# AVERAGE STRESS IN MATRIX AND AVERAGE ELASTIC ENERGY OF MATERIALS WITH MISFITTING INCLUSIONS*

## T. MORI† and K. TANAKA††

Having noted an important role of image stress in work hardening of dispersion hardened materials,[1,3] the present paper discusses a method of calculating the average internal stress in the matrix of a material containing inclusions with transformation strain. It is shown that the average stress in the matrix is uniform throughout the material and independent of the position of the domain where the average treatment is carried out. It is also shown that the actual stress in the matrix is the average stress plus the locally fluctuating stress, the average of which vanishes in the matrix. Average elastic energy is also considered by taking into account the effects of the interaction among the inclusions and of the presence of the free boundary.

## CONTRAINTE MOYENNE DANS LA MATRICE ET ENERGIE ELASTIQUE MOYENNE DES MATERIAUX CONTENANT DES INCLUSIONS IMPARFAITES

Ayant remarqué que la force image joue un rôle important dans la consolidation des matériaux durcis par dispersion, les auteurs proposent ici une méthode de calcul de la contrainte interne moyenne dans la matrice d'un matériau contenant des inclusions présentant des déformations dues à une transformation, et montrent que la contrainte moyenne dans la matrice est uniforme à travers le matériau et indépendante de la position de la zone dans laquelle le traitement moyen est effectué. Ils montrent aussi que la contrainte réelle dans la matrice est égale à la somme de la contrainte moyenne et de la contrainte locale variable dont la moyenne pour toute la matrice tend vers zéro. L'énergie élastique moyenne est également calculée en tenant compte des effets d'interaction entre les inclusions et de la présence du joint libre.

## DIE MITTLERE SPANNUNG IN DER MATRIX UND DIE MITTLERE ELASTISCHE ENERGIE VON MATERIALIEN MIT EINSCHLÜSSEN

Nachdem die große Bedeutung der Bildkraft für die Verfestigung von dispersions gehärteten Materialien[1,3] betont wurde, diskutiert die vorliegende Arbeit eine Methode zur Beschreibung der mittleren inneren Spannung in der Matrix eines Materials, das Einschlüsse mit Umwandlungsverspannungen enthält. Es wird gezeigt, daß die mittlere Spannung in der Matrix im ganzen Material gleichförmig und unabhängig von der Lage des Bereichs ist, für den die Behandlung durchgeführt wurde. Außerdem wird gezeigt, daß die aktuelle Spannung in der Matrix gleich der mittleren Spannung plus einer lokal fluktuierenden Spannung ist, deren Mittelwert über die gesamte Matrix verschwindet. Die mittlere elastische Energie wird ebenfalls diskutiert unter Berücksichtigung der Wechselwirkungseffekte zwischen den Einschlüssen und der Gegenwart der freien Oberfläche.

## 1. INTRODUCTION

As Brown pointed out in describing the present situation,[1] some papers have recently appeared which discuss work-hardening of dispersion hardened materials.[2-5] All these papers deal with internal stress which developed as a result of plastic deformation occurring only in the matrix. References (2) and (4) pursue energy consideration and agree with each other with respect to the hardening rate, while references (3) and (5) discuss the role of internal stress in the matrix in work hardening. Although Brown presented a comprehensive view of what was discussed in these papers,[1] we would like nonetheless to report our own understanding, for Brown's explanation and derivation of some stresses and strains were, in details, not completely acceptable from our point of view and we feel that an alternative treatment is possible. Therefore, the results of some of our calculations of the internal stress and of elastic energy will be presented here. Since the internal stress developed

by uniform plastic deformation occurring only in the matrix can be duplicated by giving uniform transformation strain to inclusions alone, internal stress will be characterized in the following sections by inclusions with uniform transformation strain. Thus, some of the results will be directly applicable to a material with misfitting precipitates.

## 2. AVERAGE INTERNAL STRESS IN MATRIX

For simplicity, the case where elastic constants $C_{ijkl}$ are uniform throughout a specimen $V_0$ will be considered. Suppose the specimen has $N$ inclusions which are, on the macroscopic scale, uniformly distributed in the matrix. When an inclusion $V$ acquires uniform transformation strain $\varepsilon_{ij}{}^T$, total strain $\varepsilon_{ij}{}'$ is introduced into the specimen. $\varepsilon_{ij}{}'$ is divided into two parts; $\varepsilon_{ij}{}^\infty$ and $\varepsilon_{ij}{}^{im}$. $\varepsilon_{ij}{}^\infty$ represents constrained strain* (total strain) when the inclusion is in an infinite body, and $\varepsilon_{ij}{}^{im}$ represents image strain introduced into the actual specimen by the presence of the free boundary $|V_0|$ of the specimen. $\varepsilon_{ij}{}^{im}$ is elastic in nature throughout the specimen. $\varepsilon_{ij}{}^\infty$ is elastic outside the inclusion, but $(\varepsilon_{ij}{}^\infty - \varepsilon_{ij}{}^T)$ is elastic

