Important Math Concepts

Algebra, Trigonometry, Pre-Calculus, Calculus, and More to Come

Connor Darling, Carlos Aguilera

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Contents

Chapter 1

Algebra

1.1 Quadratic Formula

The Quadratic Formula can be used to solve any quadratic equation. First, bring a quadratic equation, i.e. $ax^2 + bx + c = 0$, where a, b, and c are coefficients and $a \neq 0$. Then we plub these coefficients into the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.2 Discriminant

Used to determine the number of solutions to the quadratic equation, $ax^2 + bx + c = 0$. The formula is:

$$b^2 - 4ac$$

1.3 Slope Formula

Used to find the slope of a line when you have two points on the line: $(x_1, y_1); (x_2, y_2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

1.4 Point-Slope Form of the Equation of a Line

Used to find the equation of a line when given one point (x_1, y_1) and the slope (m).

$$y - y_1 = m(x - x_1)$$

1.5 Slope Intercept Form of the Equation of a Line

Where m = slope and b = y-intercept

$$y = mx + b$$

1.6 Midpoint Formula

Used to find the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

1.7 Distance Formula

Used to find the distance between two points: (x_1, y_1) and (x_2, y_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.8 Vertex of a Parabola

Given a quadratic formula, $ax^2 + bx + c = 0$

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

1.9 Completing the Square

Completing the square is a technique for rewriting quadratics in the form $(x + a)^2 + b$.

For example, $x^2 + 2x + 3$ can be written as $(x + 1)^2 + 2$. The two expressions are totally equivalent, but the second one is nicer to work with in some situations.

Example for Completing the Square
Let's say we are given a quadratic and asked to complete the square:

$$x^2 + 10x + 24 = 0$$

We begin by moving the constant term to the right side of the equation.

$$x^2 + 10x = -24$$

We complete the square by taking half of the coefficient of our x term, squaring it, and adding it to both sides of the equation. Since the coefficient of our x term is 10, half of it would be 5, and squaring it gives us 25.

$$x^2 + 10x + 25 = -24 + 25$$

We can now rewrite the left side of the equation as a squared term.

$$(x + 5)^2 = 1$$

Take the square root of both sides.

$$x + 5 = \pm 1$$

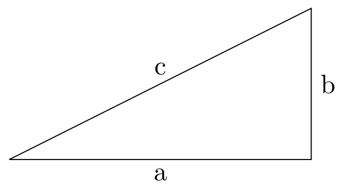
Isolate x to find the solution(s).

$$x = -5 \pm 1$$

1.10 Pythagorean Theorem

Used to find the missing side of a right triangle:
a = length of side;
b = length of side;
c = hypotenuse

$$a^2 + b^2 = c^2$$



1.11 Rules of Logarithms

Where:
$b > 0$ but $b \neq 1$, and M, N, and k are real numbers but M and N must be positive

General: Common Logarithm

$$\log_{10} x \longrightarrow \log x$$

If the logarithm is using base-10, it is common to drop the number 10 because it is understood.

General: Natural Logarithm

$$\log_e x \longrightarrow \ln x$$

If the logarithm is using base-e, you can replace log base e with just ln.

General: Natural Logarithm evaluated at e^x

$$\ln(e^x) = x$$

If the ln is evaluated at e^x the two undo eachother and it becomes just x.

General: e raised to the Natural Logarithm of x

$$e^{\ln x} = x$$

Again, these two functions undo each other and it becomes x

Rule 1: Product Rule

$$\log_b(M \cdot N) = \log_b M + \log_b N$$

The logarithm of the product is the sum of the logarithms of the factors.

Rule 2: Quotient Rule

$$\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$$

The logarithm of the ratio of two quantities is the logarithm of the numerator minus the logarithm of the denominator.

Rule 3: Power Rule

$$\log_b(M^k) = k \cdot \log_b M$$

The logarithm of an exponential number is the exponent times the logarithm of the base.

Rule 4: Zero Rule

$$\log_b(1) = 0$$

The logarithm of 1 to any base is always equal to zero. As long as b is positive but $b \neq 1$.

Rule 5: Identity Rule

$$\log_b(b) = 1$$

The logarithm of the argument (inside the parenthesis) wherein the argument equals the base is equal to 1.

Rule 6: Inverse Property of Logarithm

$$\log_b(b^k) = k$$

The logarithm of an exponential number where its base is the same as the base of the log is equal to the exponent.

