1}{3 + 2i}\frac{3 - 2i}{3 - 2i} = \frac{3 - 2i}{13}.$$

In case $a^2 + b^2 = 1$, this says that $(a + ib)^{-1}$ is $a - ib$. ***On the unit circle, $1/z$ is $\bar{z}$.*** Later we will say: $1/e^{i\theta}$ is $e^{-i\theta}$ (the conjugate). A better way to multiply and divide is to use the polar form with distance r and angle θ.

The Polar Form

The square root of a^2+b^2 is $|z|$. This is the ***absolute value*** (or ***modulus***) of the number $z = a + ib$. The same square root is also written r, because it is the distance from 0 to the complex number. The number r in the polar form gives the size of z:

The absolute value of $\quad z = a + ib \quad$ is $\quad \boldsymbol{|z| = \sqrt{a^2 + b^2}.} \quad$ This is also called r.

The absolute value of $\quad z = 3 + 2i \quad$ is $\quad |z| = \sqrt{3^2 + 2^2}. \quad$ This is $r = \sqrt{13}$.

The other part of the polar form is the angle θ. The angle for $z = 5$ is $\theta = 0$ (because this z is real and positive). The angle for $z = 3i$ is $\pi/2$ radians. The angle for $z = -9$

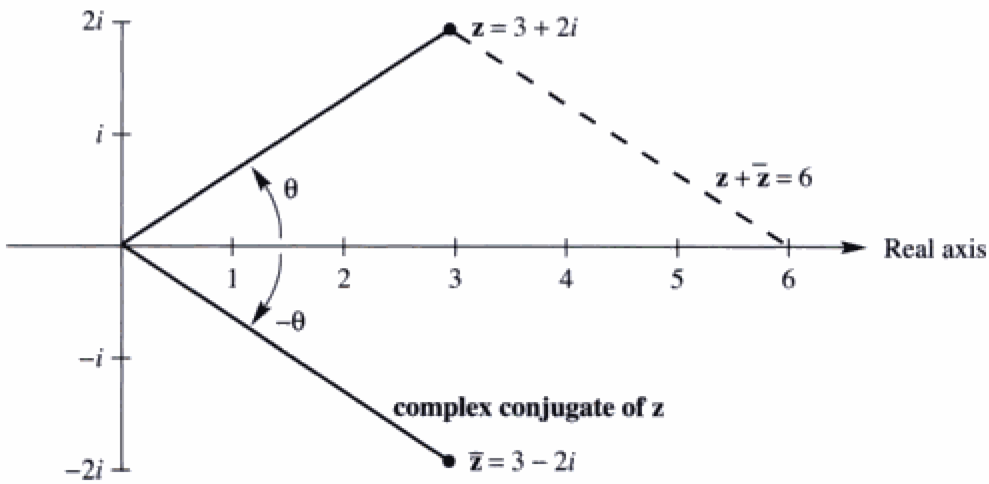


Figure 10.2 Conjugates give the mirror image of the previous figure: $z + \bar{z}$ is real.

is π radians. ***The angle doubles when the number is squared.*** This is one reason why the polar form is good for multiplying complex numbers (not so good for addition).

When the distance is r and the angle is θ, trigonometry gives the other two sides of the triangle. The real part (along the bottom) is $a = r\cos\theta$. The imaginary part (up or down) is $b = r\sin\theta$. Put those together, and the rectangular form becomes the polar form:

$$\textbf{\textit{The number}} \quad z = a + ib \quad \textbf{\textit{is also}} \quad z = r\cos\theta + ir\sin\theta.$$

Note: $\cos\theta + i\sin\theta$ ***has absolute value*** $r = 1$ ***because*** $\cos^2\theta + \sin^2\theta = 1$. Thus $\cos\theta + i\sin\theta$ lies on the circle of radius 1—*the unit circle.*

Example 1 Find r and θ for $z = 1 + i$ and also for the conjugate $\bar{z} = 1 - i$.

Solution The absolute value is the same for z and $\bar{z}$. Here it is $r = \sqrt{1+1} = \sqrt{2}$:

$$|z|^2 = 1^2 + 1^2 = 2 \qquad \text{and also} \qquad |\bar{z}|^2 = 1^2 + (-1)^2 = 2.$$

The distance from the center is $\sqrt{2}$. What about the angle? The number $1 + i$ is at the point $(1, 1)$ in the complex plane. The angle to that point is $\pi/4$ radians or $45°$. The cosine is $1/\sqrt{2}$ and the sine is $1/\sqrt{2}$. Combining r and θ brings back $z = 1 + i$:

$$r\cos\theta + ir\sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) + i\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) = 1 + i.$$

The angle to the conjugate $1 - i$ can be positive or negative. We can go to $7\pi/4$ radians which is $315°$. Or we can go *backwards through a negative angle*, to $-\pi/4$ radians or $-45°$. ***If z is at angle θ, its conjugate*** $\bar{z}$ ***is at*** $2\pi - \theta$ ***and also at*** $-\theta$.

We can freely add 2π or 4π or -2π to any angle! Those go full circles so the final point is the same. This explains why there are infinitely many choices of θ. Often we select the angle between zero and 2π radians. But $-\theta$ is very useful for the conjugate $\bar{z}$.

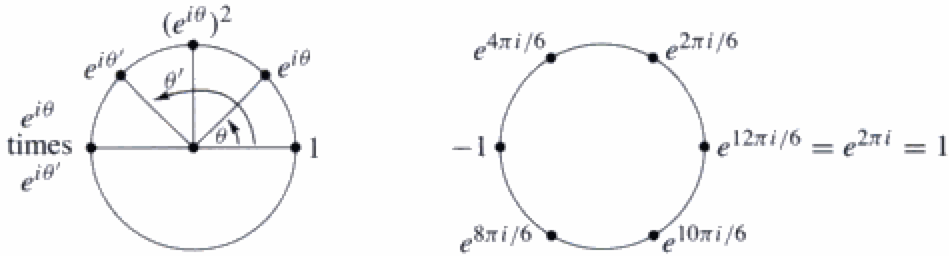


Figure 10.3 (a) Multiplying $e^{i\theta}$ times $e^{i\theta'}$. (b) The 6th power of $e^{2\pi i/6}$ is $e^{2\pi i} = 1$.

The special choice $\theta = 2\pi$ gives $\cos 2\pi + i \sin 2\pi$ which is 1. Somehow the infinite series $e^{2\pi i} = 1 + 2\pi i + \frac{1}{2}(2\pi i)^2 + \cdots$ adds up to 1.

Now multiply $e^{i\theta}$ times $e^{i\theta'}$. Angles add for the same reason that exponents add:

$$\begin{array}{r} e^2 \text{ times } e^3 \text{ is } e^5 \text{ because } (e)(e) \times (e)(e)(e) = (e)(e)(e)(e)(e) \\ e^{i\theta} \text{ times } e^{i\theta} \text{ is } e^{2i\theta} \qquad e^{i\theta} \text{ times } e^{i\theta'} \text{ is } e^{i(\theta+\theta')}. \end{array} \tag{6}$$

Every complex number $a + ib = r\cos\theta + ir\sin\theta$ now goes into its best possible form. That form is $re^{i\theta}$.

The powers $(re^{i\theta})^n$ are equal to $r^n e^{in\theta}$. They stay on the unit circle when $r = 1$ and $r^n = 1$. Then we find n different numbers whose nth powers equal 1:

Set $\boldsymbol{w = e^{2\pi i/n}}$. ***The nth powers of*** $\boldsymbol{1, w, w^2, \ldots, w^{n-1}}$ ***all equal 1.***

Those are the "nth roots of 1." They solve the equation $z^n = 1$. They are equally spaced around the unit circle in Figure 10.3b, where the full 2π is divided by n. Multiply their angles by n to take nth powers. That gives $w^n = e^{2\pi i}$ which is 1. Also $(w^2)^n = e^{4\pi i} = 1$. Each of those numbers, to the nth power, comes around the unit circle to 1.

These roots of 1 are the key numbers for signal processing. A real digital computer uses only 0 and 1. The complex Fourier transform uses w and its powers. The last section of the book shows how to decompose a vector (a signal) into n frequencies by the Fast Fourier Transform.

Problem Set 10.1

Questions 1–8 are about operations on complex numbers.

1 Add and multiply each pair of complex numbers:

(a) $2+i, 2-i$ (b) $-1+i, -1+i$ (c) $\cos\theta + i\sin\theta, \cos\theta - i\sin\theta$

2 Locate these points on the complex plane. Simplify them if necessary:

(a) $2+i$ (b) $(2+i)^2$ (c) $\frac{1}{2+i}$ (d) $|2+i|$

3 Find the absolute value $r = |z|$ of these four numbers. If θ is the angle for $6-8i$, what are the angles for the other three numbers?

(a) $6-8i$ (b) $(6-8i)^2$ (c) $\frac{1}{6-8i}$ (d) $(6+8i)^2$

4 If $|z| = 2$ and $|w| = 3$ then $|z \times w| =$ _____ and $|z+w| \le$ _____ and $|z/w| =$ _____ and $|z-w| \le$ _____.

5 Find $a+ib$ for the numbers at angles $30°, 60°, 90°, 120°$ on the unit circle. If w is the number at $30°$, check that w^2 is at $60°$. What power of w equals 1?

6 If $z = r\cos\theta + ir\sin\theta$ then $1/z$ has absolute value _____ and angle _____. Its polar form is _____. Multiply $z \times 1/z$ to get 1.

7 The 1 by 1 complex multiplication $M = (a+bi)(c+di)$ is a 2 by 2 real multiplication

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}.$$

The right side contains the real and imaginary parts of M. Test $M = (1+3i)(1-3i)$.

8 $A = A_1 + iA_2$ is a complex n by n matrix and $\boldsymbol{b} = \boldsymbol{b}_1 + i\boldsymbol{b}_2$ is a complex vector. The solution to $A\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x}_1 + i\boldsymbol{x}_2$. Write $A\boldsymbol{x} = \boldsymbol{b}$ as a real system of size $2n$:

$$\begin{bmatrix} \quad & \quad \end{bmatrix}\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{bmatrix}.$$

Questions 9–16 are about the conjugate $\bar{z} = a - ib = re^{-i\theta}$ of the number $z = a + ib = re^{i\theta}$.

9 Write down the complex conjugate of each number by changing i to $-i$:

(a) $2-i$ (b) $(2-i)(1-i)$ (c) $e^{i\pi/2}$ (which is i)
(d) $e^{i\pi} = -1$ (e) $\frac{1+i}{1-i}$ (which is also i) (f) $i^{103} =$ _____.

10 The sum $z+\bar{z}$ is always _____. The difference $z-\bar{z}$ is always _____. Assume $z \neq 0$. The product $z \times \bar{z}$ is always _____. The ratio $z/\bar{z}$ always has absolute value _____.

11 For a real 3 by 3 matrix, the numbers a_2, a_1, a_0 from the determinant are real:

$$\det(A-\lambda I) = -\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0.$$

Each root λ is an eigenvalue. Taking conjugates gives $-\bar{\lambda}^3 + a_2\bar{\lambda}^2 + a_1\bar{\lambda} + a_0 = 0$, so $\bar{\lambda}$ is also an eigenvalue. For the matrix with $a_{ij} = i - j$, find $\det(A-\lambda I)$ and the three eigenvalues.

Note The conjugate of $A\boldsymbol{x} = \lambda\boldsymbol{x}$ is $A\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}$. This proves two things: $\bar{\lambda}$ is an eigenvalue and $\bar{\boldsymbol{x}}$ is its eigenvector. Problem 11 only proves that $\bar{\lambda}$ is an eigenvalue.

12 The eigenvalues of a real 2 by 2 matrix come from the quadratic formula:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

gives the two eigenvalues (notice the $\pm$ symbol):

$$\lambda = \frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$

(a) If $a = b = d = 1$, the eigenvalues are complex when c is _____.

(b) What are the eigenvalues when $ad = bc$?

(c) The two eigenvalues (plus sign and minus sign) are not always conjugates of each other. Why not?

13 In Problem 12 the eigenvalues are not real when $(\text{trace})^2 = (a+d)^2$ is smaller than _____. Show that the λ's *are* real when $bc > 0$.

14 Find the eigenvalues and eigenvectors of this permutation matrix:

$$P_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{has} \quad \det(P_4 - \lambda I) = \_\_\_\_\_.$$

15 Extend P_4 above to P_6 (five 1's below the diagonal and one in the corner). Find $\det(P_6 - \lambda I)$ and the six eigenvalues in the complex plane.

16 A real skew-symmetric matrix ($A^{\mathrm{T}} = -A$) has pure imaginary eigenvalues. First proof: If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then block multiplication gives

$$\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ i\boldsymbol{x} \end{bmatrix} = i\lambda \begin{bmatrix} \boldsymbol{x} \\ i\boldsymbol{x} \end{bmatrix}.$$

This block matrix is symmetric. Its eigenvalues must be _____! So λ is _____.

Questions 17–24 are about the form $re^{i\theta}$ of the complex number $r\cos\theta + ir\sin\theta$.

17 Write these numbers in Euler's form $re^{i\theta}$. Then square each number:

(a) $1+\sqrt{3}i$ (b) $\cos 2\theta + i\sin 2\theta$ (c) $-7i$ (d) $5-5i$.

18 Find the absolute value and the angle for $z = \sin\theta + i\cos\theta$ (careful). Locate this z in the complex plane. Multiply z by $\cos\theta + i\sin\theta$ to get _____.

19 Draw all eight solutions of $z^8 = 1$ in the complex plane. What are the rectangular forms $a+ib$ of these eight numbers?

20 Locate the cube roots of 1 in the complex plane. Locate the cube roots of -1. Together these are the sixth roots of _____.

21 By comparing $e^{3i\theta} = \cos 3\theta + i\sin 3\theta$ with $(e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$, find the "triple angle" formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

22 Suppose the conjugate $\bar{z}$ is equal to the reciprocal $1/z$. What are all possible z's?

23 (a) Why do e^i and i^e both have absolute value 1?

(b) In the complex plane put stars near the points e^i and i^e.

(c) The number i^e could be $(e^{i\pi/2})^e$ or $(e^{5i\pi/2})^e$. Are those equal?

24 Draw the paths of these numbers from $t=0$ to $t=2\pi$ in the complex plane:

(a) e^{it} (b) $e^{(-1+i)t} = e^{-t}e^{it}$ (c) $(-1)^t = e^{t\pi i}$.

inner product of z with itself. To make that happen, the complex inner product should use the conjugate transpose (not just the transpose). There will be no effect when the vectors are real, but there is a definite effect when they are complex:

DEFINITION The inner product of real or complex vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$:

$$\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \begin{bmatrix} \overline{u}_1 & \cdots & \overline{u}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \overline{u}_1 v_1 + \cdots + \overline{u}_n v_n. \tag{3}$$

With complex vectors, $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ is different from $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$. *The order of the vectors is now important.* In fact $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u} = \overline{v}_1 u_1 + \cdots + \overline{v}_n u_n$ is the complex conjugate of $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$. We have to put up with a few inconveniences for the greater good.

Example 1 The inner product of $\boldsymbol{u} = \begin{bmatrix} 1 \\ i \end{bmatrix}$ with $\boldsymbol{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & -i \end{bmatrix}\begin{bmatrix} i \\ 1 \end{bmatrix} = 0$. *Not* $2i$.

Example 2 The inner product of $\boldsymbol{u} = \begin{bmatrix} 1+i \\ 0 \end{bmatrix}$ with $\boldsymbol{v} = \begin{bmatrix} 2 \\ i \end{bmatrix}$ is $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 2 - 2i$.

Example 1 is surprising. Those vectors $(1, i)$ and $(i, 1)$ don't look perpendicular. But they are. ***A zero inner product still means that the*** (complex) ***vectors are orthogonal.*** Similarly the vector $(1, i)$ is orthogonal to the vector $(1, -i)$. Their inner product is $1 - 1 = 0$. We are correctly getting zero for the inner product—where we would be incorrectly getting zero for the length of $(1, i)$ if we forgot to take the conjugate.

Note We have chosen to conjugate the first vector $\boldsymbol{u}$. Some authors choose the second vector $\boldsymbol{v}$. Their complex inner product would be $\boldsymbol{u}^{\mathrm{T}}\overline{\boldsymbol{v}}$. It is a free choice, as long as we stick to one or the other. We wanted to use the single symbol $^{\mathrm{H}}$ in the next formula too:

The inner product of $A\boldsymbol{u}$ with $\boldsymbol{v}$ equals the inner product of $\boldsymbol{u}$ with $A^{\mathrm{H}}\boldsymbol{v}$:

$$(A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v}). \tag{4}$$

The conjugate of $A\boldsymbol{u}$ is $\overline{A\boldsymbol{u}}$. Transposing it gives $\overline{\boldsymbol{u}}^{\mathrm{T}}\overline{A}^{\mathrm{T}}$ as usual. This is $\boldsymbol{u}^{\mathrm{H}}A^{\mathrm{H}}$. Everything that should work, does work. The rule for $^{\mathrm{H}}$ comes from the rule for $^{\mathrm{T}}$. That applies to products of matrices:

10C *The conjugate transpose of AB is $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$.*

We are constantly using the fact that $(a-ib)(c-id)$ is the conjugate of $(a+ib)(c+id)$.

Among real matrices, the *symmetric matrices* form the most important special class: $A = A^{\mathrm{T}}$. They have real eigenvalues and a full set of orthogonal eigenvectors. The diagonalizing matrix S is an orthogonal matrix Q. Every symmetric matrix can be written as $A = Q\Lambda Q^{-1}$ and also as $A = Q\Lambda Q^{\mathrm{T}}$ (because $Q^{-1} = Q^{\mathrm{T}}$). All this follows from $a_{ij} = a_{ji}$, when A is real.

Among complex matrices, the special class consists of the ***Hermitian matrices***: $A = A^{\mathrm{H}}$. The condition on the entries is now $a_{ij} = \overline{a_{ji}}$. In this case we say that "*A is* Hermitian." *Every real symmetric matrix is Hermitian*, because taking its conjugate has no effect. The next matrix is also Hermitian:

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \qquad \begin{array}{l}\text{The main diagonal is real since } a_{ii} = \overline{a_{ii}}. \\ \text{Across it are conjugates } 3+3i \text{ and } 3-3i.\end{array}$$

This example will illustrate the three crucial properties of all Hermitian matrices.

10D *If $A = A^{\mathrm{H}}$ and z is any vector, the number $z^{\mathrm{H}}Az$ is real.*

Quick proof: $z^{\mathrm{H}}Az$ is certainly 1 by 1. Take its conjugate transpose:

$$(z^{\mathrm{H}}Az)^{\mathrm{H}} = z^{\mathrm{H}}A^{\mathrm{H}}(z^{\mathrm{H}})^{\mathrm{H}} \quad \text{which is } z^{\mathrm{H}}Az \text{ again.}$$

Reversing the order has produced the same 1 by 1 matrix (this used $A = A^{\mathrm{H}}$!) For 1 by 1 matrices, the conjugate transpose is simply the conjugate. So the number $z^{\mathrm{H}}Az$ equals its conjugate and must be real. Here is $z^{\mathrm{H}}Az$ in our example:

$$\begin{bmatrix} \overline{z}_1 & \overline{z}_2 \end{bmatrix} \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2\overline{z}_1 z_1 + 5\overline{z}_2 z_2 + (3-3i)\overline{z}_1 z_2 + (3+3i) z_1 \overline{z}_2.$$

The terms $2|z_1|^2$ and $5|z_2|^2$ from the diagonal are both real. The off-diagonal terms are conjugates of each other—so their sum is real. (The imaginary parts cancel when we add.) The whole expression $z^{\mathrm{H}}Az$ is real.

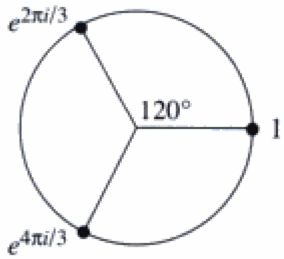
10E *Every eigenvalue of a Hermitian matrix is real.*

Proof Suppose $Az = \lambda z$. *Multiply both sides by z^{H} to get $z^{\mathrm{H}}Az = \lambda z^{\mathrm{H}}z$.* On the left side, $z^{\mathrm{H}}Az$ is real by **10D**. On the right side, $z^{\mathrm{H}}z$ is the length squared, real and positive. So the ratio $\lambda = z^{\mathrm{H}}Az / z^{\mathrm{H}}z$ is a real number. Q.E.D.

The example above has real eigenvalues $\lambda = 8$ and $\lambda = -1$. Take the determinant of $A - \lambda I$ to get $(d-8)(d+1)$:

$$\begin{vmatrix} 2-\lambda & 3-3i \\ 3+3i & 5-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10 - |3+3i|^2$$
$$= \lambda^2 - 7\lambda + 10 - 18 = (\lambda - 8)(\lambda + 1).$$

10F *The eigenvectors of a Hermitian matrix are orthogonal* (provided they correspond to different eigenvalues). If $Az = \lambda z$ and $Ay = \beta y$ then $y^{\mathrm{H}}z = 0$.



Fourier matrix $$F = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

Figure 10.4 The cube roots of 1 go into the Fourier matrix $F = F_3$.

10G *The matrix U has orthonormal columns when $U^{\mathrm{H}}U = I$.*
If U is square, it is a ***unitary matrix***. Then $\boxed{U^{\mathrm{H}} = U^{-1}.}$

$$U^{\mathrm{H}}U = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

Suppose U (with orthogonal column) multiplies any z. The vector length stays the same, because $z^{\mathrm{H}}U^{\mathrm{H}}Uz = z^{\mathrm{H}}z$. If z is an eigenvector, we learn something more: ***The eigenvalues of unitary (and orthogonal) matrices all have absolute value*** $|\lambda| = 1$.

10H *If U is unitary then* $\boxed{\|Uz\| = \|z\|.}$ Therefore $Uz = \lambda z$ leads to $\boxed{|\lambda| = 1.}$

Our 2 by 2 example is both Hermitian ($U = U^{\mathrm{H}}$) and unitary ($U^{-1} = U^{\mathrm{H}}$). That means real eigenvalues ($\lambda = \bar{\lambda}$), and it means absolute value one ($\lambda^{-1} = \bar{\lambda}$). A real number with absolute value 1 has only two possibilities: *The eigenvalues are* 1 *or* -1.

One thing more about the example: The diagonal of U adds to zero. The trace is zero. So one eigenvalue is $\lambda = 1$, the other is $\lambda = -1$. The determinant must be 1 times -1, the product of the λ's.