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* In Eshelby's notation $\varepsilon_{ij}{}^\infty$ is expressed as $\varepsilon_{ij}{}^c$.

inside the inclusion. $\varepsilon_{ij}^{\infty}$ is given by

$$\varepsilon_{ij}^{\infty} = \varepsilon_{mn}^{T}\int_{V}\{-C_{mnkl}[G_{ki,lj}(x, x')$$
$$+ G_{kj,li}(x, x')]/2\}\, dD(x'), \qquad (1)$$

where $G_{ki}$ is Green's function for an infinite homogeneous body with elastic constants $C_{ijkl}$.[6] It can be shown that when all inclusions acquire identical transformation strain $\varepsilon_{ij}^{T}$, the specimen as a whole undergoes a shape change and the average strain, $\langle\varepsilon_{ij}^{F}\rangle_{V_0}$, that descibes the specimen shape change is given by

$$\langle\varepsilon_{ij}^{F}\rangle_{V_0} = f\varepsilon_{ij}^{T}, \qquad (2)$$

where $f$ is a volume fraction of the inclusions. From this, let us assume the following: (1) the average of total strains, in domain $V_R$, from all the inclusions, is also equal to $f\varepsilon_{ij}^{T}$ if $V_R$ is a representative domain for the specimen. This implies that $V_R$ has a sufficient number of inclusions, say, $M$ inclusions, $MV/V_R = f$, and $V_R$ is not in a neighborhood close to the boundary $|V_0|$. Assumption (1) is realistic. As an example, let us consider the lattice constants of martensite. The macroscopically determined average lattice constants of a martensite single crystal (if it were present) are believed to be equal to those measured by a narrow beam X-ray technique if interstitial atoms are uniformly distributed. Assumption (1) was justified by Eshelby when $\varepsilon_{ij}^{T} = \varepsilon\delta_{ij}$ ($\delta_{ij}$ is the Kronecker delta).[7]

Let us consider the case where $V_R$ has a shape similar to that of the inclusions. Then total strain $\varepsilon_{ij}^{F}$ is expressed as

$$\varepsilon_{ij}^{F} = \sum_{P=1}^{M}\varepsilon_{ij}^{\infty}(x, x^P) + \sum_{P>M}^{N}\varepsilon_{ij}^{\infty}(x, x^P)$$
$$+ \sum_{P=1}^{N}\varepsilon_{ij}^{im}(x, x^P), \qquad (3)$$

where $\varepsilon_{ij}^{\infty}(x, x^P)$ is the quantity defined by equation (1) when the $P$-th inclusion is at $x^P$, $\varepsilon_{ij}^{im}(x, x^P)$ is the image strain due to the $P$-th inclusion, and $V_R$ is assumed to contain inclusions 1 to $M$. Apparently, the second and the third sums are elastic and do not fluctuate much in $V_R$. Therefore, it is meaningful to define the average of the second and the third sums in equation (3) in $V_R$ as the average elastic strain $\langle\varepsilon_{ij}\rangle_{V_R}$,

$$\langle\varepsilon_{ij}\rangle_{V_R} = (1/V_R)\int_{V_R}\left[\sum_{P>M}^{N}\varepsilon_{ij}^{\infty}(x, x^P)\right.$$
$$\left. + \sum_{P=1}^{N}\varepsilon_{ij}^{im}(x, x^P)\right]dD(x). \qquad (4)$$

As shown in a previous paper,[8] when a single inclusion $V$ with uniform transformation strain is within an infinite homogeneous body, volume integrals of total strain and stress vanish if the integration is carried out in the region $V' - V$, where $V'$ is a domain surrounding $V$ and is of shape similar to that of $V$. Applying this conclusion, the average of the first sum of equation (3) in $V_R$ is calculated as

$$(1/V_R)\int_{V_R}\sum_{P=1}^{M}\varepsilon_{ij}^{\infty}(x, x^P)\, dD(x) = fS_{ijmn}e_{mn}^{T}, \qquad (5)$$

where $S_{ijkl}$ are Eshelby's tensors and are equal to the integrals in equation (1) when $x$ is within $V$ in equation (1).[6] Assumption (1) and equations (3) through (5) give

$$\langle\varepsilon_{ij}\rangle_{V_R} = -f(S_{ijmn}e_{mn}^{T} - e_{ij}^{T}). \qquad (6)$$

Thus, the average elastic strain defined above is independent of the position and the size of $V_R$.