Rule 7: Inverse Property of Exponent

$$b^{\log_b(k)} = k$$

Raising the logarithm of a number to its base is equal to the number.

Rule 8: Change of Base Formula

$$\log_b x = \frac{\log_c x}{\log_c b}$$

Where the argument **x** becomes the argument of the numerator logarithm.
The base **b** becomes the argument of the denominator logarithm.
And the logs in the numerator and denominator have the same base **c**, the value we chose.

1.12 Special Factoring Formulas

Factoring Formula 1: $(a+b)^2 = a^2 + 2ab + b^2$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(a+b)^2 = (a+b)(a+b)$$

Multiply the binomials

$$= a^2 + ab + ab + b^2$$

$$= a^2 + 2ab + b^2$$

Factoring Formula 2: $(a-b)^2 = a^2 - 2ab + b^2$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(a-b)^2 = (a-b)(a-b)$$

Multiply the binomials

$$= a^2 - ab - ab + b^2$$

$$= a^2 - 2ab + b^2$$

Factoring Formula 3: $(a+b)(a-b) = a^2 - b^2$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

Multiply the binomials

$$(a+b)(a-b) = a^2 - ab + ba + b^2$$

$$= a^2 - b^2$$

Factoring Formula 4: $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(x+y)(x^2 - xy + y^2)$$
$$= x^3 - x^2y + xy^2 + x^2y - xy^2 + y^3$$
$$= x^3 + y^3$$

Factoring Formula 5: $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(x-y)(x^2 + xy + y^2)$$
$$= x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3$$
$$= x^3 - y^3$$

Factoring Formula 6: $(x+a)(x+b) = x^2 + (a+b)x + ab$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

Multiply the binomials

$$(x+a)(x+b) = x^2 + xb + ax + b^2$$

$$= x^2 + (a+b)x + ab$$

Factoring Formula 7: $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(a+b)^3 = (a+b)^2(a+b)$$
$$= (a^2 + 2ab + b^2)(a+b)$$
$$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$
$$= a^3 + b^3 + 3a^2b + 3ab^2$$

or

$$a^3 + b^3 + 3ab(a+b)$$

Factoring Formula 8: $(a-b)^3 = a^3 - b^3 - 3ab(a-b)$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(a-b)^3 = (a-b)^2(a-b)$$
$$= (a^2 - 2ab + b^2)(a-b)$$
$$= a^3 - 2a^2b + ab^2 - a^2b + 2ab^2 - b^3$$
$$= a^3 - b^3 - 3a^2b + 3ab^2$$

or

$$a^3 - b^3 - 3ab(a-b)$$

Factoring Formula 9: $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(a+b+c)^2 = (a+b+c)(a+b+c)$$
$$= a^2 + ab + ac + ba + b^2 + bc + ca + bc + c^2$$
$$= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$

Factoring Formula 10: $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - xz)$
Let us start with the left-hand side of this formula and reach the right-hand side at the end.

$$(x+y+z)(x^2 + y^2 + z^2 - xy - yz - xz)$$
$$= (x^3 + xy^2 + xz^2 - x^2y - xyz - x^2z) + (x^2y + y^3 + yz^2 - xy^2 - y^2z - xyz) + (x^2z + y^2z + z^3 - xyz - yz^2 - xz^2)$$

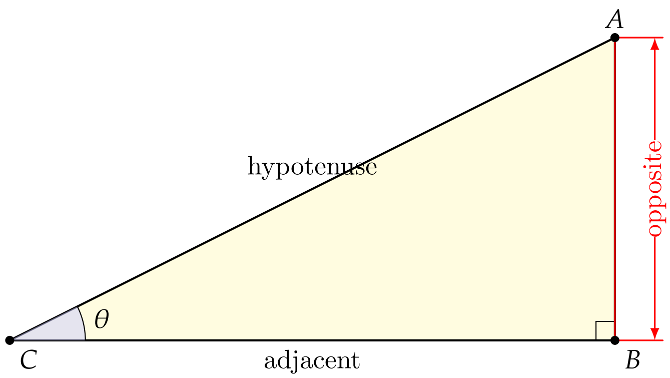
Cancel terms

$$= x^3 + y^3 + z^3 - 3xyz$$

Chapter 2

Trigonometry

2.1 Basics - SOH CAH TOA



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BA}{CA}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{CB}{CA}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{BA}{CB}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{CA}{BA}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{CA}{CB}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{CB}{BA}$$