Example 3 The 3 by 3 ***Fourier matrix*** is in Figure 10.4. Is it Hermitian? Is it unitary? The Fourier matrix is certainly symmetric. It equals its transpose. But it doesn't equal its conjugate transpose—*it is not Hermitian*. If you change i to $-i$, you get a different matrix.

Is F unitary? *Yes*. The squared length of every column is $\frac{1}{3}(1+1+1)$. The columns are unit vectors. The first column is orthogonal to the second column because $1 + e^{2\pi i/3} + e^{4\pi i/3} = 0$. This is the sum of the three numbers marked in Figure 10.4.

Notice the symmetry of the figure. If you rotate it by 120°, the three points are in the same position. Therefore their sum S also stays in the same position! The only possible sum is $S = 0$, because this is the only point that is in the same position after 120° rotation.

Is column 2 of F orthogonal to column 3? Their dot product looks like

$$\tfrac{1}{3}(1 + e^{6\pi i/3} + e^{6\pi i/3}) = \tfrac{1}{3}(1+1+1).$$

This is not zero. That is because we forgot to take complex conjugates! The complex inner product uses $^{\mathrm{H}}$ not $^{\mathrm{T}}$:

$$\begin{aligned}(\text{column } 2)^{\mathrm{H}}(\text{column } 3) &= \tfrac{1}{3}(1\cdot 1 + e^{-2\pi i/3}e^{4\pi i/3} + e^{-4\pi i/3}e^{2\pi i/3}) \\ &= \tfrac{1}{3}(1 + e^{2\pi i/3} + e^{-2\pi i/3}) = 0.\end{aligned}$$

So we do have orthogonality. ***Conclusion: F is a unitary matrix.***

The next section will study the n by n Fourier matrices. Among all complex unitary matrices, these are the most important. When we multiply a vector by F, we are computing its ***discrete Fourier transform***. When we multiply by F^{-1}, we are computing the *inverse transform*. The special property of unitary matrices is that $F^{-1} = F^{\mathrm{H}}$. The inverse transform only differs by changing i to $-i$:

$$F^{-1} = F^{\mathrm{H}} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{-2\pi i/3} & e^{-4\pi i/3} \\ 1 & e^{-4\pi i/3} & e^{-2\pi i/3}\end{bmatrix}.$$

Everyone who works with F recognizes its value. The last section of the book will bring together Fourier analysis and linear algebra.

This section ends with a table to translate between real and complex—for vectors and for matrices:

Real versus Complex

$\mathbf{R}^n$: vectors with n real components	$\leftrightarrow$	$\mathbf{C}^n$: vectors with n complex components
length: $\lVert \boldsymbol{x}\rVert^2 = x_1^2 + \cdots + x_n^2$	$\leftrightarrow$	length: $\lVert z\rVert^2 = \lvert z_1\rvert^2 + \cdots + \lvert z_n\rvert^2$
$(A^{\mathrm{T}})_{ij} = A_{ji}$	$\leftrightarrow$	$(A^{\mathrm{H}})_{ij} = \overline{A_{ji}}$
$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	$\leftrightarrow$	$(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
dot product: $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + \cdots + x_ny_n$	$\leftrightarrow$	inner product: $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = \overline{u}_1v_1 + \cdots + \overline{u}_nv_n$
$(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y})$	$\leftrightarrow$	$(A\boldsymbol{u})^{\mathrm{H}}\boldsymbol{v} = \boldsymbol{u}^{\mathrm{H}}(A^{\mathrm{H}}\boldsymbol{v})$
orthogonality: $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = 0$	$\leftrightarrow$	orthogonality: $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v} = 0$
symmetric matrices: $A = A^{\mathrm{T}}$	$\leftrightarrow$	Hermitian matrices: $A = A^{\mathrm{H}}$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$(real Λ)	$\leftrightarrow$	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric matrices: $K^{\mathrm{T}} = -K$	$\leftrightarrow$	skew-Hermitian matrices $K^{\mathrm{H}} = -K$
orthogonal matrices: $Q^{\mathrm{T}} = Q^{-1}$	$\leftrightarrow$	unitary matrices: $U^{\mathrm{H}} = U^{-1}$
orthonormal columns: $Q^{\mathrm{T}}Q = I$	$\leftrightarrow$	orthonormal columns: $U^{\mathrm{H}}U = I$
$(Q\boldsymbol{x})^{\mathrm{T}}(Q\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ and $\lVert Q\boldsymbol{x}\rVert = \lVert\boldsymbol{x}\rVert$	$\leftrightarrow$	$(U\boldsymbol{x})^{\mathrm{H}}(U\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$ and $\lVert Uz\rVert = \lVert z\rVert$

The columns and also the eigenvectors of Q and U are orthonormal. Every $|\lambda| = 1$.

Problem Set 10.2

1 Find the lengths of $\boldsymbol{u} = (1+i, 1-i, 1+2i)$ and $\boldsymbol{v} = (i, i, i)$. Also find $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{v}$ and $\boldsymbol{v}^{\mathrm{H}}\boldsymbol{u}$.

2 Compute $A^{\mathrm{H}}A$ and AA^{H}. Those are both _____ matrices:

$$A = \begin{bmatrix} i & 1 & i \\ 1 & i & i \end{bmatrix}.$$

3 Solve $Az = \mathbf{0}$ to find a vector in the nullspace of A in Problem 2. Show that z is orthogonal to the columns of A^{H}. Show that z is *not* orthogonal to the columns of A^{T}.

4 Problem 3 indicates that the four fundamental subspaces are $\boldsymbol{C}(A)$ and $\boldsymbol{N}(A)$ and _____ and _____. Their dimensions are still r and $n-r$ and r and $m-r$. They are still orthogonal subspaces. *The symbol* $^{\mathrm{H}}$ *takes the place of* $^{\mathrm{T}}$.

5 (a) Prove that $A^{\mathrm{H}}A$ is always a Hermitian matrix.

(b) If $Az = \mathbf{0}$ then $A^{\mathrm{H}}Az = \mathbf{0}$. If $A^{\mathrm{H}}Az = \mathbf{0}$, multiply by z^{H} to prove that $Az = \mathbf{0}$. The nullspaces of A and $A^{\mathrm{H}}A$ are _____. Therefore $A^{\mathrm{H}}A$ is an invertible Hermitian matrix when the nullspace of A contains only $z =$ _____.

6 True or false (give a reason if true or a counterexample if false):

(a) If A is a real matrix then $A + iI$ is invertible.

(b) If A is a Hermitian matrix then $A + iI$ is invertible.

(c) If U is a unitary matrix then $A + iI$ is invertible.

7 When you multiply a Hermitian matrix by a real number c, is cA still Hermitian? If $c = i$ show that iA is skew-Hermitian. The 3 by 3 Hermitian matrices are a subspace provided the "scalars" are real numbers.

8 Which classes of matrices does P belong to: orthogonal, invertible, Hermitian, unitary, factorizable into LU, factorizable into QR?

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

9 Compute P^2, P^3, and P^{100} in Problem 8. What are the eigenvalues of P?

10 Find the unit eigenvectors of P in Problem 8, and put them into the columns of a unitary matrix F. What property of P makes these eigenvectors orthogonal?

25 How are the eigenvalues of A^{H} related to the eigenvalues of the square complex matrix A?

26 If $\boldsymbol{u}^{\mathrm{H}}\boldsymbol{u} = 1$ show that $I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is Hermitian and also unitary. The rank-one matrix $\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}}$ is the projection onto what line in $\mathbf{C}^n$?

27 If $A + iB$ is a unitary matrix (A and B are real) show that $Q = \left[\begin{smallmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{smallmatrix}\right]$ is an orthogonal matrix.

28 If $A + iB$ is a Hermitian matrix (A and B are real) show that $\left[\begin{smallmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{smallmatrix}\right]$ is symmetric.

29 Prove that the inverse of a Hermitian matrix is a Hermitian matrix.

30 Diagonalize this matrix by constructing its eigenvalue matrix Λ and its eigenvector matrix S:

$$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix} = A^{\mathrm{H}}.$$

31 A matrix with orthonormal eigenvectors has the form $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$. *Prove that* $AA^{\mathrm{H}} = A^{\mathrm{H}}A$. These are exactly the *normal matrices.*

THE FAST FOURIER TRANSFORM ■ 10.3

Many applications of linear algebra take time to develop. It is not easy to explain them in an hour. The teacher and the author must choose between completing the theory and adding new applications. Generally the theory wins, because this course is the best chance to make it clear—and the importance of any one application seems limited. This section is almost an exception, because the importance of Fourier transforms is almost unlimited.

More than that, the algebra is basic. ***We want to multiply quickly by F and F^{-1}, the Fourier matrix and its inverse.*** This is achieved by the Fast Fourier Transform—the most valuable numerical algorithm in our lifetime.

The FFT has revolutionized signal processing. Whole industries are speeded up by this one idea. Electrical engineers are the first to know the difference—they take your Fourier transform as they meet you (if you are a function). Fourier's idea is to represent f as a sum of harmonics $c_k e^{ikx}$. The function is seen in *frequency space* through the coefficients c_k, instead of *physical space* through its values $f(x)$. The passage backward and forward between c's and f's is by the Fourier transform. Fast passage is by the FFT.

An ordinary product $F\boldsymbol{c}$ uses n^2 multiplications (the matrix has n^2 nonzero entries). The Fast Fourier Transform needs only n times $\frac{1}{2}\log_2 n$. We will see how.

Roots of Unity and the Fourier Matrix

Quadratic equations have two roots (or one repeated root). Equations of degree n have n roots (counting repetitions). This is the Fundamental Theorem of Algebra, and to make it true we must allow complex roots. This section is about the very special equation $z^n = 1$. The solutions z are the "nth roots of unity." They are n evenly spaced points around the unit circle in the complex plane.

Figure 10.5 shows the eight solutions to $z^8 = 1$. Their spacing is $\frac{1}{8}(360°) = 45°$. The first root is at $45°$ or $\theta = 2\pi/8$ radians. ***It is the complex number*** $w = e^{i\theta} = e^{i2\pi/8}$. We call this number w_8 to emphasize that it is an 8th root. You could write it in terms of $\cos\frac{2\pi}{8}$ and $\sin\frac{2\pi}{8}$, but don't do it. The seven other 8th roots are $w^2, w^3, \dots, w^8$, going around the circle. Powers of w are best in polar form, because we work only with the angle.

The fourth roots of 1 are also in the figure. They are $i, -1, -i, 1$. The angle is now $2\pi/4$ or $90°$. The first root $w_4 = e^{2\pi i/4}$ is nothing but i. Even the square roots of 1 are seen, with $w_2 = e^{i2\pi/2} = -1$. Do not despise those square roots 1 and -1. The idea behind the FFT is to go from an 8 by 8 Fourier matrix (containing powers of w_8) to the 4 by 4 matrix below (with powers of $w_4 = i$). The same idea goes from 4 to 2. By exploiting the connections of F_8 down to F_4 and up to F_{16} (and beyond), the FFT makes multiplication by F_{1024} very quick.

We describe the *Fourier matrix*, first for $n = 4$. Its rows contain powers of 1 and w and w^2 and w^3. These are the fourth roots of 1, and their powers come in a special order:

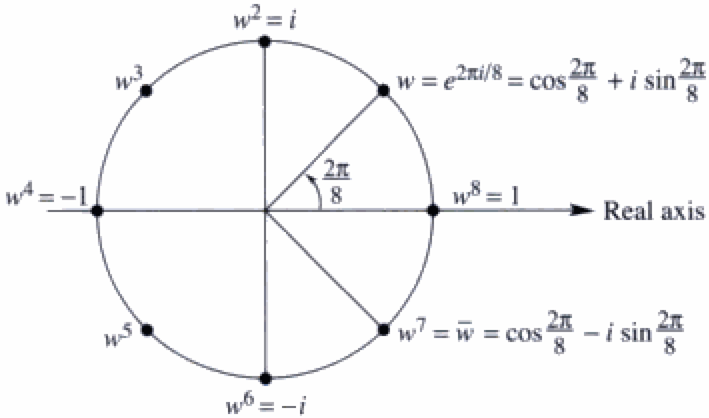


Figure 10.5 The eight solutions to $z^8 = 1$ are $1, w, w^2, \ldots, w^7$ with $w = (1+i)/\sqrt{2}$.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}.$$

The matrix is symmetric ($F = F^{\mathrm{T}}$). It is *not* Hermitian. Its main diagonal is not real. But $\frac{1}{2}F$ is a ***unitary matrix***, which means that $(\frac{1}{2}F^{\mathrm{H}})(\frac{1}{2}F) = I$:

The columns of F give $F^{\mathrm{H}}F = 4I$. The inverse of F is $\frac{1}{4}F^{\mathrm{H}}$ which is $\frac{1}{4}\overline{F}$.

The inverse changes from $w = i$ to $\overline{w} = -i$. That takes us from F to $\overline{F}$. When the Fast Fourier Transform gives a quick way to multiply by F_4, it does the same for the inverse.

The unitary matrix is $U = F/\sqrt{n}$. We prefer to avoid that $\sqrt{n}$ and just put $\frac{1}{n}$ outside F^{-1}. The main point is to multiply the matrix F times the coefficients in the Fourier series $c_0 + c_1e^{ix} + c_2e^{2ix} + c_3e^{3ix}$:

$$F\boldsymbol{c} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1w + c_2w^2 + c_3w^3 \\ c_0 + c_1w^2 + c_2w^4 + c_3w^6 \\ c_0 + c_1w^3 + c_2w^6 + c_3w^9 \end{bmatrix}. \qquad (1)$$

The input is four complex coefficients c_0, c_1, c_2, c_3. The output is four function values y_0, y_1, y_2, y_3. The first output $y_0 = c_0 + c_1 + c_2 + c_3$ is the value of the Fourier series at $x = 0$. *The second output is the value of that series $\sum c_ke^{ikx}$ at $x = 2\pi/4$:*

$$y_1 = c_0 + c_1e^{i2\pi/4} + c_2e^{i4\pi/4} + c_3e^{i6\pi/4} = c_0 + c_1w + c_2w^2 + c_3w^3.$$

The third and fourth outputs y_2 and y_3 are the values of $\sum c_ke^{ikx}$ at $x = 4\pi/4$ and $x = 6\pi/4$. These are *finite* Fourier series! They contain $n = 4$ terms and they are evaluated at $n = 4$ points. Those points $x = 0, 2\pi/4, 4\pi/4, 6\pi/4$ are equally spaced.

The next point would be $x = 8\pi/4$ which is 2π. Then the series is back to y_0, because $e^{2\pi i}$ is the same as $e^0 = 1$. Everything cycles around with period 4. In this world $2+2$ is 0 because $(w^2)(w^2) = w^0 = 1$. In matrix shorthand, F times $\boldsymbol{c}$ gives a column vector $\boldsymbol{y}$. The four y's come from evaluating the series at the four x's with spacing $2\pi/4$:

$$\boldsymbol{y} = F\boldsymbol{c} \text{ produces } y_j = \sum_{k=0}^{3} c_k e^{ik(2\pi j/4)} = \text{ the value of the series at } x = \frac{2\pi j}{4}.$$

We will follow the convention that j and k go from 0 to $n-1$ (instead of 1 to n). The "zeroth row" and "zeroth column" of F contain all ones.

The n by n Fourier matrix contains powers of $w = e^{2\pi i/n}$:

$$F_n\boldsymbol{c} = \begin{bmatrix} 1 & 1 & 1 & \cdot & 1 \\ 1 & w & w^2 & \cdot & w^{n-1} \\ 1 & w^2 & w^4 & \cdot & w^{2(n-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & w^{n-1} & w^{2(n-1)} & \cdot & w^{(n-1)^2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \cdot \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ y_{n-1} \end{bmatrix}. \tag{2}$$

F_n is symmetric but not Hermitian. ***Its columns are orthogonal***, and $F_n\overline{F}_n = nI$. *Then* F_n^{-1} *is* $\overline{F}_n/n$. The inverse contains powers of $\overline{w}_n = e^{-2\pi i/n}$. Look at the pattern in F:

The entry in row j, column k is w^{jk}. Row zero and column zero contain $w^0 = 1$.

The zeroth output is $y_0 = c_0 + c_1 + \cdots + c_{n-1}$. This is the series $\sum c_k e^{ikx}$ at $x = 0$. When we multiply $\boldsymbol{c}$ by F_n, we sum the series at n points. *When we multiply $\boldsymbol{y}$ by F_n^{-1}, we find the coefficients $\boldsymbol{c}$ from the function values $\boldsymbol{y}$.* The matrix F passes from "frequency space" to "physical space." F^{-1} returns from the function values $\boldsymbol{y}$ to the Fourier coefficients $\boldsymbol{c}$.

One Step of the Fast Fourier Transform

We want to multiply F times $\boldsymbol{c}$ as quickly as possible. Normally a matrix times a vector takes n^2 separate multiplications—the matrix has n^2 entries. You might think it is impossible to do better. (If the matrix has zero entries then multiplications can be skipped. But the Fourier matrix has no zeros!) By using the special pattern w^{jk} for its entries, F can be factored in a way that produces many zeros. This is the **FFT**.

The key idea is to connect F_n with the half-size Fourier matrix $F_{n/2}$. Assume that n is a power of 2 (say $n = 2^{10} = 1024$). We will connect F_{1024} to F_{512}—or rather to ***two copies of*** F_{512}. When $n = 4$, the key is in the relation between the matrices

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_2 & \\ & F_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix}.$$

On the left is F_4, with no zeros. On the right is a matrix that is half zero. The work is cut in half. But wait, those matrices are not the same. The block matrix with two copies of the half-size F is one piece of the picture but not the only piece. Here is the factorization of F_4 with many zeros:

$$F_4 = \begin{bmatrix} 1 & & 1 & \\ & 1 & & i \\ 1 & & -1 & \\ & 1 & & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & & \\ 1 & i^2 & & \\ & & 1 & 1 \\ & & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}. \tag{3}$$

The matrix on the right is a permutation. It puts the even c's (c_0 and c_2) ahead of the odd c's (c_1 and c_3). The middle matrix performs separate half-size transforms on the evens and odds. The matrix at the left combines the two half-size outputs—in a way that produces the correct full-size output $\boldsymbol{y} = F_4\boldsymbol{c}$. You could multiply those three matrices to see that their product is F_4.

The same idea applies when $n = 1024$ and $m = \frac{1}{2}n = 512$. The number w is $e^{2\pi i/1024}$. It is at the angle $\theta = 2\pi/1024$ on the unit circle. The Fourier matrix F_{1024} is full of powers of w. The first stage of the FFT is the great factorization discovered by Cooley and Tukey (and foreshadowed in 1805 by Gauss):

$$F_{1024} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} & \\ & F_{512} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}. \tag{4}$$

I_{512} is the identity matrix. D_{512} is the diagonal matrix with entries $(1, w, \ldots, w^{511})$. The two copies of F_{512} are what we expected. Don't forget that they use the 512th root of unity (which is nothing but w^2!!) The permutation matrix separates the incoming vector $\boldsymbol{c}$ into its even and odd parts $\boldsymbol{c}' = (c_0, c_2, \ldots, c_{1022})$ and $\boldsymbol{c}'' = (c_1, c_3, \ldots, c_{1023})$.

Here are the algebra formulas which say the same thing as the factorization of F_{1024}:

10I (**FFT**) Set $m = \frac{1}{2}n$. The first m and last m components of $\boldsymbol{y} = F_n\boldsymbol{c}$ are combinations of the half-size transforms $\boldsymbol{y}' = F_m\boldsymbol{c}'$ and $\boldsymbol{y}'' = F_m\boldsymbol{c}''$. Equation (4) shows $I\boldsymbol{y}' + D\boldsymbol{y}''$ and $I\boldsymbol{y}' - D\boldsymbol{y}''$:

$$\begin{aligned} y_j &= y'_j + w_n^j y''_j, \quad j = 0, \ldots, m-1 \\ y_{j+m} &= y'_j - w_n^j y''_j, \quad j = 0, \ldots, m-1. \end{aligned} \tag{5}$$

Thus the three steps are: split $\boldsymbol{c}$ into $\boldsymbol{c}'$ and $\boldsymbol{c}''$, transform them by F_m into $\boldsymbol{y}'$ and $\boldsymbol{y}''$, and reconstruct $\boldsymbol{y}$ from equation (5).

You might like the flow graph in Figure 10.6 better than these formulas. The graph for $n = 4$ shows $\boldsymbol{c}'$ and $\boldsymbol{c}''$ going through the half-size F_2. Those steps are called "*butterflies*," from their shape. Then the outputs from the F_2's are combined using the I and D matrices to produce $\boldsymbol{y} = F_4\boldsymbol{c}$:

One last note about this remarkable algorithm. There is an amazing rule for the order that the c's enter the **FFT**, after all the even-odd permutations. Write the numbers 0 to $n-1$ in binary (base 2). *Reverse the order of their digits.* The complete picture shows the bit-reversed order at the start, the $l = \log_2 n$ steps of the recursion, and the final output $y_0, \ldots, y_{n-1}$ which is F_n times $\boldsymbol{c}$. The book ends with that very fundamental idea, a matrix multiplying a vector.

Thank you for studying linear algebra. I hope you enjoyed it, and I very much hope you will use it. It was a pleasure to write about this neat subject.

Problem Set 10.3

1 Multiply the three matrices in equation (3) and compare with F. In which six entries do you need to know that $i^2 = -1$?

2 Invert the three factors in equation (3) to find a fast factorization of F^{-1}.

3 F is symmetric. So transpose equation (3) to find a new Fast Fourier Transform!

4 All entries in the factorization of F_6 involve powers of $w =$ sixth root of 1:

$$F_6 = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_3 & \\ & F_3 \end{bmatrix} \begin{bmatrix} P \end{bmatrix}.$$

Write down these three factors with $1, w, w^2$ in D and powers of w^2 in F_3. Multiply!

5 If $\boldsymbol{v} = (1, 0, 0, 0)$ and $\boldsymbol{w} = (1, 1, 1, 1)$, show that $F\boldsymbol{v} = \boldsymbol{w}$ and $F\boldsymbol{w} = 4\boldsymbol{v}$. Therefore $F^{-1}\boldsymbol{w} = \boldsymbol{v}$ and $F^{-1}\boldsymbol{v} =$ _____.

6 What is F^2 and what is F^4 for the 4 by 4 Fourier matrix?

7 Put the vector $\boldsymbol{c} = (1, 0, 1, 0)$ through the three steps of the **FFT** to find $\boldsymbol{y} = F\boldsymbol{c}$. Do the same for $\boldsymbol{c} = (0, 1, 0, 1)$.

