Mclauren Series

$$f(x) \approx \sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!} = f(0)\frac{x^0}{0!} + f'(0)\frac{x^1}{1!} + f''(0)\frac{x^2}{2!} + \cdots$$

$$= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \cdots$$

"the sum from zero to infinity of the k^{th} derivative of f evaluated at 0 times x to the k^{th} power divided by k factorial"

Examples (A) $f(x) = e^x$

Step 1 Find $f^{(k)}(x)$

$f'(x) = (e^x)' = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$\vdots$

$f^{(k)}(x) = e^x$

Step 2 Find $f^{(k)}(0)$

$f^{(k)}(x) = e^0 = 1$

Step 3 Plug into the formula

$$\sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!} = \sum_{k=0}^{\infty} 1\cdot\frac{x^k}{k!} = \boxed{\sum_{k=0}^{\infty}\frac{x^k}{k!}}$$

(B) $f(x) = e^{-x}$

Step 1 Find $f^{(k)}(x)$

$f'(x) = e^{-x}(-x)' = e^{-x}(-1) = -e^{x}$

chain rule (on $g(x) = -x$ and $e^g = e^{-x}$)

$f''(x) = (-e^{-x})' = (-1)(e^{-x})' = (-1)(-e^{-x}) = e^{-x}$

previous step

$f'''(x) = (e^{-x})''' = (e^{-x})' = -e^{-x}$

previous step   previous step

$$f^{(4)}(x) = e^{-x}$$
$$f^{(5)}(x) = -e^{-x}$$
$$\vdots$$
$$f^{(k)}(x) = (-1)^k e^{-x}$$

(you can check $(-1)^0 = 1$, $(-1)^1 = -1$, $(-1)^2 = 1$, $(-1)^3 = -1$, $\dots$)

Step 2 Find $f^{(k)}(0)$

$$f^{(k)}(0) = (-1)^k e^{-0} = (-1)^k$$

Step 3, Plug into the formula

$$\sum_{k=0}^{\infty} f^{(k)}(0)\frac{x^k}{k!} = \boxed{\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!}}$$

So $$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \dots$$

(C) $f(x) = \frac{1}{1-x}$

Step 1 Find $f^{(k)}(x)$

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$
$$f'(x) = \left((1-x)^{-1}\right)' = (-1)(1-x)^{-2}(0-1)$$
$$= (1-x)^{-2}$$

(chain rule on $a(x) = 1-x$, $b(x) = x^{-1}$

So $b(a(x))' = b'(a(x))\,a'(x)$
$$= (-1)\,a(x)^{-2}(0-1)$$
$$= (-1)(1-x)^{-2}(-1)$$
$$= (1-x)^{-2}$$)

$$f''(x) = \left((1-x)^{-2}\right)'$$
$$= (-2)(1-x)^{-3}(0-1)$$
$$= 2(1-x)^{-3}$$
$$f'''(x) = \left(2(1-x)^{-3}\right)'$$
$$= 2(-3)(1-x)^{-4}(-1)$$

$$= 6(1-x)^{-4}$$

$$f^{(4)}(x) = (6(1-x)^{-4})'$$

$$= 6\cdot(-4)(1-x)^{-5}(-1)$$

$$= 24(1-x)^{-5}$$

$$f^{(5)}(x) = 120(1-x)^{-6}$$

$$f^{(6)}(x) = 720(1-x)^{-7}$$

$$\vdots$$

$$f^{(k)}(x) = k!\,(1-x)^{-k-1}$$

Step 2 Find $f^{(k)}(0)$

$$f^{(k)}(0) = k!(1-0)^{-k-1} = k!\,(1)^{-k-1} = k!$$

Step 3

$$\sum_{k=0}^{\infty} k!\,\frac{x^k}{k!} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$ (this work for $-1 \leq x \leq 1$)

(D) $\sin(x)$

Step 1

$$f(x) = \sin(x)$$

$$f'(x) = \cos(x)$$

$$f''(x) = -\sin(x)$$

$$f'''(x) = -\cos(x)$$

$$f^{(4)}(x) = \sin(x)$$

$$f^{(5)}(x) = \cos(x)$$

$$\vdots$$

Step 2

$$f(0) = \sin(0) = 0$$

$$f'(0) = \cos(0) = 1$$

$$f''(0) = 0$$

$$f'''(0) = -1$$

$$f^{(4)}(0) = 0$$

$$\vdots$$

$$f^{(k)}(0) = \begin{cases} 0 & k = 0, 2, 4, \dots \\ 1 & k = 1, 5, 9, \dots \\ -1 & k = 3, 7, 11, \dots \end{cases}$$

Step 3

$$f(x) \approx \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$$
$$= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \dots$$
$$= x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \dots$$
$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

(E) $\ln(x+1)$

$$f'(x) = (x+1)^{-1}$$
$$f''(x) = -(x+1)^{-2}$$
$$f'''(x) = 2(x+1)^{-3}$$
$$f^{(4)}(x) = -6(x+1)^{-4}$$
$$f^{(5)}(x) = 24(x+1)^{-5}$$
$$f^{(6)}(x) = -120(x+1)^{-6}$$
$$\vdots$$

(compute similar to C)

$$f^{(k)}(x) = \begin{cases} \ln(x+1) & k=0 \\ (-1)^{k-1}(k-1)!\,(x+1)^{-k} & k \geq 1 \end{cases}$$

Step 2 $$f^{(k)}(0) = \begin{cases} \ln(1) & k=0 \\ (-1)^{k-1}(k-1)!\,(1)^{-k} & k \geq 1 \end{cases}$$
$$= \begin{cases} 0 & k=0 \\ (-1)^{k-1}(k-1)! & k \geq 1 \end{cases}$$

Step 3 $$\ln(x+1) = 0 + (-1)^0 0! \frac{x}{1!} + (-1)^1 1! \frac{x^2}{2!} + (-1)^2 2! \frac{x^3}{3!} + (-1)^3 3! \frac{x^4}{4!} + \dots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$
$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

Alternate way $\ln(x+1) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$

$$= f(0)\frac{x^0}{0!} + \sum_{k=1}^{\infty} f^{(k)}(0) \frac{x^k}{k!}$$

$$= 0 + \sum_{k=1}^{\infty} (-1)^{k-1}(k-1)! \frac{x^k}{k!}$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$