MATHD022: Discrete Mathematics

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Contents

1. The Language of Mathematics

1.1 Variables

Definition A **variable** is a symbol that is used as a placeholder when:

- The quantity has one of more values, but is not known.
  - For example: $2x^2 - x = 7$
- The quantity represents **any element** from a given set.
  - For example: The reciporical of any non-zero integer n is $\frac{1}{n}$.

Writing Sentences using Variables We can rewrite the following sentences using variables:

- Is there an integer n that has a remainder of 2 when it is divided by 5?
  - Is there an integer n such that $n\%5 = 2$?
- The cube root of any negative real number is negative.
  - For any real number s, if $s < 0$, then $\sqrt[3]{s} < 0$.

Types of Statements

- A **universal statement** is a statement that is true always true.
  - For example: **All** positive numbers are greater than 0.
- A **conditional statement** is a statement that is true if a certain condition is met.
  - For example: **If** 378 is divisible by 18, **then** 378 is divisible by 6.
- A **universal conditional statement** is a statement that is both conditional and universal.

  - For example: **For all** animals a, if a is a dog, **then** a is a mammal.
  - As a universal statement: **For all** dogs a, a is a mammal.
  - As a conditional statement: **If** a is a dog, **then** a is a mammal.

- An **existential statement** gives a property that is true for at least one thing.
  - **There is** a prime number that is even.

- A **universal existential statement** is a statement where the first part is universal and the second part is existential.
  - **Every** real number **has** an additive inverse.
  - **For all** real numbers r, **there is** an additive inverse $-r$.
  - **For all** real numbers r, **there is** a real number s such that $r+s = 0$.

- An **existential universal statement** is a statement where the first part is existential and the second part is universal.
  - **There is** a positive integer that is less than or equal to **every** positive integer.
  - **There is** a positive intefer m such that **every** positive integer is greater than or equal to m.
  - **There is** a positive integer m with the property that **for all** positive integers n, $m \leq n$.

1.2 Sets

Definition A **set** is a collection of objects.

Notation

- $x \in S$: x is an element of S.
- $x \notin S$: x is not an element of S.
- $S = \{1, 2, 3, \dots\}$: is **set roster notation.**

Axion of Extension A set is determined by what its elements are. Orders of elements or repeated elements can't be determine the set.
For example: $\{1, 2, 3\} = \{3, 2, 2, 1, 2, 3, 1\}$. There are 3 elements in both sets.

Common Sets

- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{Z}$: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ the set of all integers.
- $\mathbb{N}$: $\{1, 2, 3, \dots\}$ the set of all natural numbers.
- $\mathbb{Q}$: the set of all rational numbers.
- $\emptyset = \{\}$: the empty set, or null set.

The null set is a subset of every set.

Set Builder Notation Let S denote a set and let $x \in S$ be and element in S. $P(x)$ is a property that some elements of S satisfy.

$$A = \{x \in S | P(x)\}$$

A constains elements in S such that (—) P(x) is true.

Subsets

Definition Let A and B be sets. A is a **subset** ($\subseteq$) of B if every element of A is also an element of B.

Proper Subsets Let A and B be sets. A is a **proper subset** ($\subset$) of B if every element of A is also an element of B, **and** there is at least one element in B that is not in A.

Example Let $A = \mathbb{Z}^+, B = \{n \in \mathbb{Z} | 0 \leq n \leq 100\}, and C = \{100, 200, 300, 400, 500\}$.

- $B \subseteq A$ is false.
- $C \subset A$ is true.
- $C \subseteq B$ is false.
- $C \subseteq C$ is true.

