
Example 1

Evaluate the limit $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x^2-9}$ using

(a) algebraic manipulation (factor and cancel)

Solution

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x+4}{x+3} = \frac{7}{6}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} \overset{H}{=} \lim_{x\to 3} \frac{2x+1}{2x} = \frac{7}{6}$$

Example 2

Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 4x}$ using

(a) the basic trigonometric limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ together with appropriate changes of variables

Solution

Write the limit as

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} = \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)\left(\lim_{x\to 0} \frac{x\cos 4x}{\sin 4x}\right)$$

In the first limit let $u = 3x$ and in the second let $v = 4x$. Then the limit is

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} &= \left(\lim_{u\to 0} \frac{3\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v\cos v}{4\sin v}\right) \\ &= \tfrac{3}{4}\left(\lim_{u\to 0} \frac{\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v}{\sin v}\right)\left(\lim_{v\to 0} \cos v\right) = \tfrac{3}{4}\end{aligned}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} \overset{H}{=} \lim_{x\to 0} \frac{3\cos 3x}{4\sec^2 4x} = \tfrac{3}{4}$$

Example 3

Evaluate the limit $\lim_{x\to\frac{\pi}{2}} \left(x - \frac{\pi}{2}\right)\tan x$ using L'Hopital's Rule.

Solution

Write the limit as

$$\lim_{x\to\frac{\pi}{2}} \left(x - \tfrac{\pi}{2}\right)\tan x = \lim_{x\to\frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cot x}$$

Then direct substitution gives $\frac{0}{0}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\frac{\pi}{2}} \left(x - \tfrac{\pi}{2}\right)\tan x \overset{\mathrm{H}}{=} \lim_{x\to\frac{\pi}{2}} \frac{1}{\left(-\csc^2 x\right)} = -1$$

Example 4

Evaluate the limit $\lim_{x\to 1} \dfrac{\sqrt{2-x} - x}{x-1}$ using L'Hopital's Rule.

Solution

Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x\to 1} \frac{\sqrt{2-x} - x}{x-1} \overset{\mathrm{H}}{=} \lim_{x\to 1} \frac{-\dfrac{1}{\sqrt{2-x}} - 1}{1} = -\frac{3}{2}$$

Note that this result can also be obtained by rationalizing the numerator by multiplying top and bottom by the root conjugate $\sqrt{2-x} + x$.

Example 5

Evaluate the limit $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$ using L'Hopital's Rule.

Solution

Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{\sin x}{2x}$$

Again direct substitution gives $\frac{0}{0}$ so use L'Hopital's Rule a second time to give

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\mathrm{H}}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Example 6

Evaluate the limits at infinity

(a) $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$

Solution

Direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

(b) $\lim\limits_{x\to\infty} x^2 e^{-x}$

Solution

Write the limit as

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

Then direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} x^2 e^{-x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(c) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right)$

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right) = \lim_{x\to\infty} \frac{\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right)}{\frac{1}{x}}$$

Now direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\begin{aligned}\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) &\overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{-1/2}\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{1/2}} = \frac{1}{2}\end{aligned}$$

Example 7

Evaluate the limit $\lim\limits_{x\to 2^+} \dfrac{\ln(x-2)}{\ln(x^2-4)}$ using L'Hopital's Rule.

Solution

Direct substitution gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule

$$\begin{aligned}\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)} &\overset{H}{=} \lim_{x\to 2^+} \frac{\frac{1}{x-2}}{\frac{2x}{x^2-4}} = \lim_{x\to 2^+} \frac{x^2-4}{2x(x-2)} = \lim_{x\to 2^+} \frac{x^2-4}{2x^2-4x} \\ &\overset{H}{=} \lim_{x\to 2^+} \frac{2x}{4x-4} = 1\end{aligned}$$

Example 8

Evaluate the limit $\lim_{x\to\infty}(\ln x - x)$ using L'Hopital's Rule.

