RESEARCH STATEMENT

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1. Introduction

My primary research interests are in Ramsey theory and combinatorial number theory. To date, I have completed five research projects which have resulted in one publication [5] and one paper in preparation [6]. The focus of my research has been on problems in and around semigroup Ramsey theory.

In semigroup Ramsey theory, there are "density" and "partition" theorems. This refers to the given conditions in the associated theorems. Consider two cornerstone results of the field.

Theorem 1.1 (Van der Waerden's Theorem). *Let* $\mathbb{N} = \bigcup_{i=1}^{n} A_i$ *be a finite partition of* $\mathbb{N} = \{1, 2, \dots\}$. *Then there exists an* $1 \leq i \leq n$ *such that for all* $k \in \mathbb{N}$ *there are* a *and* b *in* $\mathbb{N}$ *such that*

$$\{a, a+b, a+2b, \dots, a+(k-1)\,b\} \subset A_i.$$

Definition 1.2. For $A \subseteq \mathbb{N}$, define **the upper density of** A to be

$$\overline{d}\,(A) := \limsup_{N} \frac{|A \cap [N]|}{N},$$

where $[N] = \{1, 2, \dots, N\}$.

Theorem 1.3 (Szemerédi's Theorem). *Let* $A \subseteq \mathbb{N}$ *be such that* $\overline{d}\,(A) > 0$. *Then for all* $k \in \mathbb{N}$ *there are* a *and* b *in* $\mathbb{N}$ *such that*

$$\{a, a+b, a+2b, \dots, a+(k-1)\,b\} \subset A.$$

Van der Waerden's theorem is an example of a partition result and Szemerédi's theorem is it's density analogue. It is clear that $\overline{d}\,(A) > 0$ is saying something about the size of A in $\mathbb{N}$. As it turns out, Van der Waerden's Theorem and other partition results are also concerned with large subsets of $\mathbb{N}$. The typical argument is that for any partition of $\mathbb{N}$ some cell of the partition is *large* enough – which can be interpreted using an appropriate algebraic notion of size – to guarantee the sought algebraic configuration.

As a researcher, my goal is to solve problems in a way that is complete and provides insight into the nature of the problem. To this end, I study the history, background, and development of my work and look there for connections with other areas of mathematics.

2. Research Projects

2.1. **Filter Relative Notions of Size and Combinatorial Consequences.** In four of the five research projects I have completed, I performed a careful analysis of new, broad notions of size. These ideas have a very strong connection with the Stone-Čech compactification of a semigroup, and all of the study being done is through the clarifying perspective it provides. Filter relative notions of size are generalizations of the following properties - each relevant in interconnected ways to Ramsey theory.

Definition 2.1. Let $A \subseteq \mathbb{N}$. For $k \in \mathbb{N}$, take $-k + A := \{x \in \mathbb{N} : \ x + k \in A\}$. We say that

(1) A is **syndetic** if it has bounded gaps. More technically, $\exists k \in \mathbb{N}$, $\bigcup_{i=1}^{k} -k + A = \mathbb{N}$.

(2) A is **thick** if it contains arbitrarily long sequences of consecutive numbers. More technically, $\forall k \in \mathbb{N}$, $\bigcap_{i=1}^{k} -k + A \neq \varnothing$.

(3) A is **piecewise syndetic** if there is some bound k such that A contains arbitrarily long sequences of numbers with gaps between successive numbers no longer than k. More technically,

$$\exists k \in \mathbb{N}, \{-x + \bigcup_{i=1}^{k} (-i + A) : x \in \mathbb{N}\}$$

has the finite intersection property.

The "more technical" parts of the above definitions have been appropriately generalized to arbitrary semigroups. The Stone-Čech compactification of a discrete topological space, X, is homeomorphic to the space of all ultrafilters on X with the correct topology. It is denoted βX. For an arbitrary semigroup S, the semigroup operation extends to the Stone-Čech compactification - as the space of all ultrafilters on S. This semigroup operation is called the ultrafilter product.

