1. If (X,d) is a metric space show that then (X,ρ) is a metric space with $\rho = \frac{d}{1+d}$. Show that both metrics induce the same topology.

   Proof that ρ is a metric:

   (a) We need to show $\rho(x,y) = 0$ *if and only if* $x = y$.

   If $\rho(x,y) = 0$ then $\frac{d(x,y)}{1+d(x,y)} = 0$. As $1 + d(x,y) \neq 0$ we may multiply both sides by $1 + d(x,y)$. This gives us $d(x,y) = 0$. Since d is a metric, $x = y$.

   For the converse, if $x = y$ then $d(x,y) = 0$. Thus $\rho(x,y) = 0$.

   (b) We need to show *if* $x \neq y$ *then* $\rho(x,y) = \rho(y,x)$.

   If $x \neq y$ then $d(x,y) = d(y,x)$. Consequently,

   $$\rho(x,y) = \frac{d(x,y)}{d(x,y)+1} = \frac{d(y,x)}{d(y,x)+1} = \rho(y,x).$$

   (c) Finally, we need that *for all* $x,y,z \in X$, $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$. Let $A = d(x,y)$, $B = d(y,z)$, $C = d(x,z)$. Then,

   $$\begin{aligned}\rho(x,y) + \rho(y,z) - \rho(x,z) &= \frac{A}{A+1} + \frac{B}{B+1} - \frac{C}{C+1} \\ &= \frac{A+B-C+AB[2+C]}{(A+1)(B+1)(C+1)} \\ &\geq 0.\end{aligned}$$

   This is because $d(x,y) + d(y,z) \geq d(x,z)$. In terms of our substitution, $A+B-C \geq 0$. Thus ρ satisfies the triangle inequality.

   Proof that d and ρ induce the same topology: It is enough to show that they have the same base. Let $B_d(x,\epsilon_d)$ be an open ball of radius ϵ_d in (X,d) and $B_\rho(x,\epsilon_\rho)$ be an open ball of radius ϵ_ρ in (X,ρ). Then $B_d(x,\epsilon_d) = B_\rho\left(x, \frac{\epsilon_d}{\epsilon_d+1}\right)$ and $B_\rho(x,\epsilon_\rho) = B_d\left(x, \frac{\epsilon_\rho}{1-\epsilon_\rho}\right)$. We may achieve the latter as $\rho(x,y) < 1$ meaning that we only need to consider $0 \leq \epsilon_\rho < 1$. Thus both topologies have the same base and are therefore equivalent.

2. Consider Cantor's Intersection Theorem from class. Show that if the condition that $d(F_n) \to 0$ is removed then $F = \cap_n F_n$ may be empty.

   Cantor's Intersection Theorem:

   Let (X, d) be a complete metric space and $\{F_n\}_n$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$ as $n \to \infty$. Then $F = \cap_n F_n$ contains exactly one point.

3. In a metric space (X, d), show that a set A is noowhere dense if and ony if its complement is everywhere dense.