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty}(\ln x - x) = \lim_{x\to\infty}\ln\left(xe^{-x}\right)$$

Then, as in Example 6 (a) and (b),

$$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x}{e^x} \overset{H}{=} \lim_{x\to\infty}\frac{1}{e^x} = 0$$

Now let $u = xe^{-x}$, then $x \to \infty \Rightarrow u \to 0^+$. Hence

$$\lim_{x\to\infty}(\ln x - x) = \lim_{u\to 0^+}\ln u = -\infty$$

Example 9

Evaluate the limit $\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}$ using L'Hopital's Rule.

Solution

Direct substitution gives the indeterminate form 0^0. First find the natural logarithm of the limit, as

$$\ln\left(\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}\right) = \lim_{x\to 0^+}\sqrt{x}\ln(\sin x) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{\dfrac{1}{\sqrt{x}}}$$

Now direct substitution gives $\frac{\infty}{\infty}$, so we can use L'Hopital's Rule

$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(\sin x)}{\dfrac{1}{\sqrt{x}}} &\overset{H}{=} \lim_{x\to 0^+}\frac{\dfrac{\cos x}{\sin x}}{\left(-\frac{1}{2}\right)x^{-3/2}} \\ &= -2\lim_{x\to 0^+}\frac{x^{3/2}\cos x}{\sin x} = -2\left(\lim_{x\to 0^+}\sqrt{x}\cos x\right)\left(\lim_{x\to 0^+}\frac{x}{\sin x}\right) = 0\end{aligned}$$

since

$$\lim_{x\to 0^+}\sqrt{x}\cos x = 0$$

and we can use L'Hopital's Rule on the second limit to give

$$\lim_{x\to 0^+}\frac{x}{\sin x} \overset{H}{=} \lim_{x\to 0^+}\frac{1}{\cos x} = 1$$

Hence

$$\ln\left(\lim_{x\to 0^+}(\sin x)^{\sqrt{x}}\right) = 0$$

so that

$$\lim_{x\to 0^+}(\sin x)^{\sqrt{x}} = e^0 = 1$$

7.7 Indeterminate Forms and L'Hospital's Rule

A Click here for answers.

S Click here for solutions.

1–45 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, use it. If l'Hospital's Rule doesn't apply, explain why.