Definition 2.2. Let p, q be ultrafilters on a semigroup S. Then define, taking $x^{-1}A := \{y \in S : \ xy \in A\}$,

$$p \cdot q := \{A \subseteq S : \{x \in S : \ x^{-1}A \in q\} \in p\}.$$

For many mathematicians first introduced to this product, it can be helpful to point out that from a meaningful perspective this product corresponds to convolution of measures.

Importantly, $(\beta S, \cdot)$ is a *right compact topological semigroup*. Such semigroups are guaranteed to have a unique minimal ideal, denoted $K(\beta S)$, and at least one idempotent. We call idempotents in the minimal ideal *minimal idempotent ultrafilters* and define $E(K(\beta S))$ to be the collection of all such ultrafilters.

Proposition 2.3. *Let S be a semigroup. Then $A \subseteq S$ is piecewise syndetic if and only if there is a $p \in K(\beta S)$ such that $A \in p$.*

Definition 2.4. Let S be a semigroup. We say $A \subseteq S$ is **central** if there is an $e \in E(K(\beta S))$ with $A \in e$.

It is easy to see that central sets are piecewise syndetic and hence contain what ever configurations piecewise syndetic sets contain. Central sets are far richer in combinatorial configurations. An important feature for Ramsey theory, ultrafilters are exactly the filters which are *prime*.

Definition 2.5. Let S be a set. We say $\mathcal{A} \subseteq \mathcal{P}(S)$ is **prime** if $A \cup B \in \mathcal{A}$ implies $A \in \mathcal{A}$ or $B \in \mathcal{A}$.

The following is a simple consequence of Proposition 2.3, Definition 2.4, and the fact that ultrafilters are prime.

Proposition 2.6. *The collection of all piecewise syndetic subsets of S and the collection of all central subsets of S are each a prime collection of subsets of S.*

Piecewise syndetic subsets of $\mathbb{N}$ contain arbitrarily long arithmetic progressions, and hence, by the proposition above, we have Van der Waerden's theorem.

The *filter relative* generalization of syndeticity, thickness and piecewise syndeticity allow us to localize their combinatorial structure to a particular filter – whose sets might have strong arithmetic properties themselves!

Definition 2.7. Let $\mathcal{F}$ be a filter on a semigroup S and $A \subseteq S$. We say that

(1) A is **$\mathcal{F}$-syndetic** if $\forall V \in \mathcal{F}$, $\exists H \in \mathcal{P}_f(V)$, $\bigcup_{h \in H} h^{-1}A \in \mathcal{F}$.

(2) A is **$\mathcal{F}$-thick** if $\exists V \in \mathcal{F}, \forall H \in \mathcal{P}_f(V), \{\bigcap_{h\in H} h^{-1}A\} \cup \mathcal{F}$ has the finite intersection property.
(3) A is **piecewise $\mathcal{F}$-syndetic** if

$$\forall V \in \mathcal{F},\ \exists H \in \mathcal{P}_f(V), U_V \in \mathcal{F},\ \{x^{-1}\bigcup_{h\in H} h^{-1}A : x \in U_V,\ V \in \mathcal{F}\} \cup \mathcal{F}$$

has the finite intersection property.

Definition 2.8. Let S be a set, let βS be the collection of all ultrafilters on S and let $\mathcal{A} \subseteq \mathcal{P}(S) \setminus \{\varnothing\}$.
(1) If $\mathcal{A}$ has the finite intersection property, then $\overline{\mathcal{A}} := \{p \in \beta S : \mathcal{A} \subseteq p\}$ is nonempty.
(2) $\mathcal{A}^* := \{A \subseteq S :\ A^c \notin \mathcal{A}\}$

If $\overline{\mathcal{F}}$ is a subsemigroup then it contains a minimal ideal and minimal idempotents just as we had in the case for βS. Often, any results that can be found using the Stone-Čech compactification of a semigroup can readily be extended to the filter relative version because all of the same theoretical background that the βS results rely on are true for a closed subsemigroup $\overline{\mathcal{F}}$.

