PHYS4B: Electromagnitism for Scientists and Engineers

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Contents

1. Electrostatics

Introduction to Electrostatics

Charge

Definition Much like inertia, charge is a fundamental property of matter that describes how an object interacts with electric fields. It is measured in Coulombs $[C]$. Practically, charge indicates if the object has an excess or deficiency of electrons.

Quantization of Charge Charge is quantized, meaning that it can only exist in discrete values. The smallest possible charge is the charge of an electron. Protons and electrons carry equal but opposite charges, represented by e.

$$q = \pm Ne \Longrightarrow e = 1.6 \times 10^{-19} C$$

Where N is any integer. This means that the charge of an object is always a multiple of the charge of an electron.

Conservation of Charge Much like energy and matter, charge is conserved in a closed system. This means that the total charge in a system will remain constant.

$$\sum q_i = \sum q_f$$

Conductors and Insulators Electrical conductors are materials in which some of the electrons are not bound to atoms and can move relatively freely through the material. This allows for the easy transfer of charge. Metals are the most common conductors. Electrical insulators are materials in which all of the electrons are bound to atoms and cannot move freely. This impededs the transfer of charge. Glass, rubber, and plastic are common insulators.

Coloumb's Law

Definition Coulomb's law describes the fundimental force between 2 charged objects. It is given by the equation:

$$F_e = k_e \frac{q_1 q_2}{r^2} \implies \vec{F}_{12} = k_e \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

Where F_e is the electrostatic force in Newtons, k_e is Coulomb's constant, q_1 and q_2 are the charges of the objects in Coulombs, r is the distance between the objects in meters, and $\hat{r}_{12}$ is a unit vector pointing from object 1 to object 2.

Notice the similarities between Coulomb's law and Newton's law of gravitation. These similarities are because both are fundamental forces of nature. Both are inverse square laws, meaning that the force between the objects decreases exponentially as the distance between them increases. Both forces are proportional to a property of matter (mass for gravity and charge for electrostatics) and a constant (G and k_e respectively).

The main difference is that electrostatic forces can be attractive or repulsive, while gravity is always attractive. This is because charge can be positive or negative, while it is (for our purposes) impossible to have negative mass.

Electric Fields

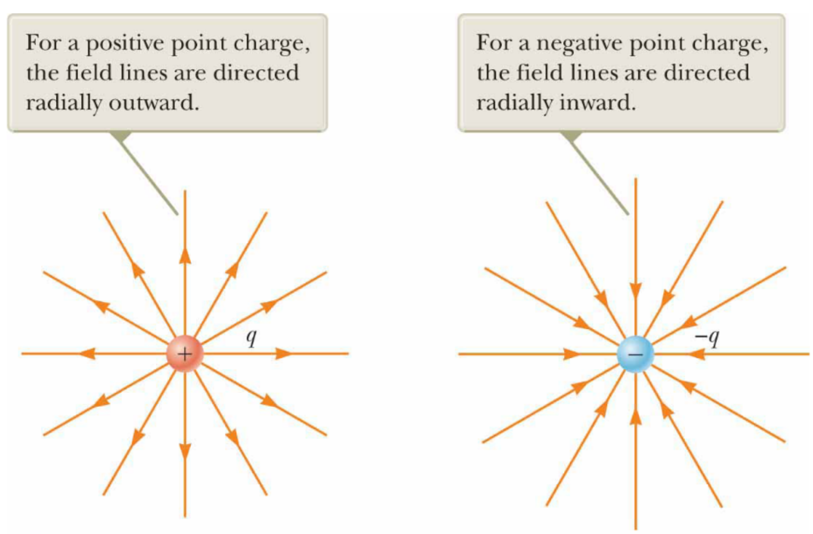
Definition The electric field is a vector field that describes the force experienced by a charge at any point in space. It is measured in Newtons per Coulomb $[\frac{N}{C}]$. It is given by the equation:

$$\vec{E} = \frac{\vec{F_e}}{q}$$

Where $\vec{E}$ is the electric field in Newtons per Coulomb, $\vec{F_e}$ is the electrostatic force experienced by the particle in Newtons, and q is the charge of the particle in Coulombs.

Electric Field Lines

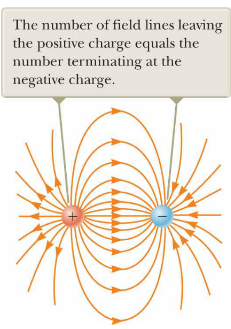
Definition Electric field lines are a visual representation of the electric field. They are drawn such that the electric field is tangent to the line at any point. The electric field lines are drawn such that they point away from positive charges and towards negative charges.



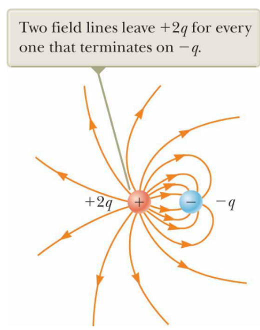
The density of the lines leaving or terminating at a particle is proportional to the charge of the particle.

$$\frac{N_2}{N_1} = |\frac{q_2}{q_1}|$$

Where N is the number of field lines coming from a charge, and q is the charge of that particle.



(a) Field lines between two equal and opposite charges



(b) Field lines between two unequal and opposite charges

Charge Density

Definition Charge density is the amount of charge per unit length (λ), area (σ), or volume (ρ) depending on the geometry of the object.

$$\text{Linear Charge Density} : \lambda = \frac{Q}{\ell} \implies dq = \lambda d\ell$$
$$\text{Surface Charge Density} : \sigma = \frac{Q}{A} \implies dq = \sigma dA$$
$$\text{Volume Charge Density} : \rho = \frac{Q}{V} \implies dq = \rho dV$$

Where Q is the total charge, ℓ is the length, A is the area, and V is the volume of the object.