The average internal stress, $\langle\sigma_{ij}^{0}\rangle_{V_R}$, defined by

$$\langle\sigma_{ij}^{0}\rangle_{V_R} = C_{ijkl}\langle\varepsilon_{kl}\rangle_{V_R}, \qquad (7)$$

is also independent of the position and the size of $V_R$, insofar as $V_R$ can be regarded as a representative volume. $\langle\sigma_{ij}^{0}\rangle_{V_R}$ is the average in $V_R$ of the sum of the image stresses of all the inclusions and the stresses of the inclusions outside $V_R$ when they are in an infinite body. However, it should be noted that this sum itself is nearly constant in $V_R$.

Next, let us consider the following sum,

$$\sigma_{ij}^{\infty}(1, M) = \sum_{P=1}^{M}\sigma_{ij}^{\infty}(x, x^P), \qquad (8)$$

where $\sigma_{ij}^{\infty}(x, x^P)$ is internal stress due to the $P$-th inclusion at $x^P$ within $V_R$ when it is in an infinite body. This sum, $\sigma_{ij}^{\infty}(1, M)$ cannot be assumed nearly constant in $V_R$; instead, it fluctuates and the wave length of the fluctuation is apparently of the order of the inter-inclusion spacing. $\sigma_{ij}^{\infty}(1, M)$ is called locally fluctuating stress, to which nearby inclusions obviously contribute predominantly. However, the average of this locally fluctuating stress in the matrix of $V_R$ can be shown to be zero. Instead of averaging equation (8) directly, let us take the following approach. Consider a fixed point in $V_R$. First imagine $V_R$ without inclusions. Next introduce an inclusion into $V_R$ such that the fixed point is always outside the inclusions and record $\sigma_{ij}^{\infty}\delta V$. Here, $\sigma_{ij}^{\infty}$ is the stress due to the inclusion when it is in an infinite body and $\delta V$ is a small volume element around the fixed point. If the introduction of the inclusion is repeated many times in a random manner, $\Sigma\sigma_{ij}^{\infty}\delta V$ becomes proportional to the integral, $\int_{V_R-V}\sigma_{ij}^{\infty}\delta V$, where the center of the inclusion is conversely fixed at the opposite

position of the above fixed point from the center of $V_R$. Because of the statement following equation (4), this integral vanishes. Since this is true for every inclusion, the average of $\sigma_{ij}{}^\infty$ from all the actually present inclusions in $V_R$ at a point within the matrix in $V_R$ becomes zero. That is,

$$\langle \sigma_{ij}{}^\infty (1, M) \rangle_M = 0. \qquad (9)$$

The total average stress, $\langle \sigma_{ij} \rangle_M$, in the matrix within $V_R$ is the sum of $\langle \sigma_{ij}{}^0 \rangle_{V_R}$ and $\langle \sigma_{ij}{}^\infty(1, M) \rangle_M$. From equations (6), (7) and (9),

$$\langle \sigma_{ij} \rangle_M = C_{ijkl} \langle \varepsilon_{kl} \rangle_{V_R}$$
$$= -f\sigma_{ij}{}^{I\infty}, \qquad (10)$$

with

$$\sigma_{ij}{}^{I\infty} = C_{ijkl}(S_{klmn}\varepsilon_{mn}{}^T - \varepsilon_{kl}{}^T), \qquad (11)$$

where $\sigma_{ij}{}^{I\infty}$ is stress within a single inclusion when it is in an infinitely extended body. $\langle \sigma_{ij} \rangle_M$ is independent of the size and position of $V_R$ and of the shape of the specimen.

In summary, the internal stress at any point in the matrix is the uniform average stress $\langle \sigma_{ij} \rangle_M$ plus the locally fluctuating stress from nearby inclusions. This locally fluctuating stress is averaged to be zero in the matrix. It is important to note that $\langle \sigma_{ij} \rangle_M$ is the average, within the matrix, of the sums of the stresses of the inclusions when they are present in an infinite body ($\Sigma\sigma_{ij}{}^\infty$) and of the image stresses of all the inclusions ($\Sigma\sigma_{ij}{}^{im}$). The relative contributions of $\Sigma\sigma_{ij}{}^\infty$ and of $\Sigma\sigma_{ij}{}^{im}$ are not generally determined. However, as shown in the appendix, the contribution from each term can be calculated, if the specimen is of ellipsoidal shape. Especially when the specimen is similar in shape to the inclusions, $\langle \sigma_{ij} \rangle_M$ is solely due to $\Sigma\sigma_{ij}{}^{im}$.

