MATHD022: Discrete Mathematics

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Contents

1. The Language of Mathematics

1.1 Variables

Definition A **variable** is a symbol that is used as a placeholder when:

- The quantity has one of more values, but is not known.
  - For example: $2x^2 - x = 7$
- The quantity represents **any element** from a given set.
  - For example: The reciporical of any non-zero integer n is $\frac{1}{n}$.

Writing Sentences using Variables We can rewrite the following sentences using variables:

- Is there an integer n that has a remainder of 2 when it is divided by 5?
  - Is there an integer n such that $n\%5 = 2$?
- The cube root of any negative real number is negative.
  - For any real number s, if $s < 0$, then $\sqrt[3]{s} < 0$.

Types of Statements

- A **universal statement** is a statement that is true always true.
  - For example: **All** positive numbers are greater than 0.
- A **conditional statement** is a statement that is true if a certain condition is met.
  - For example: **If** 378 is divisible by 18, **then** 378 is divisible by 6.
- A **universal conditional statement** is a statement that is both conditional and universal.
  - For example: **For all** animals a, if a is a dog, **then** a is a mammal.

  - As a universal statement: **For all** dogs a, a is a mammal.
  - As a conditional statement: **If** a is a dog, **then** a is a mammal.

- An **existential statement** gives a property that is true for at least one thing.

  - **There is** a prime number that is even.

- A **universal existential statement** is a statement where the first part is universal and the second part is existential.

  - **Every** real number **has** an additive inverse.
  - **For all** real numbers r, **there is** an additive inverse $-r$.
  - **For all** real numbers r, **there is** a real number s such that $r+s = 0$.

- An **existential universal statement** is a statement where the first part is existential and the second part is universal.

  - **There is** a positive integer that is less than or equal to **every** positive integer.
  - **There is** a positive intefer m such that **every** positive integer is greater than or equal to m.
  - **There is** a positive integer m with the property that **for all** positive integers n, $m \leq n$.

1.2 Sets

Definition A **set** is a collection of objects.

Notation

- $x \in S$: x is an element of S.
- $x \notin S$: x is not an element of S.
- $S = \{1, 2, 3, \dots\}$: is **set roster notation.**

Axion of Extension A set is determined by what its elements are. Orders of elements or repeated elements can't be determine the set.
For example: $\{1, 2, 3\} = \{3, 2, 2, 1, 2, 3, 1\}$. There are 3 elements in both sets.

Common Sets

- $\mathbb{R}$: the set of all real numbers.
- $\mathbb{Z}$: $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ the set of all integers.
- $\mathbb{N}$: $\{1, 2, 3, \dots\}$ the set of all natural numbers.
- $\mathbb{Q}$: the set of all rational numbers.
- $\emptyset = \{\}$: the empty set, or null set.

The null set is a subset of every set.

Set Builder Notation Let S denote a set and let $x \in S$ be and element in S. $P(x)$ is a property that some elements of S satisfy.

$$A = \{x \in S | P(x)\}$$

A constains elements in S such that (—) P(x) is true.

Subsets

Definition Let A and B be sets. A is a **subset** ($\subseteq$) of B if every element of A is also an element of B.

Proper Subsets Let A and B be sets. A is a **proper subset** ($\subset$) of B if every element of A is also an element of B, **and** there is at least one element in B that is not in A.

Example Let $A = \mathbb{Z}^+, B = \{n \in \mathbb{Z} | 0 \leq n \leq 100\}, and C = \{100, 200, 300, 400, 500\}$.

- $B \subseteq A$ is false.
- $C \subset A$ is true.
- $C \subseteq B$ is false.
- $C \subseteq C$ is true.

Cartesian Product of sets Let A and B be sets. The **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Example Let $A = \{1, 2, 3\}$ and $B = u, v$.

$$A \times B = \{(1, u), (1, v), (2, u), (2, v), (3, u), (3, v)\}$$
$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

1.3 Relations and Functions

Relations Let A and B be sets. A **relation** from A to B is a subset of the Cartesian product $A \times B$.

$$R \subseteq A \times B$$

- If (x,y) $\in R$, we say that x is related to y by R, denoted as xRy.
- **A** is in the **domain** of **R**
- **B** is the **codomain** of **R**

Example Let $A = \{1, 2, 3\}$ and $B = \{1, 2\}$ and define a relation R from A to B as follows:

$$\begin{aligned}(x, y) \in R \iff & \frac{x+y}{2} \in \mathbb{Z} \\ R = & \{(1,1), (1,2), (2,1), (2,2), (3,1)\} \\ & \text{Domain of } R = \{1, 2, 3\} \\ & \text{Codomain of } R = \{1, 2\}\end{aligned}$$

Functions

Definition Let A and B be two sets. A function F from A to B is a relation with domain A and co-domain B that satisfies the following properties:

- For every element $x \in A$, there is an element $y \in B$ such $(x, y) \in F$
- For every element $x \in A$,

Example Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations defined below are functions from A to B?

- $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$
  - Not a function because 4 is related to 1 and 3. This is not a many-to-one relationship.
- For all $(x.y) \in A \times B, (x, y) \in S \iff y = x + 1$
  - $S = \{(2, 3), (4, 5)\}$ is a function from A to B.
- $T = \{(2, 5), (4, 1), (6, 1)\}$
  - T is a function from A to B as A has a many-to-one relationshop with B.

Equivalent Functions

Let A and B be two sets. Two functions f and g from A to B:

$$f = g \iff f(x) = g(x) \quad \forall \quad x \in A$$

2. The Logic of Compound Statements

2.1 Logical Form and Equivalence

Arguments

Definition An arguement is a sequence of statements aimed at demonstrating the truth of an assertion.

- The assertion at the end of the sequence is called the conclusion.
- The statements that support the conclusion are called premises.
- If the premises are true, the conclusion must also be true.

Example

- If student A is a math major or student A is a computer science major,
- Then student A will take Discrete Math.

Logical Statements

Definition A logical statement is a declarative sentence that is either true or false, but not both.

- Not p: $\neg p$
- p and/but q: $p \wedge q$
- p or q: $p \vee q$
- Neither p nor q: $\neg p \wedge \neg q$

Example h = healthy, w = wealthy, s = wise

- John is healthy and wealthy but not wise.
  - $(h \wedge w) \wedge \neg s$
- John is neither wealthy nor wise, but he is healthy
  - $(\neg w \wedge \neg s) \wedge h$

Equivalent Statements

Definition Two logical statements are equivalent if they have the same truth tables, denoted:

$$p \equiv q$$

De Morgan's Laws The negation of an and statement is logically equivalent to the or statement of the negations. Similarly, the negation of an or statement is logically equivalent to the and statement of the negations.

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Tautological and Condtradictory Statements

- A tautological statement is a statement that is always true.
- A contradictory statement is a statement that is always false.