Comprehensive Exam

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Van der Waerden's Theorem

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Van der Waerden's Theorem (Furstenberg, Weiss 1978)

Let T be a homeomorphism of a compact metric space, (X, ρ). For any natural number k and any positive ϵ, there is an x in X and a natural number n such that

$$\rho\left(x, T^{in}x\right) < \epsilon$$

for all i in $[1, k]$.

A generalization of Van der Waerden's Theorem

Polynomial Van der Waerden's Theorem (Bergelson, Leibman 1996)

Let T be a homeomorphism of a compact metric space, (X, ρ). For any natural number k, any positive ϵ and any polynomials $(p_i)_{i=1}^{k} \subset \mathbb{Q}[x]$ with $p_i(\mathbb{Z}) \subset \mathbb{Z}$ and zero constant term, there is an x in X and a natural number n such that

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$$\rho\left(x, T^{p_i(n)}x\right) < \epsilon$$

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Corollary

If the natural numbers are finitely colored then at least one color contains $\{a, a + p_1(n), a + p_2(n), \ldots, a + p_k(n)\}$ for some natural numbers a and n.

Definitions

Gamma polynomials

Let Γ be a finitely generated torsion free nilpotent group. Define $P\Gamma$ to be the minimal subgroup of $\Gamma^{\mathbb{Z}}$ which contains the constant mappings and is closed under the following operations: if $g, h \in P\Gamma$ and $p \in \mathbb{Q}[x]$ with $p(\mathbb{Z}) \subset \mathbb{Z}$ then $g(n)\,h(n)$ and $g(n)^{p(n)}$ are in $P\Gamma$. We call the elements of $P\Gamma$ gamma polynomials. Define $P\Gamma_0 = \{g \in P\Gamma : g(0) = 1_\Gamma\}$.

Definitions continued

Mal'cev Basis

Let Γ be a finitely generated nilpotent group without torsion. Then there exists a set of elements, $\{S_1, \ldots, S_n\}$, (a Mal'cev basis) of Γ so that

- for any $1 \leq i < j \leq n$, $[S_i, S_j]$ belongs to the subgroup of Γ generated by $S_1, \ldots S_{i-1}$;
- every element, T, of Γ can be uniquely represented in the form

$$T = \prod_{j=1}^{n} S_j^{r_j(T)}, \quad r_j(T) \in \mathbb{Z} \text{ for all } j.$$

The mapping $r : \Gamma \to \mathbb{Z}^n$ defined by $r(T) = (r_1(T), \ldots r_n(T))$ is such that there are polynomial mappings $R : \mathbb{Z}^{2n} \to \mathbb{Z}^n$, $R' : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ such that for any $T, T' \in \Gamma$ and any $n \in \mathbb{N}$,

$$r\left(TT'\right) = R\left(r(T), r\left(T'\right)\right), \; r\left(T^m\right) = R'\left(r(T), m\right).$$

Characterization of gamma polynomials

Lemma

For a Mal'cev basis $S_1, \ldots, S_n$. Every gamma polynomial, g, can be uniquely represented in the form

$$g\left(m\right) = \prod_{j=1}^{n} S_j^{p_j\left(m\right)}$$

where $p_j \in \mathbb{Q}\left[x\right]$ with $p_j\left(\mathbb{Z}\right) \subset \mathbb{Z}$.

A second generalization of Van der Waerden's Theorem

Leibman's Theorem (1994)

Let Γ be a finitely generated nilpotent group of homeomorphisms of a compact metric space, (X, ρ), and let $A \subset P\Gamma_0$ be finite. Then, for all positive ϵ, there is an x in X and a natural number, n, such that

$$\rho\left(x, g\left(n\right) x\right) < \epsilon$$

for all g in A.

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Corollary

Let Γ be a finitely generated nilpotent group and let $A \subset P\Gamma_0$ be finite. If Γ is finitely colored then, for some a in Γ and n in $\mathbb{N}$, $\{g\left(n\right) a : g \in A\}$ is monochromatic.

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IP-system

An IP-system is a sequence, $\left(n_\alpha\right)_{\alpha \in \mathcal{F}} \subset \mathbb{N}$, such that $n_{\alpha \cup \beta} = n_\alpha + n_\beta$ for disjoint α and β.

The Stone-Čech compactification

Ultrafilter

A filter, p, on a set, S, is a nonempty subset of the power set of S satisfying the following

- $\varnothing$ is not in p and S is in p;
- if A, B are in p then $A \cap B$ is in p;
- if A is in p and $B \supset A$ then B is in p;

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Notes: When S is a discrete topological space, βS is the Stone-Čech compactification as it is defined in point-set topology with the topology on βS being defined with closed base consisting of all sets of the form $\{p \in \beta S : A \in p\}$ where A is some subset of S.

