Problem Set 5 (Total Points: 200), Due July 24th

Normal Problems

Problem (1)**.** (15 points) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the equation $x^2 + xy + y^2 = x + y$.

Problem (2)**.** (15 points) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the equation $\cos(xy) = 1$.

Problem (3)**.** (15 points) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the equation $x^2 + y^2 = y^4$.

Problem (4)**.** (15 points) Solve the following limit with L'Hôpital's rule:

$$\lim_{x \to \infty} \frac{e^x}{e^{e^x}}$$

Problem (5)**.** (15 points) Solve the following limit with L'Hôpital's rule:

$$\lim_{x \to \infty} x^{-1} e^{2x}$$

Problem (6)**.** (25 points) Solve the following limit with L'Hôpital's rule:

$$\lim_{x \to 0} \frac{\tan(x) - x}{x^3}$$

Problem (7)**.** (Bonus, 25 points) Find $(x^x)'$. Hint: The easiest way to solve this problem is probably to use $x^x = e^{\ln(x^x)}$ to solve this problem. Both $x \cdot x^{x-1}$ and $\ln(x)x^x$ are wrong answers to this problem. However, the right answer does look similar to these.

Mini Projects (100 points)

Choose one of the following projects. Submit your answer on a separate sheet of paper.

Project (A)**.** The formula for the curvature of the graph of a function f at a point x is

$$k_f(x) = \frac{f''(x)}{(1 + f'(x)^2)^{3/2}}.$$

(a) Find the curvature for the function $f(x) = x^2$. Describe the limits of the curvature for this function as x goes to ∞ and $-\infty$. Sketch a graph of f and interpret your results.

(b) An inflection point is a point where the curvature of a function changes sign. Another definition of an inflection point is where the $f''(x)$ changes sign. Show these two definitions are the same. Hint: $1 + f'(x)^2$ is always a positive number.

(c) Find the curvature of the function $f(x) = (x^2 - 1)^3$. Sketch a graph of $f(x)$ and of $k_f(x)$ (feel free to use Desmos). Find all the inflection points of f.

(d) (Bonus) The curvature measures the inverse of the radius of a circle tangent to the point $(x, f(x))$. Derive the formula for the curvature, $k_f(x) = \frac{f''(x)}{(1+f'(x)^2)^{3/2}}$ using the definition in terms of the inverse radius. Feel free to use the internet for this problem but you must submit an answer in your own words.

Project (B)**.** This project is about optimizing the area of various shapes.

(a) Suppose a rectangle coordinates in the plane being

$$(0,0), (x,0), (x,y), (y,0).$$

Suppose $y = 4 - x$. What is the maximum area of the rectangle. Assume x and y are positive numbers.

(b) Repeat part (a) but for the equation $y = \sqrt{8 - x^2}$.

(c) Repeat part (a) but for the equation $y = \dfrac{5}{x}$.

(d) Suppose a rectangle has perimeter 20. What is the maximum area it can have?

(e) Suppose a right triangle has perimeter $4+2\sqrt{2}$. What is the maximum area it can have?

(f) (Bonus) Suppose an isosceles triangle has a perimeter $8\sqrt{3}$. What's the maximum area it can have?

Project (C)**.** This project is about using derivative rules in other contexts.

For the first 5 questions assume f and g are functions such that

$$f(1) = 2$$
$$f'(1) = 5$$
$$f'(2) = 11$$
$$g(1) = 3$$
$$g'(1) = 7$$
$$g'(2) = 13$$

(a) Suppose $h(x) = f(x) + 2g(x)$. Find $h'(1)$.

(b) Suppose $h(x) = f(x)g(x)$. Find $h'(1)$.

(c) Suppose $h(x) = \dfrac{f(x)}{g(x)}$. Find $h'(1)$.

(d) Suppose $h(x) = g(f(x))$. Find $h'(1)$.

(e) Suppose $h(x) = f(x^2) + g(x^3)$. Find $h'(2)$.

For these next problems we are going to find and derive some derivative rules.

(f) Find a formula for $\left(\dfrac{1}{f(x)}\right)'$ in terms of $f(x)$ and $f'(x)$.

(g) Find a formula for $(f(x)^n)'$ in terms of n, $f(x)$, and $f'(x)$.

(h) In this problem you will prove the quotient rule. For functions $f(x)$ and $g(x)$ let $h(x) = \dfrac{1}{f(x)}$ so that $f(x)h(x) = \dfrac{f(x)}{g(x)}$. Using the product rule and the part (f) show that $(f(x)h(x))' = \dfrac{f(x)'g(x) - f(x)g(x)'}{g(x)^2}$. Show your work.

(i) Suppose $f(x)$ and $g(x)$ are inverse functions so in particular $f(g(x)) = x$. That the derivate of both sides of the equation and use the chain rule. Divide both sides of the equation by $f'(g(x))$ to get a formula for $g'(x)$. This formula is called the inverse function rule.

(j) (Bonus) Find a formula for $\left(f(x)^{g(x)}\right)'$ As with question 7 on the normal problems you can't solve this problem just by using the power rule or the taking a logarithm. You will probably need to use $e^{\ln(x)} = x$.