<u>Definition</u> Say f is continuous and defined on $[a,b]$.

$$\int_a^b f(x)\,dx := \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a+\left(\tfrac{b-a}{n}\right)k\right) \frac{b-a}{n}$$

"the (definite) integral of f from a to b
(in the variable x)"

<u>Theorem</u> (The fundamental theorem of calculus):

A) Suppose $F'(x) = f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

this is just notation

B) $$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

(A) is called the 2^{nd} fundamental theorem and (B) is called the 1^{st} fundamental theorem.

<u>Definition</u> Indefinite Integrals / Antiderivatives

$$\int f(x)\,dx = F(x) + C \quad \text{where} \quad F'(x) = f(x)$$

<u>Important Indefinite Integrals</u>

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \int 1/\sqrt{1-x^2}\,dx = \sin^{-1}(x) + C$$
$$\int \cos(x)\,dx = \sin(x) + C \qquad \int 1/(1+x^2)\,dx = \tan^{-1}(x) + C$$
$$\int e^x\,dx = e^x + C \qquad \int 1/x\,dx = \ln(|x|) + C$$

$\int 1\,dx = x + C$, $\int x\,dx = \frac{x^2}{2} + C$

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ (as long as $n \neq -1$)

$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$ $\int \frac{dx}{\sqrt{x}} = \frac{1}{2}\sqrt{x} + C$ $\int \frac{dx}{x^2} = -\frac{1}{x} + C$

$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

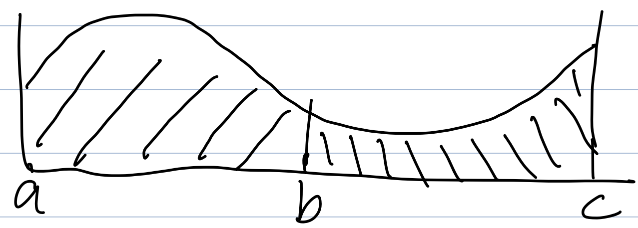
$\int k\,f(x)\,dx = k\int f(x)\,dx$ (k is a constant)

Example $\int e^x - \frac{2}{1+x^2}\,dx = \int e^x\,dx - 2\int \frac{1}{1+x^2}\,dx$

$= e^x - 2\tan^{-1}(x) + C$

Finding Definite Integrals

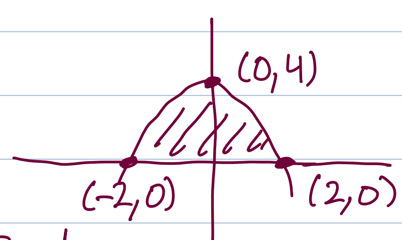
$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$



$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Example Find the area bounded by $y = 4 - x^2$ and the x-axis.

Solution: Draw a picture



So we want $\int_{-2}^{2} 4 - x^2\,dx$

We find $\int 4-x^2\,dx = 4x - \frac{x^3}{3} + C$

(we can ignore the $+C$ when doing definite integrals because they'll cancel).

$$\int_{-2}^{2} 4-x^2\,dx = 4x - \frac{x^3}{3}\Big|_{-2}^{2}$$
$$= 4(2) - \frac{(2)^3}{3} - \left(4(-2) - \frac{(-2)^3}{3}\right)$$
$$= 8 - \frac{8}{3} + 8 - \frac{8}{3}$$
$$= 32/3$$

u-Substitution

$$\int f'(u)\,u'\,dx = f(u(x)) + C$$

u is a function of x and write $du = u'\,dx$

Examples $\int x e^{x^2}\,dx$

solution: set $u = x^2$ (because x^2 is "inside")

then $du = 2x\,dx$ (because $u' = 2x$)

So $dx = \frac{du}{2x}$

thus $\int x e^{x^2}\,dx = \int x e^{u} \frac{du}{2x}$
$$= \int \frac{e^u}{2}\,du \quad \text{(cancel } x\text{'s)}$$
$$= \frac{1}{2}e^u + C$$
$$= \frac{1}{2}e^{x^2} + C$$