Cartesian Product of sets Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Example Let $A = \{1, 2, 3\}$ and $B = u, v$.

$$A \times B = \{(1, u), (1, v), (2, u), (2, v), (3, u), (3, v)\}$$
$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

1.3 Relations and Functions

Relations Let A and B be sets. A **relation** from A to B is a subset of the Cartesian product $A \times B$.

$$R \subseteq A \times B$$

- If (x,y) $\in R$, we say that x is related to y by R, denoted as xRy.
- **A** is in the **domain** of **R**
- **B** is the **codomain** of **R**

Example Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$ and define a relation R from A to B as follows:

$$\begin{aligned}(x, y) \in R \iff & \frac{x+y}{2} \in \mathbb{Z} \\ R = & \{(1,1),(1,2),(2,1),(2,2),(3,1)\} \\ & \text{Domain of } R = \{1, 2, 3\} \\ & \text{Codomain of } R = \{1, 2\}\end{aligned}$$

Functions

Definition Let A and B be two sets. A function F from A to B is a relation with domain A and co-domain B that satisfies the following properties:

- For every element $x \in A$, there is an element $y \in B$ such $(x, y) \in F$
- For every element $x \in A$,

Example Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations defined below are functions from A to B?

- $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$
  - Not a function because 4 is related to 1 and 3. This is not a many-to-one relationship.
- For all $(x.y) \in A \times B, (x, y) \in S \iff y = x + 1$
  - $S = \{(2, 3), (4, 5)\}$ is a function from A to B.
- $T = \{(2, 5), (4, 1), (6, 1)\}$
  - T is a function from A to B as A has a many-to-one relationshop with B.

Equivalent Functions

Let A and B be two sets. Two functions f and g from A to B:

$$f = g \iff f(x) = g(x) \quad \forall \quad x \in A$$

2. The Logic of Compound Statements

2.1 Logical Form and Equivalence

Arguments

Definition An arguement is a sequence of statements aimed at demonstrating the truth of an assertion.

- The assertion at the end of the sequence is called the conclusion.
- The statements that support the conclusion are called premises.
- If the premises are true, the conclusion must also be true.

Example

- If student A is a math major or student A is a computer science major,
- Then student A will take Discrete Math.

Logical Statements

Definition A logical statement is a declarative sentence that is either true or false, but not both.

- Not p: $\neg p$
- p and/but q: $p \wedge q$
- p or q: $p \vee q$
- Neither p nor q: $\neg p \wedge \neg q$

Example h = healthy, w = wealthy, s = wise

- John is healthy and wealthy but not wise.
  - $(h \wedge w) \wedge \neg s$
- John is neither wealthy nor wise, but he is healthy
  - $(\neg w \wedge \neg s) \wedge h$

Equivalent Statements

Definition Two logical statements are equivalent if they have the same truth tables, denoted:

$$p \equiv q$$

De Morgan's Laws The negation ($\neg$) of an and statement is logically equivalent to the or statement of the negations. Similarly, the negation of an or statement is logically equivalent to the and statement of the negations.

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Tautological and Condtradictory Statements

- A tautological statement is a statement that is always true.
- A contradictory statement is a statement that is always false.

2.2 Conditional Statements

Definition A Conditional statement is in the form "If p, then q" and is denoted as $p \implies q$ This is read as p implies q.

- p is the **hypothesis** of the statement.
- q is the **conclusion** of the statement.

Order of Operations

- (): parentheses
- $\neg$: negation
- $\wedge/\vee$: conjunction/disjunction
- $\implies$: implication

Equivalent of Conditional Statements

$$p \implies q \equiv \neg p \vee q$$
$$\neg(p \implies q) \equiv p \wedge \neg q$$

Example Find the negation of the following statement: "If my car is in the repair shop then I cannot go to class".

- Hypothesis (p): "My car is in the repair shop"
- Conclusion (q): "I cannot go to class"
- Convert: $p \implies q \equiv \neg p \vee q$
- Negation: $\neg(p \implies q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$
- Convert back: "My car is in the repair shop and I can go to class"

Negation vs Inverse The negation of a statement is NOT the same as the inverse of the statement.

- Negation: $\neg(p \implies q)$
- Inverse: $\neg p \implies \neg q$

Example If p is a square, then p is a rectangle.