8 Compute $\boldsymbol{y} = F_8\boldsymbol{c}$ by the three **FFT** steps for $\boldsymbol{c} = (1, 0, 1, 0, 1, 0, 1, 0)$. Repeat the computation for $\boldsymbol{c} = (0, 1, 0, 1, 0, 1, 0, 1)$.

9 If $w = e^{2\pi i/64}$ then w^2 and $\sqrt{w}$ are among the _____ and _____ roots of 1.

10 (a) Draw all the sixth roots of 1 on the unit circle. Prove they add to zero.

(b) What are the three cube roots of 1? Do they also add to zero?

11 The columns of the Fourier matrix F are the *eigenvectors* of the cyclic permutation P. Multiply PF to find the eigenvalues λ_1 to λ_4:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}.$$

This is $PF = F\Lambda$ or $P = F\Lambda F^{-1}$. The eigenvector matrix (usually S) is F.

12 The equation $\det(P - \lambda I) = 0$ is $\lambda^4 = 1$. This shows again that the eigenvalue matrix Λ is ______. Which permutation P has eigenvalues = cube roots of 1?

13 (a) Two eigenvectors of C are $(1, 1, 1, 1)$ and $(1, i, i^2, i^3)$. What are the eigenvalues?

$$\begin{bmatrix} c_0 & c_1 & c_2 & c_3 \\ c_3 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = e_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad C \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix} = e_2 \begin{bmatrix} 1 \\ i \\ i^2 \\ i^3 \end{bmatrix}.$$

(b) $P = F\Lambda F^{-1}$ immediately gives $P^2 = F\Lambda^2 F^{-1}$ and $P^3 = F\Lambda^3 F^{-1}$. Then $C = c_0 I + c_1 P + c_2 P^2 + c_3 P^3 = F(c_0 I + c_1 \Lambda + c_2 \Lambda^2 + c_3 \Lambda^3) F^{-1} = FEF^{-1}$. That matrix E in parentheses is diagonal. It contains the ______ of C.

14 Find the eigenvalues of the "periodic" $-1, 2, -1$ matrix from $E = 2I - \Lambda - \Lambda^3$, with the eigenvalues of P in Λ. The -1's in the corners make this matrix periodic:

$$C = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \quad \text{has } c_0 = 2, c_1 = -1, c_2 = 0, c_3 = -1.$$

15 To multiply C times a vector $\boldsymbol{x}$, we can multiply $F(E(F^{-1}\boldsymbol{x}))$ instead. The direct way uses n^2 separate multiplications. Knowing E and F, the second way uses only $n \log_2 n + n$ multiplications. How many of those come from E, how many from F, and how many from F^{-1}?

16 How could you quickly compute these four components of $F\boldsymbol{c}$ starting from $c_0 + c_2, c_0 - c_2, c_1 + c_3, c_1 - c_3$? You are finding the Fast Fourier Transform!

$$F\boldsymbol{c} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + ic_1 + i^2 c_2 + i^3 c_3 \\ c_0 + i^2 c_1 + i^4 c_2 + i^6 c_3 \\ c_0 + i^3 c_1 + i^6 c_2 + i^9 c_3 \end{bmatrix}.$$

SOLUTIONS TO SELECTED EXERCISES

Problem Set 1.1, page 7

4 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $\boldsymbol{v} - 3\boldsymbol{w} = (-1, -5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

6 The components of every $c\boldsymbol{v} + d\boldsymbol{w}$ add to zero. Choose $c = 4$ and $d = 10$ to get $(4, 2, -6)$.

9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$.

11 Five more corners $(0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. The centers of the six faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2})$, $(\frac{1}{2}, 1, \frac{1}{2})$.

12 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional sides and 24 two-dimensional faces and 32 one-dimensional edges. See Worked Example **2.4 A**.

13 sum = zero vector; sum = −4:00 vector; 1:00 is 60° from horizontal = $(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

16 All combinations with $c + d = 1$ are on the line through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V} = -\boldsymbol{v} + 2\boldsymbol{w}$ is on that line beyond $\boldsymbol{w}$.

17 The vectors $c\boldsymbol{v} + c\boldsymbol{w}$ fill out the line passing through $(0, 0)$ and $\boldsymbol{u} = \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w}$. It continues beyond $\boldsymbol{v} + \boldsymbol{w}$ and $(0, 0)$. With $c \geq 0$, half this line is removed and the "ray" starts at $(0, 0)$.

20 (a) $\frac{1}{3}\boldsymbol{u} + \frac{1}{3}\boldsymbol{v} + \frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{w}$ is the center of the edge between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill in the triangle keep $c \geq 0$, $d \geq 0$, $e \geq 0$, and $c + d + e = 1$.

22 The vector $\frac{1}{2}(\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{w})$ is *outside* the pyramid because $c + d + e = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} > 1$.

25 (a) Choose $\boldsymbol{u} = \boldsymbol{v} = \boldsymbol{w}$ = any nonzero vector (b) Choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions, and $\boldsymbol{w}$ to be a combination like $\boldsymbol{u} + \boldsymbol{v}$.

28 An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. The ratios a/c and b/d are equal. Then $ad = bc$. Then (divide by bd) the ratios a/b and c/d are equal!

Problem Set 1.2, page 17

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{3}{5}, \frac{4}{5}) = (.6, .8)$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (\frac{4}{5}, \frac{3}{5}) = (.8, .6)$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|} \cdot \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = \frac{24}{25}$. The vectors $\boldsymbol{w}, \boldsymbol{u}, -\boldsymbol{w}$ make $0°, 90°, 180°$ angles with $\boldsymbol{w}$.

5 (a) $\boldsymbol{v} \cdot (-\boldsymbol{v}) = -1$ (b) $(\boldsymbol{v}+\boldsymbol{w}) \cdot (\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{w} - \boldsymbol{w} \cdot \boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = 0$ so $\theta = 90°$ (c) $(\boldsymbol{v} - 2\boldsymbol{w}) \cdot (\boldsymbol{v} + 2\boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 4\boldsymbol{w} \cdot \boldsymbol{w} = -3$

7 All vectors $\boldsymbol{w} = (c, 2c)$; all vectors (x, y, z) with $x + y + z = 0$ lie on a *plane*; all vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a *line*.

9 If $v_2 w_2 / v_1 w_1 = -1$ then $v_2 w_2 = -v_1 w_1$ or $v_1 w_1 + v_2 w_2 = 0$.

11 $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ means angle $> 90°$; this is half of the plane.

12 $(1, 1)$ perpendicular to $(1, 5) - c(1, 1)$ if $6 - 2c = 0$ or $c = 3$; $\boldsymbol{v} \cdot (\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v} \cdot \boldsymbol{w} / \boldsymbol{v} \cdot \boldsymbol{v}$.

15 $\frac{1}{2}(x + y) = 5$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = .8$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

21 $2\boldsymbol{v} \cdot \boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v} + \boldsymbol{w}\|^2 = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w} \le \|\boldsymbol{v}\|^2 + 2\|\boldsymbol{v}\|\|\boldsymbol{w}\| + \|\boldsymbol{w}\|^2 = (\|\boldsymbol{v}\| + \|\boldsymbol{w}\|)^2$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta - a) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = v_1 w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2 w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$.

25 (a) $v_1^2 w_1^2 + 2v_1 w_1 v_2 w_2 + v_2^2 w_2^2 \le v_1^2 w_1^2 + v_1^2 w_2^2 + v_2^2 w_1^2 + v_2^2 w_2^2$ is true because the difference is $v_1^2 w_2^2 + v_2^2 w_1^2 - 2v_1 w_1 v_2 w_2$ which is $(v_1 w_2 - v_2 w_1)^2 \ge 0$.

26 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2 + U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2 + U_2^2)$. The whole line becomes $.96 \le (.6)(.8) + (.8)(.6) \le \frac{1}{2}(.6^2 + .8^2) + \frac{1}{2}(.8^2 + .6^2) = 1$.

28 Try $\boldsymbol{v} = (1, 2, -3)$ and $\boldsymbol{w} = (-3, 1, 2)$ with $\cos\theta = \frac{-7}{14}$ and $\theta = 120°$. Write $\boldsymbol{v} \cdot \boldsymbol{w} = xz + yz + xy$ as $\frac{1}{2}(x + y + z)^2 - \frac{1}{2}(x^2 + y^2 + z^2)$. If $x + y + z = 0$ this is $-\frac{1}{2}(x^2 + y^2 + z^2)$, so $\boldsymbol{v} \cdot \boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -\frac{1}{2}$.

31 Three vectors in the plane could make angles $> 90°$ with each other: $(1, 0), (-1, 4), (-1, -4)$. Four vectors could not do this ($360°$ total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$?

Problem Set 2.1, page 30

2 The columns are $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$ and $\boldsymbol{b} = (2, 3, 4) = 2\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$.

3 The planes are the same: $2x = 4$ is $x = 2$, $3y = 9$ is $y = 3$, and $4z = 16$ is $z = 4$. The solution is the same intersection point. The columns are changed; but same combination $\widehat{\boldsymbol{x}} = \boldsymbol{x}$.

5 If $z=2$ then $x+y=0$ and $x-y=z$ give the point $(1,-1,2)$. If $z=0$ then $x+y=6$ and $x-y=4$ give the point $(5,1,0)$. Halfway between is $(3,0,1)$.

7 Equation $1+$ equation $2-$ equation 3 is now $0=-4$. Line misses plane; *no solution.*

9 Four planes in 4-dimensional space normally meet at a *point*. The solution to $A\boldsymbol{x}=(3,3,3,2)$ is $\boldsymbol{x}=(0,0,1,2)$ if A has columns $(1,0,0,0)$, $(1,1,0,0)$, $(1,1,1,0)$, $(1,1,1,1)$. The equations are $x+y+z+t=3$, $y+z+t=3$, $z+t=3$, $t=2$.

15 $2x+3y+z+5t=8$ is $A\boldsymbol{x}=\boldsymbol{b}$ with the 1 by 4 matrix $A=[\,2\ \ 3\ \ 1\ \ 5\,]$. The solutions $\boldsymbol{x}$ fill a 3D "plane" in 4 dimensions.

17 $R=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$, $180°$ rotation from $R^2=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=-I$.

19 $E=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}$, $E=\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$.

23 The dot product $[\,1\ \ 4\ \ 5\,]\begin{bmatrix}x\\ y\\ z\end{bmatrix}=(1\text{ by }3)(3\text{ by }1)$ is zero for points (x,y,z) on a plane in three dimensions. The columns of A are one-dimensional vectors.

24 $A=[\,1\ \ 2\ \ ;\ \ 3\ \ 4\,]$ and $\boldsymbol{x}=[\,5\ \ -2\,]'$ and $\boldsymbol{b}=[\,1\ \ 7\,]'$. $\boldsymbol{r}=\boldsymbol{b}-A*\boldsymbol{x}$ prints as zero.

26 $\text{ones}(4,4)*\text{ones}(4,1)=[\,4\ \ 4\ \ 4\ \ 4\,]'$; $B*\boldsymbol{w}=[\,10\ \ 10\ \ 10\ \ 10\,]'$.

29 The row picture shows four *lines*. The column picture is in *four*-dimensional space. No solution unless the right side is a combination of *the two columns.*

31 $\boldsymbol{u}_7, \boldsymbol{v}_7, \boldsymbol{w}_7$ are all close to $(.6,.4)$. Their components still add to 1.

32 $\begin{bmatrix}.8 & .3\\ .2 & .7\end{bmatrix}\begin{bmatrix}.6\\ .4\end{bmatrix}=\begin{bmatrix}.6\\ .4\end{bmatrix}=$ *steady state* $\boldsymbol{s}$. No change when multiplied by $\begin{bmatrix}.8 & .3\\ .2 & .7\end{bmatrix}$.

34 $M=\begin{bmatrix}8 & 3 & 4\\ 1 & 5 & 9\\ 6 & 7 & 2\end{bmatrix}=\begin{bmatrix}5+u & 5-u+v & 5-v\\ 5-u-v & 5 & 5+u+v\\ 5+v & 5+u-v & 5-u\end{bmatrix}$; $M_3(1,1,1)=(15,15,15)$; $M_4(1,1,1,1)=(34,34,34,34)$ because the numbers 1 to 16 add to 136 which is 4(34).

Problem Set 2.2, page 41

3 Subtract $-\frac{1}{2}$ times equation 1 (or add $\frac{1}{2}$ times equation 1). The new second equation is $3y=3$. Then $y=1$ and $x=5$. If the right side changes sign, so does the solution: $(x,y)=(-5,-1)$.

4 Subtract $l=\frac{c}{a}$ times equation 1. The new second pivot multiplying y is $d-(cb/a)$ or $(ad-bc)/a$. Then $y=(ag-cf)/(ad-bc)$.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 2 \cdot 16 = 32$ makes the system solvable. The lines become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

13 Subtract 2 times row 1 from row 2 to reach $(d-10)y - z = 2$. Equation (3) is $y - z = 3$. If $d = 10$ exchange rows 2 and 3. If $d = 11$ the system is singular; third pivot is missing.

14 The second pivot position will contain $-2 - b$. If $b = -2$ we exchange with row 3. If $b = -1$ (singular case) the second equation is $-y - z = 0$. A solution is $(1, 1, -1)$.

16 If row 1 = row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and there is no *third* pivot. If column 1 = column 2 there is no *second* pivot.

18 Row 2 becomes $3y - 4z = 5$, then row 3 becomes $(q+4)z = t - 5$. If $q = -4$ the system is singular — no third pivot. Then if $t = 5$ the third equation is $0 = 0$. Choosing $z = 1$ the equation $3y - 4z = 5$ gives $y = 3$ and equation 1 gives $x = -9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows $1 + 2 =$ row 3 on the left side but not the right side: for example $x + y + z = 0$, $x - 2y - z = 1$, $2x - y = 1$. No parallel planes but still no solution.

24 $A = \begin{bmatrix} 1 & 1 & 1 \\ a & a+1 & a+1 \\ b & b+c & b+c+3 \end{bmatrix}$ for any a, b, c leads to $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

26 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

29 $A(2,:) = A(2,:) - 3 * A(1,:)$ Subtracts 3 times row 1 from row 2.

30 The average pivots for rand(3) *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite! *With* row exchanges in MATLAB's lu code, the averages .75 and .50 and .365 are much more stable (and should be predictable, also for randn with normal instead of uniform probability distribution).

Problem Set 2.3, page 53

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} M = E_{32}E_{31}E_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

5 $A^n = \begin{bmatrix} 1 & bn \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

11 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

15 (a) mn (every entry) (b) mnp (c) n^3 (this is n^2 dot products).

17 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

19 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix.

20 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - (\frac{a_{31}}{a_{11}})a_{12}$ (d) $a_{22} - (\frac{a_{21}}{a_{11}})a_{12}$.

23 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$.

25 $A_1^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, $A_2^n = 2^{n-1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $A_3^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

27 (a) (Row 3 of A)$\cdot$(column 1 of B) and (Row 3 of A)$\cdot$(column 2 of B) are both zero.

(b) $\begin{bmatrix} x \\ x \\ 0 \end{bmatrix}\begin{bmatrix} 0 & x & x \end{bmatrix} = \begin{bmatrix} 0 & x & x \\ 0 & x & x \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} x \\ x \\ x \end{bmatrix}\begin{bmatrix} 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & x \\ 0 & 0 & x \end{bmatrix}$: upper triangular!

28 A times B is $A\begin{bmatrix} | & | & | \end{bmatrix}$, $[\text{——}]B$, $[\text{——}]\begin{bmatrix} | & | & | \end{bmatrix}$, $\begin{bmatrix} | & | & | \end{bmatrix}\begin{bmatrix} \text{——} \\ \text{——} \end{bmatrix}$

31 In Problem 30, $\boldsymbol{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$, $D - \boldsymbol{cb}/a = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ in lower corner of EA.

33 A times $X = [\, \boldsymbol{x}_1 \;\; \boldsymbol{x}_2 \;\; \boldsymbol{x}_3 \,]$ will be the identity matrix $I = [\, A\boldsymbol{x}_1 \;\; A\boldsymbol{x}_2 \;\; A\boldsymbol{x}_3 \,]$.

34 The solution for $\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ is $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will produce those $\boldsymbol{x}_1 = (1, 1, 1)$, $\boldsymbol{x}_2 = (0, 1, 1)$, $\boldsymbol{x}_3 = (0, 0, 1)$ as columns of its "inverse".

37 $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 3 & 1 & 1 & 3 \\ 3 & 0 & 3 & 1 & 1 \\ 1 & 3 & 0 & 3 & 1 \\ 1 & 1 & 3 & 0 & 3 \\ 3 & 1 & 1 & 3 & 0 \end{bmatrix}$, $A^3 + A^2$ no zeros so diameter 3

39 If A is "northwest" and B is "southeast", AB is upper triangular and BA is lower triangular. Row i of A ends with $i-1$ zeros. Column j of B starts with $n-j$ zeros. If $i > j$ then (row i of A) $\cdot$ (column j of B) $= 0$. So AB is upper triangular. Similarly BA is lower triangular. Problem 2.7.40 asks about inverses and transposes and permutations of a northwest A and a southeast B.

Problem Set 2.5, page 78

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$, $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

7 (a) In $A\boldsymbol{x} = (1,0,0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) The right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $\boldsymbol{x} = (1,1,-1)$ solves $A\boldsymbol{x} = \mathbf{0}$ (b) Elimination keeps columns $1+2 =$ column 3. When columns 1 and 2 end in zeros so does column 3: no third pivot.

12 $C = AB$ gives $C^{-1} = B^{-1}A^{-1}$ so $A^{-1} = BC^{-1}$.

14 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract column 2 of A^{-1} from column 1.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)I$. The inverse of one matrix is the other divided by $ad-bc$.

18 $A^2B = I$ can be written as $A(AB) = I$. Therefore A^{-1} is AB.

21 6 of the 16 are invertible, including all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$;

$\begin{bmatrix} 1 & 3 & 1 & 0 \\ 3 & 8 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -8 & 3 \\ 0 & 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$.

24 $\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$.

27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the pattern); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

31 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

34 $\boldsymbol{x} = (1,1,\ldots,1)$ has $P\boldsymbol{x} = Q\boldsymbol{x}$ so $(P-Q)\boldsymbol{x} = \mathbf{0}$.

35 $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

37 A can be invertible but B is always singular. Each row of B will add to zero, from $0+1+2-3$, so the vector $\boldsymbol{x} = (1,1,1,1)$ will give $B\boldsymbol{x} = \mathbf{0}$. I thought A would be invertible as long as you put the 3's on its main diagonal, but that's wrong:

$$A\boldsymbol{x} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & 1 & 2 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \mathbf{0} \quad \text{but} \quad A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \quad \text{is invertible}$$

40 The three Pascal matrices have $S = LU = LL^{\mathrm{T}}$ and then $\mathrm{inv}(S) = \mathrm{inv}(L^{\mathrm{T}})\mathrm{inv}(L)$. Note that the triangular L is $\mathrm{abs}(\mathrm{pascal}(n,1))$ in MATLAB.

42 If $AC = I$ for square matrices then $C = A^{-1}$ (it is proved in **2I** that $CA = I$ will also be true). The same will be true for C^*. But a square matrix has only one inverse so $C = C^*$.

43
$$\begin{aligned} MM^{-1} &= (I_n - UV)\,(I_n + U(I_m - VU)^{-1}V) \\ &= I_n - UV + U(I_m - VU)^{-1}V - UVU(I_m - VU)^{-1}V \\ &= I_n - UV + U(I_m - VU)(I_m - VU)^{-1}V = I_n \qquad \text{(formulas } \mathbf{1}, \mathbf{2}, \mathbf{4} \text{ are similar)} \end{aligned}$$

Problem Set 2.6, page 91

2 $\ell_{31} = 1$ and $\ell_{32} = 2$ (and $\ell_{33} = 1$): reverse the steps to recover $x + 3y + 6z = 11$ from $U\boldsymbol{x} = \boldsymbol{c}$:

1 times $(x + y + z = 5) + 2$ times $(y + 2z = 2) + 1$ times $(z = 2)$ gives $x + 3y + 6z = 11$.

4 $L\boldsymbol{c} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{c} \\ \\ \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$; $\boldsymbol{c} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$. $U\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} \\ \boldsymbol{x} \\ \\ \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$; $\boldsymbol{x} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$.

6 $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -2 & 1 & \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6 \end{bmatrix} = U$. Then $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

$U = E_{21}^{-1}E_{32}^{-1}U = LU$.

10 $c = 2$ leads to zero in the second pivot position: exchange rows and the matrix will be OK. $c = 1$ leads to zero in the third pivot position. In this case the matrix is *singular*.

12 $A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = LDU$; notice U is L^{T}

$$A = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & -4 & \\ & & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$
$= LDL^{\mathrm{T}}$.

14 $\begin{bmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & r & r & r \\ & b-r & s-r & s-r \\ & & c-s & t-s \\ & & & d-t \end{bmatrix}$. Need $\begin{matrix} a \neq 0 \\ b \neq r \\ c \neq s \\ d \neq t \end{matrix}$

15 $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$ gives $\boldsymbol{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Then $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ gives $\boldsymbol{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$.

Check that $A = LU = \begin{bmatrix} 2 & 4 \\ 8 & 17 \end{bmatrix}$ times $\boldsymbol{x}$ is $\boldsymbol{b} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$.

18 (a) Multiply $LDU = L_1 D_1 U_1$ by inverses to get $L_1^{-1} L D = D_1 U_1 U^{-1}$. The left side is lower triangular, the right side is upper triangular $\Rightarrow$ both sides are diagonal.

(b) Since L, U, L_1, U_1 have diagonals of 1's we get $D = D_1$. Then $L_1^{-1} L$ is I and $U_1 U^{-1}$ is I.

20 A tridiagonal T has 2 nonzeros in the pivot row and only one nonzero below the pivot (so 1 operation to find the multiplier and 1 to find the new pivot!). $T =$ bidiagonal L times U:

$$T = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix} \longrightarrow U = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \text{ Reverse steps by } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

22 $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & & 1 \end{bmatrix} \begin{bmatrix} * & * & * \\ 0 & & \\ 0 & & \end{bmatrix}$ ($*$'s are all known after the first pivot is used).

25 The 2 by 2 upper submatrix B has the first two pivots 2, 7. Reason: Elimination on A starts in the upper left corner with elimination on B.

27 $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 3 & 6 \\ & & & 1 & 4 \\ & & & & 1 \end{bmatrix}$. *Pascal's triangle* in L and U. MATLAB's lu code will wreck the pattern. chol does no row exchanges for symmetric matrices with positive pivots.

32 inv$(A) * \boldsymbol{b}$ should take 3 times as long as $A \backslash \boldsymbol{b}$ (n^3 for A^{-1} *vs* $n^3/3$ multiplications for LU).

