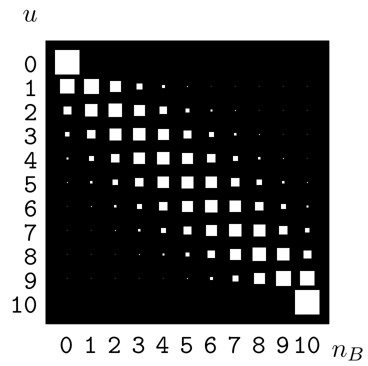


Figure 2.5. Joint probability of u and n_B for Bill and Fred's urn problem, after $N = 10$ draws.

The marginal probability of u is $P(u) = \frac{1}{11}$ for all u. You wrote down the probability of n_B given u and N, $P(n_B \,|\, u, N)$, when you solved exercise 2.4 (p.27). [You *are* doing the highly recommended exercises, aren't you?] If we define $f_u \equiv u/10$ then

$$P(n_B \,|\, u, N) = \binom{N}{n_B} f_u^{n_B}(1-f_u)^{N-n_B}. \tag{2.23}$$

What about the denominator, $P(n_B \,|\, N)$? This is the marginal probability of n_B, which we can obtain using the sum rule:

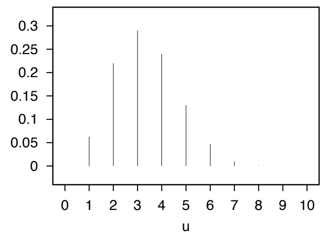
$$P(n_B \,|\, N) = \sum_u P(u, n_B \,|\, N) = \sum_u P(u)P(n_B \,|\, u, N). \tag{2.24}$$

So the conditional probability of u given n_B is

$$P(u \,|\, n_B, N) = \frac{P(u)P(n_B \,|\, u, N)}{P(n_B \,|\, N)} \tag{2.25}$$

$$= \frac{1}{P(n_B \,|\, N)} \frac{1}{11} \binom{N}{n_B} f_u^{n_B}(1-f_u)^{N-n_B}. \tag{2.26}$$

This conditional distribution can be found by normalizing column 3 of figure 2.5 and is shown in figure 2.6. The normalizing constant, the marginal probability of n_B, is $P(n_B = 3 \,|\, N = 10) = 0.083$. The posterior probability (2.26) is correct for all u, including the end-points $u = 0$ and $u = 10$, where $f_u = 0$ and $f_u = 1$ respectively. The posterior probability that $u = 0$ given $n_B = 3$ is equal to zero, because if Fred were drawing from urn 0 it would be impossible for any black balls to be drawn. The posterior probability that $u = 10$ is also zero, because there are no white balls in that urn. The other hypotheses $u = 1$, $u = 2$, ... $u = 9$ all have non-zero posterior probability. □



u	$P(u \,\|\, n_B = 3, N)$
0	0
1	0.063
2	0.22
3	0.29
4	0.24
5	0.13
6	0.047
7	0.0099
8	0.00086
9	0.0000096
10	0

Figure 2.6. Conditional probability of u given $n_B = 3$ and $N = 10$.

Terminology of inverse probability

In inverse probability problems it is convenient to give names to the probabilities appearing in Bayes' theorem. In equation (2.25), we call the marginal probability $P(u)$ the *prior* probability of u, and $P(n_B \,|\, u, N)$ is called the *likelihood* of u. It is important to note that the terms likelihood and probability are not synonyms. The quantity $P(n_B \,|\, u, N)$ is a function of both n_B and u. For fixed u, $P(n_B \,|\, u, N)$ defines a *probability* over n_B. For fixed n_B, $P(n_B \,|\, u, N)$ defines the *likelihood* of u.