- Hypothesis (p): "p is a square"
- Conclusion (q): "p is a rectangle"
- Negation: $\neg(p \implies q) \equiv p \land \neg q$
- Convert back: "p is a square and p is not a rectangle"
- Inverse: $\neg p \implies \neg q \equiv p \lor \neg q$
- Convert: "If p is not a square, then p is not a rectangle"

More statement types

- Contrapositive of $p \implies q \equiv \neg q \implies \neg p$
- Converse of $p \implies q \equiv q \implies p$
- Inverse of $p \implies q \equiv \neg p \implies \neg q$

Example If today is Easter then tomorrow is Monday.

- Hypothesis (p): "Today is Easter"
- Conclusion (q): "Tomorrow is Monday"
- Convert: $p \implies q$
- Contrapositive: $\neg q \implies \neg p \equiv$ If tomorrow is not Monday, then today is not Easter
- Converse: $q \implies p \equiv$ If tomorrow is Monday, then today is Easter
- Inverse: $\neg p \implies \neg q \equiv$ If today is not Easter, then tomorrow is not Monday

Biconditional Statements A biconditional statement is in the form "p if and only if q" and is denoted as $p \iff q$. This is read as p if and only if q.

$$p \iff q \equiv (p \implies q) \wedge (q \implies p) \tag{1}$$

Sufficient and Necessary Conditions If r and s are statements:

- r is a **sufficient condition** for s if $r \implies s$.
- r is a **necessary condition** for s if $s \implies r$ or $s \implies r$.
- r is a **necessary and sufficient condition** for s if $r \iff s$.

2.3 Valid and Invalid Arguments

Definition An **argument** is a sequence of statements, and an **argument form** is a sequence of statement form.

- The final statement or statement form is called the **conclusion**. The symbol $\therefore$ (therefore) is used to denote the conclusion.
- All the preceding statements or statement forms are called **premises**, or assumptions or hypotheses.
- An argument form is **valid** means if all premises are true, then the conclusion must also be true.

Example Determine whether the following argument form is valid or invalid:

$$\begin{aligned} p &\implies q \vee \neg r \\ q &\implies p \wedge r \\ \therefore p &\implies r \end{aligned}$$

p	q	r	$p \implies (q \vee \neg r)$	$q \implies (p \wedge r)$	$p \implies r$	Valid?
T	T	T	T	T	T	Valid
T	T	F	F	T	F	Invalid
T	F	T	T	F	T	Invalid
T	F	F	F	F	F	Invalid
F	T	T	T	F	T	Invalid
F	T	F	T	F	T	Invalid
F	F	T	T	F	T	Invalid
F	F	F	T	F	T	Invalid

Therefore the argument form is invalid.

Syllogisms

Definition An argument form with two premisies are called syllogism. The firest and second premises are called the major premise and minor premise respectively.

Modus Ponens Modus Ponens is a valid argument form that can be expressed as:

$$\begin{aligned} &p \implies q \\ &p \\ \therefore\ &q \end{aligned}$$

This means that if $p \implies q$ (if p then q) is true, and p is true, then we can conclude that q must also be true.

Example If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.
$\therefore$ At least two pigeons roost in the same hole.

Modus Tollens Modus Tollens is a valid argument form that can be expressed as:

$$\begin{aligned} &p \implies q \\ &\neg q \\ \therefore\ &\neg p \end{aligned}$$

This means that if $p \implies q$ (if p then q) is true, and q is false, then we can conclude that p must also be false.

Rules of Inference A rule of inference is a form of argument that is valid. Both modus ponens and modus tollens are rules of inference. The following are additional examples of rules of inference:

A ***rule of inference*** is a form of argument that is valid. Both modus ponens and modus tollens are rule of inference. The following are additional examples of rules of inference.

