Exponent rules:

$$a^{\log_a(x)} = x$$

$$a^n a^m = a^{n+m}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$a^{nk} = (a^n)^k$$

$$a^n b^n = (ab)^n$$

Logarithm Rules:

$$\log_a(a^n) = n$$

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

$$\log_a(x/y) = \log_a(x) - \log_a(y)$$

$$\log_a(x^k) = k\log_a(x)$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Examples

$$2^{(2^{x+1})} = 2^{2^x 2^1} = 2^{2\cdot 2^x} = (2^2)^{(2^x)} = 4^{(2^x)}$$

$$\begin{aligned}\log_{10}(400) &= \log_{10}(4\cdot 100)\\ &= \log_{10}(4) + \log_{10}(100)\\ &= \log_{10}(2^2) + \log_{10}(10^2)\\ &= 2\log_{10}(2) + 2\\ &\approx 2(0.301) + 2\\ &= 2.602\end{aligned}$$

Trigonometric Identities:

$\sin(-x) = -\sin(x)$ $\quad$ $\sin(x+2\pi n) = \sin(x)$ for all integers n

$\cos(-x) = \cos(x)$ $\quad$ $\cos(x+2\pi n) = \cos(x)$ for all integers n

$\cos(\frac{\pi}{2} - x) = \sin(x)$ $\quad$ $\sin^2(x) + \cos^2(x) = 1$

$\tan(x) = \sin(x)/\cos(x)$ $\quad$ $\tan(x+\pi n) = \tan(x)$ for all integer n

$\cot(x) = \cos(x)/\sin(x)$

$\sec(x) = 1/\cos(x)$ $\quad$ $\csc(x) = 1/\sin(x)$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$\sin(x/2) = \pm\sqrt{\frac{1-\cos(x)}{2}}$ $\quad$ $\cos(x/2) = \pm\sqrt{\frac{1+\cos(x)}{2}}$

Example: The proof of the half angle identity for sine:

Start with the angle addition formula for cosine with $x=y$:

$$\cos(2x) = \cos(x+x)$$
$$= \cos(x)\cos(x) - \sin(x)\sin(x)$$
$$= \cos^2(x) - \sin^2(x)$$
$$= (1-\sin^2(x)) - \sin^2(x) \qquad \text{(using } \sin^2(x)+\cos^2(x)=1\text{)}$$
$$= 1 - 2\sin^2(x)$$

Solve this equation for $\sin(x)$:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \Rightarrow \sin(x) = \pm\sqrt{\frac{1-\cos(2x)}{2}}$$

now substitute x with $x/2$ so

$$\sin(x/2) = \pm\sqrt{\frac{1-\cos(x)}{2}}$$

Limits

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

The squeeze theorem:

If $f(x) \le g(x) \le h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$

then $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

Derivatives

$$f'(x) = \frac{df}{dx}(x) = \frac{d}{dx} f(x)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{(The definition of the derivative)}$$

$$(f(x)+g(x))' = f'(x) + g'(x) \qquad (c\, f(x))' = c\, f'(x)$$

$$(x^n)' = n x^{n-1} \qquad \text{(The power rule)}$$

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \qquad \text{(The product rule)}$$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \qquad \text{(The quotient rule)}$$

$$f(g(x))' = f'(g(x))\, g'(x) \qquad \text{(The chain rule)}$$

$$f^{-1}(x)' = \frac{1}{f'(f^{-1}(x))} \qquad \text{(The inverse function rule)}$$

$$\sin(x)' = \cos(x), \quad \tan(x)' = \sec^2(x), \quad \sec(x)' = \tan(x)\sec(x)$$

$$\cos(x)' = -\sin(x), \quad \cot(x)' = -\csc^2(x), \quad \csc(x)' = -\cot(x)\csc(x)$$

$(e^x)' = e^x \qquad \ln(x)' = \frac{1}{x} \qquad |x|' = \mathrm{sgn}(x)$ (except at $x=0$)

