Abstract Harmonic Analysis

Conner Griffin

University of Memphis

March 25, 2022

Topological Groups

- A *topological group* is a group G with a topology on G so that the group operations are continuous. Specifically, $(x, y) \to xy$ is continuous from $G \times G$ to G and $x \to x^{-1}$ is continuous from $G \to G$.

Topological Groups

- A *topological group* is a group G with a topology on G so that the group operations are continuous. Specifically, $(x, y) \to xy$ is continuous from $G \times G$ to G and $x \to x^{-1}$ is continuous from $G \to G$.
- The topology of G is invariant under translation and inversion meaning that if U is open then so is xU, Ux, and $U^{-1} = \{x^{-1} : x \in U\}$.

Topological Groups

- A *topological group* is a group G with a topology on G so that the group operations are continuous. Specifically, $(x, y) \to xy$ is continuous from $G \times G$ to G and $x \to x^{-1}$ is continuous from $G \to G$.
- The topology of G is invariant under translation and inversion meaning that if U is open then so is xU, Ux, and $U^{-1} = \{x^{-1} : x \in U\}$.
- If H is a subgroup of G then so is $\overline{H}$.

Topological Groups

- A *topological group* is a group G with a topology on G so that the group operations are continuous. Specifically, $(x, y) \to xy$ is continuous from $G \times G$ to G and $x \to x^{-1}$ is continuous from $G \to G$.
- The topology of G is invariant under translation and inversion meaning that if U is open then so is xU, Ux, and $U^{-1} = \{x^{-1} : x \in U\}$.
- If H is a subgroup of G then so is $\overline{H}$.
- A topological space, X, is locally compact if every point $x \in X$ has a neighborhood, U, so that $x \in U \subset K$ for some compact K.

Topological Groups

- A *topological group* is a group G with a topology on G so that the group operations are continuous. Specifically, $(x, y) \to xy$ is continuous from $G \times G$ to G and $x \to x^{-1}$ is continuous from $G \to G$.
- The topology of G is invariant under translation and inversion meaning that if U is open then so is xU, Ux, and $U^{-1} = \{x^{-1} : x \in U\}$.
- If H is a subgroup of G then so is $\overline{H}$.
- A topological space, X, is locally compact if every point $x \in X$ has a neighborhood, U, so that $x \in U \subset K$ for some compact K.
- If H is a subgroup of topological group G, $q : G \to G/H$ the canonical quotient map. U in G/H is open if and only if $q^{-1}(U)$ open in G.
- If a topological group, G, is T_1 then G is Hausdorff. If G is not T_1 then $\overline{\{1\}}$ is a normal subgroup of G and $G/\overline{\{1\}}$ is Hausdorff.

Definition

Let G be a locally compact group. A *left (respectively right) Haar measure* on G is a Borel measure, m, with the following properties:

1. $m(gB) = m(B)$ (respectively $m(Bg) = m(B)$) for all Borel sets, B, and all $g \in G$
2. $m(K) < \infty$ for all compact sets $K \subset G$.
3. $m(U) > 0$ for all open $U \subseteq G$.

Haar Measure

Definition

Let G be a locally compact group. A *left (respectively right) Haar measure* on G is a Borel measure, m, with the following properties:

1. $m(gB) = m(B)$ (respectively $m(Bg) = m(B)$) for all Borel sets, B, and all $g \in G$
2. $m(K) < \infty$ for all compact sets $K \subset G$.
3. $m(U) > 0$ for all open $U \subseteq G$.

- If G is a locally compact topological group then there is a left and right Haar measure each of which is unique up to a scalar multiple.

Haar Measure

Definition

Let G be a locally compact group. A *left (respectively right) Haar measure* on G is a Borel measure, m, with the following properties:

1. $m(gB) = m(B)$ (respectively $m(Bg) = m(B)$) for all Borel sets, B, and all $g \in G$
2. $m(K) < \infty$ for all compact sets $K \subset G$.
3. $m(U) > 0$ for all open $U \subseteq G$.

- If G is a locally compact topological group then there is a left and right Haar measure each of which is unique up to a scalar multiple.
- For a discrete group, Haar measure is simply counting measure.

Unitary Representation

Definition

Let G be a locally compact group. A *unitary representation* of G is a homomorphism, π, from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on some Hilbert space $\mathcal{H}_\pi$.

Unitary Representation

Definition

Let G be a locally compact group. A *unitary representation* of G is a homomorphism, π, from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on some Hilbert space $\mathcal{H}_\pi$.

- The weak and strong operator topologies are equivalent on $U(\mathcal{H}_\pi)$.

Unitary Representation

Definition

Let G be a locally compact group. A *unitary representation* of G is a homomorphism, π, from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on some Hilbert space $\mathcal{H}_\pi$.