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	**a.** $p \vee q$ $\sim q$ $\therefore p$ **b.** $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	**a.** p $\therefore p \vee q$ **b.** q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	**a.** $p \wedge q$ $\therefore p$ **b.** $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ (contradiction) $\therefore p$

Prove by Detachment
Prove by contrapositive
Disjunctive of syllogism
Law of Syllogism

Contradictions

Definition A contradiction is a statement that is always false.

$$\neg p \implies c$$
$$\therefore p$$

2 column rule The 2 column rule is a way to prove by contradiction. For example with knights and knaves. Knights always tell the truth and knaves always lie:

- A says B is a knight
- B says A and I are of opposite types

Suppose A is a knight:

What A says must be true	By the definition of a knight
B is a knight	by given (what A says)
What B says must be true	By the definition of a knight
A and B are of opposite types	by given (what B says)
Contradiction	A is not a knight or A is a knave
The supposition is false	by rule of contradiction
A is not a knight or A is a knave	by negation of supposition.

3. The Logic of Quantified Statements

3.1 Predicates and Quantified Statements (Part 1)

Predicates

Definition A predicate is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. For example: "$P(x) : x$ is a positive integer" is a predicate. The statement $P(3)$ is true, while $P(-2)$ is false.

Domain of a Predicate The Domain of a predicate is the set of all values that can be substituted for the variable.

Example Let $P(x)$ be the predicate "$x^2 > x$." The domain of $P(x)$ is $\mathbb{R}$.

$$\begin{aligned} P(\frac{1}{2}) :& \quad (\frac{1}{2})^2 > \frac{1}{2} && = False \\ P(-\frac{1}{2}) :& \quad (-\frac{1}{2})^2 > -\frac{1}{2} && = True \\ P(2) :& \quad 2^2 > 2 && = True \end{aligned}$$

Truth Sets

Definition If $P(x)$ is a predicate with domain D, the truth set of $P(x)$ is the set of all elements in D for which $P(x)$ is true when they are substituted for x. The truth set of $P(x)$ is denoted by:

$$\{x \in D \ni P(x)\} \subseteq D$$

Example Let $P(x)$ be the predicate "$n^2 \leq 30$" with domain $\mathbb{Z}$. The truth set of $P(x)$ is:

$$\{x \in \mathbb{Z} \ni P(x)\} = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$$

Quantified Statements

Definition A quantified statement is a statement that contains a quantifier. The two most common quantifiers are:

- **Universal Quantifier** $\forall$ (for all)
- **Existential Quantifier** $\exists$ (there exists)

Universal Statements Let $P(x)$ be a predicate with domain D. A univseral statment is a statement of the form "$\forall x \in D, P(x)$" which is read as "for all x in D, $P(x)$ is true."

- It is defined to be true if and only if $P(x)$ is true for all x in D.
- It is defined to be false if and only if $P(x)$ is false for at least one x in D.
- The value of x for which $P(x)$ is false is called a counterexample.

Example Let $D = \{1, 2, 3\}$, and show that the statement "$\forall x \in D, x^2 \geq x$" is true.

$$\begin{aligned} & 1^2 \geq 1 \text{ is true} \\ & 2^2 \geq 2 \text{ is true} \\ & 3^2 \geq 3 \text{ is true} \\ \therefore \quad & \forall x \in D, x^2 \geq x \text{ is true.} \end{aligned}$$

Existential Statements Let $P(x)$ be a predicate with domain D. An existential statement is a statement of the form "$\exists x \in D \ni P(x)$" which is read as "there exists an x in D such that $P(x)$ is true."

- It is defined to be true if and only if $P(x)$ is true for at least one x in D.
- It is defined to be false if and only if $P(x)$ is false for all x in D.
- The value of x for which $P(x)$ is true is called a witness.

Example Show that the statement "$\exists x \in \mathbb{Z} \ni \frac{1}{x} = x$" is true.

$$x = 1 : \frac{1}{1} = 1 \text{ is true}$$
$$\therefore \quad \exists x \in \mathbb{Z} \ni \frac{1}{x} = x \text{ is true.}$$

Universal Conditional Statements A universal conditional statement is a statement of the form "$\forall x \in D, P(x) \implies Q(x)$" which is read as "for all x in D, if $P(x)$ is true, then $Q(x)$ is true."

