45 min, Friday July 21st

Topics on the Midterm:

- graphing (shifts, features on graphs)
- limits (elementary and L'Hopital),
- limit definition of the derivate
- derivatives (polynomials, trigonometry, product rule, chain rule, quotient rule),
- linear approximation,
- implicit differentiation,
- related rates,
- extreme value problems,
- optimization problems.

Summary Questions: (✱ questions are less likely to be tested)
(✩ questions will only appear in bonuses)

1) Suppose $f(x)$ is graphed. How are the following functions modified

a) $f(x+a)$, $f(x)+a$, $f(ax)$, $af(x)$, $f(-x)$, $-f(x)$

✩ b) $f^{-1}(x)$

2) Consider the following terms

- removable discontinuity (hole), jump discontinuity, vertical asymptote
- essential discontinuity, horizontal asymptote, corner, cusp
- local maximum, local minimum, vertical tangent
- stationary point (horizontal tangent), stationary inflection point

a) Sketch a picture of each situation

✱ b) Define each term via limits and derivatives
(you may need the one sided versions)

3) Find the following derivatives

a) $0, 1, x, x^2, x^3, x^4$

b) $1/x$, $\sqrt{x}$, $1/x^2$, $1/x^3$

* c) $x^{2/3}$, $x^{-1/2}$, $x^{8/5}$

d) e^x, $\ln(x)$

* e) 2^x, 3^x

f) $\sin(x)$, $\cos(x)$, $\tan(x)$

* g) $\cot(x)$, $\sec(x)$, $\csc(x)$

☆ h) $\sin^{-1}(x)$, $\cos^{-1}(x)$, $\tan^{-1}(x)$

4) State the sum rule, product rule, quotient rule, chain rule
☆ the inverse function rule

5) a) Define the derivative as a limit with $h \to 0$

* b) Define the derivative as a limit with $x \to a$

c) State the linear approximation formula

d) State L'Hopital's rule

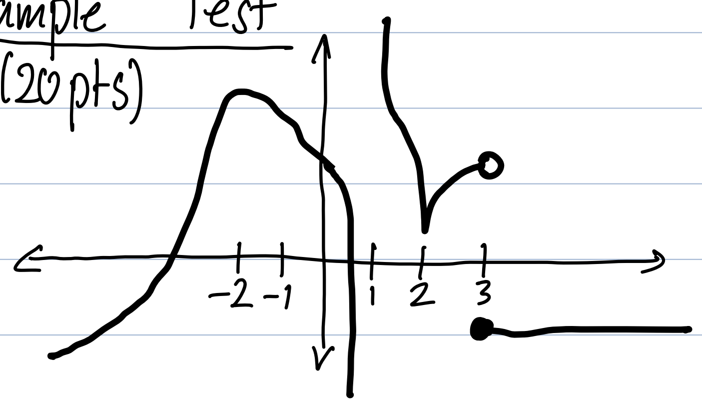
☆ e) Extend L'Hopital's rule for $\frac{\infty}{\infty}$ or $x \to \infty$

f) State the 1^{st} Derivative test.

☆ g) State the 2^{nd} Derivative test.

Sample Test

1) (20 pts)



Label the features
a) jump discontinuity
b) cusp
c) vertical asymptote
d) local maximum

for the following x values

$x = -2$ local max

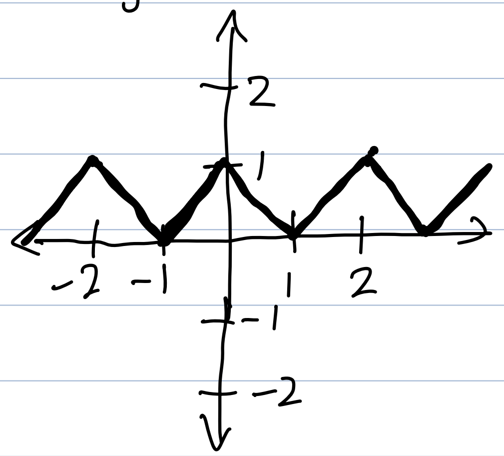
$x = 1$ VA

$x = 2$ cusp

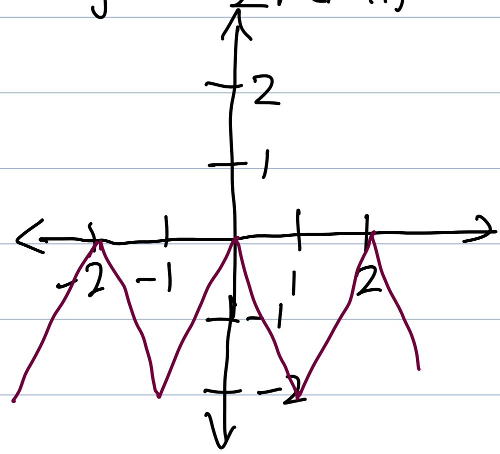
$x = 3$ jump

2) (20 pts) Consider the graph $y=f(x)$
Sketch the graph $y = 2f(x+1)$

$y=f(x)$



$y = -2f(x+1)$



3) (20pts) Fill in the missing parts of this proof that $f'(a) = 4a$
where $f(x) = 2x^2$