1. $\lim_{x \to 2} \dfrac{x - 2}{x^2 - 4}$

2. $\lim_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1}$

3. $\lim_{x \to -1} \dfrac{x^6 - 1}{x^4 - 1}$

4. $\lim_{x \to 0} \dfrac{\tan x}{x + \sin x}$

5. $\lim_{x \to 0} \dfrac{e^x - 1}{\sin x}$

6. $\lim_{x \to 0} \dfrac{x + \tan x}{\sin x}$

7. $\lim_{x \to 0} \dfrac{\sin x}{x^3}$

8. $\lim_{x \to \pi} \dfrac{\tan x}{x}$

9. $\lim_{x \to 3\pi/2} \dfrac{\cos x}{x - (3\pi/2)}$

10. $\lim_{t \to 16} \dfrac{\sqrt[4]{t} - 2}{t - 16}$

11. $\lim_{x \to a} \dfrac{\sqrt[3]{x} - \sqrt[3]{a}}{x - a}$

12. $\lim_{x \to 0} \dfrac{6^x - 2^x}{x}$

13. $\lim_{x \to \infty} \dfrac{(\ln x)^3}{x^2}$

14. $\lim_{x \to 0} \dfrac{\sin x}{e^x}$

15. $\lim_{x \to 0} \dfrac{\tan \alpha x}{x}$

16. $\lim_{x \to 0} \dfrac{\sin^2 x}{\tan(x^2)}$

17. $\lim_{x \to \infty} \dfrac{\ln \ln x}{\sqrt{x}}$

18. $\lim_{x \to \infty} \dfrac{\ln(1 + e^x)}{5x}$

19. $\lim_{x \to 0} \dfrac{\tan^{-1}(2x)}{3x}$

20. $\lim_{x \to 0} \dfrac{x}{\sin^{-1}(3x)}$

21. $\lim_{x \to 0} \dfrac{\sin mx}{\sin nx}$

22. $\lim_{x \to 0} \dfrac{\sin^{10} x}{\sin(x^{10})}$

23. $\lim_{x \to 0} \dfrac{x + \sin 3x}{x - \sin 3x}$

24. $\lim_{x \to 0} \dfrac{e^{4x} - 1}{\cos x}$

25. $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$

26. $\lim_{x \to 0} \dfrac{x + \tan 2x}{x - \tan 2x}$

27. $\lim_{x \to 0} \dfrac{\tan 2x}{\tanh 3x}$

28. $\lim_{x \to 0} \dfrac{2x - \sin^{-1} x}{2x + \cos^{-1} x}$

29. $\lim_{x \to 0} \dfrac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$

30. $\lim_{x \to -\infty} xe^x$

31. $\lim_{x \to \infty} e^{-x} \ln x$

32. $\lim_{x \to (\pi/2)^-} \sec 7x \cos 3x$

33. $\lim_{x \to 0^+} \sqrt{x} \sec x$

34. $\lim_{x \to \pi} (x - \pi) \cot x$

35. $\lim_{x \to 1^+} (x - 1) \tan(\pi x/2)$

36. $\lim_{x \to 0} \left(\dfrac{1}{x^4} - \dfrac{1}{x^2} \right)$

37. $\lim_{x \to \infty} \left(x - \sqrt{x^2 - 1} \right)$

38. $\lim_{x \to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \right)$

39. $\lim_{x \to \infty} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1} \right)$

40. $\lim_{x \to \infty} \left(\dfrac{x^3}{x^2 - 1} - \dfrac{x^3}{x^2 + 1} \right)$

41. $\lim_{x \to 0^+} (\sin x)^{\tan x}$

42. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x^2} \right)^x$

43. $\lim_{x \to 0^+} (\cot x)^{\sin x}$

44. $\lim_{x \to \infty} \left(1 + \dfrac{1}{x} \right)^{x^2}$

45. $\lim_{x \to 0^-} (-\ln x)^x$

Answers

E Click here for exercises.

S Click here for solutions.

1. $\frac{1}{4}$

2. 5

3. $\frac{3}{2}$

4. $\frac{1}{2}$

5. 1

6. 2

7. ∞

8. 0

9. 1

10. $\frac{1}{32}$

11. $\frac{1}{3a^{2/3}}$

12. $\ln 3$

13. 0

14. 0

15. α

16. 1

17. 0

18. $\frac{1}{5}$

19. $\frac{2}{3}$

20. $\frac{1}{3}$

21. $\frac{m}{n}$

22. 1

23. -2

24. 0

25. $\frac{1}{2}$

26. -3

27. $\frac{2}{3}$

28. 0

29. $\frac{1}{3}$

30. 0

31. 0

32. $\frac{3}{7}$

33. 0

34. 1

35. $-\frac{2}{\pi}$

36. ∞

37. 0

38. 1

39. 0

40. 0

41. 1

42. 1

43. 1

44. ∞

45. 1

Solutions

E Click here for exercises.

A Click here for answers.

The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$.

1. $\displaystyle\lim_{x\to 2}\frac{x-2}{x^2-4}=\lim_{x\to 2}\frac{x-2}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$

2. $\displaystyle\lim_{x\to 1}\frac{x^2+3x-4}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+4)}{x-1}=\lim_{x\to 1}(x+4)=5$

3. $\displaystyle\lim_{x\to -1}\frac{x^6-1}{x^4-1}\overset{\text{H}}{=}\lim_{x\to -1}\frac{6x^5}{4x^3}=\frac{-6}{-4}=\frac{3}{2}$

4. $\displaystyle\lim_{x\to 0}\frac{\tan x}{x+\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sec^2 x}{1+\cos x}=\frac{1}{1+1}=\frac{1}{2}$

5. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x}{\cos x}=\frac{1}{1}=1$

6. $\displaystyle\lim_{x\to 0}\frac{x+\tan x}{\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1+\sec^2 x}{\cos x}=\frac{1+1^2}{1}=2$

7. $\displaystyle\lim_{x\to 0}\frac{\sin x}{x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x}{3x^2}=\infty$

8. $\displaystyle\lim_{x\to \pi}\frac{\tan x}{x}=\frac{\tan \pi}{\pi}=\frac{0}{\pi}=0$. L'Hospital's Rule does not apply because the denominator doesn't approach 0.