2.2 General and Pythagorean Identities

General

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

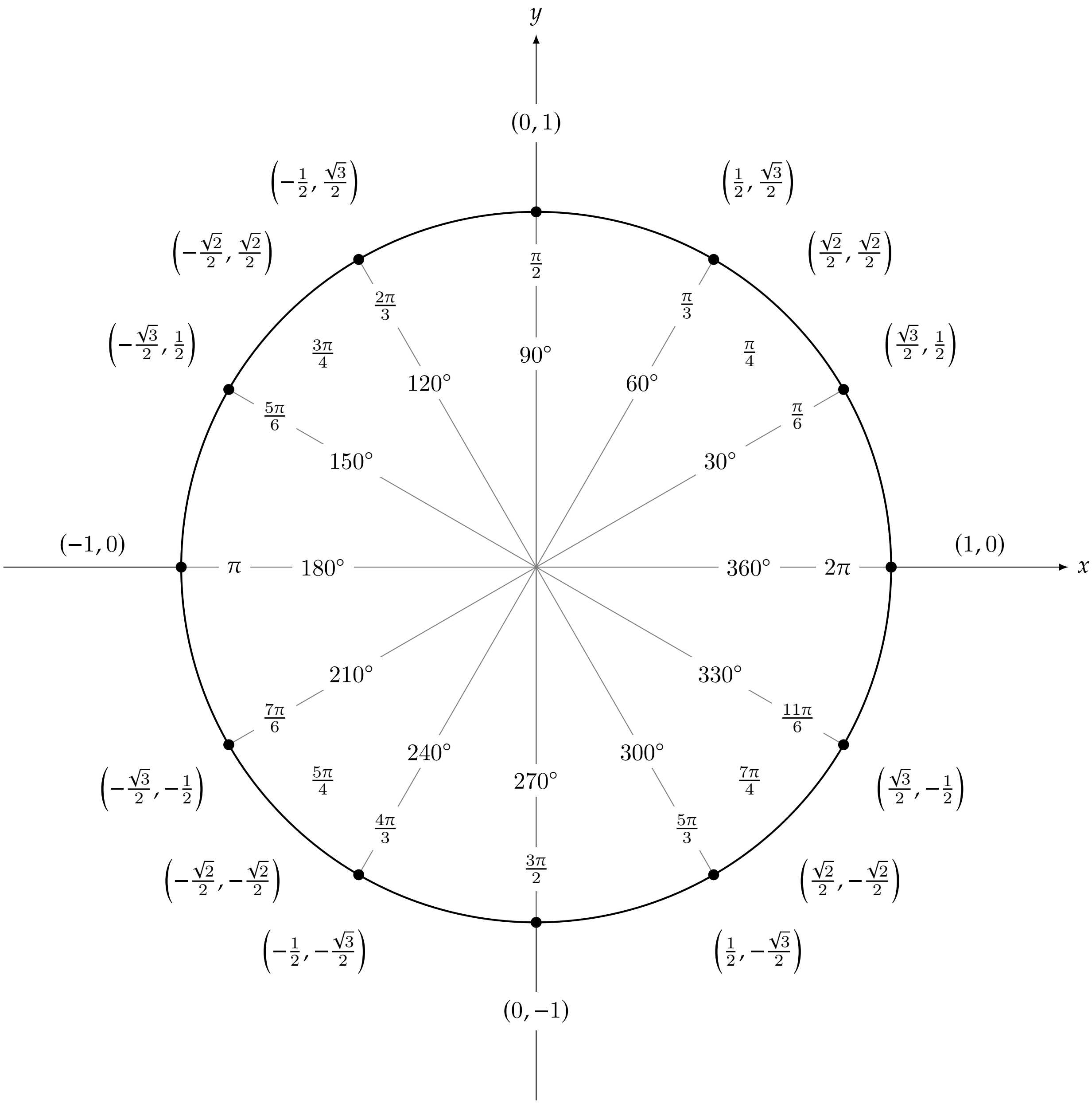
Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\csc^2\theta - \cot^2\theta = 1$$

2.3 Unit Circle



2.4 Half Angle Formulas

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$$

2.5 Trig Addition and Subtraction Formulas

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Angle addition formulas express trigonometric functions of sums of angles A $\pm$ B in terms of functions of A and B. The fundamental formulas of angle addition in trigonometry.