3.2 Predicates and Quantified Statements (Part 2)

Negations of Quantified Statements

Negation of a Universal Statement

$$\neg(\forall x \in D, P(x)) \equiv \exists x \in D \ni \neg P(x)$$

Negation of an Existential Statement

$$\neg(\exists x \in D \ni P(x)) \equiv \forall x \in D, \neg P(x)$$

Negation of a Universal Conditional Statement

$$\neg(\forall x \in D, P(x) \implies Q(x)) \equiv \exists x \in D \ni P(x) \wedge \neg Q(x)$$

Consider the statement: $\forall x \in D, P(x) \implies Q(x)$.

$$\begin{aligned} \text{It's contrapositive is:} \quad & \forall x \in D, \neg Q(x) \implies \neg P(x) \\ \text{It's converse is:} \quad & \forall x \in D, Q(x) \implies P(x) \\ \text{It's inverse is:} \quad & \forall x \in D, \neg P(x) \implies \neg Q(x) \end{aligned}$$

3.3 Statements with Multiple Quantifiers

Consider the statement: $\forall x \in D, \exists y \in E \ni P(x,y)$. To show the truth of the statement, we must show that for every x in D, there exists a y in D such that $P(x,y)$ is true.

Example Let $D = \{1,2,3\}$ and $P(x,y)$ be the predicate "$x+y = 4$". Show that the statement $\forall x \in D, \exists y \in D \ni P(x,y)$ is true.

$$\begin{aligned}
x = 1: &\quad \exists y \in D \ni 1 + y = 4 \implies y = 3 \\
x = 2: &\quad \exists y \in D \ni 2 + y = 4 \implies y = 2 \\
x = 3: &\quad \exists y \in D \ni 3 + y = 4 \implies y = 1 \\
\therefore &\quad \forall x \in D, \exists y \in D \ni P(x,y) \text{ is true.}
\end{aligned}$$

Consider the statement: $\exists x \in D \ni \forall y \in D, P(x,y)$. To show the truth of the statement, we must show that there exists an x in D such that for every y in D, $P(x,y)$ is true.

Example Let $D = \{1,2,3\}$ and $P(x,y)$ be the predicate "$x+y = 4$". Show that the statement $\exists x \in D \ni \forall y \in P, P(x,y)$ is false.

$$\begin{aligned}
x = 1: &\quad \forall y \in D, 1 + y = 4 \implies y = 3 \\
x = 2: &\quad \forall y \in D, 2 + y = 4 \implies y = 2 \\
x = 3: &\quad \forall y \in D, 3 + y = 4 \implies y = 1 \\
\therefore &\quad \exists x \in D \ni \forall y \in D, P(x,y) \text{ is false.}
\end{aligned}$$

Negation of Multiply-Quantified Statements

$$\neg(\forall x \in D, \exists y \in E \ni P(x,y)) \equiv \exists x \in D \ni \forall y \in E, \neg P(x,y))$$
$$\neg(\exists x \in D, \ni \forall y \in E, P(x,y)) \equiv \forall x \in D, \exists y \in E \ni \neg P(x,y)$$

3.4 Arguments with Quantified Statements

Universal Model Ponens (Direct Proof)

	$\forall x, P(x) \implies Q(x)$	If x makes P(x) true, then x makes Q(x) true.
	$P(a)$	Input a makes P(a) true.
$\therefore$	$Q(a)$	Therefore a makes Q(a) true.

Example Let $P(x)$ be the predicate "x is a prime number" and $Q(x)$ be the predicate "x is an odd number".

	$\forall x, P(x) \implies Q(x)$	If x is a prime number, then x is an odd number.
	$P(3)$	3 is a prime number,
$\therefore$	$Q(3)$	therefore 3 is an odd number.

Universal Modus Tollens (Prove by Contradiction)

	$\forall x, P(x) \implies Q(x)$	If x makes P(x) true, then x makes Q(x) true.
	$\neg Q(a)$	Input a makes Q(a) false.
$\therefore$	$\neg P(a)$	Therefore a does not make P(a) true.