(we can check $\left(\frac{1}{2}e^{x^2}\right)' = \frac{1}{2}2x\,e^{x^2} = xe^{x^2}$)

$$\int \frac{6x^2}{2+x^3}\,dx$$

set $u = 2+x^3$

So $du = 3x^2 dx$

So $\int \frac{6x^2}{2+x^3}\,dx = \int \frac{2\,du}{u}$

$= 2\ln|u| + C$

$= 2\ln|2+x^3| + C$

$$\int \frac{1+3x}{x+x^2}\,dx$$

set $u = x+x^2$ so $du = (1+2x)\,dx$

So we'll seperate $\frac{1+3x}{x+x^2} = \frac{1+2x+x}{x+x^2} = \frac{1+2x}{x+x^2} + \frac{x}{x+x^2}$

then $\int \frac{1+3x}{x+x^2}\,dx = \int \frac{1+2x}{x+x^2}\,dx + \int \frac{x}{x+x^2}\,dx$

$= \int \frac{1}{u}\,du + \int \frac{x}{x+x^2}\,dx$

Now we'll deal with $\int \frac{x}{x+x^2}\,dx = \int \frac{1}{1+x}\,dx$

Set $v = 1+x$ so $dv = dx$

So $\int \frac{1}{1+x}\,dx = \int \frac{1}{v}\,dv$

Hence $\int \frac{1+3x}{x+x^2}\,dx = \int \frac{1}{u}\,du + \int \frac{1}{v}\,dv$

$= \ln|u| + \ln|v| + C$

$= \ln|x+x^2| + \ln|1+x| + C$

This is a final answer but we could simplify using log rules. $\ln|x+x^2| + \ln|1+x| = \ln(|x+x^2|\cdot|1+x|)$

$= \ln|(x+x^2)(1+x)|$

$= \ln|x^3+2x^2+x|$

We can check our work

$\ln|x^3+2x^2+x|' = \frac{3x^2+4x+1}{x^3+2x^2+x}$ (chain rule)

$= \frac{(3x+1)(x+1)}{(x^2+x)(x+1)}$

$= \frac{3x+1}{x^2+x}$ ✓

$$\int \sqrt{1-x^2}\,dx$$

set $u = \sin^{-1}(x)$ so $\sin(u) = x$

$du = \frac{1}{\sqrt{1-x^2}}\,dx$ and $\sqrt{1-x^2}\,du = dx$

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-x^2}\sqrt{1-x^2}\,du$$
$$= \int (1-x^2)\,du$$
$$= \int 1-\sin^2 u\,du$$
$$= \int \cos^2 u\,du$$
$$= \int \frac{1+\cos 2u}{2}\,du$$
$$= \frac{1}{2}u + \frac{1}{2}\int \cos 2u\,du$$

Set $v = 2u$ so $\frac{1}{2}dv = du$

$$= \frac{1}{2}u + \frac{1}{2}\int \cos v\,\frac{1}{2}dv$$
$$= \frac{1}{2}u + \frac{1}{4}\sin v + C$$
$$= \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$
$$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{4}\sin(2\sin^{-1}x) + C$$
$$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{4}2\sin(\sin^{-1}x)\cos(\sin^{-1}x) + C$$
$$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-\sin^2\sin^{-1}x} + C$$
$$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$

Integration By Parts

$$\int u\,dv = uv - \int v\,du$$

Examples $\int x e^x\,dx$

$u = x$, $dv = e^x dx$

so $du = (x)'dx = dx$

and $\int dv = \int e^x dx$ so $v = e^x$

$\int x e^x dx = \int u\,dv$

$$= uv - \int v\,du$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x$$

$$\int x^2 \sin(x)\,dx$$

$u = x^2$, $dv = \sin(x)dx$

so $du = 2x\,dx$, $v = -\cos(x)$

so $\int x^2 \sin(x)\,dx = x^2(-\cos(x)) - \int (-\cos(x))\,2x\,dx$

$$= -x^2\cos(x) + 2\int \cos(x)\,x\,dx$$

(Set $w = x$, $dy = \cos(x)dx$
So $dw = dx$, $y = \sin(x)$)

$$= -x^2\cos(x) + \int w\,dy$$
$$= -x^2\cos(x) + wy - \int y\,dw$$
$$= -x^2\cos(x) + x\sin(x) - \int \sin(x)\,dx$$
$$= -x^2\cos(x) + x\sin(x) - (-\cos(x)) + C$$
$$= -x^2\cos(x) + x\sin(x) + \cos(x) + C$$

$$\int \sin^{-1}(x)\,dx$$

$u = \sin^{-1}(x)$ $dv = dx$

So $du = \frac{1}{\sqrt{1-x^2}}dx$ $v = x$

thus $\int \sin^{-1}(x)\,dx = \sin^{-1}(x)\,x - \int x \frac{1}{\sqrt{1-x^2}}\,dx$

