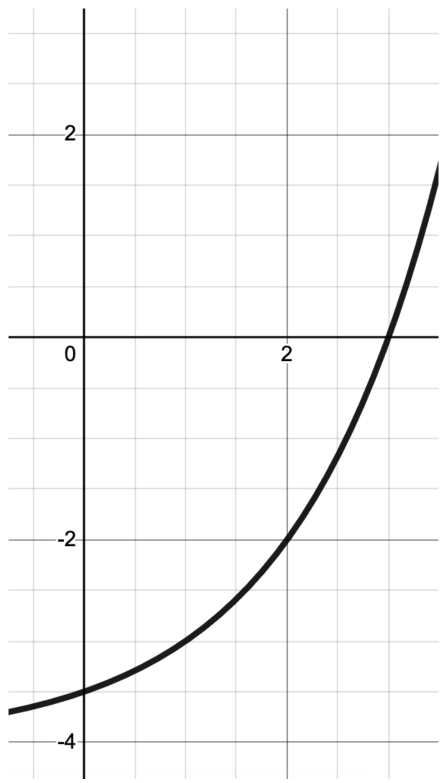
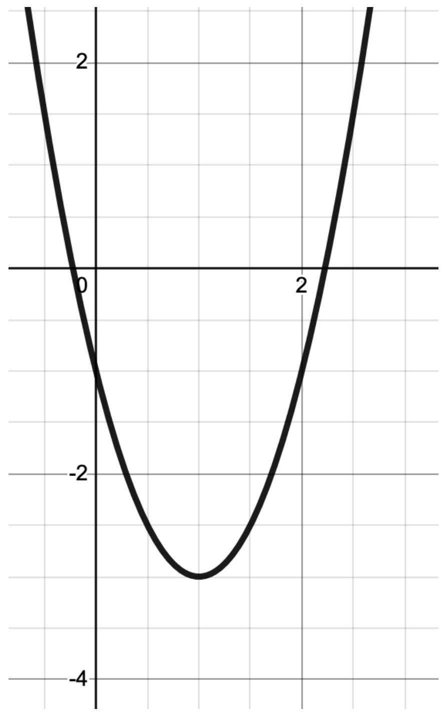
Problem Set 3 (Total Points: 435), Due July 17th

Review Questions

Problem (1)**.** (10 points) The following graph is $f(x) = a2^{b+x} + c$. Find a, b, and c.



Problem (2)**.** (10 points) The following graph is $f(x) = ax^2 + bx + c$. Find a, b, and c.



Problem (3)**.** (35 points)

(a) Sketch a graph of $f(x) = \dfrac{(x^2 - 4x + 5)(x - 2)}{(x^2 - 3x + 2)(x - 3)}$.

(b) Identify the removable discontinuity on the graph with a hole.

(c) Find $\lim_{x \to \infty} f(x)$ (your answer may be infinity).

(d) Find $\lim_{x \to 3^+} f(x)$ (your answer may be infinity).

(e) Find $\lim_{x \to 1^-} f(x)$ (your answer may be infinity).

(f) Find $\lim_{x \to 0^+} f(x)$ (your answer may be infinity).

(g) Sketch a tangent line on your graph at $x = 2$. What is $f'(2)$?

Problem (4)**.** (30 points) Here's an argument for why $(x^4 + 1)' = 4x^3$ using the limit definition of the derivative. There are three mistakes in the argument. Identify them and suggest corrections. (Please use complete sentences for this question.)

$$\begin{align}
(x^4+1)' &= \lim_{h\to 0} \frac{(x+h)^4+1-x^4+1}{h} \tag{1}\\
&= \lim_{h\to 0} \frac{(x+h)(x+h)(x+h)(x+h)-x^4+2}{h} \tag{2}\\
&= \lim_{h\to 0} \frac{(x^2+2xh+h^2)(x^2+2xh+h^2)-x^4+2}{h} \tag{3}\\
&= \lim_{h\to 0} \frac{x^4+2x^3h+x^2h^2+2x^3h+2x^2h^2+2xh^3+(x^2+2xh+h^2)(h^2)-x^4+2}{h} \tag{4}\\
&= \lim_{h\to 0} \frac{4x^3h+h^2(x^2+2x^2+2xh+x^2+2xh+h^2)+2}{h} \tag{5}\\
&= \lim_{h\to 0} 4x^3+h(4x^2+4xh+h^2)+\frac{2}{h} \tag{6}\\
&= 4x^3+\lim_{h\to 0} h(4x^2+4xh+h^2)+\lim_{h\to 0}\frac{2}{h} \tag{7}\\
&= 4x^3+0+0 \tag{8}\\
&= 4x^3 \tag{9}
\end{align}$$

Problem (4)**.** (20 points) For this problem it will be helpful to know the following limit $\lim_{h\to 0} \frac{e^h-1}{h} = 1$. (Sometimes this limit is used to define e.) Using the limit definition of the derivative show $(e^x)' = e^x$. Show your work.

Computing Derivates Questions

Problem (5)**.** (30 points) State the following derivate rules:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}x^m$ where m is a positive number.

(b) $\frac{\mathrm{d}}{\mathrm{d}x}c$ where c is a constant.

(c) $\frac{\mathrm{d}}{\mathrm{d}t}a(t)+b(t)$ where a and b are differentiable functions.

(d) $\frac{\mathrm{d}}{\mathrm{d}t}a(t)b(t)$ where a and b are differentiable functions.

