Important Math Concepts

College Level Calculus II - Integration

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Contents

Chapter 1

Calculus II

1.1 Derivatives and Integral Formulas Relevant to Calculus II

Differentiation Formula	**Corresponding Integration Formula**
$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$
$\frac{d}{dx}\ln x = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}a^x = (\ln a)a^x$	$\int a^x\,dx = \left(\frac{1}{\ln a}\right)a^x + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$
$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$	
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}\operatorname{arcsec} x = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \operatorname{arcsec}\lvert x\rvert + C$

1.2 Anti-Derivatives

Anti-derivatives, also known as indefinite integrals, are the reverse operation of finding the derivative of a function. Given a function $f(x)$, an anti-derivative of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. The symbol for anti-derivative is $\int$.

Anti-derivatives are used in many areas of mathematics, physics, and engineering to find the total amount of some quantity over an interval. For example, in physics, anti-derivatives can be used to find the distance traveled by an object given its velocity function. In economics, anti-derivatives can be used to find the total cost of producing a certain quantity of goods given the cost function.

The process of finding an anti-derivative of a function is called integration. There are several techniques for finding anti-derivatives, including integration by substitution, integration by parts, and using integration tables. It's important to note that for most functions, there is no closed-form solution for the antiderivative, in those cases numerical integration is used.

Here are some examples of finding anti-derivatives:

1. The anti-derivative of x^2 is $\frac{x^3}{3} + C$, where C is an arbitrary constant.

2. The anti-derivative of $2x$ is $x^2 + C$

3. The anti-derivative of e^x is $e^x + C$

1.3 Limit of Riemann Sums and Definite Integrals

Riemann Sum: A Riemann sum is a sum of the form:

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

A Riemann sum is a sum of areas of n rectangles with width Δx and a height $f(x_i)$. A Riemann sum can be used to approximate the area under the graph of a function $f(x)$.

Definite Integral: The definite integral $\int_a^b f(x)dx$ is the exact area under the graph of a function $f(x)$ on the interval $[a, b]$. The definite integral can be found by taking a limit of a Riemann sum as the number of rectangles used approaches infinity. That is,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Steps to Rewriting the Limit of a Riemann Sum as a Definite Integral:

1. Determine the value of Δx. Remember that $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.
2. Choose a lower bound $a = 0$ and determine the upper bound b using the fact that $\Delta x = \frac{b-a}{n}$
3. Determine the function $f(x)$ by replacing the $x_i = i\Delta x$ with an x.
4. Use the gathered information to write the definite integral $\int_a^b f(x)dx$.

Example: Rewrite the limit of the Riemann sum as a definite integral.

$$\sum_{i=1}^{n} \left(5 + \frac{3i}{n}\right)^5 \cdot \frac{3}{n}$$

1. Determine the value of Δx. Remember that $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$.

Δx is written outside of the paraentheses in the Riemann Sum. We have:

$$\Delta x = \frac{3}{n}$$

2. Choose a lower bound $a = 0$ and determine the upper bound b using the fact that $\Delta x = \frac{b-a}{n}$

Using $a = 0$ and $\Delta x = \frac{b-a}{n}$, we have:

$$\Delta x = \frac{b-a}{n}$$

$$\frac{3}{n} = \frac{b-0}{n}$$

$$\frac{3}{n} = \frac{b}{n}$$

$$3 = b$$

So we have the bounds of integration $a = 0$ and $b = 3$.

3. Determine the function $f(x)$ by replacing the $x_i = i\Delta x$ with an x.

In the Riemann sum, we have $f(x_i) = \left(5 + \frac{3i}{n}\right)^5$.

Since $\Delta x = \frac{3}{n}$, we know that $x_i = i\Delta x = \frac{3i}{n}$. Therefore:

$$f(x_i) = \left(5 + \frac{3i}{n}\right)^5$$

$$f(x_i) = (5 + x_i)^5$$

$$f(x_i) = (5 + x)^5$$

4. item Use the gathered information to write the definite integral $\int_a^b f(x)dx$.

Using $a = 0, b = 3$ and $f(x) = (5 + x)^5$, we have:

$$\lim_{n\to\infty} \sum_{i=i}^{n} \left(5 + \frac{3i}{n}\right)^5 \cdot \frac{3}{n} = \int_0^3 (5 + x)^5 dx$$

1.4 U-Substitution with Examples

u-substitution is a method for evaluating definite and indefinite integrals. The basic idea is to make a substitution of the variable of integration in order to make the integral easier to evaluate.

