

Problem: Suppose the probability space (X, m) is nonatomic. Show that there are sets with arbitrarily small positive measure, that is, for every $\epsilon > 0$ there is a set $A \subset X$ with $0 < m(A) < \epsilon$.

Solution based on notes from the course on ergodic theory I took with Máté Wierdl: Suppose, for contradiction, that there is a set, A , so that $\alpha := \inf\{m(B) : B \subset A \text{ and } m(B) \neq 0\} > 0$. We will use Ramsey's theorem from graph theory to construct a $B \subset A$ with $m(B) = \alpha$. Let $(B_n)_n \subset A$ be such that $\lim_n m(B_n) = \alpha$. We may choose this sequence so that $m(B_n) < 2\alpha$ for all n . Consider the graph with vertices (B_n) with an edge connecting two vertices, B_m and B_n , if and only if $m(B_m \cap B_n) > 0$. By our assumption, the measure of their intersection must be at least α . By Ramsey's theorem there is an infinite independent set or an infinite clique. In a probability space, a collection of disjoint sets must be finite. So there is a subsequence, $(B_{n_k})_k$ where no two of the B_{n_k} 's are disjoint. That is $m(B_{n_k} \cap B_{n_j}) \geq \alpha$ for all k, j . In fact, any finite intersection of the B_{n_k} 's must have measure at least α . Indeed, suppose $m(\cap_{k=1}^K B_{n_k}) \geq \alpha$. Then, $m(\cap_{k=1}^{K+1} B_{n_k}) = 0$ implies $m(\cap_{k=1}^{K+1} B_{n_k} \cup [B_{n_{K+1}} \cap B_1]) \geq 2\alpha$. But this union is a subset of B_1 which has measure strictly less than 2α . Contradiction. By induction $m(\cap_{k=1}^K B_{n_k}) \geq \alpha$ for all K . Then we have $\alpha \leq \lim_K m(\cap_{k=1}^K B_{n_k}) = m(\cap_k B_{n_k}) \leq \lim_k m(B_{n_k}) = \alpha$. Thus $\cap_k B_{n_k}$ is atomic.

THM: Let (X, m, T) be an ergodic, non-atomic system. Assume T is invertible and that T^{-1} is also measure preserving. Given $\epsilon > 0$, $H \in \mathbb{N}$ there is a $B \subset X$ such that $B, TB, \dots, T^{H-1}B$ are pairwise disjoint and $m(\cup_{n < H} T^n B) > 1 - \epsilon$.

Proof: Let $A_0 \subset X$ be such that $0 < m(A_0) < \frac{\epsilon}{H-1}$. Define $A_{n+1} = TA_n \setminus (\cup_{i=0}^n A_i)$ for $n \geq 0$. Note that by this construction A_0, A_1, \dots are pairwise disjoint. As T is a measure preserving transformation, we have that TA_0, TA_1, \dots are also pairwise disjoint. Suppose not. Then there are n, m so that $m(TA_m \cap TA_n) > 0$. But then we would have $m(A_m \cap A_n) > 0$. Contradiction. From this we have that $A_{n+1} = TA_n \setminus A_0$. Observe, $TA_n \cap A_i = TA_n \cap (TA_{i-1} \setminus \cup_{j=0}^{i-1} A_j) = \emptyset$ for all $0 < i \leq n$.

Define $E_{k,n} = T^{-k}(T(A_{n+k-1}) \cap A_0)$ for all n and $k \in [1, H-1]$. Define $\bar{A}_n = A_n \setminus \cup_{k=1}^{H-1} E_{k,n}$. Define out base set as

$$B = \cup_{n \equiv_H 0} \bar{A}_n.$$

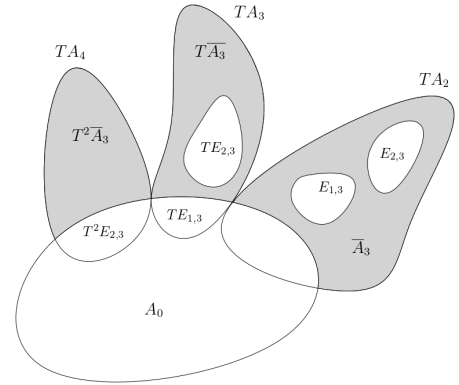
An example of what the \bar{A}_n would look like in the case when $n = 3$ and $H = 3$ is given to the right. Notice that each of our error sets as well as the first $H-2$ forward images will be removed among the A_n to get the \bar{A}_n but that only the removal of the $E_{k,Hn}$ is occurring in B . In the specific case of our illustration $TE_{1,3} = T^2 E_{2,2} = TA_3 \cap A_0$. However in B only $E_{1,3}$ will be removed.