$$f'(a) = \lim_{x \to a} \frac{2x^2 - 2a^2}{x-a}$$

$$= \lim_{x \to a} \frac{2(x-a)(x+a)}{x-a}$$

because $x \neq a$ in the limit we may cancel

$$= \lim_{x \to a} 2(x+a)$$

because $x+a$ is a continuous function we may evaluate the limit directly

$$= 4a$$

4) (10 pts) Find $\frac{d}{dt} \tan(t+4) = \sec^2(t+4)$

5) (10pts) Find $\frac{d}{dx} \ln(x) e^x = \frac{1}{x} e^x + \ln(x) e^x$

6) (10pts) Find $\frac{d}{du} \frac{1}{u^2+1} = -\frac{2u}{(u^2+1)^2}$

7) (30 pts) Recall the following formula
Volume of a sphere: $V = \frac{4}{3}\pi r^3$.
Suppose a spherical ballon is being filled. The radius is increasing at a rate 4 mm/s (millimeters per second).
Suppose the ballon currently has radius 50 mm.
How fast is the volume of the ballon increasing?

8) (30 pts) Fill in the following proof that $\lim_{x\to\infty} \frac{2^x+x}{3^x+x} = 0$

Apply L'Hopitals Rule ($\frac{\infty}{\infty}$ case)

$$\lim_{x\to\infty} \frac{2^x+x}{3^x+x} = \lim_{x\to\infty} \frac{(2^x+x)'}{(3^x+x)'}$$

$$= \lim_{x\to\infty} \frac{\ln(2)\,2^x+1}{\ln(3)\,3^x+1}$$

Apply L'Hopitals Rule ($\frac{\infty}{\infty}$ case)

$$=$$

$$=$$

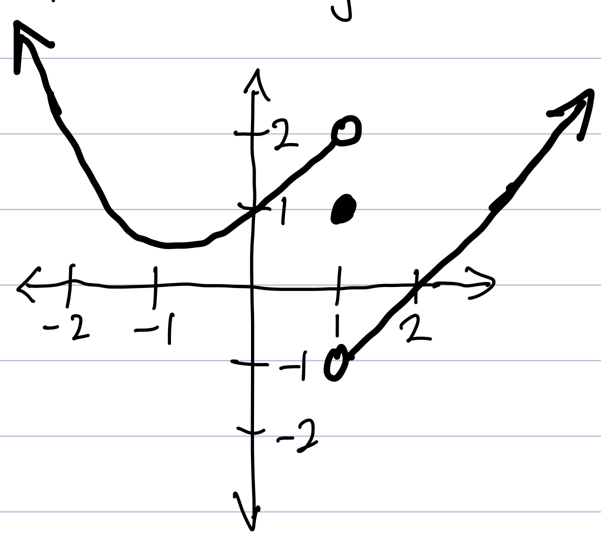
$$= \lim_{x\to\infty} \frac{\quad}{\quad} \left(\frac{2}{3}\right)^x$$

$$= \frac{\quad}{\quad} \lim_{x\to\infty} \left(\frac{2}{3}\right)^x$$

because $\frac{2}{3} < 1$ as x goes to infinity $\left(\frac{2}{3}\right)^x$ goes to 0.

$$= 0$$

9) (20 pts) Identify the following values on the given graph $y=f(x)$



$\lim_{x\to 1^-} f(x) =$ ______

$f(1) =$ ______

$\lim_{x\to\infty} f(x) =$ ______

$f'(-1) =$ ______

10) (40 pts) Suppose it costs

$$C(x) = 1 + \frac{9}{x+1} + x$$

thousand dollars for Pear Inc. to produce a laptop where x is number of millions of laptops that Pear Inc produces. Find the number of laptops Pear Inc should produce to minimize the cost of a laptop.

____________ million laptops

11 Bonus) (20 pts) Find $\frac{dy}{dx}$ for the curve $x^3 + y^3 = 1$.
Your answer may depend on x and y.

12 Bonus) (10 pts) Find $\csc\left(x^2 + \frac{1}{x^2}\right)'$