

# Calculus Problems (20 points each)

**Problem (1).** A function  $f$  is called even if  $f(-x) = f(x)$  for all  $x$ . For example  $f(x) = x^2$  or  $\cos$  are even functions. A function is called odd if  $f(-x) = -f(x)$  for all  $x$ . For example  $\sin$ ,  $\tan$ , or  $f(x) = x^3$  are odd functions. Suppose  $f$  and  $g$  are two functions. Classify if the functions  $h(x) = f(x)g(x)$  and  $i(x) = f(g(x))$  are even or odd based on when  $f$  and  $g$  are even or odd. A table is given below for your answer.

$f$	$g$	$h$	$i$
even	even		
even	odd		
odd	even		
odd	odd		

**Problem (2).** Here's an argument for why  $(x^4 + 1)' = 4x^3$  using the limit definition of the derivative. There are three mistakes in the argument. Identify them and suggest corrections.

$$(x^4 + 1)' = \lim_{h \rightarrow 0} \frac{(x+h)^4 + 1 - x^4 + 1}{h} \quad (1)$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)(x+h)(x+h)(x+h) - x^4 + 2}{h} \quad (2)$$

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2xh + h^2)(x^2 + 2xh + h^2) - x^4 + 2}{h} \quad (3)$$

$$= \lim_{h \rightarrow 0} \frac{x^4 + 2x^3h + x^2h^2 + 2x^3h + 2x^2h^2 + 2xh^3 + (x^2 + 2xh + h^2)(h^2) - x^4 + 2}{h} \quad (4)$$

$$= \lim_{h \rightarrow 0} \frac{4x^3h + h^2(x^2 + 2x^2 + 2xh + x^2 + 2xh + h^2) + 2}{h} \quad (5)$$

$$= \lim_{h \rightarrow 0} 4x^3 + h(4x^2 + 4xh + h^2) + \frac{2}{h} \quad (6)$$

$$= 4x^3 + \lim_{h \rightarrow 0} h(4x^2 + 4xh + h^2) + \lim_{h \rightarrow 0} \frac{2}{h} \quad (7)$$

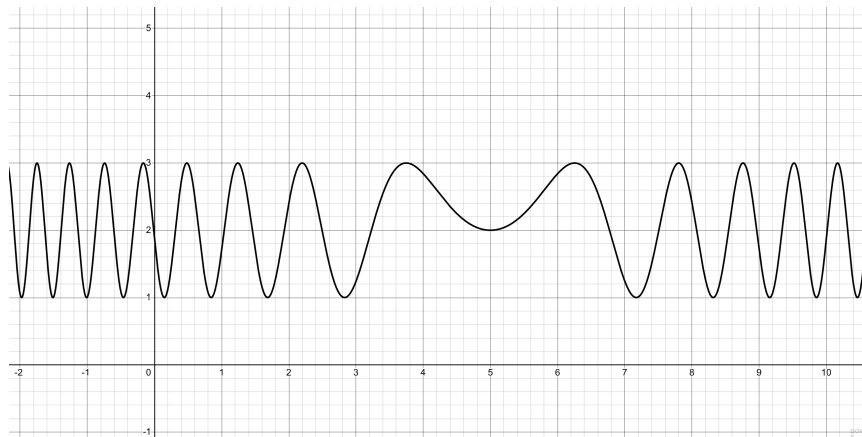
$$= 4x^3 + 0 + 0 \quad (8)$$

$$= 4x^3 \quad (9)$$

3. Let  $n$  be a positive integer and  $f(x) = \frac{1}{1+x}$ . Find the general form of  $f^{(n)}(x)$ . Here  $f^{(n)}$  denotes the  $n$ th derivative of  $f$ . Your answer will involve the factorial symbol,  $n! = 1 \cdot 2 \cdot 3 \cdots n$ .

4. Find  $(x^x)'$ . Here are two wrong answers:  $x \cdot x^{x-1}$  and  $\ln(x)x^x$ .

**Problem (5).** Below is the graph of a function  $f(x)$ . Identify the function (I recommend trying some things with Desmos for this).



### Problem Solving Problems (20 points each)

6. When the roots of the polynomial

$$P(x) = (x-1)^1(x-2)^2(x-3)^3 \cdots (x-10)^{10}$$

are removed from the number line, what remains is the union of 11 disjoint open intervals. On how many of these intervals is  $P(x)$  positive?

7. What is the area of the region in the coordinate plane defined by

$$||x| - 1| + ||y| - 1| \leq 1?$$

8. For complex numbers  $u = a+bi$  and  $v = c+di$ , define the binary operation  $\otimes$  by

$$u \otimes v = ac + bdi.$$

Suppose  $z$  is a complex number such that  $z \otimes z = z^2 + 40$ . What is  $|z|$ ?

9. Cyrus the frog jumps 2 units in a direction, then 2 more in another direction. What is the probability that he lands less than 1 unit away from his starting position?
10. A real-valued function  $f$  has the property that for all real numbers  $a$  and  $b$ ,

$$f(a+b) + f(a-b) = 2f(a)f(b).$$

Which one of the following cannot be the value of  $f(1)$ ?

- (A) 0      (B) 1      (C) -1      (D) 2      (E) -2