9. $\displaystyle\lim_{x\to 3\pi/2}\frac{\cos x}{x-3\pi/2}\overset{\text{H}}{=}\lim_{x\to 3\pi/2}\frac{-\sin x}{1}=-\sin\tfrac{3\pi}{2}=1$

10. $$\begin{aligned}\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{t-16}&=\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{(\sqrt{t}+4)(\sqrt{t}-4)}\\&=\lim_{t\to 16}\frac{\sqrt[4]{t}-2}{(\sqrt{t}+4)(\sqrt[4]{t}+2)(\sqrt[4]{t}-2)}\\&=\lim_{t\to 16}\frac{1}{(\sqrt{t}+4)(\sqrt[4]{t}+2)}\\&=\frac{1}{(4+4)(2+2)}=\frac{1}{32}\end{aligned}$$

11. $\displaystyle\lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a}\overset{\text{H}}{=}\lim_{x\to a}\frac{(1/3)x^{-2/3}}{1}=\frac{1}{3a^{2/3}}$

12. $$\begin{aligned}\lim_{x\to 0}\frac{6^x-2^x}{x}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{6^x(\ln 6)-2^x(\ln 2)}{1}\\&=\ln 6-\ln 2=\ln\tfrac{6}{2}=\ln 3\end{aligned}$$

13. $$\begin{aligned}\lim_{x\to\infty}\frac{(\ln x)^3}{x^2}&\overset{\text{H}}{=}\lim_{x\to\infty}\frac{3(\ln x)^2(1/x)}{2x}=\lim_{x\to\infty}\frac{3(\ln x)^2}{2x^2}\\&\overset{\text{H}}{=}\lim_{x\to\infty}\frac{6(\ln x)(1/x)}{4x}=\lim_{x\to\infty}\frac{3\ln x}{2x^2}\\&\overset{\text{H}}{=}\lim_{x\to\infty}\frac{3/x}{4x}=\lim_{x\to\infty}\frac{3}{4x^2}=0\end{aligned}$$

14. $\displaystyle\lim_{x\to 0}\frac{\sin x}{e^x}=\frac{0}{1}=0$. L'Hospital's Rule does not apply.

15. $\displaystyle\lim_{x\to 0}\frac{\tan\alpha x}{x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\alpha\sec^2\alpha x}{1}=\alpha$

16. $$\begin{aligned}\lim_{x\to 0}\frac{\sin^2 x}{\tan(x^2)}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{2\sin x\cos x}{2x\sec^2(x^2)}\\&=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{\cos x}{\sec^2(x^2)}=1\cdot 1=1\end{aligned}$$

17. $\displaystyle\lim_{x\to\infty}\frac{\ln\ln x}{\sqrt{x}}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{1/(x\ln x)}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}\ln x}=0$

18. $$\begin{aligned}\lim_{x\to\infty}\frac{\ln(1+e^x)}{5x}&\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x/(1+e^x)}{5}\\&=\lim_{x\to\infty}\frac{e^x}{5(1+e^x)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{5e^x}=\frac{1}{5}\end{aligned}$$

19. $\displaystyle\lim_{x\to 0}\frac{\tan^{-1}(2x)}{3x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{2/(1+4x^2)}{3}=\frac{2}{3}$

20. $$\begin{aligned}\lim_{x\to 0}\frac{x}{\sin^{-1}(3x)}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{1}{3/\sqrt{1-(3x)^2}}\\&=\lim_{x\to 0}\frac{1}{3}\sqrt{1-9x^2}=\frac{1}{3}\end{aligned}$$

21. $\displaystyle\lim_{x\to 0}\frac{\sin mx}{\sin nx}\overset{\text{H}}{=}\lim_{x\to 0}\frac{m\cos mx}{n\cos nx}=\frac{m}{n}$