Examples: $(2^x)' = ((e^{\ln 2})^x)'$ (exponent rule)

$= (e^{\ln(2)x})'$ (exponent rule)

$= (\ln(2)x)' e^{\ln(2)x}$ (via the chain rule)

$= \ln(2)\, e^{\ln(2)x}$

$= \ln(2)\,(e^{\ln(2)})^x$

$= \ln(2)\, 2^x$

$\tan(x)' = \left(\frac{\sin(x)}{\cos(x)}\right)'$

$= \frac{\sin(x)'\cos(x) - \sin(x)\cos(x)'}{\cos^2(x)}$ (quotient rule)

$= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)}$

$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$

$= \frac{1}{\cos^2(x)}$ (pythagorean identity)

$= \sec^2(x)$

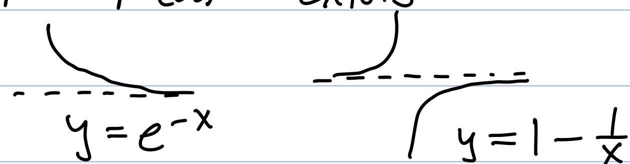
Features of Functions:

f is continuous at a if $\lim_{x\to a} f(x) = f(a)$

f is differentiable at a if $f'(a)$ exists

Horizontal Asymptote

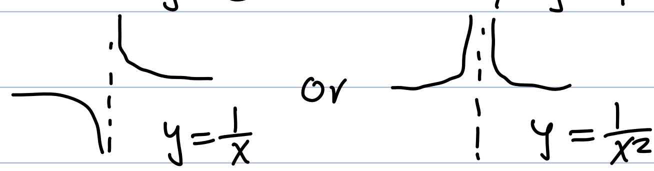
$\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ exists



Vertical Asymptote

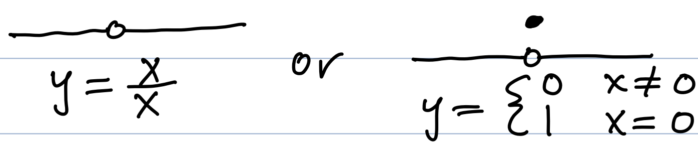
$\lim_{x\to a^+} f(x) = \pm\infty$

or $\lim_{x\to a^-} f(x) = \pm\infty$



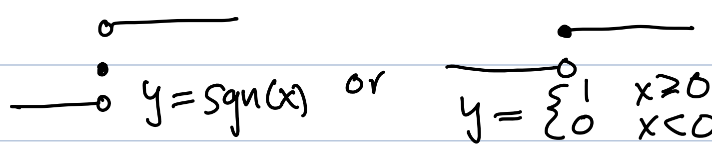
Hole

$\lim_{x\to a} f(x) \neq f(a)$

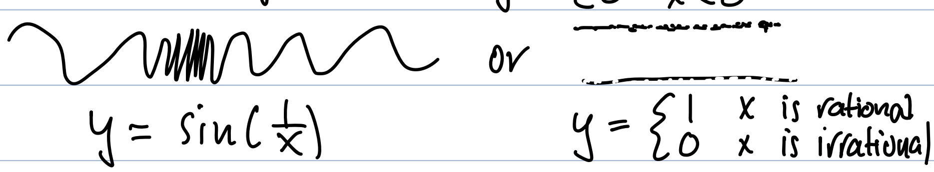


Jump discontinuity

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$



Essential discontinuity



Applications of Derivatives

$y = f'(x_0)(x-x_0) + f(x_0)$ (The tangent line at x_0)

$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$ (Linear approximation formula)

$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2}$ (Quadratic approximation)

<u>Extreme value theorem:</u> Suppose f is differentiable on the interval $[a,b]$, if c is an extreme value of f then $c=a$, $c=b$, or $f'(c)=0$.