2.6 Double Angle Formulas

Recalling the addition formula described below,

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

We consider what happens if we let B equal to A. then the first of these formulas becomes

$$\sin(A+A) = \sin A\cos A + \cos A\sin A$$

so that

$$\sin 2A = 2\sin A\cos A$$

If we do the same for the second equation

$$\cos(A+A) = \cos A\cos A - \sin A\sin A$$

so that

$$\cos 2A = \cos^2 A - \sin^2 A$$

Similarly

$$\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A\tan A}$$

so that

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

2.7 Power Reducing Formulas

For $\sin^2\theta$:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = (1 - \sin^2\theta) - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\frac{2sin^2\theta}{2} = \frac{1 - \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

For $\cos^2\theta$:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta)$$

$$\cos 2\theta = \cos^2\theta - 1 + \cos^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\frac{\cos 2\theta + 1}{2} = \frac{2\cos^2\theta}{2}$$

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

For $\tan^2\theta$:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\tan^2\theta = \sec^2\theta - 1$$

Chapter 3

Calculus I

This chapter will embody most of calculus I topics, excluding basic integration which will be saved for the following chapter calculus II

3.1 Definition of the Limit

Limit
The limit of a function is the value the function approaches at a given value of x, regardless of whether the function actually reaches that value.

Example: Limit

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

This represents the limit of the function $(x^2 - 4)/(x - 2)$ as x approaches 2.

One-sided limits
The one-sided limits are the left- and right-hand limits. The left-hand limit is the limit of the function as we approach from the left side (or negative side), whereas the right-hand limit is the limit of the function as we approach from the right side (or positive side).

Example: One Sided Limit

$$\lim_{x \to 0^+} \frac{1}{x}$$

This represents the limit of the function $1/x$ as x approaches 0 from the right (or positive direction).

Another Example: One Sided Limit

$$\lim_{x \to 0^-} \frac{1}{x}$$

This represents the limit of the function $1/x$ as x approaches 0 from the left (or negative direction).

General Limit
The general limit exists at a point $x = c$ if

1. the left-hand limit exists at $x = c$,
2. the right-hand limit exists at $x = c$, and
3. those left and right-hand limits are equal to one another.

The general limit does not exist (DNE) at $x = c$ if

1. the left-hand limit does not exist at $x = c$, and/or
2. the right-hand limit does not exist at $x = c$, and/or
3. the left- and right-hand limits both exist, but aren't equal to one another.

3.2 Continuity

Continuity
If we can draw the graph of the function without ever lifting our pencil off the paper as we sketch it out from left to right, then the function is continuous everywhere. At any point where we have to lift our pencil off the paper in order to continue sketching it, there must be a discontinuity at that point.

Point (removable) discontinuity
A point discontinuity exists wherever there's a hole in the graph at one specific point. These are also called "removable discontinuities" because we can "remove" the discontinuity redefining the function at that particular point. Point discontinuities exist when a factor in rational functions when a factor that would have made the denominator 0 is cancelled from the function. The general limit always exists at a point discontinuity.

Jump discontinuity
A jump discontinuity exists wherever there's a big break in the graph that isn't caused by an asymptote. Jump discontinuities usually occur in piecewise-defined functions. The general limit never exists at a jump discontinuity.

Infinite (essential) discontinuity
An infinite discontinuity is the kind of discontinuity that occurs at an asymptote. Infinite discontinuities exist in rational functions in factors that make the denominator equal to 0 and can't be cancelled from the denominator.

Endpoint discontinuity
Endpoint discontinuities exist at a and b when a function is defined over a particular interval $[a, b]$. The general limit never exists at an endpoint discontinuity.

3.3 Solving Limits

Process for solving limits
Try direct substitution first, then factoring, then conjugate method.

Conjugate
The conjugate of an expression is an expression with the same two terms, but with the opposite sign between the terms.

Limits at infinity
The limit at infinity is the limit of the function as we approach ∞ or $-\infty$.

Infinite Limits
A limit is infinite when the value of the limit is ∞ or $-\infty$ as we approach a particular point.

Degree in a Rational Function
The degree of the numerator or denominator is the exponent on the term with the largest exponent.

- $N < D$: If the degree of the numerator is less than the degree of the denominator, then the horizontal asymptote is given by $y = 0$.
- $N > D$: If the degree of the numerator is greater than the degree of the denominator, then the function doesn't have a horizontal asymptote.
- $N = D$: If the degree of the numerator is equal to the degree of the denominator, then the horizontal asymptote is given by the ratio of the coefficients on the highest-degree terms.