22. $$\begin{aligned}\lim_{x\to 0}\frac{\sin^{10}x}{\sin(x^{10})}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{10\sin^9 x\cos x}{10x^9\cos(x^{10})}\\&=\left[\lim_{x\to 0}\frac{\sin x}{x}\right]^9\lim_{x\to 0}\frac{\cos x}{\cos(x^{10})}\\&=1^9\cdot 1=1\end{aligned}$$

23. $\displaystyle\lim_{x\to 0}\frac{x+\sin 3x}{x-\sin 3x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1+3\cos 3x}{1-3\cos 3x}=\frac{1+3}{1-3}=-2$

24. $\displaystyle\lim_{x\to 0}\frac{e^{4x}-1}{\cos x}=\frac{0}{1}=0$

25. $$\begin{aligned}\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sec^2 x-\cos x}{3x^2}\\&\overset{\text{H}}{=}\lim_{x\to 0}\frac{2\sec^2 x\tan x+\sin x}{6x}\\&\overset{\text{H}}{=}\lim_{x\to 0}\frac{4\sec^2 x\tan^2 x+2\sec^4 x+\cos x}{6}\\&=\frac{0+2+1}{6}=\frac{1}{2}\end{aligned}$$

26. $$\begin{aligned}\lim_{x\to 0}\frac{x+\tan 2x}{x-\tan 2x}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{1+2\sec^2 2x}{1-2\sec^2 2x}\\&=\frac{1+2(1)^2}{1-2(1)^2}=-3\end{aligned}$$

27. $\displaystyle\lim_{x\to 0}\frac{\tan 2x}{\tanh 3x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{2\sec^2 2x}{3\,\mathrm{sech}^2\,3x}=\frac{2}{3}$

28. $\displaystyle\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\cos^{-1}x}=\frac{2(0)-0}{2(0)+\pi/2}=0$. L'Hospital's Rule does not apply.

29. $\displaystyle \lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{2-1/\sqrt{1-x^2}}{2+1/(1+x^2)}=\frac{2-1}{2+1}=\frac{1}{3}$

30. $\displaystyle \lim_{x\to-\infty}xe^x=\lim_{x\to-\infty}\frac{x}{e^{-x}}\overset{\text{H}}{=}\lim_{x\to-\infty}\frac{1}{-e^{-x}}=\lim_{x\to-\infty}-e^x=0$

31. $\displaystyle \lim_{x\to\infty}e^{-x}\ln x=\lim_{x\to\infty}\frac{\ln x}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{1/x}{e^x}=\lim_{x\to\infty}\frac{1}{xe^x}=0$

32. $\displaystyle \lim_{x\to(\pi/2)^-}\sec 7x\cos 3x=\lim_{x\to(\pi/2)^-}\frac{\cos 3x}{\cos 7x}\overset{\text{H}}{=}\lim_{x\to(\pi/2)^-}\frac{-3\sin 3x}{-7\sin 7x}=\frac{3(-1)}{7(-1)}=\frac{3}{7}$

33. $\displaystyle \lim_{x\to0^+}\sqrt{x}\sec x=0\cdot 1=0$

34. $\displaystyle \lim_{x\to\pi}(x-\pi)\cot x=\lim_{x\to\pi}\frac{x-\pi}{\tan x}\overset{\text{H}}{=}\lim_{x\to\pi}\frac{1}{\sec^2 x}=\frac{1}{(-1)^2}=1$

35. $\displaystyle \lim_{x\to1^+}(x-1)\tan(\pi x/2)=\lim_{x\to1^+}\frac{x-1}{\cot(\pi x/2)}\overset{\text{H}}{=}\lim_{x\to1^+}\frac{1}{-\csc^2(\pi x/2)\frac{\pi}{2}}=-\frac{2}{\pi}$

36. $\displaystyle \lim_{x\to0}\left(\frac{1}{x^4}-\frac{1}{x^2}\right)=\lim_{x\to0}\frac{1-x^2}{x^4}=\infty$