- The weak and strong operator topologies are equivalent on $U(\mathcal{H}_\pi)$.
  Proof: Let $\{T_\alpha\}$ is a net in $U(\mathcal{H}_\pi)$ which converges weakly to T. For any $u \in \mathcal{H}_\pi$,

$$\begin{aligned} \|(T_\alpha - T)u\|^2 &= \|T_\alpha u\|^2 + \|Tu\|^2 - 2\,\mathrm{Re}\langle T_\alpha u, Tu\rangle \\ &= 2\|u\|^2 - 2\,\mathrm{Re}\langle T_\alpha u, Tu\rangle \end{aligned}$$

  This converges to $2\|u\|^2 + 2\|Tu\|^2$ with α. Then $2\|u\|^2 + 2\|Tu\|^2$ converges to zero with u.

If $\mathcal{M}$ is a closed subset of $\mathcal{H}_\pi$ such that $\pi(x)\mathcal{M} \subset \mathcal{M}$ for all $x \in G$, then we say that $\mathcal{M}$ is invariant. We say that π is irreducible if it's only invariant subspaces are $\{0\}$ and $\mathcal{H}_\pi$.

Unitary Representations

If $\mathcal{M}$ is a closed subset of $\mathcal{H}_\pi$ such that $\pi(x)\mathcal{M} \subset \mathcal{M}$ for all $x \in G$, then we say that $\mathcal{M}$ is invariant. We say that π is irreducible if it's only invariant subspaces are $\{0\}$ and $\mathcal{H}_\pi$.

Proposition

If G is Abelian, then every irreducible representation of G is one-dimensional.

Unitary Representations

If $\mathcal{M}$ is a closed subset of $\mathcal{H}_\pi$ such that $\pi(x)\mathcal{M} \subset \mathcal{M}$ for all $x \in G$, then we say that $\mathcal{M}$ is invariant. We say that π is irreducible if it's only invariant subspaces are $\{0\}$ and $\mathcal{H}_\pi$.

Proposition

If G is Abelian, then every irreducible representation of G is one-dimensional.

Gelfand-Raikov

If G is any locally compact group, the irreducible unitary representations of G separate points on G. That is, if x and y are distinct points of G, there is an irreducible representation π such that $\pi(x) \neq \pi(y)$.

Gelfand-Raikov guarantees that we have an irreducible unitary representations of G other than the trivial one $(\pi(x) = I.)$

The Dual Group

By the previous proposition, when π is irreducible, $\mathcal{H}_\pi$ may be taken to be $\mathbb{C}$. Then $\pi(x)(z)$ is a unitary operator on $\mathbb{C}$. That is, it has to be multiplication by an element of the unit circle. So we have $\pi(x)(z) = \xi(x)z$ where ξ is a continuous homomorphism from G into the torus, $\mathbb{T}$.

The Dual Group

By the previous proposition, when π is irreducible, $\mathcal{H}_\pi$ may be taken to be $\mathbb{C}$. Then $\pi(x)(z)$ is a unitary operator on $\mathbb{C}$. That is, it has to be multiplication by an element of the unit circle. So we have $\pi(x)(z) = \xi(x)z$ where ξ is a continuous homomorphism from G into the torus, $\mathbb{T}$. Such homomorphisms are called the unitary characters of G.

The Dual Group

By the previous proposition, when π is irreducible, $\mathcal{H}_\pi$ may be taken to be $\mathbb{C}$. Then $\pi(x)(z)$ is a unitary operator on $\mathbb{C}$. That is, it has to be multiplication by an element of the unit circle. So we have $\pi(x)(z) = \xi(x)z$ where ξ is a continuous homomorphism from G into the torus, $\mathbb{T}$. Such homomorphisms are called the unitary characters of G.

Definition

The *dual group* of a topological group, G, is the set of all unitary characters of G. It is denoted $\widehat{G}$.

The Dual Group

By the previous proposition, when π is irreducible, $\mathcal{H}_\pi$ may be taken to be $\mathbb{C}$. Then $\pi(x)(z)$ is a unitary operator on $\mathbb{C}$. That is, it has to be multiplication by an element of the unit circle. So we have $\pi(x)(z) = \xi(x) z$ where ξ is a continuous homomorphism from G into the torus, $\mathbb{T}$. Such homomorphisms are called the unitary characters of G.

Definition

The *dual group* of a topological group, G, is the set of all unitary characters of G. It is denoted $\widehat{G}$.

The dual group plays an important role in defining the Fourier transform. As such, to reflect the typical Fourier transform, we use the following notation:

$$\langle x, \xi \rangle = \xi(x).$$

Examples

- $\widehat{\mathbb{R}} \cong \mathbb{R}$ with the following familiar definition, $\langle x, \xi \rangle = e^{2\pi i \xi x}$.
- $\widehat{\mathbb{T}} \cong \mathbb{Z}$ with $\langle \alpha, n \rangle = \alpha^n$.
- $\widehat{\mathbb{Z}} \cong \mathbb{T}$ with $\langle n, \alpha \rangle = \alpha^n$.
- $\widehat{\mathbb{Z}/k\mathbb{Z}} \cong \mathbb{Z}/k\mathbb{Z}$ with $\langle m, n \rangle = e^{2\pi i mn/k}$.