(now set $w = 1-x^2$ so $dw = -2x\,dx$)

$$\int x \frac{1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{w}} \frac{dw}{-2}$$
$$= -\frac{1}{2}\int \frac{dw}{\sqrt{w}}$$
$$= -\frac{1}{2}\int w^{-1/2}\,dw$$
$$= -\frac{1}{2}\frac{1}{1/2} w^{1/2} + C$$
$$= -\sqrt{1-x^2} + C$$

putting everything together $\int \sin^{-1}(x)\,dx = \sin^{-1}(x)\,x - \left(-\sqrt{1-x^2}\right) + C$

$$= \sin^{-1}(x)\,x + \sqrt{1-x^2} + C$$

$$\int \sin(x)\, e^x\, dx$$

$u = \sin(x)$ $\quad e^x dx = dv$ $\quad$ so $\quad v = e^x,\ du = \cos x\, dx$

then $\quad \int \sin(x)\, e^x dx = \sin(x) e^x - \int \cos(x)\, e^x dx$

$w = \cos(x)$ $\quad e^x dx = dv$ $\quad$ So $\quad v = e^x,\ dw = -\sin x\, dx$

$$\int \cos(x)\, e^x dx = \cos(x) e^x - \int (-\sin(x)) e^x dx$$

putting it together

$$\int \sin(x) e^x dx = \sin(x) e^x - \int \cos(x) e^x dx$$
$$= \sin(x) e^x - \cos(x) e^x + \int (-\sin(x)) e^x dx$$
$$= \sin(x) e^x - \cos(x) e^x - \int \sin(x) e^x dx$$

adding $\int \sin(x) e^x dx$ to both sides

$$2\int \sin(x) e^x dx = \sin(x) e^x - \cos(x) e^x$$

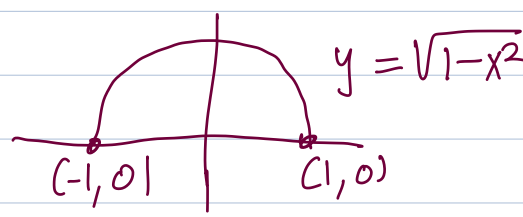
divide by 2

$$\int \sin(x) e^x dx = \frac{1}{2}\left(\sin(x) e^x - \cos(x) e^x\right)$$

Using Integrals

Example Find the area of a semicircle of radius 1.

Solution:



(solve $x^2+y^2=1$ for y)

so we want $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

$$= \frac{1}{2}\sin^{-1}(x) + \frac{1}{2}x\sqrt{1-x^2}\ \Big|_{-1}^{1} \qquad \text{(see above)}$$

$$= \frac{1}{2}\sin^{-1}(1) + \frac{1}{2}\,1\sqrt{1-1} - \left(\frac{1}{2}\sin^{-1}(-1) + \frac{1}{2}(-1)\sqrt{1-(-1)^2}\right)$$

$= \frac{1}{2}\sin^{-1}(1) + 0 \quad - \frac{1}{2}\sin^{-1}(-1) + 0$

$= \frac{1}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}\left(\frac{\pi}{2}\right)$

$= \frac{\pi}{2}$

<u>Example</u> Suppose a ball is falling with acceleration $a(t) = -10$, initial velocity $v(0) = -1$ and initial height $h(0) = 10$. Find its height over time.

Solution: $v'(t) = a(t)$ so $\int a(t)\,dt = v(t)$

thus $v(t) = \int (-10)\,dt = -10t + C$

Solving for C use $v(0) = -1 = -10 \cdot 0 + C$

so $C = -1$ so $v(t) = -10t - 1$

$h'(t) = v(t)$ so $\int v(t)\,dt = h(t)$

thus $h(t) = \int (-10t - 1)\,dt$

$= -10\frac{t^2}{2} - t + C$

$= -5t^2 - t + C$

set $t=0$ $h(0) = 10 = -5(0)^2 - 0 + C$

so $C = 10$ so $\boxed{h(t) = -5t^2 - t + 10}$