36 This L comes from the $-1, 2, -1$ tridiagonal $A = LDL^{\mathrm{T}}$. (Row i of L) $\cdot$ (Column j of L^{-1}) $= \left(\frac{1-i}{i}\right)\left(\frac{j}{i-1}\right) + (1)\left(\frac{j}{i}\right) = 0$ for $i > j$ so $LL^{-1} = I$. Then L^{-1} leads to $A^{-1} = (L^{-1})^{\mathrm{T}} D^{-1} L^{-1}$. ***The*** $-1, 2, -1$ ***matrix has inverse*** $A_{ij}^{-1} = j(n-i+1)/(n+1)$ for $i \geq j$ (reverse for $i \leq j$).

Problem Set 2.7, page 104

2 $(AB)^{\mathrm{T}}$ is not $A^{\mathrm{T}}B^{\mathrm{T}}$ *except when* $AB = BA$. In that case transpose to find: $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$.

4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. But the diagonal entries of $A^{\mathrm{T}}A$ are dot products of columns of A with themselves. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

6 $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$, $B^{\mathrm{T}} = C$, $D^{\mathrm{T}} = D$.

8 The 1 in row 1 has n choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ choices overall).

10 $(3,1,2,4)$ and $(2,3,1,4)$ keep only 4 in position; 6 more even P's keep 1 or 2 or 3 in position; $(2,1,4,3)$ and $(3,4,1,2)$ exchange 2 pairs. Then $(1,2,3,4)$ and $(4,3,2,1)$ make 12 even P's.

14 There are $n!$ permutation matrices of order n. Eventually two powers of P *must be the same*: If $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^6 = I$.

18 (a) $5+4+3+2+1 = 15$ independent entries if $A = A^{\mathrm{T}}$ (b) L has 10 and D has 5: total 15 in LDL^{T} (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

20 $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = LDL^{\mathrm{T}}$.

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}\begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix}.$$

22 $\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$

24 $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and use a_{12} as pivot then $A = L_1P_1U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

29 One way to decide even vs. odd is to count all pairs that P has in the wrong order. Then P is even or odd when that count is even or odd. Hard step: show that an exchange always reverses that count! Then 3 or 5 exchanges will leave that count odd.

32 Inputs $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x}$; $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix}$ 1 truck 1 plane

33 $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the *cost* of inputs while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the *value* of outputs.

34 $P^3 = I$ so three rotations for 360°; P rotates around $(1, 1, 1)$ by 120°.

37 These are groups: Lower triangular with diagonal 1's, diagonal invertible D, permutations P, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

40 Certainly B^{T} is northwest. B^2 is a full matrix! B^{-1} is southeast: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$. The rows of B are in reverse order from a lower triangular L, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the *columns* in reverse order from L^{-1}. So B^{-1} is *southeast*. Northwest times southeast is upper triangular! $B = PL$ and $C = PU$ give $BC = (PLP)U =$ upper times upper.

41 The i, j entry of PAP is the $n - i + 1, n - j + 1$ entry of A. The main diagonal reverses order.

Problem Set 3.1, page 118

1 $\boldsymbol{x} + \boldsymbol{y} \neq \boldsymbol{y} + \boldsymbol{x}$ and $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) \neq (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ and $(c_1 + c_2)\boldsymbol{x} \neq c_1\boldsymbol{x} + c_2\boldsymbol{x}$.

3 (a) $c\boldsymbol{x}$ may not be in our set: not closed under scalar multiplication. Also no $\mathbf{0}$ and no $-\boldsymbol{x}$

(b) $c(\boldsymbol{x} + \boldsymbol{y})$ is the usual $(xy)^c$, while $c\boldsymbol{x} + c\boldsymbol{y}$ is the usual $(x^c)(y^c)$. Those are equal. With $c = 3, x = 2, y = 1$ they equal 8. This is $3(\mathbf{2} + \mathbf{1})$!! The zero vector is the number 1.

5 (a) One possibility: The matrices cA form a subspace not containing B (b) Yes: the subspace must contain $A - B = I$ (c) All matrices whose main diagonal is all zero.

9 (a) The vectors with integer components allow addition, but not multiplication by $\frac{1}{2}$

(b) Remove the x axis from the xy plane (but leave the origin). Multiplication by any c is allowed but not all vector additions.

11 (a) All matrices $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ (b) All matrices $\begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}$ (c) All diagonal matrices.

15 (a) Two planes through $(0, 0, 0)$ probably intersect in a line through $(0, 0, 0)$ (b) The plane and line probably intersect in the point $(0, 0, 0)$ (c) Suppose $\boldsymbol{x}$ is in $\boldsymbol{S} \cap \boldsymbol{T}$ and $\boldsymbol{y}$ is in $\boldsymbol{S} \cap \boldsymbol{T}$. Both vectors are in both subspaces, so $\boldsymbol{x} + \boldsymbol{y}$ and $c\boldsymbol{x}$ are in both subspaces.

20 (a) Solution only if $b_2 = 2b_1$ and $b_3 = -b_1$ (b) Solution only if $b_3 = -b_1$.

23 The extra column $\boldsymbol{b}$ enlarges the column space unless $\boldsymbol{b}$ is *already in* the column space of A: $[\, A \;\; \boldsymbol{b} \,] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (larger column space) (no solution to $A\boldsymbol{x} = \boldsymbol{b}$) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ ($\boldsymbol{b}$ already in column space) ($A\boldsymbol{x} = \boldsymbol{b}$ has a solution)

25 The solution to $Az = \boldsymbol{b} + \boldsymbol{b}^*$ is $z = \boldsymbol{x} + \boldsymbol{y}$. If $\boldsymbol{b}$ and $\boldsymbol{b}^*$ are in the column space so is $\boldsymbol{b} + \boldsymbol{b}^*$.

5 I think this is true.

7 Special solutions are columns of $N = [-2 \;\; -1 \;\; 1 \;\; 0; \;\; -3 \;\; -5 \;\; 0 \;\; 1]$ and $[1 \;\; 0 \;\; 0; \;\; 0 \;\; -2 \;\; 1]$.

13 P has rank r (the same as A) because elimination produces the same pivot columns.

14 The rank of R^{T} is also r, and the example matrix A has rank 2:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

16 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

18 If we know that $\operatorname{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \le \operatorname{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, we have $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.

20 Certainly A and B have at most rank 2. Then their product AB has at most rank 2. Since BA is 3 by 3, it cannot be I even if $AB = I$.

21 (a) A and B will both have the same nullspace and row space as R (same R for both matrices). (b) A equals an *invertible* matrix times B, when they share the same R. A key fact!

23 $$A = \text{(pivot columns)(nonzero rows of } R) = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}.$$

24 The m by n matrix Z has r ones at the start of its main diagonal. Otherwise Z is all zeros.

Problem Set 3.4, page 152

2 $$\begin{bmatrix} 2 & 1 & 3 & \boldsymbol{b}_1 \\ 6 & 3 & 9 & \boldsymbol{b}_2 \\ 4 & 2 & 6 & \boldsymbol{b}_3 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 3 & \boldsymbol{b}_1 \\ 0 & 0 & 0 & \boldsymbol{b}_2 - \boldsymbol{3b}_1 \\ 0 & 0 & 0 & \boldsymbol{b}_3 - \boldsymbol{2b}_1 \end{bmatrix} \quad \text{Then } [\, R \;\; \boldsymbol{d} \,] = \begin{bmatrix} 1 & 1/2 & 3/2 & \mathbf{5} \\ 0 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

$A\boldsymbol{x} = \boldsymbol{b}$ has a solution when $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the column space is the line through $(2, 6, 4)$ which is the intersection of the planes $b_2 - 3b_1 = 0$ and $b_3 - 2b_1 = 0$; the nullspace contains all combinations of $\boldsymbol{s}_1 = (-1/2, 1, 0)$ and $\boldsymbol{s}_2 = (-3/2, 0, 1)$; particular solution $\boldsymbol{x}_p = \boldsymbol{d} = (5, 0, 0)$ and complete solution $\boldsymbol{x}_p + c_1\boldsymbol{s}_1 + c_2\boldsymbol{s}_2$.

4 $\boldsymbol{x}_{complete} = (\frac{1}{2}, 0, \frac{1}{2}, 0) + x_2(-3, 1, 0, 0) + x_4(0, 0, -2, 1)$.

6 (a) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \end{bmatrix}$ (no free variables)

(b) Solvable if $b_2 = 2b_1$ and $3b_1 - 3b_3 + b_4 = 0$. Then $\boldsymbol{x} = \begin{bmatrix} 5b_1 - 2b_3 \\ b_3 - 2b_1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

8 (a) Every $\boldsymbol{b}$ is in the column space: *independent rows.* (b) Need $b_3 = 2b_2$. Row 3 − 2 row 2 = 0.

12 (a) $\boldsymbol{x}_1 - \boldsymbol{x}_2$ and $\boldsymbol{0}$ solve $A\boldsymbol{x} = \boldsymbol{0}$ (b) $2\boldsymbol{x}_1 - 2\boldsymbol{x}_2$ solves $A\boldsymbol{x} = \boldsymbol{0}$; $2\boldsymbol{x}_1 - \boldsymbol{x}_2$ solves $A\boldsymbol{x} = \boldsymbol{b}$.

13 (a) The particular solution $\boldsymbol{x}_p$ is always multiplied by 1 (b) Any solution can be the particular solution (c) $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is shorter (length $\sqrt{2}$) than $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

(d) The "homogeneous" solution in the nullspace is $\boldsymbol{x}_n = \boldsymbol{0}$ when A is invertible.

14 If column 5 has no pivot, x_5 is a free variable. The zero vector *is not* the only solution to $A\boldsymbol{x} = \boldsymbol{0}$. If $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, it has *infinitely many* solutions.

16 The largest rank is 3. Then there is a pivot in every *row*. The solution *always exists*. The column space is $\mathbf{R}^3$. An example is $A = [\,I \quad F\,]$ for any 3 by 2 matrix F.

18 Rank = 3; rank = 3 unless $q = 2$ (then rank = 2).

25 (a) $r < m$, always $r \le n$ (b) $r = m$, $r < n$ (c) $r < m$, $r = n$ (d) $r = m = n$.

28 $\begin{bmatrix} 1 & 2 & 3 & \mathbf{0} \\ 0 & 0 & 4 & \mathbf{0} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \end{bmatrix}$; $\boldsymbol{x}_n = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$; $\begin{bmatrix} 1 & 2 & 3 & \mathbf{5} \\ 0 & 0 & 4 & \mathbf{8} \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & \mathbf{-1} \\ 0 & 0 & 1 & \mathbf{2} \end{bmatrix} \boldsymbol{x}_p = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

The pivot columns contain I so -1 and 2 go into $\boldsymbol{x}_p$.

30 $\begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 1 & 3 & 2 & 0 & \mathbf{5} \\ 2 & 0 & 4 & 9 & \mathbf{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 3 & \mathbf{2} \\ 0 & 3 & 0 & -3 & \mathbf{3} \\ 0 & 0 & 0 & 3 & \mathbf{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & \mathbf{-4} \\ 0 & 1 & 0 & 0 & \mathbf{3} \\ 0 & 0 & 0 & 1 & \mathbf{2} \end{bmatrix}$; $\boldsymbol{x}_p = \begin{bmatrix} -4 \\ 3 \\ 0 \\ 2 \end{bmatrix}$

and $\boldsymbol{x}_n = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Problem Set 3.5, page 167

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent. All six vectors are on the plane $(1,1,1,1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column 1 = $\boldsymbol{0}$; if $d = 0$ then b(column 1) − a(column 2) = $\boldsymbol{0}$; if $f = 0$ then all columns end in zero (all are perpendicular to $(0, 0, 1)$, all in the xy plane, must be dependent).

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A.

8 If $c_1(\boldsymbol{w}_2+\boldsymbol{w}_3)+c_2(\boldsymbol{w}_1+\boldsymbol{w}_3)+c_3(\boldsymbol{w}_1+\boldsymbol{w}_2) = \boldsymbol{0}$ then $(c_2+c_3)\boldsymbol{w}_1+(c_1+c_3)\boldsymbol{w}_2+(c_1+c_2)\boldsymbol{w}_3 = \boldsymbol{0}$. Since the $\boldsymbol{w}$'s are independent this requires $c_2 + c_3 = 0, c_1 + c_3 = 0, c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives zero.

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) Plane in $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when there is a solution to $A\boldsymbol{x} = \boldsymbol{b}$; $\boldsymbol{c}$ is in the row space when there is a solution to $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$. *False*. The zero vector is always in the row space.

14 The dimension of $\mathbf{S}$ is (a) zero when $\boldsymbol{x} = \mathbf{0}$ (b) one when $\boldsymbol{x} = (1, 1, 1, 1)$ (c) three when $\boldsymbol{x} = (1, 1, -1, -1)$ because all rearrangements of this $\boldsymbol{x}$ are perpendicular to $(1, 1, 1, 1)$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** $\dim \mathbf{S} = 2$.

16 The n independent vectors span a space of dimension n. They are a *basis* for that space. If they are the columns of A then m is *not less* than n ($m \geq n$).

19 (a) The 6 vectors *might not* span $\mathbf{R}^4$ (b) The 6 vectors *are not* independent

(c) Any four *might be* a basis.

21 One basis is $(2, 1, 0)$, $(-3, 0, 1)$. The vector $(2, 1, 0)$ is a basis for the intersection with the xy plane. The normal vector $(1, -2, 3)$ is a basis for the line perpendicular to the plane.

23 (a) True (b) False because the basis vectors may not be in $\mathbf{S}$.

26 Rank 2 if $c = 0$ and $d = 2$; rank 2 except when $c = d$ or $c = -d$.

29 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

30 $-\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} + \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} = 0$

34 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

36 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

40 The subspace of matrices that have $AS = SA$ has dimension *three*.

42 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent, $\boldsymbol{b}$ *is* a combination of those columns.

Problem Set 3.6, page 180

1 (a) Row and column space dimensions = 5, nullspace dimension = 4, left nullspace dimension = 2 sum = 16 = $m + n$ (b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r + (n - r)$ must be 3 (c) $[1 \;\; 1]$ (d) $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

(e) Row space = column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension and actually the same vectors ($A\boldsymbol{x} = \mathbf{0}$ means $\boldsymbol{x} \perp$ row space, $A^{\mathrm{T}}\boldsymbol{x} = \mathbf{0}$ means $\boldsymbol{x} \perp$ column space).

6 A: Row space $(0,3,3,3)$ and $(0,1,0,1)$; column space $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; left nullspace $(0,1,0)$. B: Row space (1), column space $(1,4,5)$, nullspace: empty basis, left nullspace $(-4,1,0)$ and $(-5,0,1)$.

9 (a) Same row space and nullspace. Therefore rank (dimension of row space) is the same (b) Same column space and left nullspace. Same rank (dimension of column space).

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n (b) If $m - r > 0$, the left nullspace contains a nonzero vector.

12 $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ but $2 + 2$ is 4.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$.

18 Row $3 - 2$ row $2 +$ row $1 =$ zero row so the vectors $c(1,-2,1)$ are in the left nullspace. The same vectors happen to be in the nullspace.

20 (a) All combinations of $(-1,2,0,0)$ and $(-\frac{1}{4},0,-3,1)$ (b) One (c) $(1,2,3)$, $(0,1,4)$.

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the *row space* of A; unique solution if the *left nullspace* (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

26 The rows of $AB = C$ are combinations of the rows of B. So rank $C \le$ rank B. Also rank $C \le$ rank A. (The columns of C are combinations of the columns of A).

29 $a_{11} = 1, a_{12} = 0, a_{13} = 1, a_{22} = 0, a_{32} = 1, a_{31} = 0, a_{23} = 1, a_{33} = 0, a_{21} = 1$ (not unique).

Problem Set 4.1, page 191

1 Both nullspace vectors are orthogonal to the row space vector in $\mathbf{R}^3$. Column space is perpendicular to the nullspace of A^{T} in $\mathbf{R}^2$.

3 (a) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ (b) Impossible, $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ not orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in $\boldsymbol{N}(A^{\mathrm{T}})$ is impossible: not perpendicular (d) This asks for $A^2 = 0$; take $A = \left[\begin{smallmatrix} 1 & -1 \\ 1 & -1 \end{smallmatrix}\right]$ (e) $(1,1,1)$ will be in the nullspace and row space; no such matrix.

6 Multiply the equations by $y_1 = 1$, $y_2 = 1$, $y_3 = -1$. They add to $0 = 1$ so no solution: $\boldsymbol{y} = (1,1,-1)$ is in the left nullspace. Can't have $0 = (\boldsymbol{y}^{\mathrm{T}}A)\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{b} = 1$.

9 Since A is invertible, $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = AA^{-1}(A^{\mathrm{T}})^{-1}A^{\mathrm{T}} = I$: project onto all of $\mathbf{R}^2$.

11 (a) $\boldsymbol{p} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = (2,3,0)$ and $\boldsymbol{e} = (0,0,4)$ (b) $\boldsymbol{p} = (4,4,6)$ and $\boldsymbol{e} = (0,0,0)$.

15 The column space of $2A$ is the same as the column space of A. $\widehat{\boldsymbol{x}}$ for $2A$ is *half* of $\widehat{\boldsymbol{x}}$ for A.

16 $\frac{1}{2}(1,2,-1) + \frac{3}{2}(1,0,1) = (2,1,1)$. Therefore $\boldsymbol{b}$ is in the plane. Projection shows $P\boldsymbol{b} = \boldsymbol{b}$.

18 (a) $I - P$ is the projection matrix onto $(1,-1)$ in the perpendicular direction to $(1,1)$

(b) $I - P$ is the projection matrix onto the plane $x + y + z = 0$ perpendicular to $(1,1,1)$.

20 $$\boldsymbol{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad Q = \boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}/\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} = \begin{bmatrix} 1/6 & -1/6 & -1/3 \\ -1/6 & 1/6 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{bmatrix}, \quad P = I - Q = \begin{bmatrix} 5/6 & 1/6 & 1/3 \\ 1/6 & 5/6 & -1/3 \\ 1/3 & -1/3 & 1/3 \end{bmatrix}.$$

21 $\left(A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\right)^2 = A(A^{\mathrm{T}}A)^{-1}(A^{\mathrm{T}}A)(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$. Therefore $P^2 = P$. $P\boldsymbol{b}$ is always in the column space (where P projects). Therefore its projection $P(P\boldsymbol{b})$ is $P\boldsymbol{b}$.

24 The nullspace of A^{T} is *orthogonal* to the column space $\boldsymbol{C}(A)$. So if $A^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{0}$, the projection of $\boldsymbol{b}$ onto $\boldsymbol{C}(A)$ should be $\boldsymbol{p} = \boldsymbol{0}$. Check $P\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = A(A^{\mathrm{T}}A)^{-1}\boldsymbol{0} = \boldsymbol{0}$.

28 $P^2 = P = P^{\mathrm{T}}$ give $P^{\mathrm{T}}P = P$. Then the $(2,2)$ entry of P equals the $(2,2)$ entry of $P^{\mathrm{T}}P$ which is the length squared of column 2.

29 Set $A = B^{\mathrm{T}}$. Then A has independent columns. By **4G**, $A^{\mathrm{T}}A = BB^{\mathrm{T}}$ is invertible.

30 (a) The column space is the line through $\boldsymbol{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ so $P_C = \dfrac{\boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}} = \dfrac{1}{25}\begin{bmatrix} 9 & 12 \\ 12 & 25 \end{bmatrix}$. We can't use $(A^{\mathrm{T}}A)^{-1}$ because A has dependent columns. (b) The row space is the line through $\boldsymbol{v} = (1,2,2)$ and $P_R = \boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}/\boldsymbol{v}^{\mathrm{T}}\boldsymbol{v}$. Always $P_C A = A$ and $AP_R = A$ and then $P_C A P_R = A$!

Problem Set 4.3, page 215

1 $$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} \text{ give } A^{\mathrm{T}}A = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \text{ and } A^{\mathrm{T}}\boldsymbol{b} = \begin{bmatrix} 36 \\ 112 \end{bmatrix}.$$

$$A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b} \text{ gives } \widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \text{ and } \boldsymbol{p} = A\widehat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 5 \\ 13 \\ 17 \end{bmatrix} \text{ and } \boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p} = \begin{bmatrix} -1 \\ 3 \\ -5 \\ 3 \end{bmatrix}.$$

$E = \|\boldsymbol{e}\|^2 = 44$.

5 $E = (C-0)^2 + (C-8)^2 + (C-8)^2 + (C-20)^2$. $A^{\mathrm{T}} = [\,1 \;\; 1 \;\; 1 \;\; 1\,]$, $A^{\mathrm{T}}A = [\,4\,]$ and $A^{\mathrm{T}}\boldsymbol{b} = [\,36\,]$ and $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}\boldsymbol{b} = 9 =$ best height C. Errors $\boldsymbol{e} = (-9,-1,-1,11)$.

7 $A = [\,0 \;\; 1 \;\; 3 \;\; 4\,]^{\mathrm{T}}$, $A^{\mathrm{T}}A = [\,26\,]$ and $A^{\mathrm{T}}\boldsymbol{b} = [\,112\,]$. Best $D = 112/26 = 56/13$.

13 The multiple to subtract is $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$. Then $\boldsymbol{B} = \boldsymbol{b} - \dfrac{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}}{\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}}\boldsymbol{a} = (4,0) - 2\cdot(1,1) = (2,-2)$.

14 $\begin{bmatrix}1 & 4\\ 1 & 0\end{bmatrix} = \begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2\end{bmatrix}\begin{bmatrix}\|\boldsymbol{a}\| & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}\\ 0 & \|\boldsymbol{B}\|\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 2\sqrt{2}\\ 0 & 2\sqrt{2}\end{bmatrix} = QR.$

15 (a) $\boldsymbol{q}_1 = \frac{1}{3}(1,2,-2)$, $\boldsymbol{q}_2 = \frac{1}{3}(2,1,2)$, $\boldsymbol{q}_3 = \frac{1}{3}(2,-2,-1)$ (b) The nullspace of A^{T} contains $\boldsymbol{q}_3$ (c) $\widehat{\boldsymbol{x}} = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}(1,2,7) = (1,2)$.

16 The projection $\boldsymbol{p} = (\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})\boldsymbol{a} = 14\boldsymbol{a}/49 = 2\boldsymbol{a}/7$ is closest to $\boldsymbol{b}$; $\boldsymbol{q}_1 = \boldsymbol{a}/\|\boldsymbol{a}\| = \boldsymbol{a}/7$ is $(4,5,2,2)/7$. $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (-1,4,-4,-4)/7$ has $\|B\| = 1$ so $\boldsymbol{q}_2 = \boldsymbol{B}$.