3 Clues we need for u-sub

1. Product of 2 Functions
2. One factor is a composition
3. The other factor is the derivative of the inside factor

Here's an example of how to use u-substitution to evaluate an indefinite integral:

$$\int 2x\sqrt{x^2+5}\,dx$$

$$\begin{aligned}
u &= x^2+5 \\
du &= 2x\,dx \\
&= \int \sqrt{u}\,du \\
&= \int (u)^{\frac{1}{2}}\,du \\
&= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C \\
&= \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{2}{3}(x^2+5)^{\frac{3}{2}} + C
\end{aligned}$$

Another Example:

$$\int x^2 e^{x^3}\,dx$$

$$\begin{aligned}
u &= x^3 \\
u' &= 3x^2 \\
&= \int 3\left(\frac{1}{3}\right) x^2 e^{x^3}\,dx \\
&= \frac{1}{3}\int 3x^2 e^{x^3}\,dx \\
du &= 3x^2\,dx \\
&= \frac{1}{3}\int e^u\,du \\
&= \frac{1}{3}e^u + C \\
&= \frac{1}{3}e^{x^3} + C
\end{aligned}$$

Another Example:

$$\int \frac{y^5}{(1-y^3)^{\frac{3}{2}}}\,dy$$

$$= \int y^5 \cdot \frac{1}{(1-y^3)^{\frac{3}{2}}}\,dy$$

$$\begin{aligned} u &= 1-y^3 \quad \text{so,} \quad y^3 = 1-u \\ du &= -3y^2 dy \\ -\frac{1}{3}du &= y^2 dy \end{aligned}$$

$$\begin{aligned}
&= \int \left(y^2 \cdot y^3\right) \cdot \frac{1}{(1-y^3)^{\frac{3}{2}}}\,dy \\
&= -\frac{1}{3}\int (1-u) \cdot \frac{1}{u^{\frac{3}{2}}}\,du \\
&= -\frac{1}{3}\int (1-u) \cdot \left(u^{-\frac{3}{2}}\right)\,du \\
&= -\frac{1}{3}\int u^{-\frac{3}{2}} - u^{-\frac{1}{2}}\,du \\
&= -\frac{1}{3}\left[\frac{u^{-\frac{1}{2}}}{-\frac{1}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right] + C \\
&= -\frac{1}{3}\left[-2u^{-\frac{1}{2}} - 2u^{\frac{1}{2}}\right] + C \\
&= \frac{2u^{-\frac{1}{2}}}{3} + \frac{2u^{\frac{1}{2}}}{3} + C \\
&= \frac{2}{3u^{\frac{1}{2}}} + \frac{2u^{\frac{1}{2}}}{3} + C \\
&= \frac{2}{3\sqrt{u}} + \frac{2\sqrt{u}}{3} + C \\
&= \frac{2}{3\sqrt{1-y^3}} + \frac{2\sqrt{1-y^3}}{3} + C
\end{aligned}$$

1.5 U-Substitution with Logs

When using logs, the substitution is typically made with the variable $u = \ln(x)$. This is because the derivative of $\ln(x)$ is $\frac{1}{x}$, which makes it easy to integrate when the function contains x in the denominator. By making this substitution, the integral can be rewritten in terms of u, making it easier to evaluate.

It's important to remember that when making a substitution with logs, you need to change the limits of integration as well, and you need to take the absolute value of u when evaluating the integral.

For example:

$$\int_{e^2}^{e^3} \frac{\ln x}{x}\, dx$$

$$\begin{aligned}
&= \int_{e^2}^{e^3} \ln x \cdot \frac{1}{x}\, dx \\
u &= \ln x \\
u' &= \frac{1}{x} \\
du &= \frac{1}{x}\, dx \\
&= \int_{u(e^2)}^{u(e^3)} u\, du
\end{aligned}$$

$$\begin{aligned}
u(e^3) &= ln(e^3) = 3 \\
u(e^2) &= ln(e^2) = 2
\end{aligned}$$

$$\begin{aligned}
\int_2^3 u\, du &= \left.\frac{u^2}{2}\right|_2^3 \\
&= \frac{(3)^2}{2} - \frac{(2)^2}{2} \\
&= \frac{9}{2} - \frac{4}{2} \\
&= \frac{5}{2}
\end{aligned}$$

Another example:

Here $\frac{1}{x\cos x}$ is a function composition because of the reciprocal function. Meaning we can try this for u. That then means we need to use the product rule to find du.