37. $\displaystyle \lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)$

$$=\lim_{x\to\infty}\left(x-\sqrt{x^2-1}\right)\frac{x+\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{x^2-(x^2-1)}{x+\sqrt{x^2-1}}=\lim_{x\to\infty}\frac{1}{x+\sqrt{x^2-1}}=0$$

38. $\displaystyle \lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)$

$$=\lim_{x\to\infty}\left(\sqrt{x^2+x+1}-\sqrt{x^2-x}\right)\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{(x^2+x+1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$$

$$=\lim_{x\to\infty}\frac{2+1/x}{\sqrt{1+1/x+1/x^2}+\sqrt{1-1/x}}=\frac{2}{1+1}=1$$

39. $\displaystyle \lim_{x\to\infty}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to\infty}\frac{1}{x}-\lim_{x\to\infty}\frac{1}{e^x-1}$ (since both limits exist) $=0-0=0$

40. $\displaystyle \lim_{x\to\infty}\left(\frac{x^3}{x^2-1}-\frac{x^3}{x^2+1}\right)=\lim_{x\to\infty}\frac{x^3(x^2+1)-x^3(x^2-1)}{(x^2-1)(x^2+1)}=\lim_{x\to\infty}\frac{2x^3}{x^4-1}=\lim_{x\to\infty}\frac{2/x}{1-1/x^4}=0$

41. $y=(\sin x)^{\tan x}\ \Rightarrow\ \ln y=\tan x\ln(\sin x)$, so

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\tan x\ln(\sin x)=\lim_{x\to0^+}\frac{\ln(\sin x)}{\cot x}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{(\cos x)/\sin x}{-\csc^2 x}=\lim_{x\to0^+}(-\sin x\cos x)=0\ \Rightarrow$$

$$\lim_{x\to0^+}(\sin x)^{\tan x}=\lim_{x\to0^+}e^{\ln y}=e^0=1.$$

42. Let $y=\left(1+\dfrac{1}{x^2}\right)^x$. Then $\ln y=x\ln\left(1+\dfrac{1}{x^2}\right)\ \Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x\ln\left(1+\frac{1}{x^2}\right)=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x^2}\right)}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\left(-\frac{2}{x^3}\right)\Big/\left(1+\frac{1}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{2/x}{1+1/x^2}=0,$$

so $\displaystyle \lim_{x\to\infty}\left(1+1/x^2\right)^x=\lim_{x\to\infty}e^{\ln y}=e^0=1.$

43. $y=(\cot x)^{\sin x}\ \Rightarrow\ \ln y=\sin x\ln(\cot x)\ \Rightarrow$

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln(\cot x)}{\csc x}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{(-\csc^2 x)/\cot x}{-\csc x\cot x}=\lim_{x\to0^+}\frac{\csc x}{\cot^2 x}=\lim_{x\to0^+}\frac{\sin x}{\cos^2 x}=0$$

so $\displaystyle \lim_{x\to0^+}(\cot x)^{\sin x}=\lim_{x\to0^+}e^{\ln y}=e^0=1.$

44. Let $y=(1+1/x)^{x^2}$. Then $\ln y=x^2\ln(1+1/x)\ \Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}x^2\ln(1+1/x)=\lim_{x\to\infty}\frac{\ln(1+1/x)}{1/x^2}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{(-1/x^2)/(1+1/x)}{-2/x^3}=\lim_{x\to\infty}\frac{x}{2(1+1/x)}=\infty\ \Rightarrow$$

$$\lim_{x\to\infty}(1+1/x)^{x^2}=\lim_{x\to\infty}e^{\ln y}=\infty.$$

45. $y=(-\ln x)^x\ \Rightarrow\ \ln y=x\ln(-\ln x)$, so

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}x\ln(-\ln x)=\lim_{x\to0^+}\frac{\ln(-\ln x)}{1/x}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{(1/-\ln x)(-1/x)}{-1/x^2}=\lim_{x\to0^+}\frac{-x}{\ln x}=0\ \Rightarrow$$

$$\lim_{x\to0^+}(-\ln x)^x=e^0=1.$$