18 $\boldsymbol{A} = \boldsymbol{a} = (1,-1,0,0)$; $\boldsymbol{B} = \boldsymbol{b} - \boldsymbol{p} = (\frac{1}{2},\frac{1}{2},-1,0)$; $\boldsymbol{C} = \boldsymbol{c} - \boldsymbol{p}_A - \boldsymbol{p}_B = (\frac{1}{3},\frac{1}{3},\frac{1}{3},-1)$. Notice the pattern in those orthogonal vectors $\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}$.

20 (a) True (b) True. $Q\boldsymbol{x} = x_1\boldsymbol{q}_1 + x_2\boldsymbol{q}_2$. $\|Q\boldsymbol{x}\|^2 = x_1^2 + x_2^2$ because $\boldsymbol{q}_1\cdot\boldsymbol{q}_2 = 0$.

21 The orthonormal vectors are $\boldsymbol{q}_1 = (1,1,1,1)/2$ and $\boldsymbol{q}_2 = (-5,-1,1,5)/\sqrt{52}$. Then $\boldsymbol{b} = (-4,-3,3,0)$ projects to $\boldsymbol{p} = (-7,-3,-1,3)/2$. Check that $\boldsymbol{b} - \boldsymbol{p} = (-1,-3,7,-3)/2$ is orthogonal to both $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$.

22 $A = (1,1,2)$, $B = (1,-1,0)$, $C = (-1,-1,1)$. Not yet orthonormal.

26 $(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{C}^*)\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{c}}{\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}}\boldsymbol{B}$ because $\boldsymbol{q}_2 = \dfrac{\boldsymbol{B}}{\|\boldsymbol{B}\|}$ and the extra $\boldsymbol{q}_1$ in $\boldsymbol{C}^*$ is orthogonal to $\boldsymbol{q}_2$.

29 There are mn multiplications in (11) and $\frac{1}{2}m^2n$ multiplications in each part of (12).

30 The columns of the wavelet matrix W are orthonormal. Then $W^{-1} = W^{\mathrm{T}}$. See Section 7.3 for more about wavelets.

33 $Q_1 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ reflects across x axis, $Q_2 = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{bmatrix}$ across plane $y+z=0$.

36 Orthogonal and lower triangular $\Rightarrow \pm 1$ on the main diagonal, 0 elsewhere.

Problem Set 5.1, page 240

1 $\det(2A) = 8$ and $\det(-A) = (-1)^4\det A = \frac{1}{2}$ and $\det(A^2) = \frac{1}{4}$ and $\det(A^{-1}) = 2$.

5 $|J_5| = 1$, $|J_6| = -1$, $|J_7| = -1$. The determinants are 1, 1, -1, -1 repeating, so $|J_{101}| = 1$.

8 $Q^{\mathrm{T}}Q = I \Rightarrow |Q|^2 = 1 \Rightarrow |Q| = \pm 1$; Q^n stays orthogonal so can't blow up. Same for Q^{-1}.

10 If the entries in every row add to zero, then $(1,1,\ldots,1)$ is in the nullspace: singular A has det $= 0$. (The columns add to the zero column so they are linearly dependent.) If every row adds to one, then rows of $A - I$ add to zero (not necessarily $\det A = 1$).

11 $CD = -DC \Rightarrow |CD| = (-1)^n|DC|$ and *not* $-|DC|$. If n is even we can have $|CD| \neq 0$.

14 $\det(A) = 24$ and $\det(A) = 5$.

15 det $= 0$ and det $= 1 - 2t^2 + t^4 = (1 - t^2)^2$.

17 Any 3 by 3 skew-symmetric K has $\det(K^{\mathrm{T}}) = \det(-K) = (-1)^3\det(K)$. This is $-\det(K)$. But also $\det(K^{\mathrm{T}}) = \det(K)$, so we must have $\det(K) = 0$.

21 Rules 5 and 3 give Rule 2. (Since Rules 4 and 3 give 5, they also give Rule 2.)

23 $\det(A) = 10$, $A^2 = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$, $\det(A^2) = 100$, $A^{-1} = \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix}$, $\det(A^{-1}) = \frac{1}{10}$.
$\det(A - \lambda I) = \lambda^2 - 7\lambda + 10 = 0$ when $\lambda = 2$ or $\lambda = 5$.

27 $\det A = abc$, $\det B = -abcd$, $\det C = a(b - a)(c - b)$.

30 $\begin{bmatrix} \partial f/\partial a & \partial f/\partial c \\ \partial f/\partial b & \partial f/\partial d \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$.

32 Typical determinants of rand(n) are $10^6, 10^{25}, 10^{79}, 10^{218}$ for $n = 50, 100, 200, 400$. Using randn(n) with normal bell-shaped probabilities these are $10^{31}, 10^{78}, 10^{186}$, Inf means $\geq 2^{1024}$. MATLAB computes $1.999999999999999 \times 2^{1023} \approx 1.8 \times 10^{308}$ but one more 9 gives Inf!

34 Reduce B to [row 3 : row 2; row 1]. Then $\det B = -6$.

Problem Set 5.2, page 253

2 $\det A = -2$, independent; $\det B = 0$, dependent; $\det C = (-2)(0)$, dependent.

4 (a) The last three rows must be dependent (b) In each of the 120 terms: Choices from the last 3 rows must use 3 columns; at least one choice will be zero.

5 $a_{11}a_{23}a_{32}a_{44}$ gives -1, $a_{14}a_{23}a_{32}a_{41}$ gives $+1$ so $\det A = 0$;
$\det B = 2 \cdot 4 \cdot 4 \cdot 2 - 1 \cdot 4 \cdot 4 \cdot 1 = 48$.

7 (a) If $a_{11} = a_{22} = a_{33} = 0$ then 4 terms are sure zeros (b) 15 terms are certainly zero.

9 Some term $a_{1\alpha}a_{2\beta}\cdots a_{n\omega}$ is not zero! Move rows $1, 2, \ldots, n$ into rows $\alpha, \beta, \ldots, \omega$. Then these nonzero a's will be on the main diagonal.

10 To get $+1$ for the even permutations the matrix needs an *even* number of -1's. For the odd P's the matrix needs an *odd* number of -1's. So six 1's and det $= 6$ are impossible: max(det) $= 4$.

12 $C = \begin{bmatrix} 6 & -3 \\ -1 & 2 \end{bmatrix}$. $C = \begin{bmatrix} 0 & 42 & -35 \\ 0 & -21 & 14 \\ -3 & 6 & -3 \end{bmatrix}$. $\det B = 1(0) + 2(42) + 3(-35) = -21$.

13 $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Therefore $A^{-1} = \frac{1}{4}C^{\mathrm{T}}$.

15 (a) $C_1 = 0$, $C_2 = -1$, $C_3 = 0$, $C_4 = 1$ (b) $C_n = -C_{n-2}$ by cofactors of row 1 then cofactors of column 1. Therefore $C_{10} = -C_8 = C_6 = -C_4 = -1$.

17 The 1, 1 cofactor is E_{n-1}. The 1, 2 cofactor has a single 1 in its first column, with cofactor E_{n-2}. Signs give $E_n = E_{n-1} - E_{n-2}$. Then $1, 0, -1, -1, 0, 1$ repeats by sixes; $E_{100} = -1$.

18 The 1, 1 cofactor is F_{n-1}. The 1, 2 cofactor has a 1 in column 1, with cofactor F_{n-2}. Multiply by $(-1)^{1+2}$ and also (-1) from the 1, 2 entry to find $F_n = F_{n-1} + F_{n-2}$ (so Fibonacci).

20 Since x, x^2, x^3 are all in the same row, they are never multiplied in $\det V_4$. The determinant is zero at $x = a$ or b or c, so $\det V$ has factors $(x-a)(x-b)(x-c)$. Multiply by the cofactor V_3. Any Vandermonde matrix $V_{ij} = (c_i)^{j-1}$ has $\det V =$ product of all $(c_l - c_k)$ for $l > k$.

21 $G_2 = -1$, $G_3 = 2$, $G_4 = -3$, and $G_n = (-1)^{n-1}(n-1) =$ (product of the n eigenvalues!)

23 The problem asks us to show that $F_{2n+2} = 3F_{2n} - F_{2n-2}$. Keep using the Fibonacci rule:

$$F_{2n+2} = F_{2n+1} + F_{2n} = F_{2n} + F_{2n-1} + F_{2n} = F_{2n} + (F_{2n} - F_{2n-2}) + F_{2n} = 3F_{2n} - F_{2n-2}.$$

26 (a) All L's have det $= 1$; $\det U_k = \det A_k = 2, 6, -6$ for $k = 1, 2, 3$ (b) Pivots $2, \frac{3}{2}, -\frac{1}{3}$.

27 Problem 25 gives $\det \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} = 1$ and $\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = |A|$ times $|D - CA^{-1}B|$ which is $|AD - ACA^{-1}B|$. If $AC = CA$ this is $|AD - CAA^{-1}B| = \det(AD - CB)$.

29 (a) $\det A = a_{11}C_{11} + \cdots + a_{1n}C_{1n}$. The derivative with respect to a_{11} is the cofactor C_{11}.

31 There are five nonzero products, all 1's with a plus or minus sign. Here are the (row, column) numbers and the signs: $+(1,1)(2,2)(3,3)(4,4) + (1,2)(2,1)(3,4)(4,3) - (1,2)(2,1)$ $(3,3)(4,4) - (1,1)(2,2)(3,4)(4,3) - (1,1)(2,3)(3,2)(4,4)$. Total $1 + 1 - 1 - 1 - 1 = -1$.

34 With $a_{11} = 1$, the $-1, 2, -1$ matrix has det $= 1$ and inverse $(A^{-1})_{ij} = n + 1 - \max(i, j)$.

35 With $a_{11} = 2$, the $-1, 2, -1$ matrix has det $= n + 1$ and $(n+1)(A^{-1})_{ij} = i(n - j + 1)$ for $i \le j$ and symmetrically $(n+1)(A^{-1})_{ij} = j(n - i + 1)$ for $i \ge j$.

Problem Set 5.3, page 269

2 (a) $y = -c/(ad - bc)$ (b) $y = (fg - id)/D$.

3 (a) $x_1 = 3/0$ and $x_2 = -2/0$: no solution (b) $x_1 = 0/0$ and $x_2 = 0/0$: *undetermined.*

4 (a) $x_1 = \det([\,\boldsymbol{b} \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3\,])/\det A$, if $\det A \neq 0$ (b) The determinant is linear in its first column so $x_1|\boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3| + x_2|\boldsymbol{a}_2 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3| + x_3|\boldsymbol{a}_3 \;\; \boldsymbol{a}_2 \;\; \boldsymbol{a}_3|$. The last two determinants are zero.

6 (a) $\begin{bmatrix} 1 & -\frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{4}{3} & 1 \end{bmatrix}$ (b) $\dfrac{1}{4}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. The inverse of a symmetric matrix is symmetric.

8 $C = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}$ and $AC^{\mathrm{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Therefore $\det A = 3$. Cofactor of 100 is 0.

9 If we know the cofactors and $\det A = 1$ then $C^{\mathrm{T}} = A^{-1}$ and $\det A^{-1} = 1$. Now A is the inverse of A^{-1}, so A is the cofactor matrix for C.

11 We find $\det A = (\det C)^{\frac{1}{n-1}}$ with $n = 4$. Then $\det A^{-1}$ is $1/\det A$. Construct A^{-1} using the cofactors. Invert to find A.

12 The cofactors of A are integers. Division by $\det A = \pm 1$ gives integer entries in A^{-1}.

16 For $n = 5$ the matrix C contains *25* cofactors and each 4 by 4 cofactor contains *24* terms and each term needs *3* multiplications: total 1800 multiplications vs. 125 for Gauss-Jordan.

18 Volume $= \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 20$. Area of faces $=$ length of cross product $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{vmatrix} = -2\boldsymbol{i} - 2\boldsymbol{j} + 8\boldsymbol{k} =$ $6\sqrt{2}$.

19 (a) Area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 3 & 4 & 1 \\ 0 & 5 & 1 \end{vmatrix} = 5$ (b) 5 + new triangle area $\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 5 + 7 = 12$.

22 The maximum volume is $L_1 L_2 L_3 L_4$ reached when the four edges are orthogonal in $\mathbf{R}^4$. With entries 1 and -1 all lengths are $\sqrt{1+1+1+1} = 2$. The maximum determinant is $2^4 = 16$, achieved by Hadamard above. For a 3 by 3 matrix, $\det A = (\sqrt{3})^3$ can't be achieved.

24 $A^{\mathrm{T}}A = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}} \\ \boldsymbol{b}^{\mathrm{T}} \\ \boldsymbol{c}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{a} & 0 & 0 \\ 0 & \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} & 0 \\ 0 & 0 & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{c} \end{bmatrix}$ has $\begin{matrix} \det A^{\mathrm{T}}A & = & (\|\boldsymbol{a}\|\|\boldsymbol{b}\|\|\boldsymbol{c}\|)^2 \\ \det A & = & \pm\|\boldsymbol{a}\|\|\boldsymbol{b}\|\|\boldsymbol{c}\| \end{matrix}$.

26 The n-dimensional cube has 2^n corners, $n2^{n-1}$ edges and $2n$ $(n-1)$-dimensional faces. Coefficients from $(2+x)^n$ in Worked Example **2.4A**. The cube from $2I$ has volume 2^n.

27 The pyramid has volume $\frac{1}{6}$. The 4-dimensional pyramid has volume $\frac{1}{24}$.

32 Base area 10, height 2, volume 20.

36 $S = (2, 1, -1)$. The area is $\|PQ \times PS\| = \|(-2, -2, -1)\| = 3$. The other four corners could be $(0,0,0)$, $(0,0,2)$, $(1,2,2)$, $(1,1,0)$. The volume of the tilted box is $|det| = 1$.

Problem Set 6.1, page 283

1 A and A^2 and A^∞ all have the same eigenvectors. The eigenvalues are 1 and 0.5 for A, 1 and 0.25 for A^2, 1 and 0 for A^∞. Therefore A^2 is halfway between A and A^∞.

Exchanging the rows of A changes the eigenvalues to 1 and -0.5 (it is still a Markov matrix with eigenvalue 1, and the trace is now $0.2+0.3$—so the other eigenvalue is -0.5).

Singular matrices stay singular during elimination, so $\lambda = 0$ does not change.

3 A has $\lambda_1 = 4$ and $\lambda_2 = -1$ (check trace and determinant) with $\boldsymbol{x}_1 = (1, 2)$ and $\boldsymbol{x}_2 = (2, -1)$. A^{-1} has the same eigenvectors as A, with eigenvalues $1/\lambda_1 = 1/4$ and $1/\lambda_2 = -1$.

8 $A = \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix}$ has trace 6, det 9, $\lambda = 3$ and 3 with only one independent eigenvector $(1, 3)$.

9 $my'' + by' + ky = 0$ is $\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y' \\ y \end{bmatrix}$.

10 When A is skew-symmetric, $\|\boldsymbol{u}(t)\| = \|e^{At}\boldsymbol{u}(0)\| = \|\boldsymbol{u}(0)\|$. So e^{At} is an *orthogonal* matrix.

13 $\boldsymbol{u}_p = A^{-1}\boldsymbol{b} = 4$ and $u(t) = ce^{2t} + 4$; $\boldsymbol{u}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $\boldsymbol{u}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

14 Substituting $\boldsymbol{u} = e^{ct}\boldsymbol{v}$ gives $ce^{ct}\boldsymbol{v} = Ae^{ct}\boldsymbol{v} - e^{ct}\boldsymbol{b}$ or $(A - cI)\boldsymbol{v} = \boldsymbol{b}$ or $\boldsymbol{v} = (A - cI)^{-1}\boldsymbol{b} =$ particular solution. If c is an eigenvalue then $A - cI$ is not invertible.

18 The solution at time $t + T$ is also $e^{A(t+T)}\boldsymbol{u}(0)$. Thus e^{At} times e^{AT} equals $e^{A(t+T)}$.

19 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$; $e^{At} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} e^t & e^t - 1 \\ 0 & 1 \end{bmatrix}$.

20 If $A^2 = A$ then $e^{At} = I + At + \frac{1}{2}At^2 + \frac{1}{6}At^3 + \cdots = I + (e^t - 1)A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} e^t - 1 & e^t - 1 \\ 0 & 0 \end{bmatrix}$.

22 $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}$, then $e^{At} = \begin{bmatrix} e^t & \frac{1}{2}(e^{3t} - e^t) \\ 0 & e^{3t} \end{bmatrix}$.

24 (a) The inverse of e^{At} is e^{-At} (b) If $A\boldsymbol{x} = \lambda\boldsymbol{x}$ then $e^{At}\boldsymbol{x} = e^{\lambda t}\boldsymbol{x}$ and $e^{\lambda t} \neq 0$.

25 $x(t) = e^{4t}$ and $y(t) = -e^{4t}$ is a growing solution. The correct matrix for the exchanged unknown $\boldsymbol{u} = (y, x)$ is $\begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$ and it *does* have the same eigenvalues as the original matrix.

Problem Set 6.4, page 326

3 $\lambda = 0, 2, -1$ with unit eigenvectors $\pm(0, 1, -1)/\sqrt{2}$ and $\pm(2, 1, 1)/\sqrt{6}$ and $\pm(1, -1, -1)/\sqrt{3}$.

5 $Q = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ -1 & -2 & 2 \end{bmatrix}$.

8 If $A^3 = 0$ then all $\lambda^3 = 0$ so all $\lambda = 0$ as in $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If A is symmetric then $A^3 = Q\Lambda^3 Q^{\mathrm{T}} = 0$ gives $\Lambda = 0$ and the only symmetric possibility is $A = Q\,0\,Q^{\mathrm{T}} =$ zero matrix.

10 If $\boldsymbol{x}$ is not real then $\lambda = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ is *not* necessarily real. Can't assume real eigenvectors!

13 (1) Choose M_i = reverse diagonal matrix to get $M_i^{-1}J_iM_i = M_i^{\mathrm{T}}$ in each block (2) M_0 has those blocks M_i on its block diagonal to get $M_0^{-1}JM_0 = J^{\mathrm{T}}$. (3) $A^{\mathrm{T}} = (M^{-1})^{\mathrm{T}}J^{\mathrm{T}}M^{\mathrm{T}}$ is $(M^{-1})^{\mathrm{T}}M_0^{-1}JM_0M^{\mathrm{T}} = (MM_0M^{\mathrm{T}})^{-1}A(MM_0M^{\mathrm{T}})$, and A^{T} is similar to A.

17 (a) True: One has $\lambda = 0$, the other doesn't (b) False. Diagonalize a nonsymmetric matrix and Λ is symmetric (c) False: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are similar (d) True: All eigenvalues of $A + I$ are increased by 1, so different from the eigenvalues of A.

18 $AB = B^{-1}(BA)B$ so AB is similar to BA. *Also* $AB\boldsymbol{x} = \lambda\boldsymbol{x}$ leads to $BA(B\boldsymbol{x}) = \lambda(B\boldsymbol{x})$.

19 Diagonals 6 by 6 and 4 by 4; AB has all the same eigenvalues as BA plus $6 - 4$ zeros.

Problem Set 6.7, page 360

2 (a) $AA^{\mathrm{T}} = \begin{bmatrix} 17 & 34 \\ 34 & 68 \end{bmatrix}$ has $\sigma_1^2 = 85$, $\boldsymbol{u}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, $\boldsymbol{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$.

(b) $A\boldsymbol{v}_1 = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix} = \begin{bmatrix} \sqrt{17} \\ 2\sqrt{17} \end{bmatrix} = \sqrt{85}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \sigma_1\boldsymbol{u}_1$.

4 $A^{\mathrm{T}}A = AA^{\mathrm{T}} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues $\sigma_1^2 = \dfrac{3+\sqrt{5}}{2}$ and $\sigma_2^2 = \dfrac{3-\sqrt{5}}{2}$.

Since $A = A^{\mathrm{T}}$ the eigenvectors of $A^{\mathrm{T}}A$ are the same as for A. Since $\lambda_2 = \frac{1-\sqrt{5}}{2}$ is *negative*, $\sigma_1 = \lambda_1$ but $\sigma_2 = -\lambda_2$. The eigenvectors are the same as in Section 6.2 for A, except for the effect of this minus sign:

$$\boldsymbol{u}_1 = \boldsymbol{v}_1 = \begin{bmatrix} \lambda_1/\sqrt{1+\lambda_1^2} \\ 1/\sqrt{1+\lambda_1^2} \end{bmatrix} \text{ and } \boldsymbol{u}_2 = -\boldsymbol{v}_2 = \begin{bmatrix} \lambda_2/\sqrt{1+\lambda_2^2} \\ 1/\sqrt{1+\lambda_2^2} \end{bmatrix}.$$

6 A proof that ***eigshow*** finds the SVD for 2 by 2 matrices. Starting at the orthogonal pair $\boldsymbol{V}_1 = (1, 0)$, $\boldsymbol{V}_2 = (0, 1)$ the demo finds $A\boldsymbol{V}_1$ and $A\boldsymbol{V}_2$ at angle θ. After a 90° turn by the mouse to $\boldsymbol{V}_2, -\boldsymbol{V}_1$ the demo finds $A\boldsymbol{V}_2$ and $-A\boldsymbol{V}_1$ at angle $\pi - \theta$. Somewhere between, the constantly orthogonal $\boldsymbol{v}_1, \boldsymbol{v}_2$ must have produced $A\boldsymbol{v}_1$ and $A\boldsymbol{v}_2$ at angle $\theta = \pi/2$. Those are the orthogonal directions for $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

8 $A = UV^{\mathrm{T}}$ since all $\sigma_j = 1$.

14 The smallest change in A is to set its smallest singular value σ_2 to zero.

16 The singular values of $A+I$ are not σ_j+1. They come from eigenvalues of $(A+I)^{\mathrm{T}}(A+I)$.

Problem Set 7.1, page 367

4 (a) $S(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $S(T(\boldsymbol{v}_1) + T(\boldsymbol{v}_2)) = S(T(\boldsymbol{v}_1)) + S(T(\boldsymbol{v}_2))$.

5 Choose $\boldsymbol{v} = (1, 1)$ and $\boldsymbol{w} = (-1, 0)$. Then $T(\boldsymbol{v}) + T(\boldsymbol{w}) = \boldsymbol{v} + \boldsymbol{w}$ but $T(\boldsymbol{v} + \boldsymbol{w}) = (0, 0)$.