3.4 Trignometric Limits

Limit problems with trigonometric functions usually revolve around three key limit values.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \cos x = 1$$

$$\lim_{x\to 0} \sin x = 0$$

3.5 Squeeze Theorem

The Squeeze Theorem allows us to find the limit of a function at a particular point, even when the function is undefined at that point. The way that we do it is by showing that our function can be "squeezed" between two other functions at the given point, and proving that the limits of these other functions are equal.

3.6 Definition of the Derivative

Secant Line, Average Rate of Change
A secant line is a line that runs right through the graph, crossing it at a point. The slope of the secant line is the average rate of change of the function over the points $(x, f(x))$ and $(x + h, f(x + h))$ at which the secant line intersects the function.

$$m = \frac{f(x+h) - f(x)}{h}$$

Tangent Line, Instantaneous Rate of Change, Difference Quotient
A tangent line is a line that just barely touches the edge of a graph, intersecting it at exactly one specific point. The line doesn't cross the graph, it skims along the graph and stays along the same side of the graph. The slope of the tangent line is the instantaneous rate of change of the function at the point at which the tangent line intersects the function.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

3.7 Derivative Power Rules

Power Rule
The power rule lets us take the derivative of power functions.

$$(a \cdot n)x^{n-1}$$

Derivative of a Constant
The derivative of a constant is 0
Given the a constant $y = 5$ for example:

$$y = 5$$

Since five is a constant it can be seen as:

$$y = 5x^0$$

So when you use the power rule on it:

$$y' = (5 \cdot 0)x^{0-1}$$

And of course anything times 0 is 0

$$y' = 0$$

Thus, the derivative of a constant is 0

Power Rule for Negative Powers
When you need to take the derivative of a negative power, you can use a simple rearrangement of terms, to make using the power rule easier.

$$x^{-a} = \frac{1}{x^a}$$

or

$$\frac{1}{x^{-a}} = x^a$$

Power Rule for Fractional Powers
If you need to take the derivative of the function involving a root, you can use a simple rearrangement of terms, to make using the power rule easier.

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

3.8 Product Rule

For $y = f(x)g(x)$, the derivative is

$$y' = f'(x)g(x) + g'(x)f(x)$$

For $y = f(x)g(x)h(x)$, the derivative is

$$y' = f'(x)g(x)h(x) + g'(x)f(x)h(x) + f(x)g(x)h'(x)$$

3.9 Quotient Rule

For $y = \frac{f(x)}{g(x)}$, the derivative is:

$$y' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

3.10 Reciprocal Rule

For $y = \frac{a}{g(x)}$, the derivative is:

$$y' = \frac{-ag'(x)}{[g(x)]^2}$$

3.11 Derivatives of the Six Trig Functions

Trigonometric Function	**Derivative**
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$

3.12 Derivative of Exponential Functions

First note that the constant $e \approx 2.718281828459045...$

Functions	**Derivative**	**Derivative with g(x) argument**
$y = e^x$	$y' = e^x$	$y' = e^{g(x)}g'(x)$
$y = a^x$	$y' = a^x(\ln a)$	$y' = a^{g(x)}(\ln a)g'(x)$

3.13 Derivative of Logarithmic Functions

First note that the $\ln a$ is the natural logarithm, evaluated at a

Functions	**Derivative**	**Derivative with g(x) argument**
$y = \log_a x$	$y' = \frac{1}{x \ln a}$	$y' = \frac{1}{g(x)\ln a}g'(x)$
$y = \ln x$	$y' = \frac{1}{x}$	$y' = \frac{1}{g(x)}g'(x)$

3.14 The Chain Rule

The chain rule is a fundamental concept in calculus that allows us to find the derivative of a composite function. It states that if $y = f(u)$ and $u = g(x)$, then the derivative of y with respect to x is given by:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{3.1}$$

This can be written more compactly as $dy/dx = dy/du \cdot du/dx$.

The chain rule is extremely useful in solving problems involving composite functions. For example, consider the function $y = (3x^2 + 4x)^5$. To find the derivative of this function, we can first find the derivative of the inner function $u = 3x^2 + 4x$:

$$\frac{du}{dx} = 6x + 4$$
$$\Rightarrow dy/du = (3x^2 + 4x)^4$$

Next, we can use the chain rule to find the derivative of the outer function y with respect to x:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= 5(3x^2 + 4x)^4 \cdot (6x + 4)\\ &= 5(3x^2 + 4x)^4(6x + 4)\end{aligned}$$

We can also use chain rule to find the derivative of multi variable function, for example $F(x, y) = x^2 * sin(y)$, if we want to find $\frac{\partial F}{\partial x}$

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x}\\ &= 2x * sin(y)\end{aligned}$$

In summary, the chain rule is a powerful tool that allows us to find the derivative of composite functions by breaking them down into simpler parts. It is essential for solving many types of problems in calculus.