Proposition 2.9. *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a subsemigroup. Then $A \subseteq S$ is piecewise $\mathcal{F}$-syndetic if and only if there is a $p \in K\left(\overline{\mathcal{F}}\right)$ with $A \in p$.*

Definition 2.10. Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a subsemigroup. We say $A \subseteq S$ is **$\mathcal{F}$-central** if there is an idempotent, e, in the minimal ideal of $\overline{\mathcal{F}}$ such that $A \in e$.

2.1.1. ***An infinitary nilpotent polynomial Hales-Jewett theorem.*** The first project I undertook for my dissertation was an infinitary polynomial Hales-Jewett. In [9], they prove a polynomial Hales-Jewett theorem making extensive use of piecewise $\mathcal{F}$-syndetic sets. Of course $\mathcal{F}$-central sets are piecewise $\mathcal{F}$-syndetic and thus must contain whatever configurations piecewise $\mathcal{F}$-syndetic sets contain. An infinitary generalization of their result follows from an induction argument which relies on the minimal idempotent containing the given $\mathcal{F}$-central set.

Definition 2.11. Let X be a set and define $\mathcal{P}_f(X) := \{\alpha \subseteq X : 0 < |\alpha| < \infty\}$. A set $A \subset \mathbb{N}$ is an IP-set if there is an infinite set $X \subseteq \mathbb{N}$ such that $\{\sum_{x\in F} x :\ F \in \mathcal{P}_f(X)\} \subseteq A$.

An IP-ring is a collection of sets with certain properties that allows it to naturally and usefully index IP-sets.

Theorem 2.12 (Infinitary nilpotent polynomial Hales-Jewett (Griffin, 2023))**.** *Let $\mathcal{F}$ be an idempotent filter on a nilpotent group Γ and let $\mathbb{P}$ be a good collection of $\mathcal{F}$-measurable polynomial mappings. Then, for all $\beta \in \mathcal{P}_f(\mathbb{N})$, $V \in \mathcal{F}$ and $\mathcal{F}$-central sets E, there is an IP-ring $\mathcal{F}^{(1)}$, with $\alpha > \beta$ for all $\alpha \in \mathcal{F}^{(1)}$, and an IP-set $(v_\alpha)_{\alpha\in\mathcal{F}^{(1)}} \subset V$ with*

$$\{g(\alpha)\, v_\alpha :\ g \in \mathbb{P},\ \alpha \in \mathcal{F}^{(1)}\} \subset E.$$

2.1.2. ***A combinatorial characterization of $\mathcal{F}$-central sets.*** In [4], they prove an $\mathcal{F}$-central sets theorem. A natural next step is a combinatorial characterization of $\mathcal{F}$-central sets which I took on as my second project. This is a filter relative generalization of [7].

An FP-tree in $A \subseteq S$ is a tree in $\Lambda := \cup_{n=1}^{\infty} S^n$ such that the image of all $f \in T$ is in A and for all $f \in T$, $B_f := \{x : f^\frown x \in T\} = \cup_{f^\frown g\in T} \mathrm{FP}(g)$ where $^\frown$ represents concatenation of sequences and $\mathrm{FP}(\cdot)$ is all finite products over a sequence.

Definition 2.13. Let S be a semigroup. Then $\mathcal{A} \subset \mathcal{P}(S)$ is **collectionwise piecewise $\mathcal{F}$-syndetic** if there is a $p \in K\left(\overline{\mathcal{F}}\right)$ such that $\mathcal{A} \subseteq p$.

Theorem 2.14 (Combinatorial Characterization of $\mathcal{F}$-Central Sets (Griffin, 2023))**.** *Let S be a semigroup, let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a subsemigroup of βS, and let $A \subset S$. Then A is $\mathcal{F}$-central if and only if there is an FP-tree, T, in A such that $\{B_f : f \in T\}$ is collectionwise piecewise $\mathcal{F}$-syndetic.*

Leveraging the full strength of [8, Theorem 4.20], I also prove the following, which is new in even the non-filter-relative case established in [7].