Example Consider the statement "All irrational numbers are real numbers.":

	$\forall x \in \mathbb{R} - \mathbb{Q}, x \in \mathbb{R}$	If x is an irrational number, then x is a real number.
	$\frac{1}{0} \notin \mathbb{R}$	$\frac{1}{0}$ is not a real number,
$\therefore$	$\frac{1}{0} \notin \mathbb{R} - \mathbb{Q}$	therefore $\frac{1}{0}$ is not an irrational number.

Converse and Inverse Errors

Converse Error

$$\forall x, P(x) \implies Q(x) \qquad Q(a) \therefore \quad P(a) \text{(Invalid Arguement)}$$

Inverse Error

$$\forall x, P(x) \implies Q(x) \qquad \neg P(a) \therefore \quad \neg Q(a)(\text{Invalid Arguement})$$

4. Elementary Number Theory and Methods of Proof

4.1 Direct Proof and Counterexample (Part 1)

Definitions Let P(n) be the predicate "n is an even number".

$$\forall n \in \mathbb{Z}, P(n) \iff \exists k \in \mathbb{Z} \ni n = 2k.$$
$$\forall n \in \mathbb{Z}, \neg P(n) \iff \exists k \in \mathbb{Z} \ni n = 2k + 1.$$

Example Is -301 even or odd?

$$-301 = 2k + 1 \text{ for } k = -151$$

Example If $a, b \in \mathbb{Z}$, is $6a^2b$ even?

$$\begin{aligned} & \exists a, b \in \mathbb{Z} \ni 6a^2b = 2(k) + 1 \\ & 6a^2b = 2(3a^2b) \text{ for } k = 3a^2b \\ \therefore \quad & 6a^2b \text{ is even.} \end{aligned}$$

Prime and Composite Number Definition Let P(n) be the predicate "n is a prime number".

$$\forall n \in \mathbb{Z}_{>1}, P(n) \iff \forall r, s \in \mathbb{Z}_{>1}, n = rs \implies r = n \vee s = n$$
$$\forall n \in \mathbb{Z}_{>1}, \neg P(n) \iff \exists r, s \in \mathbb{Z}_{>1} \ni n = rs \wedge 1 < r < n \wedge 1 < s < n$$

Constructive Proof of Existential Statement

$$\exists x in D \ni Q(x)$$

- Find an x in D that makes !(x) true.
- Give a set of directions for finding such an x in D

Example Prove there is and even integer n such that n can be written in two ways as a sum of two prime numbers.

$$\begin{aligned} &\text{Let } n = 10, \\ &\quad 10 = 3 + 7 \\ &\quad 10 = 5 + 5 \\ &\therefore \quad \text{the statement is true.} \end{aligned}$$

Disproving Universal Statement by Counterexample

$$\forall x in D, P(x) \implies Q(x)$$

- Find an x in D that makes P(x) true, but Q(x) false.

Method of Exhaustion of Proving Universal Statement

$$\forall x in D, P(x) \implies Q(x)$$

- Check all x in D to make sure that when P(x) is true, Q(x) is false.

Direct Proof of Universal Statement

$$\forall x \in D, P(x) \implies Q(x)$$

- Suppose x is an arbitrary element in D for which the hypothesis P(x) is true.
- Using definitions or previously established results and rules to conclude Q(x) is true.

Example Prove the statement "the sum of any two even integers is even."

$$\begin{aligned} & \text{Suppose a and b are two even integers} \\ \therefore \quad & a = 2k, \exists k_1 \in \mathbb{Z} \\ \therefore \quad & b = 2k, \exists k_2 \in \mathbb{Z} \\ \therefore \quad & a + b = 2k_1 + 2k_2 \\ \therefore \quad & a + b = 2(k_1 + k_2) \\ \therefore \quad & a + b \text{ is even} \end{aligned}$$