7 (a) $T(T(\boldsymbol{v})) = \boldsymbol{v}$ (b) $T(T(\boldsymbol{v})) = \boldsymbol{v}+(2, 2)$ (c) $T(T(\boldsymbol{v})) = -\boldsymbol{v}$ (d) $T(T(\boldsymbol{v})) = T(\boldsymbol{v})$.

10 (a) $T(1,0)=\mathbf{0}$ (b) $(0,0,1)$ is not in the range (c) $T(0,1)=\mathbf{0}$.

12 $T(\boldsymbol{v})=(4,4)$; $(2,2)$; $(2,2)$; if $\boldsymbol{v}=(a,b)=b(1,1)+\frac{a-b}{2}(2,0)$ then $T(\boldsymbol{v})=b(2,2)+(0,0)$.

16 No matrix A gives $A\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}$. To professors: The matrix space has dimension 4. Linear transformations come from 4 by 4 matrices. Those in Problems 13–15 were special.

17 (a) True (b) True (c) True (d) False.

20 $T(T^{-1}(M))=M$ so $T^{-1}(M)=A^{-1}MB^{-1}$.

21 (a) Horizontal lines stay horizontal, vertical lines stay vertical (b) House squashes onto a line (c) Vertical lines stay vertical.

24 (a) $ad-bc=0$ (b) $ad-bc>0$ (c) $|ad-bc|=1$. If vectors to two corners transform to themselves then by linearity $T=I$. (Fails if one corner is $(0,0)$.)

27 This emphasizes that circles are transformed to ellipses (figure in Section 6.7).

Problem Set 7.2, page 380

3 $A^2=B$ when $T^2=S$ and output basis = input basis.

6 $T(\boldsymbol{v}_1+\boldsymbol{v}_2+\boldsymbol{v}_3)=2\boldsymbol{w}_1+\boldsymbol{w}_2+2\boldsymbol{w}_3$; A times $(1,1,1)$ gives $(2,1,2)$.

7 $\boldsymbol{v}=c(\boldsymbol{v}_2-\boldsymbol{v}_3)$ gives $T(\boldsymbol{v})=\mathbf{0}$; nullspace is $(0,c,-c)$; solutions are $(1,0,0)$ + any $(0,c,-c)$.

9 We don't know $T(\boldsymbol{w})$ unless the $\boldsymbol{w}$'s are the same as the $\boldsymbol{v}$'s. In that case the matrix is A^2.

13 (c) is wrong because $\boldsymbol{w}_1$ is not generally in the input space.

15 (a) $\begin{bmatrix}2&1\\5&3\end{bmatrix}$ (b) $\begin{bmatrix}3&-1\\-5&2\end{bmatrix}$ = inverse of (a) (c) $A\begin{bmatrix}2\\6\end{bmatrix}$ must be $2A\begin{bmatrix}1\\3\end{bmatrix}$.

17 $MN=\begin{bmatrix}1&0\\1&2\end{bmatrix}\begin{bmatrix}2&1\\5&3\end{bmatrix}^{-1}=\begin{bmatrix}3&-1\\-7&3\end{bmatrix}$.

19 $(a,b)=(\cos\theta,-\sin\theta)$. Minus sign from $Q^{-1}=Q^{\mathrm{T}}$.

21 $\boldsymbol{w}_2(x)=1-x^2$; $\boldsymbol{w}_3(x)=\frac{1}{2}(x^2-x)$; $\boldsymbol{y}=4\boldsymbol{w}_1+5\boldsymbol{w}_2+6\boldsymbol{w}_3$.

24 The matrix M with these nine entries must be invertible.

28 If T is not invertible, $T(\boldsymbol{v}_1),\ldots,T(\boldsymbol{v}_n)$ will not be a basis. We couldn't choose $\boldsymbol{w}_i=T(\boldsymbol{v}_i)$.

31 $S(T(\boldsymbol{v}))=(-1,2)$ but $S(\boldsymbol{v})=(-2,1)$ and $T(S(\boldsymbol{v}))=(1,-2)$.

Problem Set 7.3, page 389

2 The last step writes 6, 6, 2, 2 as the overall average 4, 4, 4, 4 plus the difference 2, 2, -2, -2. Therefore $c_1 = 4$ and $c_2 = 2$ and $c_3 = 1$ and $c_4 = 1$.

3 The wavelet basis is $(1, 1, 1, 1, 1, 1, 1, 1)$ and the long wavelet and two medium wavelets $(1, 1,$
$-1, -1, 0, 0, 0, 0)$ and $(0, 0, 0, 0, 1, 1, -1, -1)$ and 4 short wavelets with a single pair $1, -1$.

6 If $V\boldsymbol{b} = W\boldsymbol{c}$ then $\boldsymbol{b} = V^{-1}W\boldsymbol{c}$. The change of basis matrix is $V^{-1}W$.

7 The transpose of $WW^{-1} = I$ is $(W^{-1})^{\mathrm{T}}W^{\mathrm{T}} = I$. So the matrix W^{T} (which has the $\boldsymbol{w}$'s in its rows) is the inverse to the matrix that has the $\boldsymbol{w}^*$'s in its columns.

Problem Set 7.4, page 397

1 $A^{\mathrm{T}}A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$ has $\lambda = 50$ and 0, $\boldsymbol{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\boldsymbol{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$; $\sigma_1 = \sqrt{50}$.

5 $A = QH = \frac{1}{\sqrt{50}}\begin{bmatrix} 7 & -1 \\ 1 & 7 \end{bmatrix} \frac{1}{\sqrt{50}}\begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$. H is semidefinite because A is singular.

6 $A^+ = V\begin{bmatrix} 1/\sqrt{50} & 0 \\ 0 & 0 \end{bmatrix}U^{\mathrm{T}} = \frac{1}{50}\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$; $A^+A = \begin{bmatrix} .2 & .4 \\ .4 & .8 \end{bmatrix}$, $AA^+ = \begin{bmatrix} .1 & .3 \\ .3 & .9 \end{bmatrix}$.

9 $\begin{bmatrix} \sigma_1\boldsymbol{u}_1 & \sigma_2\boldsymbol{u}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1^{\mathrm{T}} \\ \boldsymbol{v}_2^{\mathrm{T}} \end{bmatrix} = \sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \sigma_2\boldsymbol{u}_2\boldsymbol{v}_2^{\mathrm{T}}$. In general this is $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^{\mathrm{T}} + \cdots + \sigma_r\boldsymbol{u}_r\boldsymbol{v}_r^{\mathrm{T}}$.

11 A^+ is A^{-1} because A is invertible.

13 $A = [\,1\,]\,[\,5 \;\; 0 \;\; 0\,]V^{\mathrm{T}}$ and $A^+ = V\begin{bmatrix} .2 \\ 0 \\ 0 \end{bmatrix}[\,1\,] = \begin{bmatrix} .12 \\ .16 \\ 0 \end{bmatrix}$; $AA^+ = [\,1\,]$;

$$A^+A = \begin{bmatrix} .36 & .48 & 0 \\ .48 & .64 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

15 If $\det A = 0$ then $\operatorname{rank}(A) < n$; thus $\operatorname{rank}(A^+) < n$ and $\det A^+ = 0$.

18 $\boldsymbol{x}^+$ in the row space of A is perpendicular to $\widehat{\boldsymbol{x}} - \boldsymbol{x}^+$ in the nullspace of $A^{\mathrm{T}}A$ = nullspace of A. The right triangle has $c^2 = a^2 + b^2$.

19 $AA^+\boldsymbol{p} = \boldsymbol{p}$, $AA^+\boldsymbol{e} = \boldsymbol{0}$, $A^+A\boldsymbol{x}_r = \boldsymbol{x}_r$, $A^+A\boldsymbol{x}_n = \boldsymbol{0}$.

21 L is determined by ℓ_{21}. Each eigenvector in S is determined by one number. The counts are $1+3$ for LU, $1+2+1$ for LDU, $1+3$ for QR, $1+2+1$ for $U\Sigma V^{\mathrm{T}}$, $2+2+0$ for $S\Lambda S^{-1}$.

24 Keep only the r by r invertible corner Σ_r of Σ (the rest is all zero). Then $A = U\Sigma V^{\mathrm{T}}$ has the required form $A = \widehat{U}M_1\Sigma_r M_2^{\mathrm{T}}\widehat{V}^{\mathrm{T}}$ with an invertible $M = M_1\Sigma_r M_2^{\mathrm{T}}$ in the middle.

11 $A^{\mathrm{T}}A = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$ diagonal entry = number of edges into the node off-diagonal entry = -1 if nodes are connected.

13 $A^{\mathrm{T}}CA\boldsymbol{x} = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix}\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ gives potentials $\boldsymbol{x} = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$ (grounded $x_4 = 0$ and solved 3 equations); $\boldsymbol{y} = -CA\boldsymbol{x} = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$.

17 (a) 8 independent columns (b) $\boldsymbol{f}$ must be orthogonal to the nullspace so $f_1 + \cdots + f_9 = 0$ (c) Each edge goes into 2 nodes, 12 edges make diagonal entries sum to 24.

Problem Set 8.3, page 428

2 $A = \begin{bmatrix} .6 & -1 \\ .4 & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ & .75 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix}$;

A^k approaches $\begin{bmatrix} .6 & -1 \\ .4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -.4 & .6 \end{bmatrix} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}$.

3 $\lambda = 1$ and $.8$, $\boldsymbol{x} = (1, 0)$; $\lambda = 1$ and $-.8$, $\boldsymbol{x} = (\frac{5}{9}, \frac{4}{9})$; $\lambda = 1, \frac{1}{4}$, and $\frac{1}{4}$, $\boldsymbol{x} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

5 The steady state is $(0, 0, 1)$ = all dead.

6 If $A\boldsymbol{x} = \lambda\boldsymbol{x}$, add components on both sides to find $s = \lambda s$. If $\lambda \neq 1$ the sum must be $s = 0$.

8 $(.5)^k \to 0$ gives $A^k \to A^\infty$; any $A = \begin{bmatrix} .6 + .4a & .6 - .6a \\ .4 - .4a & .4 + .6a \end{bmatrix}$ with $-\frac{2}{3} \le a \le 1$.

10 M^2 is still nonnegative; $[1 \ \cdots \ 1]M = [1 \ \cdots \ 1]$ so multiply by M to find $[1 \ \cdots \ 1]M^2 = [1 \ \cdots \ 1] \Rightarrow$ columns of M^2 add to 1.

11 $\lambda = 1$ and $a + d - 1$ from the trace; steady state is a multiple of $\boldsymbol{x}_1 = (b, 1 - a)$.

13 B has $\lambda = 0$ and $-.5$ with $\boldsymbol{x}_1 = (.3, \ .2)$ and $\boldsymbol{x}_2 = (-1, 1)$; $e^{-.5t}$ approaches zero and the solution approaches $c_1 e^{0t}\boldsymbol{x}_1 = c_1\boldsymbol{x}_1$.

15 The eigenvector is $\boldsymbol{x} = (1, 1, 1)$ and $A\boldsymbol{x} = (.9, .9, .9)$.

18 $\boldsymbol{p} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 130 \\ 32 \end{bmatrix}$; $I - \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$ has no inverse.

19 $\lambda = 1$ (Markov), 0 (singular), $.2$ (from trace). Steady state $(.3, .3, .4)$ and $(30, 30, 40)$.

20 *No*, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

Problem Set 8.4, page 436

1 Feasible set = line segment from $(6, 0)$ to $(0, 3)$; minimum cost at $(6, 0)$, maximum at $(0, 3)$.

2 Feasible set is 4-sided with corners $(0, 0)$, $(6, 0)$, $(2, 2)$, $(0, 6)$. Minimize $2x - y$ at $(6, 0)$.

3 Only two corners $(4, 0, 0)$ and $(0, 2, 0)$; choose x_1 very negative, $x_2 = 0$, and $x_3 = x_1 - 4$.

4 From $(0, 0, 2)$ move to $\boldsymbol{x} = (0, 1, 1.5)$ with the constraint $x_1 + x_2 + 2x_3 = 4$. The new cost is $3(1) + 8(1.5) = \$15$ so $r = -1$ is the reduced cost. The simplex method also checks $\boldsymbol{x} = (1, 0, 1.5)$ with cost $5(1) + 8(1.5) = \$17$ so $r = 1$ (more expensive).

5 Cost = 20 at start $(4, 0, 0)$; keeping $x_1 + x_2 + 2x_3 = 4$ move to $(3, 1, 0)$ with cost 18 and $r = -2$; or move to $(2, 0, 1)$ with cost 17 and $r = -3$. Choose x_3 as entering variable and move to $(0, 0, 2)$ with cost 14. Another step to reach $(0, 4, 0)$ with minimum cost 12.

6 $\boldsymbol{c} = [\,3 \quad 5 \quad 7\,]$ has minimum cost 12 by the Ph.D. since $\boldsymbol{x} = (4, 0, 0)$ is minimizing. The dual problem maximizes $4y$ subject to $y \le 3$, $y \le 5$, $y \le 7$. Maximum = 12.

Problem Set 8.5, page 442

1 $\int_0^{2\pi} \cos(j + k)x\,dx = \left[\frac{\sin(j+k)x}{j+k}\right]_0^{2\pi} = 0$ and similarly $\int_0^{2\pi} \cos(j - k)x\,dx = 0$ (in the denominator notice $j - k \ne 0$). If $j = k$ then $\int_0^{2\pi} \cos^2 jx\,dx = \pi$.

4 $\int_{-1}^{1} (1)(x^3 - cx)\,dx = 0$ and $\int_{-1}^{1} (x^2 - \frac{1}{3})(x^3 - cx)\,dx = 0$ for all c (integral of an odd function). Choose c so that $\int_{-1}^{1} x(x^3 - cx)\,dx = [\frac{1}{5}x^5 - \frac{c}{3}x^3]_{-1}^{1} = \frac{2}{5} - c\frac{2}{3} = 0$. Then $c = \frac{3}{5}$.

5 The integrals lead to $a_1 = 0$, $b_1 = 4/\pi$, $b_2 = 0$.

6 From equation (3) the a_k are zero and $b_k = 4/\pi k$. The square wave has $\|f\|^2 = 2\pi$. Then equation (6) is $2\pi = \pi(16/\pi^2)(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots)$ so this infinite series equals $\pi^2/8$.

8 $\|\boldsymbol{v}\|^2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$ so $\|\boldsymbol{v}\| = \sqrt{2}$; $\|\boldsymbol{v}\|^2 = 1 + a^2 + a^4 + \cdots = 1/(1 - a^2)$ so $\|\boldsymbol{v}\| = 1/\sqrt{1 - a^2}$; $\int_0^{2\pi} (1 + 2\sin x + \sin^2 x)\,dx = 2\pi + 0 + \pi$ so $\|f\| = \sqrt{3\pi}$.

9 (a) $f(x) = \frac{1}{2} + \frac{1}{2}$ (square wave) so a's are $\frac{1}{2}$, 0, 0, . . ., and b's are $2/\pi$, 0, $-2/3\pi$, 0, $2/5\pi$, . . . (b) $a_0 = \int_0^{2\pi} x\,dx/2\pi = \pi$, other $a_k = 0$, $b_k = -2/k$.

11 $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$; $\cos(x + \frac{\pi}{3}) = \cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3} = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$.

13 $dy/dx = \cos x$ has $y = y_p + y_n = \sin x + C$.

Problem Set 8.6, page 448

1 (x, y, z) has homogeneous coordinates $(x, y, z, 1)$ and also (cx, cy, cz, c) for any nonzero c.

4 $S = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ & & & 1 \end{bmatrix}$, $ST = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ 1 & 4 & 3 & 1 \end{bmatrix}$, $TS = \begin{bmatrix} c & & & \\ & c & & \\ & & c & \\ c & 4c & 3c & 1 \end{bmatrix}$, use $\boldsymbol{v}TS$.

5 $S = \begin{bmatrix} 1/8.5 & & \\ & 1/11 & \\ & & 1 \end{bmatrix}$ for a 1 by 1 square.

9 $\boldsymbol{n} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ has $\|\boldsymbol{n}\| = 1$ and $P = I - \boldsymbol{n}\boldsymbol{n}^{\mathrm{T}} = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$.

10 Choose $(0, 0, 3)$ on the plane and multiply $T_- P T_+ = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 6 & 6 & 3 & 9 \end{bmatrix}$.

11 $(3, 3, 3)$ projects to $\frac{1}{3}(-1, -1, 4)$ and $(3, 3, 3, 1)$ projects to $(\frac{1}{3}, \frac{1}{3}, \frac{5}{3}, 1)$.

13 The projection of a cube is a hexagon.

14 $(3, 3, 3)(I - 2\boldsymbol{n}\boldsymbol{n}^{\mathrm{T}}) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\begin{bmatrix} 1 & -8 & -4 \\ -8 & 1 & -4 \\ -4 & -4 & 7 \end{bmatrix} = (-\frac{11}{3}, -\frac{11}{3}, -\frac{1}{3})$.

15 $(3, 3, 3, 1) \to (3, 3, 0, 1) \to (-\frac{7}{3}, -\frac{7}{3}, -\frac{8}{3}, 1) \to (-\frac{7}{3}, -\frac{7}{3}, \frac{1}{3}, 1)$.

17 Rescaled by $1/c$ because (x, y, z, c) is the same point as $(x/c, y/c, z/c, 1)$.

Problem Set 9.1, page 457

1 Without exchange, pivots .001 and 1000; with exchange, pivots 1 and -1. When the pivot is larger than the entries below it, ℓ_{ij} = entry/pivot has $|\ell_{ij}| \le 1$. $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$.

4 The largest $\|\boldsymbol{x}\| = \|A^{-1}\boldsymbol{b}\|$ is $1/\lambda_{\min}$; the largest error is $10^{-16}/\lambda_{\min}$.

5 Each row of U has at most w entries. Then w multiplications to substitute components of $\boldsymbol{x}$ (already known from below) and divide by the pivot. Total for n rows is less than wn.

6 L, U, and R need $\frac{1}{2}n^2$ multiplications to solve a linear system. Q needs n^2 to multiply the right side by $Q^{-1} = Q^{\mathrm{T}}$. So QR takes 1.5 times longer than LU to reach $\boldsymbol{x}$.

7 On column j of I, back substitution needs $\frac{1}{2}j^2$ multiplications (only the j by j upper left block is involved). Then $\frac{1}{2}(1^2 + 2^2 + \cdots + n^2) \approx \frac{1}{2}(\frac{1}{3}n^3)$.

10 With 16-digit floating point arithmetic the errors $\|x - y_{\text{computed}}\|$ for $\varepsilon = 10^{-3}$, 10^{-6}, 10^{-9}, 10^{-12}, 10^{-15} are of order 10^{-16}, 10^{-11}, 10^{-7}, 10^{-4}, 10^{-3}.

11 $\cos\theta = 1/\sqrt{10}$, $\sin\theta = -3/\sqrt{10}$, $R = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 & 14 \\ 0 & 8 \end{bmatrix}$.

14 $Q_{ij}A$ uses $4n$ multiplications (2 for each entry in rows i and j). By factoring out $\cos\theta$, the entries 1 and $\pm\tan\theta$ need only $2n$ multiplications, which leads to $\frac{2}{3}n^3$ for QR.

Problem Set 9.2, page 463

1 $\|A\| = 2$, $c = 2/.5 = 4$; $\|A\| = 3$, $c = 3/1 = 3$; $\|A\| = 2+\sqrt{2}$, $c = (2+\sqrt{2})/(2-\sqrt{2}) = 5.83$.

3 For the first inequality replace x by Bx in $\|Ax\| \le \|A\|\|x\|$; the second inequality is just $\|Bx\| \le \|B\|\|x\|$. Then $\|AB\| = \max(\|ABx\|/\|x\|) \le \|A\|\|B\|$.

7 The triangle inequality gives $\|Ax + Bx\| \le \|Ax\| + \|Bx\|$. Divide by $\|x\|$ and take the maximum over all nonzero vectors to find $\|A + B\| \le \|A\| + \|B\|$.

8 If $Ax = \lambda x$ then $\|Ax\|/\|x\| = |\lambda|$ for that particular vector x. When we maximize the ratio over all vectors we get $\|A\| \ge |\lambda|$.

13 The residual $b - Ay = (10^{-7}, 0)$ is much smaller than $b - Az = (.0013, .0016)$. But z is much closer to the solution than y.

14 $\det A = 10^{-6}$ so $A^{-1} = \begin{bmatrix} 659{,}000 & -563{,}000 \\ -913{,}000 & 780{,}000 \end{bmatrix}$. Then $\|A\| > 1$, $\|A^{-1}\| > 10^6$, $c > 10^6$.

16 $x_1^2 + \cdots + x_n^2$ is not smaller than $\max(x_i^2)$ and not larger than $x_1^2 + \cdots + x_n^2 + 2|x_1||x_2| + \cdots = \|x\|_1^2$. Certainly $x_1^2 + \cdots + x_n^2 \le n\max(x_i^2)$ so $\|x\| \le \sqrt{n}\|x\|_\infty$. Choose $y_i = \operatorname{sign} x_i = \pm 1$ to get $x \cdot y = \|x\|_1$. By Schwarz this is at most $\|x\|\|y\| = \sqrt{n}\|x\|$. Choose $x = (1, 1, \ldots, 1)$ for $\sqrt{n}$.

Problem Set 9.3, page 473

2 If $Ax = \lambda x$ then $(I - A)x = (1 - \lambda)x$. Real eigenvalues of $B = I - A$ have $|1 - \lambda| < 1$ provided λ is between 0 and 2.

6 Jacobi has $S^{-1}T = \frac{1}{3}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{3}$.

7 Gauss-Seidel has $S^{-1}T = \begin{bmatrix} 0 & \frac{1}{3} \\ 0 & \frac{1}{9} \end{bmatrix}$ with $|\lambda|_{\max} = \frac{1}{9} = (|\lambda|_{\max} \text{ for Jacobi})^2$.

9 Set the trace $2 - 2\omega + \frac{1}{4}\omega^2$ equal to $(\omega - 1) + (\omega - 1)$ to find $\omega_{\text{opt}} = 4(2 - \sqrt{3}) \approx 1.07$. The eigenvalues $\omega - 1$ are about .07.

15 The jth component of $A\boldsymbol{x}_1$ is $2\sin\frac{j\pi}{n+1} - \sin\frac{(j-1)\pi}{n+1} - \sin\frac{(j+1)\pi}{n+1}$. The last two terms, using $\sin(a+b) = \sin a\cos b + \cos a\sin b$, combine into $-2\sin\frac{j\pi}{n+1}\cos\frac{\pi}{n+1}$. The eigenvalue is $\lambda_1 = 2 - 2\cos\frac{\pi}{n+1}$.