3.15 Derivatives of Inverse Trig Functions

Function	**Derivative**	**With g(x) argument**
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arcsin g(x)) = \frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arccos g(x)) = -\frac{g'(x)}{\sqrt{1-g(x)^2}}$
$\arctan x$	$\frac{1}{1+x^2}$	$\frac{d}{dx}(\arctan g(x)) = \frac{g'(x)}{1+g(x)^2}$
$\text{arcsec}\, x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\text{arcsec}\, g(x)) = \frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}}$
$\text{arccsc}\, x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\frac{d}{dx}(\text{arccsc}\, g(x)) = -\frac{g'(x)}{\lvert g(x)\rvert\sqrt{g(x)^2-1}}$
$\text{arccot}\, x$	$-\frac{1}{1+x^2}$	$\frac{d}{dx}(\text{arccot}\, g(x)) = -\frac{g'(x)}{1+g(x)^2}$

3.16 Average Rate of Change

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

3.17 Implicit Differentiation

Implicit differentiation is a method of finding the derivative of an implicit function, which is a function that is defined implicitly rather than explicitly. In other words, the dependent variable is not isolated on one side of the equation.

Given an implicit function $F(x, y) = 0$, the derivative of y with respect to x can be found by taking the derivative of both sides of the equation with respect to x, and then solving for $\frac{dy}{dx}$. The process is summarized by the following formula:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Where $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are the partial derivatives of F with respect to x and y, respectively.

For example, consider the implicit function $x^2 + y^2 = 16$. To find $\frac{dy}{dx}$, we take the derivative of both sides with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(16)$$

$$2x + \frac{dy}{dx} \cdot 2y = 0$$

Solving for $\frac{dy}{dx}$ gives us:

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Another example is the implicit function $x^3 + y^3 = 3xy$,

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2 - 3y^2}{3y^2 - 3x^2}$$

3.18 Higher Order Derivatives

Higher Order Derivative	**Form**	**Notation**
First Derivative	y'	$f'(x)$ or $\frac{dy}{dx}$
Second Derivative	y''	$f''(x)$ or $\frac{d^2y}{dx^2}$
Third Derivative	y'''	$f'''(x)$ or $\frac{d^3y}{dx^3}$

Chapter 4

Calculus II

4.1 Derivatives and Integral Formulas Relevant to Calculus II

Differentiation Formula	**Corresponding Integration Formula**
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}a^x = (\ln a)a^x$	$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$	$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \arccos x + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \operatorname{arcsec}\lvert x\rvert + C$

4.2 Anti-Derivatives

Anti-derivatives, also known as indefinite integrals, are the reverse operation of finding the derivative of a function. Given a function $f(x)$, an anti-derivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. The symbol for anti-derivative is $\int$.

Anti-derivatives are used in many areas of mathematics, physics, and engineering to find the total amount of some quantity over an interval. For example, in physics, anti-derivatives can be used to find the distance traveled by an object given its velocity function. In economics, anti-derivatives can be used to find the total cost of producing a certain quantity of goods given the cost function.

The process of finding an anti-derivative of a function is called integration. There are several techniques for finding anti-derivatives, including integration by substitution, integration by parts, and using integration tables. It's important to note that for most functions, there is no closed-form solution for the antiderivative, in those cases numerical integration is used.

Here are some examples of finding anti-derivatives:

1. The anti-derivative of x^2 is $\frac{x^3}{3} + C$, where C is an arbitrary constant.

2. The anti-derivative of $2x$ is $x^2 + C$

3. The anti-derivative of e^x is $e^x + C$

4.3 Limit of Riemann Sums and Definite Integrals

Riemann Sum: A Riemann sum is a sum of the form:

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

A Riemann sum is a sum of areas of n rectangles with width Δx and a height $f(x_i)$. A Riemann sum can be used to approximate the area under the graph of a function $f(x)$.