Definition 2.15 (Simple product tree)**.** Let S be a semigroup and T a tree in $A \subset S$. Then T is a **simple product tree** if and only if $f \in T$ implies $\mathrm{FP}(f) \subseteq A$.

Theorem 2.16 (Simplified Combinatorial Characterization (Griffin, 2023))**.** *Let S be a semigroup, let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a subsemigroup of βS, and let $A \subset S$. Then A is $\mathcal{F}$-central if and only if there is a simple product tree, T, in A such that $\{B_f : f \in T\}$ is collectionwise piecewise $\mathcal{F}$-syndetic.*

2.1.3. ***A characterization of piecewise $\mathcal{F}$-syndetic sets.*** The following theorem is well known and incredibly useful. The known proofs used only elementary set theory which meant that proving a filter relative generalization required an original argument.

Theorem 2.17. *Let S be a semigroup. Then $A \subseteq S$ is piecewise syndetic if and only if there is a syndetic sets B and a thick set C such that $A = B \cap C$.*

In [2], they prove that the intersection of an $\mathcal{F}$-syndetic set and an $\mathcal{F}$-thick set is piecewise $\mathcal{F}$-syndetic. They left the converse as an open question which I took on as my third project. In order to achieve a solution to this problem, I developed a new perspective on relative notions of size. We are taking $\mathrm{Syn}(\mathcal{F})$, $\mathrm{Thick}(\mathcal{F})$, and $\mathrm{PS}(\mathcal{F})$ to be the collection of all $\mathcal{F}$-syndetic, $\mathcal{F}$-thick, and piecewise $\mathcal{F}$-syndetic subsets of S, respectively.

Theorem 2.18. *Let S be a set, $\mathcal{F}$ a filter on S. Then*

$$\bigcup_{p \in \overline{\mathcal{F}}} p = \mathcal{F}^* \text{ and } \bigcap_{p \in \overline{\mathcal{F}}} p = \mathcal{F}$$

Theorem 2.19 (Griffin, 2023)**.** *Let $\mathcal{F}$ be a filter on a semigroup S. Then*

$$\mathrm{Syn}(\mathcal{F}) = \bigcap_{q \in \overline{\mathcal{F}}} \bigcup_{p \in \overline{\mathcal{F}}} (p \cdot q) \text{ and } \mathrm{Thick}(\mathcal{F}) = \bigcup_{q \in \overline{\mathcal{F}}} \bigcap_{p \in \overline{\mathcal{F}}} (p \cdot q).$$

Theorem 2.20 (Griffin, 2023)**.** *Let $\mathcal{F}$ be a filter on a semigroup S. Then*

$$\mathrm{PS}(\mathcal{F}) = \bigcup_{r \in \overline{\mathcal{F}}} \bigcap_{q \in \overline{\mathcal{F}}} \bigcup_{p \in \overline{\mathcal{F}}} (p \cdot q \cdot r)$$

The completion of the filter relative version of Theorem 2.17 followed from this new perspective.

Theorem 2.21 (Griffin, 2023)**.** *Let $\mathcal{F}$ be a filter on S such that $\overline{\mathcal{F}}$ is a subsemigroup of βS. If A is piecewise $\mathcal{F}$-syndetic, then A is the intersection of an $\mathcal{F}$-syndetic set and an $\mathcal{F}$-thick set.*

2.1.4. ***Subsemigroup actions and $(\mathcal{F}, \mathcal{G})$-central sets.*** This is joint work with John H. Johnson, Dennis Davenport, et. al. It was supported, in part, by the American Institute for Mathematics' SQuaRE program. It is currently in preparation. All of the following have been defined previously, with the exception of piecewise $(\mathcal{F}, \mathcal{G})$-syndeticity. Such a notion has been defined in other work but in a way that is inequivalent.