Example: The p-adic numbers

$\mathbb{Q}_p$

Let p be a prime and r a rational number. Then there is an $m \in \mathbb{Z}$ with $r = p^m q$, where q is a rational number whose numerator and denominator are not divisible by p. This representation of r is unique. The *p-adic norm* of r is defined as $|r|_p = p^{-m}$. The *field of p-adic numbers* is the completion (in terms of Cauchy sequences) of $\mathbb{Q}$ with the metric induced by the p-adic norm. We denote it $\mathbb{Q}_p$

- If $m \in \mathbb{Z}$ and $c_j \in \{0, 1, \ldots, p-1\}$ for $j \geq m$, then $\sum_m^\infty c_j p^j$ converges in $\mathbb{Q}_p$. Additionally, every p-adic number can be represented by such a series.

We want to compute $\widehat{\mathbb{Q}_p}$. Let ξ_1 be a character of $\mathbb{Q}_p$. Then define

$$\langle \sum_{-\infty}^{\infty} c_j p^j, \xi_1 \rangle = \exp\left(2\pi i \sum_{-\infty}^{-1} c_j p^j\right)$$

It is easily checked that ξ_1 is a unitary character whose kernel is $\mathbb{Z}_p$. Define ξ_y by $\langle x, \xi_y \rangle = \langle xy, \xi_1 \rangle$. Then the map $y \to \xi_y$ is an isomorphism from $\mathbb{Q}_p$ to $\widehat{\mathbb{Q}_p}$.

The Fourier Transform

We first associate to $\xi \in \widehat{G}$ the functional

$$f \to \overline{\xi}\,(f) = \int \overline{\langle x, \xi \rangle} f\,(x)\,dx.$$

The Fourier Transform

We first associate to $\xi \in \widehat{G}$ the functional

$$f \to \overline{\xi}\left(f\right) = \int \overline{\langle x, \xi \rangle} f\left(x\right) dx.$$

This gives us the Fourier transform, $\mathcal{F} : L_1\left(G\right) \to C\left(\widehat{G}\right)$, defined by

$$\mathcal{F}f\left(\xi\right) = \widehat{f}\left(\xi\right) = \int \overline{\langle x, \xi \rangle} f\left(x\right) dx.$$

The Fourier Transform

We first associate to $\xi \in \widehat{G}$ the functional

$$f \to \overline{\xi}(f) = \int \overline{\langle x, \xi \rangle} f(x)\, dx.$$

This gives us the Fourier transform, $\mathcal{F} : L_1(G) \to C\left(\widehat{G}\right)$, defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int \overline{\langle x, \xi \rangle} f(x)\, dx.$$

Hausdorff-Young Inequality:
Suppose $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p(G)$ and $\widehat{f} \in L_q\left(\widehat{G}\right)$ then $\|\widehat{f}\|_q \leq \|f\|_p$

Pontryagin Duality

The essence of Pontryagin duality is this, the elements of G are also unitary characters on $\widehat{G}$. That is all topological groups are reflexive.

Pontryagin Duality

The essence of Pontryagin duality is this, the elements of G are also unitary characters on $\widehat{G}$. That is all topological groups are reflexive. In precise symbols, for each x in G, $\Phi(x)$, defined by $\langle \xi, \Phi(x) \rangle = \langle x, \xi \rangle$, is a unitary character of $\widehat{G}$.

Pontryagin Duality

The essence of Pontryagin duality is this, the elements of G are also unitary characters on $\widehat{G}$. That is all topological groups are reflexive. In precise symbols, for each x in G, $\Phi(x)$, defined by $\langle \xi, \Phi(x) \rangle = \langle x, \xi \rangle$, is a unitary character of $\widehat{G}$.

Pontryagin Duality

The map $\Phi : G \to \widehat{\widehat{G}}$ defined as above is an isomorphism of topological groups.

Pontryagin Duality

The essence of Pontryagin duality is this, the elements of G are also unitary characters on $\widehat{G}$. That is all topological groups are reflexive. In precise symbols, for each x in G, $\Phi(x)$, defined by $\langle \xi, \Phi(x) \rangle = \langle x, \xi \rangle$, is a unitary character of $\widehat{G}$.

Pontryagin Duality

The map $\Phi : G \to \widehat{\widehat{G}}$ defined as above is an isomorphism of topological groups.

The Fourier Inversion Theorem

If $f \in L_1(G)$ and $\widehat{f} \in L_1\left(\widehat{G}\right)$ then $f(x) = \left(\widehat{f}\right)^{\widehat{}}\left(x^{-1}\right)$ for almost every x; that is,

$$f(x) = \int \langle x, \xi \rangle \widehat{f}(\xi)\, d\xi \text{ for a.e. } x.$$

If f is continuous, then this holds for every x.