The fundamental theorem of calculus

Suppose $F'(x) = f(x)$

Then $\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$

Example $\int_0^{\pi} \sin(x)\,dx = (-\cos(\pi)) - (-\cos(0)) = --1 --1 = 2$

Because $(-\cos x)' = \sin(x)$

Practice

- $\int_1^2 e^x\,dx$
- $\int_0^1 -e^x\,dx$
- $\int_{\pi/2}^{\pi} \cos(x)\,dx$
- $\int_0^{\pi/4} \sec^2(t)\,dt$
- $\int_1^{e^2} \frac{1}{x}\,dx$
- $\int_{-\pi}^{0} 2\sin(t)\,dt$

Notation $\int f(x)\,dx = F(x) + C$ "indefinite integral"

reason: $\int_0^x f(t)\,dx = F(x) - F(0)$ (call $-F(0) = C$)

Example $\int x^3 + 2x^2 + 3x - 1\,dx$

$= \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{3}{2}x^2 - x + C$

because $\left(\frac{x^4}{4} + \frac{2}{3}x^3 + \frac{3}{2}x^2 - x + C\right)' = x^3 + 2x^2 + 3x - 1$

Practice:

- $\int x^5\,dx$
- $\int (x+1)^2\,dx$
- $\int \sec(x)\tan(x)\,dx$
- $\int \frac{1}{x^3}\,dx$
- $\int \frac{1}{2\sqrt{t}}\,dt$
- $\int 2u^3\,du$
- $\int \csc^2(u)\,du$
- $\int v + v^3 + v^5\,dv$

Hint $\left(\frac{1}{n+1}x^{n+1}\right)' = x^n$ $(n \neq -1)$ "The inverse power rule"

Modifying integrals

Suppose $\int_a^b f(x)\,dx = Q$ and $\int_a^b g(x)\,dx = R$

then $\int_a^b kf(x)\,dx = kQ$ for any k

and $\int_a^b f(x)+g(x)\,dx = Q+R$

Practice

- Find $\int_0^1 x+\sqrt{1-x^2}\,dx$
- Suppose $\int_2^3 f(x)\,dx = 5$ find $\int_2^3 f(x)+x^3\,dx$ and $\int_2^3 -3f(x)\,dx$

Facts

$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Practice

- Suppose $\int_1^{-1} f(x)\,dx = -2$, $\int_{-1}^{7} f(x)\,dx = 12$ find $\int_{-1}^{1} f(x)\,dx$ and $\int_1^7 f(x)\,dx$

Compressing and Shifting with integrals

$\int_a^b f(x+k)\,dx = \int_{a+k}^{b+k} f(x)\,dx$

$\int_a^b f(kx)\,dx = \frac{1}{k}\int_{ak}^{bk} f(x)\,dx$

Practice:

Find

- $\int_0^9 e^{3x}\,dx$
- $\int_0^1 2^x\,dx$
- $\int_0^1 \frac{1}{x-3}\,dx$
- $\int_{-1/5}^{0} \sqrt{1-(5x)^2}\,dx$
- $\int_0^{1/2} (2x+1)^{100}\,dx$
- $\int_{\pi/24}^{\pi/12} 4\csc(4x)\cot(4x)\,dx$

Suppose $\int_0^2 f(x)\,dx = -2$, find $\int_2^4 -f(x-2)\,dx$

Challenge $\int xe^x dx$, $\int x\sin(x)\,dx$, $\int xe^{-x}dx$, $\int \frac{dx}{2x+x^2}$, $\int \frac{dx}{x^2-3x+2}$