Definition 2.22. Let $\mathcal{F},\mathcal{G}$ be a filter on a semigroup S and $A\subseteq S$. We say that

(1) A is $(\mathcal{F},\mathcal{G})$-**syndetic** if $\forall V\in\mathcal{F}$, $\exists H\in\mathcal{P}_f(V)$, $\bigcup_{h\in H}h^{-1}A\in\mathcal{G}$.
(2) A is $(\mathcal{F},\mathcal{G})$-**thick** if $\exists V\in\mathcal{F}$, $\forall H\in\mathcal{P}_f(V)$, $\{\bigcap_{h\in H}h^{-1}A\}\cup\mathcal{G}$ has the finite intersection property.
(3) A is **piecewise** $(\mathcal{F},\mathcal{G})$-**syndetic** if

$$\forall V\in\mathcal{F},\ \exists H\in\mathcal{P}_f(V),\ U_V\in\mathcal{G},\ \{x^{-1}\bigcup_{h\in H}h^{-1}A : x\in U_V,\ V\in\mathcal{F}\}\cup\mathcal{G}$$

has the finite intersection property.

In this work we define $K(\mathcal{F},\mathcal{G})=\cup\{\overline{\mathcal{F}}\cdot e:\ e\in E\left(K\left(\overline{\mathcal{G}}\right)\right)\}$, and $\mathrm{PS}(\mathcal{F},\mathcal{G})$ to be the set of all piecewise $(\mathcal{F},\mathcal{G})$-syndetic sets. Assuming $\overline{\mathcal{G}}$ is a subsemigroup and $\overline{\mathcal{F}}\cdot\overline{\mathcal{G}}\subseteq\overline{\mathcal{F}}$, $K(\mathcal{F},\mathcal{G})$ may be shown to contain idempotents.

Definition 2.23. A set $A\subseteq S$ is $(\mathcal{F},\mathcal{G})$-**central** if there is an idempotent $e\in K(\mathcal{F},\mathcal{G})$ such that $A\in e$.

We have shown that $(\mathcal{F},\mathcal{G})$-central sets satisfy a strong central sets theorem and can be combinatorially characterized.

2.2. **Density Ramsey Theorems.** For the fourth project in my dissertation, I prove a density Ramsey theorem. It is a partial result towards a two dimensional Szemerédi theorem for a finitely generated free semigroup. The main tool in this case is ergodic theory. I follow the well laid path of proving a recurrence theorem corresponding to the configuration we are seeking.

Definition 2.24. Let Λ be a finite set. The **free semigroup generated by** Λ is $\Lambda^*=\bigcup_{n=0}^{\infty}\Lambda^n$ with the semigroup operation being concatenation.

Definition 2.25. Let $S_r(\Lambda^*):=\{w\in\Lambda^*:\ |w|=r\}$. Then define, for $A\subseteq\Lambda^*$, the upper density of A to be

$$\overline{d}(A)=\limsup_{N\to\infty}\frac{1}{N}\sum_{r=0}^{N-1}\frac{|A\cap S_r(\Lambda^*)|}{|S_r(\Lambda^*)|}$$

Definition 2.26. We say that $w'\in\Lambda^*$ is a descendant of w if there exists a $u\in\Lambda^*$ such that $w'=wu$. An **arithmetic subtree** in $A\subseteq\Lambda^*$ of order r is a map $\phi:B_r(\Lambda^*)\to A$ such that there exists $q\in\mathbb{Z}_{>0}$ such that for all $w\in B_{r-1}(\Lambda^*)$ and $\lambda\in\Lambda$ we have

(1) $|\phi(w\lambda)|=|\phi(w)|+q$
(2) The element $\phi(w\lambda)$ is a descendent of $\phi(w)\lambda$.

We call q the gap of this arithmetic subtree.

