u-Substitution

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$
$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Think $F'(x) = f(x)$ Then $F'(u(x))\,u'(x) = F(u(x))'$
by the chain rule so $F(u(x)) = \int f(u(x))\,u'(x)\,dx$

Examples

$$\begin{aligned}
&\int \sin(x^2)\,2x\,dx \\
&= \int \sin(u)\,u'\,dx \qquad \text{where } u = x^2 \\
&= \int \sin(u)\,du \\
&= -\cos(u) + C \\
&= -\cos(x^2) + C
\end{aligned}$$

$$\int_1^e \frac{\sqrt{\ln(x)}}{x}\,dx$$

Set $u = \ln(x)$ then $u' = \frac{1}{x}$ or $du = u'\,dx = \frac{dx}{x}$
So $\frac{\sqrt{\ln(x)}}{x}\,dx = \sqrt{u}\,du$.
Thus

$$\begin{aligned}
\int_1^e \frac{\sqrt{\ln(x)}}{x}\,dx &= \int_{\ln(1)}^{\ln(e)} \sqrt{u}\,du \\
&= \int_0^1 u^{1/2}\,du \\
&= \frac{1}{1+1/2}\,u^{1+1/2}\Big|_0^1 \\
&= \frac{1}{3/2}\,u^{3/2}\Big|_0^1 \\
&= \frac{2}{3}u^{3/2}\Big|_0^1 \\
&= \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} \\
&= \frac{2}{3}
\end{aligned}$$

Notation $du = u'\,dx = \frac{du}{dx}\,dx$

1–6 Evaluate the integral by making the given substitution.

1. $\int e^{-x}\,dx, \quad u = -x$

2. $\int x^3(2 + x^4)^5\,dx, \quad u = 2 + x^4$

3. $\int x^2\sqrt{x^3 + 1}\,dx, \quad u = x^3 + 1$

4. $\int \frac{dt}{(1 - 6t)^4}, \quad u = 1 - 6t$

5. $\int \cos^3\theta \sin\theta\,d\theta, \quad u = \cos\theta$

6. $\int \frac{\sec^2(1/x)}{x^2}\,dx, \quad u = 1/x$

7–36 Evaluate the indefinite integral.

7. $\int x \sin(x^2)\,dx$

8. $\int x^2(x^3 + 5)^9\,dx$

9. $\int (3x - 2)^{20}\,dx$

10. $\int (3t + 2)^{2.4}\,dt$

11. $\int \sin \pi t\,dt$

12. $\int e^x \cos(e^x)\,dx$

13. $\int \frac{(\ln x)^2}{x}\,dx$

14. $\int \frac{x}{(x^2 + 1)^2}\,dx$

15. $\int \frac{dx}{5 - 3x}$

16. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

17. $\int \frac{a + bx^2}{\sqrt{3ax + bx^3}}\,dx$

18. $\int \frac{z^2}{z^3 + 1}\,dz$

19. $\int e^x\sqrt{1 + e^x}\,dx$

20. $\int \sec 2\theta \tan 2\theta\,d\theta$

21. $\int \frac{\cos x}{\sin^2 x}\,dx$

22. $\int \frac{\tan^{-1}x}{1 + x^2}\,dx$

23. $\int (x^2 + 1)(x^3 + 3x)^4\,dx$

24. $\int \frac{\sin(\ln x)}{x}\,dx$

25. $\int \sqrt{\cot x}\,\csc^2 x\,dx$

26. $\int \frac{\cos(\pi/x)}{x^2}\,dx$

27. $\int \frac{dx}{\sqrt{1 - x^2}\,\sin^{-1}x}$

28. $\int \frac{dt}{\cos^2 t\sqrt{1 + \tan t}}$

29. $\int \sec^3 x \tan x\,dx$

30. $\int x^2\sqrt{2 + x}\,dx$

31. $\int x(2x + 5)^8\,dx$

32. $\int \frac{e^x}{e^x + 1}\,dx$

33. $\int \frac{\sin 2x}{1 + \cos^2 x}\,dx$

34. $\int \frac{\sin x}{1 + \cos^2 x}\,dx$

35. $\int \frac{1 + x}{1 + x^2}\,dx$

36. $\int \frac{x}{1 + x^4}\,dx$

Challenge Problems

Below are some harder problems that require a little more thinking/algebraic manipulation to make the substitutions work.

1. $\displaystyle\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$

2. $\displaystyle\int \frac{1}{2x^2 - 12x + 26}\,dx$

3. $\displaystyle\int \frac{x}{1+x^4}\,dx$

4. $\displaystyle\int (x+3)\sqrt{x-1}\,dx$

5. $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\,dx$

6. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

7. $\displaystyle\int \frac{1}{\sqrt{21 - 4x - x^2}}\,dx$

8. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\sin(x)}{1+x^6}\,dx$

9. $\displaystyle\int \frac{3x-1}{x^2+10x+28}\,dx$

10. $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

11. $\displaystyle\int_{-1}^1 \frac{\sin(x)}{1+x^2}\,dx.$

12. $\displaystyle\int \frac{1}{e^x+1}\,dx$