As Brown mentioned,[1] when it moves in the matrix as a straight line, a dislocation feels, as a whole, only the average stress $\langle \sigma_{ij} \rangle_M$, since the average of the locally fluctuating stress in the matrix is zero. In such a case, an extra applied stress equal to $-\langle \sigma_{ij} \rangle_M$ is needed to move a dislocation from the case where there is no internal stress. If $\varepsilon_{ij}{}^T$ is regarded as produced by plastic deformation, the above hardening is work hardening which Brown and Stobbs identified as due to the image stress[3] and to which energy balance consideration was applied.[2] The equivalency of both treatments has been discussed by Hart.[5]

From the present understanding of internal stress, Hart's treatment of work hardening of dispersion hardened materials[5] is identified as that which emphasizes and estimates the role of the locally fluctuating stress. As treated by Hart, a dislocation feels obstruction from the locally fluctuating stress

when it is flexible. Brown and Stobbs also considered the role of the locally fluctuating stress in work hardening in a somewhat different manner.[3] It is important to note that hardening due to this factor operates as an energy dissipation mechanism.[1,5] Because of its fluctuating and position-dependent character, estimation of hardening due to the locally fluctuating stress involves certain approximations and seems to depend on the choice of particular situations to be considered; thus, it would not seem to be as uniquely and simply performable as estimation of hardening due to the average stress in the matrix, $\langle \sigma_{ij} \rangle_M$.

## 3. INTERACTION AMONG INCLUSIONS AND ENERGY CONSIDERATION

That the average internal stress in the matrix is equal to $-f\sigma_{ij}{}^{I\infty}$ naturally indicates that the actual stress within an inclusion is, on the average, equal to $\sigma_{ij}{}^{I\infty} - f\sigma_{ij}{}^{I\infty}$. Thus, elastic energy per unit volume of the specimen, $E_{el}{}^0$, is given by

$$E_{el}{}^0 = -f(1-f)\sigma_{ij}{}^{I\infty}\varepsilon_{ij}{}^T/2. \qquad (12)$$

In Reference (2), the term $(1 - f)$ was omitted on the assumption that the interaction among inclusions can be ignored. However, equation (12) is a correct expression which takes into account interaction among inclusions together with the effect of the presence of the free boundary of the specimen. If $f$ is small, $(1 - f)$ in equation (12) can be, for all practical purposes, replaced by unity. However, when $f$ is large, ignoring the effects of the interaction among inclusions and the presence of the free boundary, as was done previously,[9] may lead to a significant error.

As already mentioned, Hart clarified,[5] in the case of work hardening of dispersion hardened materials, that a hardening rate derived from consideration of energy balance[2] is essentially equivalent to that derived from consideration of the average internal stress in the matrix.[1,3] If equation (12) is used for an elastic energy calculation, exact agreement of the hardening rates calculated by the two approaches is verified.

### REFERENCES

1. L. M. BROWN, Private communication (to be published in *Acta Met.*).
2. K. TANAKA and T. MORI, *Acta Met.* **18**, 931 (1970).
3. L. M. BROWN and W. M. STOBBS, *Phil. Mag.* **23**, 1185 (1971).

4. T. MURA, *Int. Conf. Mech. Properties of Materials at Kyoto*, Vol. 2. (1971).
5. E. W. HART, *Acta Met.* **20**, 275 (1972).
6. J. D. ESHELBY, *Proc. R. Soc.* **A241**, 376 (1957).
7. J. D. ESHELBY, *J. appl. Phys.* **25**, 255 (1954).
8. K. TANAKA and T. MORI, *J. Elasticity*. To be published.
9. K. TANAKA and T. MORI, *Phil. Mag.* **23**, 737 (1971).
10. N. KINOSHITA and T. MURA, *Phys. Status Solidi*. **5**, 759 (1971).

