I said that we often define an ensemble in terms of a collection of conditional probabilities. The following example illustrates this idea.

Example 2.3. Jo has a test for a nasty disease. We denote Jo's state of health by the variable a and the test result by b.

$$\begin{array}{ll} a=1 & \text{Jo has the disease} \\ a=0 & \text{Jo does not have the disease.} \end{array} \tag{2.12}$$

The result of the test is either 'positive' ($b = 1$) or 'negative' ($b = 0$); the test is 95% reliable: in 95% of cases of people who really have the disease, a positive result is returned, and in 95% of cases of people who do not have the disease, a negative result is obtained. The final piece of background information is that 1% of people of Jo's age and background have the disease.

OK – Jo has the test, and the result is positive. What is the probability that Jo has the disease?

Solution. We write down all the provided probabilities. The test reliability specifies the conditional probability of b given a:

$$\begin{array}{ll} P(b=1 \mid a=1) = 0.95 & P(b=1 \mid a=0) = 0.05 \\ P(b=0 \mid a=1) = 0.05 & P(b=0 \mid a=0) = 0.95; \end{array} \tag{2.13}$$

and the disease prevalence tells us about the marginal probability of a:

$$P(a=1) = 0.01 \qquad P(a=0) = 0.99. \tag{2.14}$$

From the marginal $P(a)$ and the conditional probability $P(b \mid a)$ we can deduce the joint probability $P(a, b) = P(a)P(b \mid a)$ and any other probabilities we are interested in. For example, by the sum rule, the marginal probability of $b=1$ – the probability of getting a positive result – is

$$P(b=1) = P(b=1 \mid a=1)P(a=1) + P(b=1 \mid a=0)P(a=0). \tag{2.15}$$

Jo has received a positive result $b=1$ and is interested in how plausible it is that she has the disease (i.e., that $a=1$). The man in the street might be duped by the statement 'the test is 95% reliable, so Jo's positive result implies that there is a 95% chance that Jo has the disease', but this is incorrect. The correct solution to an inference problem is found using Bayes' theorem.

$$\begin{aligned} P(a=1 \mid b=1) &= \frac{P(b=1 \mid a=1)P(a=1)}{P(b=1 \mid a=1)P(a=1) + P(b=1 \mid a=0)P(a=0)} && (2.16) \\ &= \frac{0.95 \times 0.01}{0.95 \times 0.01 + 0.05 \times 0.99} && (2.17) \\ &= 0.16. && (2.18) \end{aligned}$$

So in spite of the positive result, the probability that Jo has the disease is only 16%. □

▶ 2.2 The meaning of probability

Probabilities can be used in two ways.

Probabilities can describe *frequencies of outcomes in random experiments*, but giving noncircular definitions of the terms 'frequency' and 'random' is a challenge – what does it mean to say that the frequency of a tossed coin's