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$$A \in p \cdot q \iff \{x : \{y : xy \in A\} \in q\} \in p.$$

A different configuration

Hindman's Theorem (1974)

If the natural numbers are finitely colored then at least one color contains, for some infinite $A \subset \mathbb{N}$,

$$\mathrm{FS}\,(A) := \{\sum_{x \in E} x : \; E \subset A \text{ and } E \text{ finite, non-empty}\}.$$

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$$\mathrm{FS}\,(A) := \{\sum_{x \in E} x : \; E \subset A \text{ and } E \text{ finite, non-empty}\}.$$

IP-set

An IP-set in $\mathbb{N}$ is a subset of the natural numbers which contains $\mathrm{FS}\,(A)$ for some infinite set $A \subset \mathbb{N}$. An IP-system is a sequence, $(n_\alpha)_{\alpha \in \mathcal{F}} \subset \mathbb{N}$, such that $n_{\alpha \cup \beta} = n_\alpha + n_\beta$ for disjoint α and β.

IP-ring

Define a well ordering on $\mathcal{F}$, the finite and nonempty subsets of the naturals, by $\alpha < \beta$ if $\max(\alpha) < \min(\beta)$. An *IP-ring* is defined as

$$\mathcal{F}^{(1)} = \mathrm{FU}\left(\{\alpha_i\}_{i=1}^{\infty}\right) = \{\alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_r} : r < \infty\}$$

where $\alpha_i \subset \mathbb{N}$ is finite, nonempty, and $\alpha_i < \alpha_{i+1}$ for all i.

A third generalization of Van der Waerden's Theorem

Central Sets Theorem (Furstenberg, 1981)

If $\mathbb{N} = \cup_{i=1}^{r} C_i$ is a partition of the natural numbers, k is an element of $\mathbb{N}$ and $(n_\alpha)_{\alpha \in \mathcal{F}}$ is an IP-system then there is an i_0 and an IP-ring $\mathcal{F}^{(1)} \subset \mathcal{F}$ and an IP-system $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ such that for all $\alpha \in \mathcal{F}^{(1)}$

$$\{a_\alpha, a_\alpha + n_\alpha, a + 2n_\alpha, \ldots, a_\alpha + (k-1)\, n_\alpha\} \subset C_{i_0}.$$

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Central Set

Let S be a semigroup. A set $A \subset S$ is *central* if and only if there is some idempotent ultrafilter p in the minimal ideal of βS such that $A \in p$.

A fourth generalization of Van der Waerden's Theorem

Infinitary Polynomial Van der Waerden (McCutcheon, 1999)

If $C \subset \mathbb{N}$ is central set, $\{p_i(x)\}_{i=1}^{k}$ are in $\mathbb{Z}[x]$ with $p_i(0) = 0$ and $(n_\alpha)_{\alpha \in \mathcal{F}}$ is an IP-system then there exists an IP-ring $\mathcal{F}^{(1)}$ and an IP-system $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ so that for all $\alpha \in \mathcal{F}^{(1)}$,

$$\{a_\alpha, a_\alpha + p_1(n_\alpha), \ldots, a_\alpha + p_k(n_\alpha)\} \subset C.$$

A new generalization of Van der Waerden's Theorem

Infinitary Polynomial Nilpotent Van der Waerden (Griffin, 2022)

If $r \in \mathbb{N}$, $\Gamma = \cup_{i=1}^{r} C_i$ is a partition of a nilpotent group Γ, $(n_\alpha)_{\alpha \subset \mathcal{F}}$ is an IP-system in $\mathbb{N}$ and $A \in \mathcal{P}_f(P\Gamma_0)$ then there is a j with $1 \leq j \leq r$ and an IP-ring $\mathcal{F}^{(1)}$ and an IP-system $(a_\alpha)_{\alpha \in \mathcal{F}^{(1)}}$ in Γ such that for all $\alpha \in \mathcal{F}^{(1)}$ and all $g \in A$ we have $g(n_\alpha) a_\alpha \in C_j$.

What's next

Polynomial Hales-Jewett (Bergelson, Leibman, 1999)

Let $d, k, r \in \mathbb{N}$ and let $V = \mathbb{N}^d \times \{1, \ldots, k\}$. For any r-coloring of $\mathcal{P}_f(V)$ there exists $b \in \mathcal{P}_f(V)$ and $\alpha \in \mathcal{P}_f(\mathbb{N})$ such that $b \cap \left(\alpha^d \times \{1, \ldots, k\}\right) = \varnothing$ and

$$\left\{b, b \cup \left(\alpha^d \times \{1\}\right), b \cup \left(\alpha^d \times \{2\}\right), \ldots, b \cup \left(\alpha^d \times \{k\}\right)\right\}$$

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Nilpotent Polynomial Hales-Jewett (Johnson, Richter, 2018)

Let $\mathcal{F}$ be an idempotent filter on a nilpotent group G and let $\mathbf{P}$ be a good collection of $\mathcal{F}$-measurable polynomial mappings. Then for all $\{P_1, \ldots, P_k\} \subset \mathbf{P}$ and any $\mathcal{F}$-syndetic set A there is an $\alpha \in \mathcal{P}_f(\mathbb{N})$ and an $a \in G$ such that $\{P_i(\alpha)a\}_{i=1}^k \subset A$.

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We are pursuing a common generalization of the result of Johnson and Richter and my result.