Theorem 2.27 (Szemerédi's theorem for arithmetic trees (Furstenberg and Weiss,[3]))**.** *If $A\subset\Lambda^*$ is such that $\overline{d}(A)>0$, then for all $r\in\mathbb{N}$ there is an arithmetic tree of order r in A.*

A significant partial result toward finding the product of two arbitrarily long arithmetic trees with the same gap in positive density subsets of $\Gamma:=\Lambda^*\oplus\Lambda^*$ has been found in [1].

Definition 2.28. (1) The free product $\widetilde{\Gamma}=\Lambda^* * \Lambda^*$ is the free semigroup generated by $\{X_\lambda\}_{\lambda\in\Lambda}\bigsqcup\{Y_\lambda\}_{\lambda\in\Lambda}$.
(2) Take an arbitrary $\gamma\in\widetilde{\Gamma}$. Suppose $X_{\lambda_1},\dots,X_{\lambda_n}$ are the letters of the form X_λ in γ and that $Y_{\omega_1},\dots,Y_{\omega_m}$ are the letters of the form Y_ω in γ. The level of γ is $\mathrm{L}(\gamma):=(n,m)$.
(3) $B_r\left(\widetilde{\Gamma}\right)=\{\gamma\in\widetilde{\Gamma}:\ |\gamma|\le r\}$.

(4) Given $A \subseteq \Gamma$ and $u, v \in \mathbb{Z}^2_{\geq 0}$, we define a (u,v)-arithmetic product tree of order r in A to be a map $\phi : B_r\left(\widetilde{\Gamma}\right) \to A$ such that:
   (a) For $\gamma \in B_{r-1}\left(\widetilde{\Gamma}\right)$ and $\lambda \in \Lambda$ we have
$$\mathrm{L}\left(\phi\left(\gamma X_\lambda\right)\right) = \mathrm{L}\left(\phi\left(\gamma\right)\right) + u$$
   and
$$\mathrm{L}\left(\phi\left(\gamma Y_\lambda\right)\right) = \mathrm{L}\left(\phi\left(\gamma\right)\right) + v$$
   (b) For $\gamma \in B_{r-1}\left(\widetilde{\Gamma}\right)$ and $\lambda \in \Lambda$ we have that $\phi\left(\gamma X_\lambda\right)$ is a descendent of $\phi\left(\gamma\right) X_\lambda$ and $\phi\left(\gamma Y_\lambda\right)$ is a descendent of $\phi\left(\gamma\right) Y_\lambda$

Theorem 2.29. [1, Theorem 1.7] *Let $A \subseteq \Gamma$ be a subset with $\overline{\mathrm{d}}(A) > 0$ and let $u, v \in \mathbb{Z}^2_{>0}$. Then for $r > 0$ there exists $n \in \mathbb{Z}_{>0}$ such that there is an (nu, nv)-arithmetic product tree of order r in A.*

2.2.1. ***A two dimensional corner theorem and more for arithmetic trees.***

Lemma 2.30 (Griffin, 2024)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space and let T and S be two invertible measure preserving transformations on X which commute with each other. Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are sub-σ-algebras of $\mathcal{A}$. Let $A \in \mathcal{A}$ be such that $\mu(A) > 0$. Let f_1 be the orthogonal projection of $\mathbb{1}_A$ onto $L^2(X, \mathcal{B}_1, \mu)$ and f_2 be the orthogonal projection of $\mathbb{1}_A$ onto $L^2(X, \mathcal{B}_2, \mu)$. Then $\int \mathbb{1}_A T^n f_1 S^n f_2 d\mu$ is bounded away from zero for a syndetic set of n.*

This lemma is critical to the proof of an improvement on Theorem 2.29.

Theorem 2.31 (Griffin, 2024)**.** *Let $A \subseteq \Gamma$ be a subset with $\overline{\mathrm{d}}(A) > 0$. Then for $r > 0$ there exists $n \in \mathbb{Z}_{>0}$ such that there is an $((n,0),(0,n))$-arithmetic product tree of order r in A.*

We then have a corner theorem for arithmetic trees as a corollary. We single out this much smaller structure because of the role a corner theorem plays in Gowers' proof of the multidimensional Szemerédi theorem.

