The limit is the value of a function as it approaches a value.

Normal limit $\lim_{x\to a} f(x)$

Right limit (from above) $\lim_{x\to a^+} f(x)$ } One sided limits

Left limit (from below) $\lim_{x\to a^-} f(x)$

Limit to infinity (or minus infinity) $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$ $\xrightarrow{\infty}$, $\xleftarrow{-\infty}$

Fact $\lim_{x\to a} f(x)$ exists if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Finding limits of polynomial quotients:

$$\lim_{x\to\infty} \frac{ax^n+\cdots}{bx^m+\cdots} = \begin{cases} a/b & \text{if } n=m \\ 0 & \text{if } m<n \\ \infty & \text{if } m>n \end{cases}$$

$$\lim_{x\to r} \frac{ax^n+\cdots}{bx^m+\cdots} \underset{\text{factor}}{=} \lim_{x\to r} \frac{(x-r)^p f(x)}{(x-r)^q g(x)} = \begin{cases} \frac{f(r)}{g(r)} & \text{if } p=q \\ 0 & \text{if } p>q \\ \text{DNE} & \text{if } q-p>0 \text{ is odd} \\ \pm\infty & \text{if } q-p>0 \text{ is even} \end{cases}$$

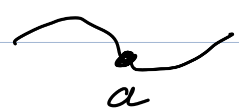
Useful limits:

$\lim_{h\to 0} \frac{\sin(h)}{h} = 1$, $\lim_{h\to 0} \frac{e^h-1}{h} = 1$, $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 0$

$\lim_{x\to 0^+} \text{sgn}(x) = 1$, $\lim_{x\to 0^-} \text{sgn}(x) = -1$, $\lim_{x\to 0} \frac{x^n}{e^x} = 0$ (for all n)

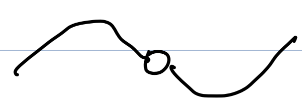
Continuity:

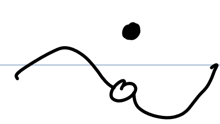
- f is continuous at a: $f(a) = \lim_{x\to a} f(x)$



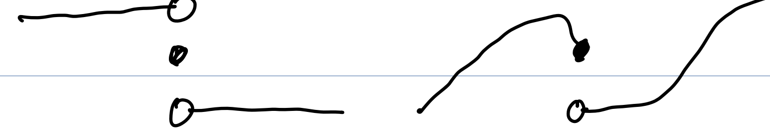
"drawing without lifting pencil."

- f has a removable discontinuity at a: $f(a) \neq \lim_{x\to a} f(x)$
  (f(a) may not exist but $\lim_{x\to a} f(x)$ exists



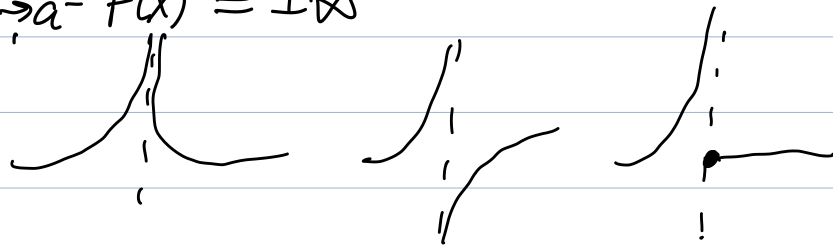


- f has a jump discontinuity at a: $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$
  (the one sided limits exist)

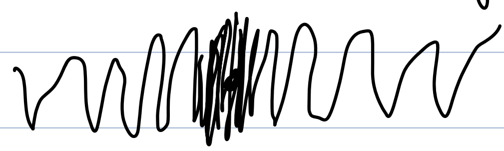


Essential discontinuities

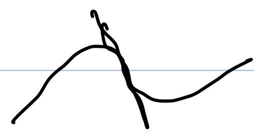
- f has a vertical asymptote at a: $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$



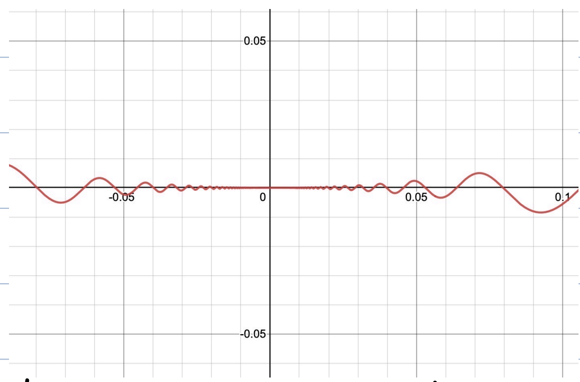
- other discontinuities (eg $\sin(\frac{1}{x})$)



Differentiability: • f is differentiable at a if $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.



"look straight zooming in"

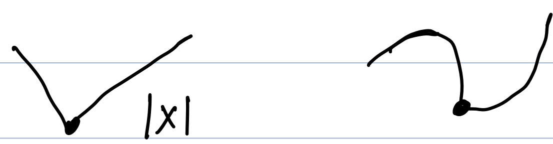


$f(x) = x^2 \sin\frac{1}{x}$

$f'(0) = 0$

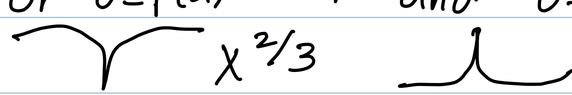
- f has a corner at a if $\partial_+ f(a) \neq \partial_- f(a)$ but exist and one is finite.

$\partial_\pm f(a) = \lim_{x\to a^\pm} \frac{f(x)-f(a)}{x-a}$ (left/right derivatives)



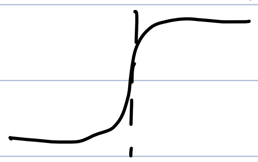
$|x|$

- f has a cusp at a if $\partial_+ f(a) = \infty$ and $\partial_- f(a) = -\infty$ or $\partial_- f(a) = -\infty$ and $\partial_- f(a) = \infty$

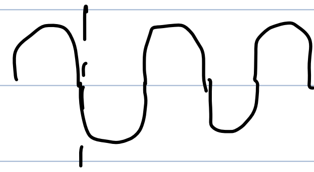


$x^{2/3}$

• f has a vertical tangent at a if $f'(a) = \pm\infty$



$x^{1/3}$



• other points of non-differentiability

$x\sin\left(\frac{1}{x}\right)$

Indeterminate Forms: $\infty - \infty$, $0 \cdot \infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞, 0^∞

where you need limits

L'Hopital's Rule: If $f(a)=0$, $g(a)=0$

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$