$$\begin{aligned}
\int \frac{\cos x - x\sin x}{x\cos x}\,dx \\
&= \int \frac{1}{x\cos x}\cdot(\cos x - x\sin x)\,dx
\end{aligned}$$

$$\begin{aligned}
u &= x\cos x \\
u' &= 1\cdot\cos x + (-\sin x)\cdot x \\
du &= (\cos x - x\sin x)\,dx
\end{aligned}$$

$$\begin{aligned}
&= \int \frac{1}{u}\,du \\
&= \ln|u| + C \\
&= \ln|x\cos x| + C
\end{aligned}$$

1.6 U-Substitution with Inverse Trig

u-substitution can also be used to evaluate integrals involving inverse trigonometric functions. Here's an example:

$$\begin{aligned}
&\int \frac{1}{4+x^2}\,dx \\
&= \int \frac{1}{4\left(1+\frac{x^2}{4}\right)}\,dx \\
&= \frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx
\end{aligned}$$

$$\begin{aligned}
u &= \frac{x}{2} \\
u' &= \frac{1}{2}
\end{aligned}$$

$$= \frac{1}{4}(\cdot 2)\int \left(\frac{1}{2}\right)\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$\begin{aligned}
du &= \frac{1}{2}\,dx \\
&= \frac{1}{2}\int \frac{1}{1+u^2}\,du \\
&= \frac{1}{2}\arctan(u) + C \\
&= \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C
\end{aligned}$$

1.7 Area Between Curves

In Calculus 2, the area between two curves refers to the region enclosed by two functions and the x-axis on a coordinate plane. The area between the curves can be found by subtracting the area under one function (the "bottom" function) from the area under the other function (the "top" function).

One way to find the area between the curves is by using definite integrals. The definite integral of a function gives the signed area under the curve of that function. The definite integral of a function over an interval [a, b] gives the signed area between the curve of that function and the x-axis over that interval. To find the area between two curves, we can find the definite integral of each function over the same interval, and then subtract the definite integral of the "bottom" function from the definite integral of the "top" function.

For example, if we have two functions f(x) and g(x), and we want to find the area between the curves for the interval [a, b], we can use the following formula: Area = $\int_a^b$ [f(x) - g(x)] dx

It is important to keep in mind that the definite integral gives the signed area, so if the "top" function is below the "bottom" function over the interval, the area between the curves will be negative.

Another way to find the area between the curves is by using Riemann Sums, which is a method that approximates the area between the curves by dividing the interval into smaller subintervals, and computing the area of rectangles with heights at the points of the function at the right endpoint of each subinterval.

It is important to note that when finding the area between curves, it is important to check the function and make sure that the area we are trying to find is the area between the curves and not the area of one of the functions.

In some cases, the area between the curves might not be continuous, in those cases we will have to split the region into smaller regions and find the area for each of them separately.

$$\text{Area} = \int_{left}^{right} (\text{top} - \text{bottom})\, dx$$

$$\text{Area} = \int_{bottom}^{top} (\text{right} - \text{left})\, dy$$

1.8 Volume by Slicing

Disks

$$\text{x-axis} = \int_a^b \pi[f(x)]^2\,dx$$

$$\text{y-axis} = \int_c^d \pi[f(y)]^2\,dy$$

Washers/Donuts

$$\text{x-axis} = \int_a^b \pi[f(x)]^2 - \pi[g(x)]^2\,dx$$

$$\text{y-axis} = \int_c^d \pi[f(y)]^2 - \pi[g(y)]^2\,dy$$

1.9 Volume of Revolution By Cylindrical Shells

$$\int \text{circumference height/width thickness}$$

Shells

$$\text{x-axis} = \int_c^d 2\pi y \left[f(y) - g(y) \right] dy$$

$$\text{y-axis} = \int_a^b 2\pi x \left[f(x) - g(x) \right] dx$$

1.10 Arc Length of a Curve

$$\text{Arc Length } = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$\text{Arc Length } = \int_c^d \sqrt{1 + [f'(y)]^2}\, dy$$

$$\text{Riemann sum version} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\, \Delta x$$

Where, $x \in [a, b]$. and $y \in [c, d]$.