17 $A^{-1} = \frac{1}{3}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$ gives $\boldsymbol{u}_0 = \begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{u}_1 = \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}$, $\boldsymbol{u}_2 = \frac{1}{9}\begin{bmatrix}5\\4\end{bmatrix}$, $\boldsymbol{u}_3 = \frac{1}{27}\begin{bmatrix}14\\13\end{bmatrix} \to \begin{bmatrix}1\\1\end{bmatrix}$.

18 $R = Q^{\mathrm{T}}A = \begin{bmatrix}1 & \cos\theta\sin\theta\\ 0 & -\sin^2\theta\end{bmatrix}$ and $A_1 = RQ = \begin{bmatrix}\cos\theta(1+\sin^2\theta) & -\sin^3\theta\\ -\sin^3\theta & -\cos\theta\sin^2\theta\end{bmatrix}$.

20 If $A - cI = QR$ then $A_1 = RQ + cI = Q^{-1}(QR + cI)Q = Q^{-1}AQ$. No change in eigenvalues.

21 Multiply $A\boldsymbol{q}_j = b_{j-1}\boldsymbol{q}_{j-1} + a_j\boldsymbol{q}_j + b_j\boldsymbol{q}_{j+1}$ by $\boldsymbol{q}_j^{\mathrm{T}}$ to find $\boldsymbol{q}_j^{\mathrm{T}}A\boldsymbol{q}_j = a_j$ (because the $\boldsymbol{q}$'s are orthonormal). The matrix form (multiplying by columns) is $AQ = QT$ where T is *tridiagonal*. Its entries are the a's and b's.

23 If A is symmetric then $A_1 = Q^{-1}AQ = Q^{\mathrm{T}}AQ$ is also symmetric. $A_1 = RQ = R(QR)R^{-1} = RAR^{-1}$ has R and R^{-1} upper triangular, so A_1 cannot have nonzeros on a lower diagonal than A. If A is tridiagonal and symmetric then (by using symmetry for the upper part of A_1) the matrix $A_1 = RAR^{-1}$ is also tridiagonal.

27 From the last line of code, $\boldsymbol{q}_2$ is in the direction of $\boldsymbol{v} = A\boldsymbol{q}_1 - h_{11}\boldsymbol{q}_1 = A\boldsymbol{q}_1 - (\boldsymbol{q}_1^{\mathrm{T}}A\boldsymbol{q}_1)\boldsymbol{q}_1$. The dot product with $\boldsymbol{q}_1$ is zero. This is Gram-Schmidt with $A\boldsymbol{q}_1$ as the second input vector.

28 $\boldsymbol{r}_1 = \boldsymbol{b} - \alpha_1 A\boldsymbol{b} = \boldsymbol{b} - (\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b})A\boldsymbol{b}$ is orthogonal to $\boldsymbol{r}_0 = \boldsymbol{b}$: *the residuals* $\boldsymbol{r} = \boldsymbol{b} - A\boldsymbol{x}$ *are orthogonal at each step*. To show that $\boldsymbol{p}_1$ is orthogonal to $A\boldsymbol{p}_0 = A\boldsymbol{b}$, simplify $\boldsymbol{p}_1$ to $c\boldsymbol{P}_1$: $\boldsymbol{P}_1 = \|A\boldsymbol{b}\|^2\boldsymbol{b} - (\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b})A\boldsymbol{b}$ and $c = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}/(\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b})^2$. Certainly $(A\boldsymbol{b})^{\mathrm{T}}\boldsymbol{P}_1 = 0$ because $A^{\mathrm{T}} = A$. (That simplification put α_1 into $\boldsymbol{p}_1 = \boldsymbol{b} - \alpha_1 A\boldsymbol{b} + (\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b} - 2\alpha_1\boldsymbol{b}^{\mathrm{T}}A\boldsymbol{b} + \alpha_1^2\|A\boldsymbol{b}\|^2)\boldsymbol{b}/\boldsymbol{b}^{\mathrm{T}}\boldsymbol{b}$. For a good discussion see *Numerical Linear Algebra* by Trefethen and Bau.)

Problem Set 10.1, page 483

2 In polar form these are $\sqrt{5}e^{i\theta}$, $5e^{2i\theta}$, $\frac{1}{\sqrt{5}}e^{-i\theta}$, $\sqrt{5}$.

4 $|z \times w| = 6$, $|z + w| \le 5$, $|z/w| = \frac{2}{3}$, $|z - w| \le 5$.

5 $a + ib = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, i, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$; $w^{12} = 1$.

9 $2+i$; $(2+i)(1+i) = 1+3i$; $e^{-i\pi/2} = -i$; $e^{-i\pi} = -1$; $\frac{1-i}{1+i} = -i$; $(-i)^{103} = (-i)^3 = i$.

10 $z + \bar{z}$ is real; $z - \bar{z}$ is pure imaginary; $z\bar{z}$ is positive; $z/\bar{z}$ has absolute value 1.

12 (a) When $a = b = d = 1$ the square root becomes $\sqrt{4c}$; λ is complex if $c < 0$ (b) $\lambda = 0$ and $\lambda = a + d$ when $ad = bc$ (c) the λ's can be real and different.

13 Complex λ's when $(a+d)^2 < 4(ad - bc)$; write $(a+d)^2 - 4(ad - bc)$ as $(a-d)^2 + 4bc$ which is positive when $bc > 0$.

22 $R + iS = (R + iS)^{\mathrm{H}} = R^{\mathrm{T}} - iS^{\mathrm{T}}$; R is symmetric but S is skew-symmetric.

24 $[\,1\,]$ and $[\,-1\,]$; any $[\,e^{i\theta}\,]$; $\begin{bmatrix} a & b+ic \\ b-ic & d \end{bmatrix}$; $\begin{bmatrix} w & e^{i\phi}\overline{z} \\ -z & e^{i\phi}\overline{w} \end{bmatrix}$ with $|w|^2 + |z|^2 = 1$.

27 Unitary means $U^{\mathrm{H}}U = I$ or $(A^{\mathrm{T}} - iB^{\mathrm{T}})(A + iB) = (A^{\mathrm{T}}A + B^{\mathrm{T}}B) + i(A^{\mathrm{T}}B - B^{\mathrm{T}}A) = I$. Then $A^{\mathrm{T}}A + B^{\mathrm{T}}B = I$ and $A^{\mathrm{T}}B - B^{\mathrm{T}}A = 0$ which makes the block matrix orthogonal.

30 $A = \begin{bmatrix} 1-i & 1-i \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 2+2i & -2 \\ 1+i & 2 \end{bmatrix} = S\Lambda S^{-1}$.

Problem Set 10.3, page 500

8 $\boldsymbol{c} \to (1,1,1,1,0,0,0,0) \to (4,0,0,0,0,0,0,0) \to (4,0,0,0,4,0,0,0)$ which is $F_8\boldsymbol{c}$. The second vector becomes $(0,0,0,0,1,1,1,1) \to (0,0,0,0,4,0,0,0) \to (4,0,0,0,-4,0,0,0)$.

9 If $w^{64} = 1$ then w^2 is a 32nd root of 1 and $\sqrt{w}$ is a 128th root of 1.

13 $e_1 = c_0 + c_1 + c_2 + c_3$ and $e_2 = c_0 + c_1 i + c_2 i^2 + c_3 i^3$; E contains the four eigenvalues of C.

14 Eigenvalues $e_1 = 2 - 1 - 1 = 0$, $e_2 = 2 - i - i^3 = 2$, $e_3 = 2 - (-1) - (-1) = 4$, $e_4 = 2 - i^3 - i^9 = 2$. Check trace $0 + 2 + 4 + 2 = 8$.

15 Diagonal E needs n multiplications, Fourier matrix F and F^{-1} need $\frac{1}{2}n \log_2 n$ multiplications each by the **FFT**. Total much less than the ordinary n^2.

16 $(c_0 + c_2) + (c_1 + c_3)$; then $(c_0 - c_2) + i(c_1 - c_3)$; then $(c_0 + c_2) - (c_1 + c_3)$; then $(c_0 - c_2) - i(c_1 - c_3)$. These steps are the **FFT**!

A FINAL EXAM

***This was the final exam on December 17, 2002 in* MIT's linear algebra course 18.06**

1 The 4 by 6 matrix A has all 2's below the diagonal and elsewhere all 1's:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$$

(a) By elimination factor A into L (4 by 4) times U (4 by 6).

(b) Find the rank of A and a basis for its nullspace (the special solutions would be good).

2 Suppose you know that the 3 by 4 matrix A has the vector $\boldsymbol{s} = (2, 3, 1, 0)$ as a basis for its nullspace.

(a) What is the *rank* of A and the complete solution to $A\boldsymbol{x} = \mathbf{0}$?

(b) What is the exact row reduced echelon form R of A?

3 The following matrix is a *projection matrix*:

$$P = \frac{1}{21}\begin{bmatrix} 1 & 2 & -4 \\ 2 & 4 & -8 \\ -4 & -8 & 16 \end{bmatrix}.$$

(a) What subspace does P project onto?

(b) What is the *distance* from that subspace to $\boldsymbol{b} = (1, 1, 1)$?

(c) What are the three eigenvalues of P? Is P diagonalizable?

4 **(a)** Suppose the product of A and B is the zero matrix: $AB = 0$. Then the **(1)** space of A contains the **(2)** space of B. Also the **(3)** space of B contains the **(4)** space of A. Those blank words are

(1) ________________ **(2)** ________________

(3) ________________ **(4)** ________________

(b) Suppose that matrix A is 5 by 7 with rank r, and B is 7 by 9 of rank s. What are the dimensions of spaces **(1)** and **(2)**? From the fact that space **(1)** contains space **(2)**, what do you learn about $r+s$?

5 Suppose the 4 by 2 matrix Q has orthonormal columns.

(a) Find the least squares solution $\widehat{\boldsymbol{x}}$ to $Q\boldsymbol{x}=\boldsymbol{b}$.

(b) Explain why QQ^{T} is not positive definite.

(c) What are the (nonzero) singular values of Q, and why?

6 Let $\boldsymbol{S}$ be the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix}1\\2\\2\end{bmatrix}$ and $\begin{bmatrix}5\\4\\-2\end{bmatrix}$.

(a) Find an orthonormal basis $\boldsymbol{q}_1, \boldsymbol{q}_2$ for $\boldsymbol{S}$ by Gram-Schmidt.

(b) Write down the 3 by 3 matrix P which projects vectors perpendicularly onto $\boldsymbol{S}$.

(c) Show how the properties of P (what are they?) lead to the conclusion that $P\boldsymbol{b}$ is orthogonal to $\boldsymbol{b}-P\boldsymbol{b}$.

7 **(a)** If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a basis for $\mathbf{R}^3$ then the matrix with those three columns is _____.

(b) If $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ span $\mathbf{R}^3$, give all possible ranks for the matrix with those four columns. ____________________.

(c) If $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ form an orthonormal basis for $\mathbf{R}^3$, and T is the transformation that projects every vector $\boldsymbol{v}$ onto the plane of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, what is the matrix for T in this basis? Explain.

8 Suppose the n by n matrix A_n has 3's along its main diagonal and 2's along the diagonal below and the $(1, n)$ position:

$$A_4=\begin{bmatrix}3&0&0&2\\2&3&0&0\\0&2&3&0\\0&0&2&3\end{bmatrix}.$$

Find by cofactors of row 1 or otherwise the determinant of A_4 and then the determinant of A_n for $n>4$.

9 There are six 3 by 3 permutation matrices P.

(a) What numbers can be the *determinant* of P? What numbers can be *pivots*?

(b) What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

MATRIX FACTORIZATIONS

1. $\mathbf{A = LU} = \begin{pmatrix}\text{lower triangular } L \\ \text{1's on the diagonal}\end{pmatrix}\begin{pmatrix}\text{upper triangular } U \\ \text{pivots on the diagonal}\end{pmatrix}$ *Section 2.6*

   Requirements: No row exchanges as Gaussian elimination reduces A to U.

2. $\mathbf{A = LDU} = \begin{pmatrix}\text{lower triangular } L \\ \text{1's on the diagonal}\end{pmatrix}\begin{pmatrix}\text{pivot matrix} \\ D \text{ is diagonal}\end{pmatrix}\begin{pmatrix}\text{upper triangular } U \\ \text{1's on the diagonal}\end{pmatrix}$

   Requirements: No row exchanges. The pivots in D are divided out to leave 1's in U. If A is symmetric then U is L^{T} and $A = LDL^{\mathrm{T}}$. *Section 2.6 and 2.7*

3. $\mathbf{PA = LU}$ (permutation matrix P to avoid zeros in the pivot positions).

   Requirements: A is invertible. Then P, L, U are invertible. P does the row exchanges in advance. Alternative: $A = L_1 P_1 U_1$. *Section 2.7*

4. $\mathbf{EA = R}$ (m by m invertible E) (any A) = rref(A).

   Requirements: None! *The reduced row echelon form* R has r pivot rows and pivot columns. The only nonzero in a pivot column is the unit pivot. The last $m - r$ rows of E are a basis for the left nullspace of A, and the first r columns of E^{-1} are a basis for the column space of A. *Sections 3.2–3.3.*

5. $\mathbf{A = CC^T}$ = (lower triangular matrix C) (transpose is upper triangular)

   Requirements: A is symmetric and positive definite (all n pivots in D are positive). This *Cholesky factorization* has $C = L\sqrt{D}$. *Section 6.5*

6. $\mathbf{A = QR}$ = (orthonormal columns in Q) (upper triangular R)

   Requirements: A has independent columns. Those are *orthogonalized* in Q by the Gram-Schmidt process. If A is square then $Q^{-1} = Q^{\mathrm{T}}$. *Section 4.4*

7. $\mathbf{A = S\Lambda S^{-1}}$ = (eigenvectors in S)(eigenvalues in Λ)(left eigenvectors in S^{-1}).

   Requirements: A must have n linearly independent eigenvectors. *Section 6.2*

8. $\mathbf{A = Q\Lambda Q^T}$ =(orthogonal matrix Q)(real eigenvalue matrix Λ)(Q^{T} is Q^{-1}).

   Requirements: A is *symmetric*. This is the Spectral Theorem. *Section 6.4*

9. $\mathbf{A = MJM^{-1}}$ = (generalized eigenvectors in M)(Jordan blocks in J)(M^{-1}).

Requirements: A is any square matrix. *Jordan form* J has a block for each independent eigenvector of A. Each block has one eigenvalue. *Section 6.6*

10. $\mathbf{A = U\,\Sigma V^T} = \begin{pmatrix}\text{orthogonal}\\ U \text{ is } m \times m\end{pmatrix}\begin{pmatrix} m \times n \text{ singular value matrix}\\ \sigma_1, \ldots, \sigma_r \text{ on its diagonal}\end{pmatrix}\begin{pmatrix}\text{orthogonal}\\ V \text{ is } n \times n\end{pmatrix}.$

Requirements: None. This *singular value decomposition* (SVD) has the eigenvectors of AA^{T} in U and of $A^{\mathrm{T}}A$ in V; $\sigma_i = \sqrt{\lambda_i(A^{\mathrm{T}}A)} = \sqrt{\lambda_i(AA^{\mathrm{T}})}$. *Sections 6.7 and 7.4*

11. $\mathbf{A^+ = V\Sigma^+U^T} = \begin{pmatrix}\text{orthogonal}\\ n \times n\end{pmatrix}\begin{pmatrix} n \times m \text{ pseudoinverse of } \Sigma\\ 1/\sigma_1, \ldots, 1/\sigma_r \text{ on diagonal}\end{pmatrix}\begin{pmatrix}\text{orthogonal}\\ m \times m\end{pmatrix}.$

Requirements: None. The *pseudoinverse* has A^+A = projection onto row space of A and AA^+ = projection onto column space. The shortest least-squares solution to $A\mathbf{x} = \mathbf{b}$ is $\hat{\boldsymbol{x}} = A^+\mathbf{b}$. This solves $A^{\mathrm{T}}A\hat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. *Section 7.4*

12. $\mathbf{A = QH}$ = (orthogonal matrix Q)(symmetric positive definite matrix H).

Requirements: A is invertible. This *polar decomposition* has $H^2 = A^{\mathrm{T}}A$. The factor H is semidefinite if A is singular. The reverse polar decomposition $A = KQ$ has $K^2 = AA^{\mathrm{T}}$. Both have $Q = UV^{\mathrm{T}}$ from the SVD. *Section 7.4*

13. $\mathbf{A = U\Lambda U^{-1}}$ = (unitary U)(eigenvalue matrix Λ)(U^{-1} which is $U^{\mathrm{H}} = \overline{U}^{\mathrm{T}}$).

Requirements: A is *normal*: $A^{\mathrm{H}}A = AA^{\mathrm{H}}$. Its orthonormal (and possibly complex) eigenvectors are the columns of U. Complex λ's unless $A = A^{\mathrm{H}}$. *Section 10.2*

14. $\mathbf{A = UTU^{-1}}$ = (unitary U)(triangular T with λ's on diagonal)($U^{-1} = U^{\mathrm{H}}$).

Requirements: *Schur triangularization* of any square A. There is a matrix U with orthonormal columns that makes $U^{-1}AU$ triangular. *Section 10.2*

15. $\mathbf{F_n} = \begin{bmatrix} I & D \\ I & -D\end{bmatrix}\begin{bmatrix}\mathbf{F_{n/2}} & \\ & \mathbf{F_{n/2}}\end{bmatrix}\begin{bmatrix}\text{even-odd}\\ \text{permutation}\end{bmatrix}$ = one step of the **FFT**.

Requirements: F_n = Fourier matrix with entries w^{jk} where $w^n = 1$. Then $\mathbf{F_n}\overline{\mathbf{F}}_\mathbf{n} = nI$. D has $1, w, w^2, \ldots$ on its diagonal. For $n = 2^l$ the *Fast Fourier Transform* has $\frac{1}{2}nl$ multiplications from l stages of D's. *Section 10.3*

CONCEPTUAL QUESTIONS FOR REVIEW

Chapter 1

1.1 Which vectors are linear combinations of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$?

1.2 Compare the dot product of $\boldsymbol{v} = (3, 1)$ and $\boldsymbol{w} = (4, 3)$ to the product of their lengths. Which is larger? Whose inequality?

1.3 What is the cosine of the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in Question 1.2? What is the cosine of the angle between the x-axis and $\boldsymbol{v}$?

Chapter 2

2.1 Multiplying a matrix A times the column vector $\boldsymbol{x} = (2, -1)$ gives what combination of the columns of A? How many rows and columns in A?

2.2 If $A\boldsymbol{x} = \boldsymbol{b}$ then the vector $\boldsymbol{b}$ is a linear combination of what vectors from the matrix A? In vector space language, $\boldsymbol{b}$ lies in the _____ space of A.

2.3 If A is the 2 by 2 matrix $\begin{bmatrix} 2 & 1 \\ 6 & 6 \end{bmatrix}$ what are its pivots?

2.4 If A is the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ how does elimination proceed? What permutation matrix P is involved?

2.5 If A is the matrix $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ find $\boldsymbol{b}$ and $\boldsymbol{c}$ so that $A\boldsymbol{x} = \boldsymbol{b}$ has no solution and $A\boldsymbol{x} = \boldsymbol{c}$ has a solution.

2.6 What 3 by 3 matrix L adds 5 times row 2 to row 3 and then adds 2 times row 1 to row 2, when it multiplies a matrix with three rows?

2.7 What 3 by 3 matrix E subtracts 2 times row 1 from row 2 and then subtracts 5 times row 2 from row 3? How is E related to L in Question 2.6?

2.8 If A is 4 by 3 and B is 3 by 7, how many *row times column* products go into AB? How many *column times row* products go into AB? How many separate small multiplications are involved (the same for both)?

2.9 Suppose $A = \begin{bmatrix} \mathbf{I} & \mathbf{U} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$ is a matrix with 2 by 2 blocks. What is the inverse matrix?

2.10 How can you find the inverse of A by working with $[\,A \quad I\,]$? If you solve the n equations $A\boldsymbol{x} =$ columns of I then the solutions $\boldsymbol{x}$ are columns of _____.

2.11 How does elimination decide whether a square matrix A is invertible?

2.12 Suppose elimination takes A to U (upper triangular) by row operations with the multipliers in L (lower triangular). Why does the last row of A agree with the last row of L times U?

2.13 What is the factorization (from elimination with possible row exchanges) of any square invertible matrix?

2.14 What is the transpose of the inverse of AB?

2.15 How do you know that the inverse of a permutation matrix is a permutation matrix? How is it related to the transpose?

Chapter 3

3.1 What is the column space of an invertible n by n matrix? What is the nullspace of that matrix?

3.2 If every column of A is a multiple of the first column, what is the column space of A?

3.3 What are the two requirements for a set of vectors in $\mathbf{R}^n$ to be a subspace?

3.4 If the row reduced form R of a matrix A begins with a row of ones, how do you know that the other rows of R are zero and what is the nullspace?

3.5 Suppose the nullspace of A contains only the zero vector. What can you say about solutions to $A\boldsymbol{x} = \boldsymbol{b}$?

3.6 From the row reduced form R, how would you decide the rank of A?

3.7 Suppose column 4 of A is the sum of columns 1, 2, and 3. Find a vector in the nullspace.

3.8 Describe in words the complete solution to a linear system $A\boldsymbol{x} = \boldsymbol{b}$.

3.9 If $A\boldsymbol{x} = \boldsymbol{b}$ has exactly one solution for every $\boldsymbol{b}$, what can you say about A?

3.10 Give an example of vectors that span $\mathbf{R}^2$ but are not a basis for $\mathbf{R}^2$.

3.11 What is the dimension of the space of 4 by 4 symmetric matrices?

3.12 Describe the meaning of *basis* and *dimension* of a vector space.

Chapter 5

5.1 What is the determinant of the matrix $-I$?

5.2 Explain how the determinant is a linear function of the first row.

5.3 How do you know that $\det A^{-1} = 1/\det A$?

5.4 If the pivots of A (with no row exchanges) are 2, 6, 6, what submatrices of A have known determinants?

5.5 Suppose the first row of A is $0, 0, 0, 3$. What does the "big formula" for the determinant of A reduce to in this case?

5.6 Is the ordering $(2, 5, 3, 4, 1)$ even or odd? What permutation matrix has what determinant, from your answer?

5.7 What is the cofactor C_{23} in the 3 by 3 elimination matrix E that subtracts 4 times row 1 from row 2? What entry of E^{-1} is revealed?

