Bayesians also use probabilities to describe *inferences.*

▶ 2.3 Forward probabilities and inverse probabilities

Probability calculations often fall into one of two categories: *forward probability* and *inverse probability*. Here is an example of a forward probability problem:

Exercise 2.4.[*2*, p.40] An urn contains K balls, of which B are black and $W = K - B$ are white. Fred draws a ball at random from the urn and replaces it, N times.

(a) What is the probability distribution of the number of times a black ball is drawn, n_B?

(b) What is the expectation of n_B? What is the variance of n_B? What is the standard deviation of n_B? Give numerical answers for the cases $N = 5$ and $N = 400$, when $B = 2$ and $K = 10$.

Forward probability problems involve a *generative model* that describes a process that is assumed to give rise to some data; the task is to compute the probability distribution or expectation of some quantity that depends on the data. Here is another example of a forward probability problem:

Exercise 2.5.[*2*, p.40] An urn contains K balls, of which B are black and $W = K - B$ are white. We define the fraction $f_B \equiv B/K$. Fred draws N times from the urn, exactly as in exercise 2.4, obtaining n_B blacks, and computes the quantity

$$z = \frac{(n_B - f_B N)^2}{N f_B(1 - f_B)}. \tag{2.19}$$

What is the expectation of z? In the case $N = 5$ and $f_B = 1/5$, what is the probability distribution of z? What is the probability that $z < 1$? [Hint: compare z with the quantities computed in the previous exercise.]

Like forward probability problems, *inverse probability problems* involve a generative model of a process, but instead of computing the probability distribution of some quantity *produced* by the process, we compute the conditional probability of one or more of the *unobserved variables* in the process, *given* the observed variables. This invariably requires the use of Bayes' theorem.

Example 2.6. There are eleven urns labelled by $u \in \{0, 1, 2, \ldots, 10\}$, each containing ten balls. Urn u contains u black balls and $10 - u$ white balls. Fred selects an urn u at random and draws N times with replacement from that urn, obtaining n_B blacks and $N - n_B$ whites. Fred's friend, Bill, looks on. If after $N = 10$ draws $n_B = 3$ blacks have been drawn, what is the probability that the urn Fred is using is urn u, from Bill's point of view? (Bill doesn't know the value of u.)

Solution. The joint probability distribution of the random variables u and n_B can be written

$$P(u, n_B \,|\, N) = P(n_B \,|\, u, N) P(u). \tag{2.20}$$

From the joint probability of u and n_B, we can obtain the conditional distribution of u given n_B:

$$\begin{aligned} P(u \,|\, n_B, N) &= \frac{P(u, n_B \,|\, N)}{P(n_B \,|\, N)} && (2.21) \\ &= \frac{P(n_B \,|\, u, N) P(u)}{P(n_B \,|\, N)}. && (2.22) \end{aligned}$$