Corollary 2.32 (Griffin, 2024)**.** *Suppose $A \subseteq \Gamma$ is such that $\overline{\mathrm{d}}(A) > 0$. Then there exist $u, v \in \Lambda^*$ and $n \in \mathbb{N}$ such that for all $i \in \Lambda$ there exists $w_i, x_i \in \Lambda^n$ such that $\{(u,v), (uiw_i, v), (u, vix_i)\}_{i \in \Lambda} \subseteq A$.*

3. Current Projects and Future Work

3.1. **Filter Relative Notions of Size and Combinatorial Consequences.**

3.1.1. ***Multiple structures and relative notions.*** This is joint work with Sayan Goswami and Sourav Patra. We are developing a finite sum/product theorem for measurable and Baire partitions of the interval $(0,1)$ which is based on the observation in the following theorem.

Theorem 3.1 (Griffin, Goswami, Patra, 2024)**.** *Let $\mathfrak{L} \subseteq \mathcal{L}$ be a minimal left ideal of $\beta(0,1)_d$ and let $\mathcal{G}$ be the filter such that $\mathfrak{L} = \overline{\mathcal{G}}$. Then $p \in K(\mathfrak{L})$ implies that for all $A \in p$, $\{x \in (0,1) :\ x^{-1}A \in p \text{ and } -x + A \in p\}$ is (multiplicatively) $\mathcal{G}$-syndetic.*

Where $(0,1)_d$ is the interval $(0,1)$ equipped with the discrete topology and $\mathcal{L} = \{p \in \beta(0,1)_d : (\forall A \in p)\ \overline{\mathrm{d}}(A) > 0\}$.

3.1.2. ***A hierarchy of notions of size in a semigroup.*** In the analytic and arithmetic hierarchies, the number of times one alternates quantifiers in a formula the higher the complexity of the sets definable by those formulas. This is reflected in the new perspective I introduced of syndeticity, thickness and piecewise syndeticity. Inspired by this we define the following.

Definition 3.2. Let $n \in \mathbb{Z}_{\geq 0}$ and S be a semigroup. Let $\mathcal{F}$ be a filter on S. We define

$$\sigma^n = \{A \subseteq S : \left(\exists p_{2n+1} \in \overline{\mathcal{F}}\right), \ \left(\forall p_{2n} \in \overline{\mathcal{F}}\right), \ldots, \left(\exists p_1 \in \overline{\mathcal{F}}\right) \ A \in p_1 \cdot p_2 \cdot \ldots \cdot p_{2n+1}\}$$

and

$$\tau^n = \{A \subseteq S : \left(\forall p_{2n+1} \in \overline{\mathcal{F}}\right), \ \left(\exists p_{2n} \in \overline{\mathcal{F}}\right), \ldots, \left(\forall p_1 \in \overline{\mathcal{F}}\right) \ A \in p_1 \cdot p_2 \cdot \ldots \cdot p_{2n+1}\}$$

Notice that $\sigma^0 = \mathcal{F}^*$, $\tau^0 = \mathcal{F}$, $\sigma^1 = \mathrm{PS}\left(\mathcal{F}\right)$ and that for all n, $\left(\sigma^n\right)^* = \tau^n$. It is also the case that τ^n is a filter and σ^n is prime.

Proposition 3.3 (Griffin, 2024)**.** *For all $n \geq 0$, $\sigma^{n+1} \subseteq \sigma^n$, $\tau^n \subseteq \tau^{n+1}$, and $\tau^n \subseteq \sigma^{n+1}$.*

Corollary 3.4 (Griffin, 2024)**.** *Let $\mathcal{F}$ be a filter on a semigroup S. Then $\bigcap_{n=0}^{\infty} \sigma^n$ is prime.*

We can define σ^n recursively, combinatorially.