Definite Integral: The definite integral $\int_a^b f(x)dx$ is the exact area under the graph of a function $f(x)$ on the interval $[a, b]$. The definite integral can be found by taking a limit of a Riemann sum as the number of rectangles used approaches infinity. That is,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Steps to Rewriting the Limit of a Riemann Sum as a Definite Integral:

1. Determine the value of Δx. Remember that $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.
2. Choose a lower bound $a = 0$ and determine the upper bound b using the fact that $\Delta x = \frac{b-a}{n}$
3. Determine the function $f(x)$ by replacing the $x_i = i\Delta x$ with an x.
4. Use the gathered information to write the definite integral $\int_a^b f(x)dx$.

Example: Rewrite the limit of the Riemann sum as a definite integral.

$$\sum_{i=1}^{n} \left(5 + \frac{3i}{n}\right)^5 \cdot \frac{3}{n}$$

1. Determine the value of Δx. Remember that $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.

Δx is written outside of the paraentheses in the Riemann Sum. We have:

$$\Delta x = \frac{3}{n}$$

2. Choose a lower bound $a = 0$ and determine the upper bound b using the fact that $\Delta x = \frac{b-a}{n}$

Using $a = 0$ and $\Delta x = \frac{b-a}{n}$, we have:

$$\Delta x = \frac{b-a}{n}$$

$$\frac{3}{n} = \frac{b-0}{n}$$

$$\frac{3}{n} = \frac{b}{n}$$

$$3 = b$$

So we have the bounds of integration $a = 0$ and $b = 3$.

3. Determine the function $f(x)$ by replacing the $x_i = i\Delta x$ with an x.

In the Riemann sum, we have $f(x_i) = \left(5 + \frac{3i}{n}\right)^5$.

Since $\Delta x = \frac{3}{n}$, we know that $x_i = i\Delta x = \frac{3i}{n}$. Therefore:

$$f(x_i) = \left(5 + \frac{3i}{n}\right)^5$$

$$f(x_i) = (5 + x_i)^5$$

$$f(x_i) = (5 + x)^5$$

4. item Use the gathered information to write the definite integral $\int_a^b f(x)dx$.

Using $a = 0, b = 3$ and $f(x) = (5 + x)^5$, we have:

$$\lim_{n\to\infty} \sum_{i=i}^{n} \left(5 + \frac{3i}{n}\right)^5 \cdot \frac{3}{n} = \int_0^3 (5 + x)^5 dx$$

4.4 U-Substitution with Examples

u-substitution is a method for evaluating definite and indefinite integrals. The basic idea is to make a substitution of the variable of integration in order to make the integral easier to evaluate.

3 Clues we need for u-sub

1. Product of 2 Functions
2. One factor is a composition
3. The other factor is the derivative of the inside factor

Here's an example of how to use u-substitution to evaluate an indefinite integral:

$$\int 2x\sqrt{x^2+5}\,dx$$

$$\begin{aligned}
u &= x^2+5 \\
du &= 2x\,dx \\
&= \int \sqrt{u}\,du \\
&= \int (u)^{\frac{1}{2}}\,du \\
&= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{2}{3}(x^2+5)^{\frac{3}{2}} + C
\end{aligned}$$

Another Example:

$$\int x^2\,e^{x^3}\,dx$$

$$\begin{aligned}
u &= x^3 \\
u' &= 3x^2 \\
&= \int 3\left(\frac{1}{3}\right)x^2\,e^{x^3}\,dx \\
&= \frac{1}{3}\int 3x^2\,e^{x^3}\,dx \\
du &= 3x^2\,dx \\
&= \frac{1}{3}\int e^u\,du \\
&= \frac{1}{3}e^u + C \\
&= \frac{1}{3}e^{x^3} + C
\end{aligned}$$

4.5 U-Substitution with Logs

When using logs, the substitution is typically made with the variable $u = \ln(x)$. This is because the derivative of $\ln(x)$ is $\frac{1}{x}$, which makes it easy to integrate when the function contains x in the denominator. By making this substitution, the integral can be rewritten in terms of u, making it easier to evaluate.

It's important to remember that when making a substitution with logs, you need to change the limits of integration as well, and you need to take the absolute value of u when evaluating the integral.