5.8 Explain the meaning of the cofactor formula for $\det A$ using column 1.

5.9 How does Cramer's Rule give the first component in the solution to $I\boldsymbol{x} = \boldsymbol{b}$?

5.10 If I combine the entries in row 2 with the cofactors from row 1, why is $a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$ automatically zero?

5.11 What is the connection between determinants and volumes?

5.12 Find the cross product of $\boldsymbol{u} = (0, 0, 1)$ and $\boldsymbol{v} = (0, 1, 0)$ and its direction.

5.13 If A is n by n, why is $\det(A - \lambda I)$ a polynomial in λ of degree n?

Chapter 6

6.1 What equation gives the eigenvalues of A without involving the eigenvectors? How would you then find the eigenvectors?

6.2 If A is singular what does this say about its eigenvalues?

6.3 If A times A equals $4A$, what numbers can be eigenvalues of A?

6.4 Find a real matrix that has no real eigenvalues or eigenvectors.

6.5 How can you find the sum and product of the eigenvalues directly from A?

6.6 What are the eigenvalues of the rank one matrix $[\,1\ \ 2\ \ 1\,]^{\mathrm{T}}[\,1\ \ 1\ \ 1\,]$?

6.7 Explain the diagonalization formula $A = S\Lambda S^{-1}$. Why is it true and when is it true?

GLOSSARY

Adjacency matrix of a graph. Square matrix with $a_{ij} = 1$ when there is an edge from node i to node j; otherwise $a_{ij} = 0$. $A = A^{\mathrm{T}}$ for an undirected graph.

Affine transformation $T(\boldsymbol{v}) = A\boldsymbol{v} + \boldsymbol{v}_0$ = linear transformation plus shift.

Associative Law $(AB)C = A(BC)$. Parentheses can be removed to leave ABC.

Augmented matrix $[\,A \;\; \boldsymbol{b}\,]$. $A\boldsymbol{x} = \boldsymbol{b}$ is solvable when $\boldsymbol{b}$ is in the column space of A; then $[\,A \;\; \boldsymbol{b}\,]$ has the same rank as A. Elimination on $[\,A \;\; \boldsymbol{b}\,]$ keeps equations correct.

Back substitution. Upper triangular systems are solved in reverse order x_n to x_1.

Basis for $\boldsymbol{V}$. Independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d$ whose linear combinations give every $\boldsymbol{v}$ in $\boldsymbol{V}$. A vector space has many bases!

Big formula for n by n determinants. Det(A) is a sum of $n!$ terms, one term for each permutation P of the columns. That term is the product $a_{1\alpha} \cdots a_{n\omega}$ down the diagonal of the reordered matrix, times $\det(P) = \pm 1$.

Block matrix. A matrix can be partitioned into matrix blocks, by cuts between rows and/or between columns. **Block multiplication** of AB is allowed if the block shapes permit (the columns of A and rows of B must be in matching blocks).

Cayley-Hamilton Theorem. $p(\lambda) = \det(A - \lambda I)$ has $p(A) =$ *zero matrix*.

Change of basis matrix M. The old basis vectors $\boldsymbol{v}_j$ are combinations $\sum m_{ij}\boldsymbol{w}_i$ of the new basis vectors. The coordinates of $c_1\boldsymbol{v}_1 + \cdots + c_n\boldsymbol{v}_n = d_1\boldsymbol{w}_1 + \cdots + d_n\boldsymbol{w}_n$ are related by $\boldsymbol{d} = M\boldsymbol{c}$. (For $n = 2$ set $\boldsymbol{v}_1 = m_{11}\boldsymbol{w}_1 + m_{21}\boldsymbol{w}_2$, $\boldsymbol{v}_2 = m_{12}\boldsymbol{w}_1 + m_{22}\boldsymbol{w}_2$.)

Characteristic equation $\det(A - \lambda I) = 0$. The n roots are the eigenvalues of A.

Cholesky factorization $A = CC^{\mathrm{T}} = (L\sqrt{D})(L\sqrt{D})^{\mathrm{T}}$ for positive definite A.

Circulant matrix C. Constant diagonals wrap around as in cyclic shift S. Every circulant is $c_0 I + c_1 S + \cdots + c_{n-1}S^{n-1}$. $C\boldsymbol{x} =$ **convolution** $\boldsymbol{c} * \boldsymbol{x}$. Eigenvectors in F.

Cofactor C_{ij}. Remove row i and column j; multiply the determinant by $(-1)^{i+j}$.

Column picture of $A\boldsymbol{x} = \boldsymbol{b}$. The vector $\boldsymbol{b}$ becomes a combination of the columns of A. The system is solvable only when $\boldsymbol{b}$ is in the column space $\boldsymbol{C}(A)$.

Column space $\boldsymbol{C}(A)$ = space of all combinations of the columns of A.

Commuting matrices $AB = BA$. If diagonalizable, they share n eigenvectors.

Companion matrix. Put $c_1, \ldots, c_n$ in row n and put $n - 1$ 1's along diagonal 1. Then $\det(A - \lambda I) = \pm(c_1 + c_2\lambda + c_3\lambda^2 + \cdots)$.

Complete solution $\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n$ to $A\boldsymbol{x} = \boldsymbol{b}$. (Particular $\boldsymbol{x}_p$) + ($\boldsymbol{x}_n$ in nullspace).

Complex conjugate $\overline{z} = a - ib$ for any complex number $z = a + ib$. Then $z\overline{z} = |z|^2$.

Condition number $cond(A) = \kappa(A) = \|A\|\|A^{-1}\| = \sigma_{\max}/\sigma_{\min}$. In $A\boldsymbol{x} = \boldsymbol{b}$, the relative change $\|\delta\boldsymbol{x}\|/\|\boldsymbol{x}\|$ is less than $cond(A)$ times the relative change $\|\delta\boldsymbol{b}\|/\|\boldsymbol{b}\|$. Condition numbers measure the *sensitivity* of the output to change in the input.

Conjugate Gradient Method. A sequence of steps (end of Chapter 9) to solve positive definite $A\boldsymbol{x} = \boldsymbol{b}$ by minimizing $\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}\boldsymbol{b}$ over growing Krylov subspaces.

Covariance matrix Σ. When random variables x_i have mean = average value = 0, their covariances Σ_{ij} are the averages of $x_i x_j$. With means $\overline{x}_i$, the matrix Σ = mean of $(\boldsymbol{x} - \overline{\boldsymbol{x}})(\boldsymbol{x} - \overline{\boldsymbol{x}})^{\mathrm{T}}$ is positive (semi)definite; it is diagonal if the x_i are independent.

Cramer's Rule for $A\boldsymbol{x} = \boldsymbol{b}$. B_j has $\boldsymbol{b}$ replacing column j of A, and $x_j = |B_j|/|A|$.

Cross product $\boldsymbol{u} \times \boldsymbol{v}$ in $\mathbf{R}^3$. Vector perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$, length $\|\boldsymbol{u}\|\|\boldsymbol{v}\||\sin\theta|$ = parallelogram area, computed as the "determinant" of $[\,\boldsymbol{i}\ \ \boldsymbol{j}\ \ \boldsymbol{k}\,;\ u_1\ \ u_2\ \ u_3\,;\ v_1\ \ v_2\ \ v_3\,]$.

Cyclic shift S. Permutation with $s_{21} = 1, s_{32} = 1, \ldots$, finally $s_{1n} = 1$. Its eigenvalues are nth roots $e^{2\pi ik/n}$ of 1; eigenvectors are columns of the Fourier matrix F.

Determinant $|A| = \det(A)$. Defined by $\det I = 1$, sign reversal for row exchange, and linearity in each row. Then $|A| = 0$ when A is singular. Also $|AB| = |A||B|$ and $|A^{-1}| = 1/|A|$ and $|A^{\mathrm{T}}| = |A|$. The big formula for $\det(A)$ has a sum of $n!$ terms, the cofactor formula uses determinants of size $n - 1$, volume of box = $|\det(A)|$.

Diagonal matrix D. $d_{ij} = 0$ if $i \neq j$. **Block-diagonal**: zero outside square blocks D_{ii}.

Diagonalizable matrix A. Must have n independent eigenvectors (in the columns of S; automatic with n different eigenvalues). Then $S^{-1}AS = \Lambda$ = eigenvalue matrix.

Diagonalization $\Lambda = S^{-1}AS$. Λ = eigenvalue matrix and S = eigenvector matrix. A must have n independent eigenvectors to make S invertible. All $A^k = S\Lambda^k S^{-1}$.

Dimension of vector space $\dim(\boldsymbol{V})$ = number of vectors in any basis for $\boldsymbol{V}$.

Distributive Law $A(B + C) = AB + AC$. Add then multiply, or multiply then add.

Dot product $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} = x_1y_1 + \cdots + x_ny_n$. Complex dot product is $\overline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{y}$. Perpendicular vectors have zero dot product. $(AB)_{ij}$ = (row i of A)·(column j of B).

Echelon matrix U. The first nonzero entry (the pivot) in each row comes after the pivot in the previous row. All zero rows come last.

Eigenvalue λ and eigenvector $\boldsymbol{x}$. $A\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\boldsymbol{x} \neq \boldsymbol{0}$ so $\det(A - \lambda I) = 0$.

Eigshow. Graphical 2 by 2 eigenvalues and singular values (MATLAB or Java).

Elimination. A sequence of row operations that reduces A to an upper triangular U or to the reduced form $R = \mathrm{rref}(A)$. Then $A = LU$ with multipliers ℓ_{ij} in L, or $PA = LU$ with row exchanges in P, or $EA = R$ with an invertible E.

Elimination matrix = Elementary matrix E_{ij}. The identity matrix with an extra $-\ell_{ij}$ in the i, j entry ($i \neq j$). Then $E_{ij}A$ subtracts ℓ_{ij} times row j of A from row i.

Ellipse (or ellipsoid) $x^{\mathrm{T}}Ax = 1$. A must be positive definite; the axes of the ellipse are eigenvectors of A, with lengths $1/\sqrt{\lambda}$. (For $\|x\| = 1$ the vectors $y = Ax$ lie on the ellipse $\|A^{-1}y\|^2 = y^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}y = 1$ displayed by eigshow; axis lengths σ_i.)

Exponential $e^{At} = I + At + (At)^2/2! + \cdots$ has derivative Ae^{At}; $e^{At}u(0)$ solves $u' = Au$.

Factorization $A = LU$. If elimination takes A to U *without row exchanges*, then the lower triangular L with multipliers ℓ_{ij} (and $\ell_{ii} = 1$) brings U back to A.

Fast Fourier Transform (FFT). A factorization of the Fourier matrix F_n into $\ell = \log_2 n$ matrices S_i times a permutation. Each S_i needs only $n/2$ multiplications, so $F_n x$ and $F_n^{-1}c$ can be computed with $n\ell/2$ multiplications. Revolutionary.

Fibonacci numbers 0, 1, 1, 2, 3, 5, ... satisfy $F_n = F_{n-1} + F_{n-2} = (\lambda_1^n - \lambda_2^n)/(\lambda_1 - \lambda_2)$. Growth rate $\lambda_1 = (1+\sqrt{5})/2$ is the largest eigenvalue of the Fibonacci matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Four fundamental subspaces of $A = C(A), N(A), C(A^{\mathrm{T}}), N(A^{\mathrm{T}})$.

Fourier matrix F. Entries $F_{jk} = e^{2\pi ijk/n}$ give orthogonal columns $\overline{F}^{\mathrm{T}}F = nI$. Then $y = Fc$ is the (inverse) Discrete Fourier Transform $y_j = \sum c_k e^{2\pi ijk/n}$.

Free columns of A. Columns without pivots; combinations of earlier columns.

Free variable x_i. Column i has no pivot in elimination. We can give the $n-r$ free variables any values, then $Ax = b$ determines the r pivot variables (if solvable!).

Full column rank $r = n$. Independent columns, $N(A) = \{0\}$, no free variables.

Full row rank $r = m$. Independent rows, at least one solution to $Ax = b$, column space is all of $\mathbf{R}^m$. *Full rank* means full column rank or full row rank.

Fundamental Theorem. The nullspace $N(A)$ and row space $C(A^{\mathrm{T}})$ are orthogonal complements (perpendicular subspaces of $\mathbf{R}^n$ with dimensions r and $n-r$) from $Ax = 0$. Applied to A^{T}, the column space $C(A)$ is the orthogonal complement of $N(A^{\mathrm{T}})$.

Gauss-Jordan method. Invert A by row operations on $[\,A \quad I\,]$ to reach $[\,I \quad A^{-1}\,]$.

Gram-Schmidt orthogonalization $A = QR$. Independent columns in A, orthonormal columns in Q. Each column q_j of Q is a combination of the first j columns of A (and conversely, so R is upper triangular). Convention: $\mathrm{diag}(R) > 0$.

Graph G. Set of n nodes connected pairwise by m edges. A **complete graph** has all $n(n-1)/2$ edges between nodes. A **tree** has only $n-1$ edges and no closed loops. A **directed graph** has a direction arrow specified on each edge.

Hankel matrix H. Constant along each antidiagonal; h_{ij} depends on $i+j$.

Hermitian matrix $A^{\mathrm{H}} = \overline{A}^{\mathrm{T}} = A$. Complex analog of a symmetric matrix: $\overline{a_{ji}} = a_{ij}$.

Hessenberg matrix H. Triangular matrix with one extra nonzero adjacent diagonal.

Hilbert matrix hilb(n). Entries $H_{ij} = 1/(i+j-1) = \int_0^1 x^{i-1}x^{j-1}dx$. Positive definite but extremely small $\lambda_{\min}$ and large condition number.

Hypercube matrix P_L^2. Row $n+1$ counts corners, edges, faces, ... of a cube in $\mathbf{R}^n$.

Identity matrix I **(or** I_n**).** Diagonal entries $= 1$, off-diagonal entries $= 0$.

Incidence matrix of a directed graph. The m by n edge-node incidence matrix has a row for each edge (node i to node j), with entries -1 and 1 in columns i and j.

Indefinite matrix. A symmetric matrix with eigenvalues of both signs ($+$ and $-$).

Independent vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k$. No combination $c_1\boldsymbol{v}_1 + \cdots + c_k\boldsymbol{v}_k =$ zero vector unless all $c_i = 0$. If the $\boldsymbol{v}$'s are the columns of A, the only solution to $A\boldsymbol{x} = \boldsymbol{0}$ is $\boldsymbol{x} = \boldsymbol{0}$.

Inverse matrix A^{-1}. Square matrix with $A^{-1}A = I$ and $AA^{-1} = I$. No inverse if $\det A = 0$ and $\operatorname{rank}(A) < n$ and $A\boldsymbol{x} = \boldsymbol{0}$ for a nonzero vector $\boldsymbol{x}$. The inverses of AB and A^{T} are $B^{-1}A^{-1}$ and $(A^{-1})^{\mathrm{T}}$. Cofactor formula $(A^{-1})_{ij} = C_{ji}/\det A$.

Iterative method. A sequence of steps intended to approach the desired solution.

Jordan form $J = M^{-1}AM$. If A has s independent eigenvectors, its "generalized" eigenvector matrix M gives $J = \operatorname{diag}(J_1, \ldots, J_s)$. The block J_k is $\lambda_k I_k + N_k$ where N_k has 1's on diagonal 1. Each block has one eigenvalue λ_k and one eigenvector $(1, 0, \ldots, 0)$.

Kirchhoff's Laws. *Current law*: net current (in minus out) is zero at each node. *Voltage law*: Potential differences (voltage drops) add to zero around any closed loop.

Kronecker product (tensor product) $A \otimes B$. Blocks $a_{ij}B$, eigenvalues $\lambda_p(A)\lambda_q(B)$.

Krylov subspace $K_j(A, \boldsymbol{b})$. The subspace spanned by $\boldsymbol{b}, A\boldsymbol{b}, \ldots, A^{j-1}\boldsymbol{b}$. Numerical methods approximate $A^{-1}\boldsymbol{b}$ by $\boldsymbol{x}_j$ with residual $\boldsymbol{b} - A\boldsymbol{x}_j$ in this subspace. A good basis for K_j requires only multiplication by A at each step.

Least squares solution $\widehat{\boldsymbol{x}}$. The vector $\widehat{\boldsymbol{x}}$ that minimizes the error $\|\boldsymbol{e}\|^2$ solves $A^{\mathrm{T}}A\widehat{\boldsymbol{x}} = A^{\mathrm{T}}\boldsymbol{b}$. Then $\boldsymbol{e} = \boldsymbol{b} - A\widehat{\boldsymbol{x}}$ is orthogonal to all columns of A.

Left inverse A^+. If A has full column rank n, then $A^+ = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$ has $A^+A = I_n$.

Left nullspace $N(A^{\mathrm{T}})$. Nullspace of $A^{\mathrm{T}} =$ "left nullspace" of A because $\boldsymbol{y}^{\mathrm{T}}A = \boldsymbol{0}^{\mathrm{T}}$.

Length $\|\boldsymbol{x}\|$. Square root of $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ (Pythagoras in n dimensions).

Linear combination $c\boldsymbol{v} + d\boldsymbol{w}$ or $\sum c_j\boldsymbol{v}_j$. Vector addition and scalar multiplication.

Linear transformation T. Each vector $\boldsymbol{v}$ in the input space transforms to $T(\boldsymbol{v})$ in the output space, and linearity requires $T(c\boldsymbol{v} + d\boldsymbol{w}) = c\,T(\boldsymbol{v}) + d\,T(\boldsymbol{w})$. Examples: Matrix multiplication $A\boldsymbol{v}$, differentiation in function space.

Linearly dependent $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. A combination other than all $c_i = 0$ gives $\sum c_i\boldsymbol{v}_i = \boldsymbol{0}$.

Lucas numbers $L_n = 2, 1, 3, 4, \ldots$ satisfy $L_n = L_{n-1} + L_{n-2} = \lambda_1^n + \lambda_2^n$, with eigenvalues $\lambda_1, \lambda_2 = (1 \pm \sqrt{5})/2$ of the Fibonacci matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Compare $L_0 = 2$ with Fibonacci.

Markov matrix M. All $m_{ij} \geq 0$ and each column sum is 1. Largest eigenvalue $\lambda = 1$. If $m_{ij} > 0$, the columns of M^k approach the steady state eigenvector $M\boldsymbol{s} = \boldsymbol{s} > \boldsymbol{0}$.

Permutation matrix P. There are $n!$ orders of $1, \ldots, n$; the $n!$ P's have the rows of I in those orders. PA puts the rows of A in the same order. P is a product of row exchanges P_{ij}; P is *even* or *odd* ($\det P = 1$ or -1) based on the number of exchanges.

Pivot columns of A. Columns that contain pivots after row reduction; not combinations of earlier columns. The pivot columns are a basis for the column space.

Pivot d. The diagonal entry (*first nonzero*) when a row is used in elimination.

Plane (or hyperplane) in $\mathbf{R}^n$. Solutions to $\boldsymbol{a}^{\mathrm{T}}\boldsymbol{x} = 0$ give the plane (dimension $n-1$) perpendicular to $\boldsymbol{a} \neq \mathbf{0}$.

Polar decomposition $A = QH$. Orthogonal Q, positive (semi)definite H.

Positive definite matrix A. Symmetric matrix with positive eigenvalues and positive pivots. Definition: $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x} > 0$ unless $\boldsymbol{x} = \mathbf{0}$.

Projection $\boldsymbol{p} = \boldsymbol{a}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a})$ onto the line through $\boldsymbol{a}$. $P = \boldsymbol{a}\boldsymbol{a}^{\mathrm{T}}/\boldsymbol{a}^{\mathrm{T}}\boldsymbol{a}$ has rank 1.

Projection matrix P onto subspace S. Projection $\boldsymbol{p} = P\boldsymbol{b}$ is the closest point to $\boldsymbol{b}$ in $\boldsymbol{S}$, error $\boldsymbol{e} = \boldsymbol{b} - P\boldsymbol{b}$ is perpendicular to $\boldsymbol{S}$. $P^2 = P = P^{\mathrm{T}}$, eigenvalues are 1 or 0, eigenvectors are in $\boldsymbol{S}$ or $\boldsymbol{S}^{\perp}$. If columns of A = basis for $\boldsymbol{S}$ then $P = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}$.

Pseudoinverse A^+ (Moore-Penrose inverse). The n by m matrix that "inverts" A from column space back to row space, with $\boldsymbol{N}(A^+) = \boldsymbol{N}(A^{\mathrm{T}})$. A^+A and AA^+ are the projection matrices onto the row space and column space. $\mathrm{Rank}(A^+) = \mathrm{rank}(A)$.

Random matrix rand(n) or randn(n). MATLAB creates a matrix with random entries, uniformly distributed on $[\,0 \quad 1\,]$ for rand and standard normal distribution for randn.

Rank one matrix $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} \neq 0$. Column and row spaces = lines $c\boldsymbol{u}$ and $c\boldsymbol{v}$.

Rank $r(A)$ = number of pivots = dimension of column space = dimension of row space.

Rayleigh quotient $q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}/\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$ for symmetric A: $\lambda_{\min} \leq q(\boldsymbol{x}) \leq \lambda_{\max}$. Those extremes are reached at the eigenvectors $\boldsymbol{x}$ for $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$.

Reduced row echelon form $R = \mathrm{rref}(A)$. Pivots = 1; zeros above and below pivots; r nonzero rows of R give a basis for the row space of A.

Reflection matrix $Q = I - 2\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$. The unit vector $\boldsymbol{u}$ is reflected to $Q\boldsymbol{u} = -\boldsymbol{u}$. All vectors $\boldsymbol{x}$ in the plane mirror $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{x} = 0$ are unchanged because $Q\boldsymbol{x} = \boldsymbol{x}$. The "Householder matrix" has $Q^{\mathrm{T}} = Q^{-1} = Q$.

Right inverse A^+. If A has full row rank m, then $A^+ = A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}$ has $AA^+ = I_m$.

Rotation matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates the plane by θ and $R^{-1} = R^{\mathrm{T}}$ rotates back by $-\theta$. Orthogonal matrix, eigenvalues $e^{i\theta}$ and $e^{-i\theta}$, eigenvectors $(1, \pm i)$.

Row picture of $A\boldsymbol{x} = \boldsymbol{b}$. Each equation gives a plane in $\boldsymbol{R}^n$; planes intersect at $\boldsymbol{x}$.

Row space $\boldsymbol{C}(A^{\mathrm{T}})$ = all combinations of rows of A. Column vectors by convention.

Saddle point of $f(x_1, \ldots, x_n)$. A point where the first derivatives of f are zero and the second derivative matrix ($\partial^2 f/\partial x_i \partial x_j$ = **Hessian matrix**) is indefinite.

Schur complement $S = D - CA^{-1}B$. Appears in block elimination on $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{smallmatrix}\right]$.