Definition 3.5. A collection $\mathcal{A} \subseteq \mathcal{P}\left(S\right) \setminus \{\varnothing\}$ is **collectionwise** σ^n if

$$\left(\exists p_{2n+1} \in \overline{\mathcal{F}}\right), \ \left(\forall p_{2n} \in \overline{\mathcal{F}}\right), \ldots, \left(\exists p_1 \in \overline{\mathcal{F}}\right) \ \mathcal{A} \subseteq p_1 \cdot p_2 \cdot \cdots \cdot p_{2n+1}.$$

Theorem 3.6 (Griffin, 2024)**.** *Let S be a semigroup. $A \subseteq S$ is in σ^n if and only if*

$$\left(\forall V \in \mathcal{F}\right), \ \left(\exists H_V \in \mathcal{P}_f\left(V\right)\right), \left(\exists W_V \in \mathcal{F}\right)$$

such that

$$\{x^{-1} \bigcup_{h \in H_V} h^{-1} A : \ V \in \mathcal{F}, \ x \in W_V\}$$

is collectionwise σ^{n-1}.

Theorem 3.7 (Griffin, 2024)**.** *If $\overline{\mathcal{F}}$ is a subsemigroup then, for all $n \geq 1$,*

$$\tau^1 = \tau^n \text{ and } \sigma^1 = \sigma^n.$$

Some interesting partial results follow, both when $\overline{\mathcal{F}}$ is a semigroup and when it is *not* a semigroup.

3.2. **Density Ramsey Theorems.**

3.2.1. ***A multidimensional Szemerédi theorem for arithmetic trees.*** It is well known that the multidimensional Szemerédi theorem follows from the multidimensional version of the corner theorem of Ajtai and Szemerédi. Thus Corollary 2.32 might make one hopeful that there is a multidimensional Szemerédi theorem for arithmetic trees. There are quite a few substantial improvements which must be made to Corollary 2.32 to achieve something analogous to the multidimensional corner theorem of Ajtai and Szemerédi. Even if we take the correct multidimensional corner theorem for our context, we could not directly apply the same proof technique because of the same difficulty that Furstenberg and Weiss point out in [3].

"Without insisting on the immediate branching, as well as the uniform spacing between generations, our results could be deduced from the classical versions using standard combinatorial considerations."

References

[1] K. Bulinski and A. Fish. Arithmetic subtrees in large subsets of products of trees. *Journal D'Analyse Mathématique*, 150(3):359– 381, 2023.

[2] C. Christopherson and J. H. Johnson. Algebraic characterizations of some relative notions of size. *Semigroup Forum*, 104:28–44, 2022.

[3] H. Furstenberg and B. Weiss. Markov processes and ramsey theory for trees. *Combinatorics, Probability and Computing*, 12(5–6):547–563, 2003.

[4] S. Goswami and J. Poddar. Central sets theorem along filters and some combinatorial consequences. *Indagationes Mathematicae*, 33(6):1312–1325, 2022.

[5] C. Griffin. A characterization of piecewise $\mathcal{F}$-syndetic sets. *Semigroup Forum*, 2024.

[6] C. Griffin, J. H. Johnson, and et. al. Subsemigroup actions of βS and $(\mathcal{F}, \mathcal{G})$-central sets (working title). In preparation.

[7] N. Hindman, A. Maleki, and D. Strauss. Central sets and their combinatorial characterization. *Journal of Combinatorial Theory, Series A*, 74(2):188–208, 1996.

[8] N. Hindman and D. Strauss. *Algebra in the Stone-Čech Compactification*. volume 27 of De Gruyter Expositions in Mathematics. Walter de Gruyter, Berlin – New York, 2012.

[9] J. H. Johnson and F. K. Richter. Revisiting the nilpotent polynomial Hales–Jewett theorem. *Advances in Mathematics*, 321:269–286, 2017.