For example:

$$\int_{e^2}^{e^3} \frac{\ln x}{x}\, dx$$

$$\begin{aligned}
&= \int_{e^2}^{e^3} \ln x \cdot \frac{1}{x}\, dx \\
u &= \ln x \\
u' &= \frac{1}{x} \\
du &= \frac{1}{x}\, dx \\
&= \int_{u(e^2)}^{u(e^3)} u\, du
\end{aligned}$$

$$\begin{aligned}
u(e^3) &= ln(e^3) = 3 \\
u(e^2) &= ln(e^2) = 2
\end{aligned}$$

$$\begin{aligned}
\int_2^3 u\, du &= \left.\frac{u^2}{2}\right|_2^3 \\
&= \frac{(3)^2}{2} - \frac{(2)^2}{2} \\
&= \frac{9}{2} - \frac{4}{2} \\
&= \frac{5}{2}
\end{aligned}$$

4.6 U-Substitution with Inverse Trig

u-substitution can also be used to evaluate integrals involving inverse trigonometric functions. Here's an example:

$$\int \frac{1}{4+x^2}\,dx$$

$$= \int \frac{1}{4\left(1+\frac{x^2}{4}\right)}\,dx$$

$$= \frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$u = \frac{x}{2}$$

$$u' = \frac{1}{2}$$

$$= \frac{1}{4}(\cdot 2)\int \left(\frac{1}{2}\right)\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$du = \frac{1}{2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{1+u^2}\,du$$

$$= \frac{1}{2}\arctan(u) + C$$

$$= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$$

4.7 Area Between Curves

In Calculus 2, the area between two curves refers to the region enclosed by two functions and the x-axis on a coordinate plane. The area between the curves can be found by subtracting the area under one function (the "bottom" function) from the area under the other function (the "top" function).

One way to find the area between the curves is by using definite integrals. The definite integral of a function gives the signed area under the curve of that function. The definite integral of a function over an interval [a, b] gives the signed area between the curve of that function and the x-axis over that interval. To find the area between two curves, we can find the definite integral of each function over the same interval, and then subtract the definite integral of the "bottom" function from the definite integral of the "top" function.

For example, if we have two functions f(x) and g(x), and we want to find the area between the curves for the interval [a, b], we can use the following formula: Area = $\int_a^b$ [f(x) - g(x)] dx

It is important to keep in mind that the definite integral gives the signed area, so if the "top" function is below the "bottom" function over the interval, the area between the curves will be negative.

Another way to find the area between the curves is by using Riemann Sums, which is a method that approximates the area between the curves by dividing the interval into smaller subintervals, and computing the area of rectangles with heights at the points of the function at the right endpoint of each subinterval.

It is important to note that when finding the area between curves, it is important to check the function and make sure that the area we are trying to find is the area between the curves and not the area of one of the functions.

In some cases, the area between the curves might not be continuous, in those cases we will have to split the region into smaller regions and find the area for each of them separately.

$$\text{Area} = \int_{left}^{right} (\text{top} - \text{bottom})\, dx$$

$$\text{Area} = \int_{bottom}^{top} (\text{right} - \text{left})\, dy$$

4.8 Volume by Slicing

Disks

$$\text{x-axis} = \int_a^b \pi[f(x)]^2\, dx$$

$$\text{y-axis} = \int_c^d \pi[f(y)]^2\, dy$$

Washers

$$\begin{aligned} \text{x-axis} &= \int_a^b \pi[f(x)]^2 - \pi[g(x)]^2\, dx \\ \text{When } y = -k, &= \int_a^b \pi[k + f(x)]^2 - \pi[k + g(x)]^2\, dx \\ \text{When } y = k, &= \int_a^b \pi[k - f(x)]^2 - \pi[k - g(x)]^2\, dx \end{aligned}$$

$$\begin{aligned} \text{y-axis} &= \int_c^d \pi[f(y)]^2 - \pi[g(y)]^2\, dy \\ \text{When } x = -k, &= \int_c^d \pi[k + f(y)]^2 - \pi[k + g(y)]^2\, dy \\ \text{When } x = k, &= \int_c^d \pi[k - f(y)]^2 - \pi[k - g(y)]^2\, dy \end{aligned}$$

4.9 Volume of Revolution By Cylindrical Shells

$$\int \text{circumference height width}$$

Shells

$$\text{x-axis} = \int_c^d 2\pi y \, [f(y) - g(y)] \, dy$$

$$\text{When } y = -k, \; = \int_c^d 2\pi(y + k) \, [f(y) - g(y)] \, dy$$

$$\text{When } y = k, \; = \int_c^d 2\pi(k - y) \, [f(y) - g(y)] \, dy$$

$$\text{y-axis} = \int_a^b 2\pi x \, [f(x) - g(x)] \, dx$$

$$\text{When } x = -k, \; = \int_a^b 2\pi(x + k) \, [f(x) - g(x)] \, dx$$

$$\text{When } x = k, \; = \int_a^b 2\pi(k - x) \, [f(x) - g(x)] \, dx$$