We have four things to prove to get our inequality:

1. $m(\cup_n A_n) = 1$
2. $m(\cup_{k=1}^{H-1} \cup_n E_{k,n}) < \epsilon$
3. $(\cup_n A_n) \setminus (\cup_m \cup_{k=1}^{H-1} E_{k,m}) = \cup_n \bar{A}_n$
4. $m(\cup_n \bar{A}_n) \leq m(\cup_{n=0}^{H-1} T^n B)$

By the injectivity of T , we have $T(A \setminus B) = TA \setminus TB$. This allows us the following,

$$A_i = TA_{i-1} \setminus A_0 \subset TA_{i-1} = T^2 A_{i-2} \setminus TA_0 \subset T^2 A_{i-2} = \dots \subset T^i A_0 \subset \cup_{n=0} T^n A_0$$



for all $i > 0$. Clearly, $A_0 \subset \cup_{n=0} T^n A_0$. Therefore, $\cup_n A_n \subset \cup_n T^n A_0$. We go by induction for the converse. The basecase is $A_0 \subset \cup_{n=0} T^n A_0$. Suppose $T^k A_0 \subset \cup_{n=0} A_n$. Then

$$T^{k+1} A_0 = T(T^k A_0) \subset T \cup_{n=0} A_n = \cup_{n=0} T A_n \subset \cup_{n=0} (A_{n+1} \cup A_0) = \cup_{n=0} A_n.$$

Thus by induction, $T^k A_0 \subset \cup_n A_n$ for all k . So we have proved one. Consequently, as $\cup_n T^n A_0$ is invariant and has positive measure, $m(\cup_n A_n) = m(\cup_n T^n A_0) = 1$.

We move on to two. By definition, $T^k E_{k,n} = T(A_{n+k-1} \cap A_0) \subset A_0$ for all n and $1 \leq k \leq H-1$. As the T^{-1} is measure preserving and since the $T^k E_{k,n}$ are disjoint in k , $m(\cup_{k=1}^n \cup_n E_{k,n}) = m(\cup_{k=1}^{H-1} \cup_n T^k E_{k,n}) = \sum_{k=1}^{H-1} m(\cup_n T^k E_{k,n}) \leq (H-1)m(A_0) < \epsilon$.

For three, note that $E_{k,n} \cap A_m = \emptyset$ for all $m \neq n$ and $1 \leq k \leq H-1$. Thus we have,

$$\cup_n A_n \setminus (\cup_m \cup_{k=1}^{H-1} E_{k,m}) = \cup_n (A_n \setminus (\cup_m \cup_{k=1}^{H-1} E_{k,m})) = \cup_n (A_n \setminus \cup_{k=1}^{H-1} E_{k,n}) = \cup_n \bar{A}_n.$$

The last thing we need to show is that $m(\cup \bar{A}_n) \leq m(\cup_{n=0}^{H-1} T^n B)$. For this it is important to realize that $m(A_n) \geq m(A_{n+1})$ and $T^{H-1} \bar{A}_n = A_{n+H-1}$. We then have

$$m(\cup_{n=0} \bar{A}_n) = \sum_{n=0} m(\bar{A}_n) = \sum_{n=H-1}^{\infty} m(A_n) \leq H \sum_{n=0}^{\infty} m(A_{nH-1}) = Hm(B) = m(\cup_{n=0}^{H-1} T^n B).$$

Putting all of this together yields

$$1 - \epsilon < 1 - m(\cup_{k=1}^{H-1} \cup_m E_{k,m}) = m(\cup_n A_n \setminus \cup_{k=1}^{H-1} \cup_m E_{k,m}) = m(\cup \bar{A}_n) \leq m(\cup_{n=0}^{H-1} T^n B).$$

Finally, if $n \neq m$ with $0 < n, m \leq H-1$ then

$$T^n B \cap T^m B = (\cup_{k \equiv_H 0} T^n \bar{A}_k) \cap (\cup_{k \equiv_H 0} T^m \bar{A}_k) \subset (\cup_{k \equiv_H 0} A_{n+k}) \cap (\cup_{k \equiv_H 0} A_{m+k}) = \emptyset.$$

□

Note: I initially tried to define $A_n = TA_{n-1} \setminus A_0$ and then show that the A_n were disjoint. However my attempt at proving the disjointness was messy and hard to read. Dr. Wierdl sent me an email with a solution that I think reads well.

Proof of disjointness from Mate: Let $A_n = TA_{n-1} \setminus A_0$. If we had $A_2 \cap A_1 \neq \emptyset$ this would mean that for some $x_1 \in A_1$ and $x_0 \in A_0$ we'd have $Tx_1 = Tx_0$ which implies $x_0 = x_1$. If we had $A_3 \cap A_1 \neq \emptyset$ then for some $x_2 \in A_2$ and $x_0 \in A_0$ we'd have $Tx_2 = Tx_0$ hence $x_2 = x_0$ but then $A_2 \cap A_0 \neq \emptyset$. If we had $A_3 \cap A_2 \neq \emptyset$ then for some $x_1 \in A_1$ and $x_2 \in A_2$, we'd have $Tx_1 = Tx_2$ so $x_1 = x_2$ so $A_1 \cap A_2 \neq \emptyset$ which we already is not the case. Etc.

Note: If I now take an arbitrary $x_0 \in A_0$ and I find that $T^n x_0 \notin A_0$ for all n , then $x_0 \in T^{-n} A_n$ for all n , that is, $x_0 \in \cap_n T^{-n} A_n$. This implies that the measure of such x_0 is 0. So almost all points $x \in A_0$ return to A_0 , that is, $T^n x \in A_0$. This we knew (by Poincaré recurrence) but I haven't seen it this way. This leads into approximation of dynamical systems. The last part of the text by Cornfeld, Fomin and Sinai has some interesting things about this.