1.11 Work

The most basic form of work that can be described by math is: "A constant force applied in the direction in which an object moves."

$$\text{Work} = (\text{Force})(\text{Distance})$$

For example, let's say I apply a force of 10 lbs to move an object 3 ft. In this case:

$$\text{Work} = (10\ lbs)(3\ ft) = 30\ ft - lbs$$

Here we see that Units = (Units of Force)(Units of Distance) which is why we end up with $ft - lbs$. Other common units are $N - m$ which come from Newton-meters

The general rule of thumb when dealing with work problems involving pumps is:

- We will call where the work begins: ϕ
- We will call where the work ends: Θ
- And in most cases weight denisty is a constant, but we'll call it ρ
- And for the volume, in most cases it involve a varying function; for the sake of example we will call the cross sections circular

$$\begin{aligned} \text{Work} = \int_{\phi}^{\Theta} (\text{force}) \cdot (\text{distance}) \\ &= \int_{\phi}^{\Theta} (\text{weight}) \cdot (\text{x}) \\ &= \int_{\phi}^{\Theta} (\rho) \cdot (\text{volume}) \cdot (\text{x}) \\ &= \int_{\phi}^{\Theta} (\rho) \cdot [\pi(r(x))^2\, dx] \cdot (\text{x}) \end{aligned}$$

Along with pump problems, are spring problems. This is typically where we use **Hooke's Law** which states: <u>The force required to hold a spring x units from its rest position is **proportional** to the distance x.</u>

In this context, proportional means "related by a constant."

- k is the spring constant

$$F(x) = kx$$

Example of hooke's law:

Problem: It takes $5N$ of force to hold a spring $3m$ from rest.
a.) Find the spring constant k.
b.) Find the work required to move the spring and additional $2m$ away from its rest position.

$$F(x) = kx$$

$$\begin{aligned} 5 &= k \cdot 3 \\ k &= \frac{5}{3} \\ F(x) &= \frac{5}{3} \cdot x \end{aligned}$$

$$\begin{aligned} \text{Work} &= \int_3^5 \frac{5}{3} \cdot x \, dx \\ &= \frac{5}{3} \int_3^5 x dx \\ &= \ldots \\ &= \frac{40}{3} N \cdot m \end{aligned}$$

1.12 Integration by Parts

$$= uv - \int u'v \, dx$$

Logs
Inverse Trig
Algebraic
Trig
Exponential

LIATE

Example:

$$\int \arctan x \, dx$$

$$= \int \arctan x \, dx$$

$$\begin{aligned} u &= \arctan x \qquad v' = 1 \\ u' &= \frac{1}{1+x^2} \qquad v = x \end{aligned}$$

$$\begin{aligned} &= uv - \int u'v \, dx \\ &= x \cdot \arctan x - \int \frac{1}{1+x^2} \cdot x \, dx \end{aligned}$$

$$\begin{aligned} w &= 1 + x^2 \\ dw &= 2x dx \end{aligned}$$

$$\begin{aligned} &= x \cdot \arctan x - {\color{red}\frac{1}{2}} \int \frac{1}{1+x^2} \cdot {\color{red}2}x \, dx \\ &= x \cdot \arctan x - \frac{1}{2} \int \frac{1}{w} \, dw \\ &= x \cdot \arctan x - \frac{1}{2} [\ln |w|] + C \\ &= x \cdot \arctan x - \frac{1}{2} \left[\ln |1 + x^2|\right] + C \\ &= x \cdot \arctan x - \frac{1}{2} \left[\ln(1 + x^2)\right] + C \end{aligned}$$

Chapter 2

Gabriel's Horn

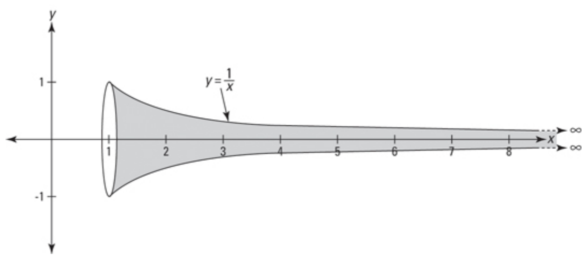

Figure 2.1: Gabriel's Horn

Finding the volume of Gabriel's Horn using the formula to find the volume of a solid of revolution:

$$\begin{aligned} V &= \pi \int_1^\infty \frac{1}{x^2}\,dx \\ &= \pi \int_1^\infty x^{-2}\,dx \\ &= \pi \left[-\frac{1}{x}\right]_1^\infty \\ &= \pi\,(0+1) \\ &= \pi \end{aligned}$$

Therefore, the volume of Gabriel's Horn is π.

Finding the surface area of Gabriel's Horn using the formula to find the surface area of a solid of revolution:

$$\begin{aligned} SA &= \int_1^\infty 2\pi \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx \\ &= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx \\ &= 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi \int_1^\infty \frac{1}{x}\,dx \\ &= [2\pi \ln|x|]_1^\infty \\ &= 2\pi(\infty - 0) \\ SA &= \infty \end{aligned}$$

Since the integral does not converge, the surface area of Gabriel's Horn is infinite.