Problem Set 3 (20 points each)

Problem (1)**.** Suppose $f(x) = \cos(x), g(x) = \dfrac{e^x + e^{-x}}{2}$. Recall that $f^{(n)}$ denotes the nth derivative of f. Find $f^{(2024)}(x), g^{(2024)}(x), f^{(2025)}(0)$, and $g^{(2025)}(1)$.

2. Find the following limit:

$$\lim_{x \to 0} \frac{\sqrt[3]{1+2x} - 1}{x}$$

3. For what value of k does the equation $e^{2x} = k\sqrt{x}$ have exactly one solution?

4. Find the two points on the curve $y = x^4 - 2x^2 - x$ that have a common tangent line.

Problem (5)**.** Consider the three points $(-3, 9), (1, 1), (a, a^2)$ on the graph of $y = x^2$. Suppose that the three normal lines at each of these points intersect at a common point. Find a. (A normal line for a point on a curve is the line that is perpendicular to the tangent line)

Problem (6)**.** The Hubble Space Telescope was deployed on April 24, 1990, by the space shuttle *Discovery.* A model for the velocity of the shuttle during this mission, from liftoff at $t = 0$ until the solid rocket boosters were jettisoned at $t = 126$s, is given by

$$v(t) = 0.001302t^3 - 0.09029t^2 + 23.61t - 3.083$$

(in feet per second). Using this model, estimate the absolute maximum and minimum values of the *acceleration* of the shuttle between liftoff and the jettisoning of the boosters. (These values are important quantities in ensuring that a rocket isn't damaged during the launch.)

Problem (7)**.** Suppose f is a function that satisfies the equation

$$f(x+y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$

Find the following quantities, $f(0), f'(0), f'(x)$.

8. Evaluate the following:

$$\frac{d}{dx} \int_{x^2}^{x^3} \frac{\ln(t)}{t} dt$$

9. Evaluate the following:

$$\int_0^z \int_0^y x^2 e^x \, dx \, dy$$

10. Find the highest and lowest points on the curve $x^2 + xy + y^2 = 12$.

Problem (10)**.** For this problem your answer should be a proof. If you haven't written a proof before, write a paragraph explaining why the result this true (this is what a proof is). Show that $|\sin(x) - \cos(x)| \leq \sqrt{2}$ for all x.