(e) $\frac{\mathrm{d}}{\mathrm{d}t}\frac{a(t)}{b(t)}$ where a and b are differentiable functions and b is nonzero.

(f) $\frac{\mathrm{d}}{\mathrm{d}t}b(a(t))$ where a and b are differentiable functions.

Problem (6)**.** (30 points) State the following standard derivates:

(a) $(1/x)'$

(b) $\sqrt{(}x)'$

(c) $(e^x)'$

(d) $\ln(x)'$

(e) $\sin(x)'$

(f) $\cos(x)'$

(g) $\tan(x)'$

(h) $\cot(x)'$

(i) $\csc(x)'$

(j) $\sec(x)'$

For **these problems** you may use the product rule, chain rule, etc. Compute derivatives of the following expressions (5 points each):

(7) $2x^2 + 1$

(8) $z^4 + z^2 - 2$

(9) $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

(10) $-\frac{1}{x^3}$

(11) $3x^{-2/3} - x^{1/5}$

(12) $\sin(x)\cos(x)$

(13) $x\ln(x) - x$

(14) $e^x(\sin(x) - \cos(x))$

(15) $\dfrac{2x^2+1}{x}$

(16) $\dfrac{\tan(x)+1}{\cos(x)+1}$

(17) $\ln(\sec(x))$

(18) e^{-x^2+x+1}

Problem (20)**.** (10 points) Find the derivative of $\ln(\sec(x)+\tan(x))$. Make sure to fully simplify your answer.

Problem (21)**.** (60 points) For the following two functions find $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$. Then guess the general form of $f^{(n)}(0)$ (the n^{th} derivative of f and 0) where n is a natural number. You may need to compute even more derivatives to see the pattern for the general case. You may not be able to write a specific formula if you can't write out a couple sentences to explain the pattern. For the second problem it will be useful to know the factorial symbol $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$.

(a) $\sin(x)$

(b) $\ln(x+1)$

Problem (22)**.** (30 points) Let

$$f(x) = \begin{cases} \operatorname{sgn}(x+2) & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \text{ and } x < 2 \\ \dfrac{2x-8}{x^2-9x+20} & \text{if } x \geq 2 \text{ and } x \neq 4 \\ 0 & \text{if } x = 4. \end{cases}$$

Sketch a graph of $f(x)$ (feel free to use Desmos). There are 4 points where f is not continuous. Point them out on the graph and label them as either jump discontinuities, removable discontinuities, or essential discontinuities.

Problem (23)**.** (40 points) Let

$$f(x) = \begin{cases} |x+2| & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \text{ and } x < 1 \\ (x-2)^{\frac{2}{5}} - \dfrac{2}{5}x & \text{if } x \geq 2 \text{ and } x < 3 \\ 2.2\sqrt{4-x} & \text{if } x \geq 3 \text{ and } x < 4 \\ -\sqrt{\left(\sin\left(\dfrac{\pi}{2}x\right)\right)} & \text{if } x \geq 4 \text{ and } x \leq 5 \\ (x-5)\sin\left(\dfrac{1}{x-5}\right) - 1 & \text{if } x > 5. \end{cases}$$

Sketch a graph of $f(x)$ (feel free to use Desmos). There are 6 points where f is not differentiable. Point them out on the graph and label them as either cusps, corners, vertical tangents, or other points of non-differentiability.

Problem (24)**.** (40 points) Suppose the price of bar of Connor's chocolate is increasing at a rate of 0.1 dollars per month in the US. To import one bar of Connor's chocolate to Brazil it costs 3 Brazilian Reais and the exchange rate of Brazilian Real always to US Dollars is 1 dollar is 4.8 Brazilian Reais. Suppose the Brazilian central bank keeps the exchange rate fixed (this is called pegging).

(a) If the price of a bar of Connor's chocolate is x US Dollars find a formula for the price of Connor's chocolate in Brazilian Reais.

(b) Find the inflation rate of Connor's chocolate in Brazil. Express your answer in terms of Brazilian Reais per month.

Suppose now that Brazil goes through a debt crisis, the peg breaks, and Brazil starts to experience inflation. Now the exchange rate is $6.1 + 0.7t$ Reais to 1 US Dollar where t is the number of months since the debt crisis.

(c) Find the new inflation rate of Connor's chocolate. Express your answer in Brazilian Reais per month. Your answer will depend on t.

Now suppose a severe forest fire burns down large parts of the Amazon leading to soy farms being destroyed. This causes the debt crisis to worsen. Now Brazil experiences hyperinflation. The new exchange rate is $16 \cdot 2^t$ Reais to 1 US Dollar where t is the number of months since the forest fire.

(d) Find the new inflation of Connor's chocolate. Express your answer in Brazilian Reais per month. Your answer will depend on t.

Problem (25)**.** (30 points) Connor's Electric Car Company is developing an experimental ion powered car. In a test run the distance it travels along a road is $d(t) = 20t^{2.5} + 30t^2$ meters (t has units in seconds).

(a) Find the velocity of the ion car over time. Express your answer in terms of meters per second.

(b) Find the acceleration of the ion car over time. Express your answer in terms of meters per second squared.

Suppose the energy the engine of the ion car consumes is given by $E(t) = 40t^{0.5} + 100$ Megawatt hours. Recall that power is the derivate of energy over time.

(c) Find the power output of the ion car's engine over time. Express your answer in terms Megawatt hours per second.