## APPENDIX

### AVERAGE STRESS AND IMAGE STRESS IN A UNIFORM ELLIPSOIDAL BODY CONTAINING ELLIPSOIDAL INCLUSIONS WITH TRANSFORMATION STRAIN

Suppose that an ellipsoidal body, $V_0$, contains an ellipsoidal inclusion $V$ with transformation strain $\varepsilon_{ij}{}^T$. Elastic constants $C_{ijkl}$ are uniform throughout the body. Let $\sigma_{ij}{}^\infty$ be stress, assuming the inclusion to be in an infinite body, and let $\sigma_{ij}{}^{im}$ be image stress to reflect the presence of the boundary $|V_0|$ in the actual body. The total internal stress in the actual body, $\sigma_{ij}$, is $\sigma_{ij}{}^\infty + \sigma_{ij}{}^{im}$. Since for static equilibrium,

$$\int_{V_0} \sigma_{ij} \, dD = \int_{V_0} (\sigma_{ij}{}^\infty + \sigma_{ij}{}^{im}) \, dD = 0,$$

$$\int_{V_0} \sigma_{ij}{}^{im} \, dD = -\int_V \sigma_{ij}{}^\infty \, dD - \int_{V_0-V} \sigma_{ij}{}^\infty dD. \tag{A1}$$

As $\sigma_{ij}{}^\infty$ in $V$ is uniform,[6] the first term in equation (A1) is expressed as

$$-\int_V \sigma_{ij}{}^\infty \, dD = -VC_{ijkl}[S_{klmn}(v)\varepsilon_{mn}{}^T - \varepsilon_{kl}], \tag{A2}$$

where $S_{klmn}(V)$ are Eshelby's tensors defined for the shape of the inclusion. Reference (8) show that the second term in equation (A1) is independent of the size of the body $V_0$ and of the position of the inclusion and is given by

$$-\int_{V_0-V} \sigma_{ij}{}^\infty \, dD = -VC_{ijkl}[S_{klmn}(V_0)$$
$$- S_{klmn}(V)]\varepsilon_{mn}{}^T, \tag{A3}$$

where $S_{klmn}(V_0)$ are Eshelby's tensors appropriate for the shape of the body $V_0$. Thus, the average, in the matrix, of $\sigma_{ij}{}^\infty$ of a single inclusion is written as

$$\langle \sigma_{ij}{}^\infty \rangle_M = \langle \sigma_{ij}{}^\infty \rangle_{V_0-V}$$
$$= \{V/(V_0 - V)\}C_{ijkl}$$
$$\times [S_{klmn}(V_0) - S_{klmn}(V)]\varepsilon_{mn}{}^T. \tag{A4}$$

Equations (A3) and (A4) *do not depend on the position of the inclusion.* From equations (A1) and (A3),

$$\int_{V_0} \sigma_{ij}{}^{im} dD = -VC_{ijkl}[S_{klmn}(V_0)\varepsilon_{mn}{}^T - \varepsilon_{kl}{}^T]. \tag{A5}$$

Consequently, the average of the image stress in the body, $V_0$, is calculated as

$$\langle \sigma_{ij}{}^{im} \rangle_{V_0} = -(V/V_0)C_{ijkl}[S_{klmn}(V_0)\varepsilon_{mn}{}^T - \varepsilon_{kl}{}^T]. \tag{A6}$$

Equations (A5) and (A6) are independent of the shape and the position of the inclusion. What is used in the above derivation is uniformity of stress and strain within an ellipsoidal inclusion with uniform transformation strain when the inclusion is in an infinite body. This uniformity is assured in an anisotropic case.[10] Thus, the above conclusions are valid when the body is anisotropic.

Finally, the average stress in the matrix will be considered when the body has many inclusions, the number of which is $N$, all of which are identical in shape and volume. When the body is similar in shape to the inclusions, the result of equation (A4) becomes zero and the average stress in the matrix becomes identical to the sum of the average image stress given by equation (A5). However, when the body has a different shape, the average stress in the matrix is, from equations (A4) and (A6),

$$\langle \sigma_{ij} \rangle_M = -(NV/V_0)C_{ijkl}[S_{klmn}(V_0)\varepsilon_{mn}{}^T - \varepsilon_{kl}{}^T]$$
$$+ (NV/V_0)(1 - V/V_0)^{-1}C_{ijkl}$$
$$\times [S_{klmn}(V_0) - S_{klmn}(V)]\varepsilon_{mn}{}^T.$$

Here, $NV/V_0$ is equal to $f$, the volume fraction of the inclusion, and is finite. Since $N$ is a large number, $V/V_0$ must be negligibly small compared to unity. Consequently, the above value becomes

$$\langle \sigma_{ij} \rangle_M = -fC_{ijkl}[S_{klmn}(V)\varepsilon_{mn}{}^T - \varepsilon_{kl}{}^T].$$

In terms of the stress, defined in equation (11), which is the internal stress within a single inclusion in an infinitely extended body, $\langle \sigma_{ij} \rangle$ is written as

$$\langle \sigma_{ij} \rangle_M = -f\sigma_{ij}{}^{I\infty},$$

which is, of course, identical to equation (10).