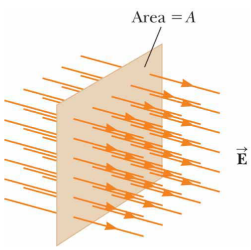
Electric Field Caused by Different Charged Geometry

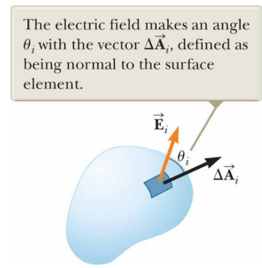
Charged objects can have different geometries resulting in different electric fields. Here are some common geometries and their electric fields.

$$
\begin{aligned}
\text{Infinite Line} &: E_{line} = \frac{\lambda}{2\pi\epsilon_0 r} \\
\text{Infinite Plane} &: E_{plane} = \frac{\sigma}{2\epsilon_0} \\
\text{Parallel Plates} &: E_{\parallel,plate} = \frac{\sigma}{\epsilon_0} \\
\text{Ring} &: E_{ring} = \frac{k_e Q x}{(x^2 + a^2)^{3/2}}
\end{aligned}
$$

Where λ is the linear charge density, σ is the surface charge density, k_e is Coulomb's constant, a is the radius of the ring, x is the distance from the ring, Q is the total charge of the object, and a is the radius of the ring.

Electric Flux





Definition Electric flux is defined as the number of electric field lines passing through a surface. It is measured in Volt meters $[Vm]$. It is given by the equation:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = EA\cos(\theta)$$

Where Φ_E is the electric flux in Newtons per Coulomb, $\vec{E}$ is the electric field in Newtons per Coulomb, $d\vec{A}$ is the differential area vector, and θ is the angle between $\vec{E}$ and $d\vec{A}$.

Gauss's Law

Definition Gauss's Law states that electric flux through a closed surface is equal to the charge enclosed by the surface divided by the permittivity of free space. This is to say that any flux generated by electric fields originating from charges outside the surface will cancel out, thus we only care about the enclosed charge.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Where Φ_E is the electric flux in Vm, $\vec{E}$ is the electric field in Newtons per Coulomb, $d\vec{A}$ is the differential area vector, Q_{enc} is the charge enclosed by the surface, and ϵ_0 is the permittivity of free space.

Electric Potential

Definition Electric potential is the amount of work needed to move a unit charge from a reference point to a specific point in space. It is measured in volts. The electric potential at a point in space is given by the following equation.

$$V = \frac{U_e}{q} = \frac{k_e q}{r}$$

Where V is the electric potential in volts, U_e is the electric potential energy in joules, and q is the charge in coulombs.

Electric Potential Energy Electric potential energy is the energy stored in a system of charges. It is measured in joules. The electric potential energy of a system of 2 charges is given by the following equation.

$$U_e = q_1 V = k_e \frac{q_1 q_2}{r}$$

Where U_e is the electric potential energy in joules, k_e is Coulomb's constant, q_1 and q_2 are the charges in coulombs, and r is the distance between the charges in meters.

Forces are Derivatives of Potentials and the electric force is no different. The electric force is the derivative of the electric potential energy with respect to the distance between the charges. We can derive this relationship as follows.

$$\vec{F}_e = -\frac{dU_e}{d\vec{\ell}} \quad \Rightarrow \quad \vec{F}_e \cdot d\vec{\ell} = -dU_e \quad \Rightarrow \quad \int \vec{F}_e d\vec{\ell} = -\int dU_e$$
$$\Rightarrow \quad \int \vec{F}_e d\vec{\ell} = -\Delta U_e \quad \Rightarrow \quad W = -\Delta U_e$$

Where $\vec{F}_e$ is the electric force in newtons, $\vec{\ell}$ is radial distance from the charge in meters, ΔU_e is the change in electric potential energy in joules, and W is the work done in joules.

Voltage

Voltage (ΔV) is the electric potential difference between two points in space. It is measured in volts. The voltage between two points in space is given by the following equation.

$$\Delta V = V_b - V_a = \int \vec{E} \cdot d\vec{\ell}$$

Where ΔV is the voltage in volts, V_b and V_a are the electric potentials at points b and a respectively, $\vec{E}$ is the electric field in newtons per coulomb, and $\vec{\ell}$ is the distance between the points in meters.

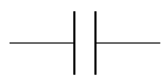
Electric Field as a function of Voltage The electric field is the derivative of the electric potential with respect to the distance between the charges. We can derive this relationship as follows.

$$\vec{E} = \vec{\nabla} V \implies \begin{cases} Ex = \frac{\partial V}{\partial x} \\ Ey = \frac{\partial V}{\partial y} \\ Ez = \frac{\partial V}{\partial z} \end{cases}$$

2. Circuits

Capacitors

Definition Capacitors are devices that can store charge. They are made of one or more pairs of conductors seperated by an insulator. In circuit diagrams, they are denoted using the following symbol:



Capacitance

Definition Capacitance is the ability of a capacitor to store charge. It is measured in farads.

$$C = \frac{Q}{\Delta V} = \epsilon_0 \frac{A}{d}$$

Where C is capacitance in farads, Q is charge in Coulombs, ΔV is voltage in volts, ϵ_0 is the permittivity of free space, A is the area of the plates in m^2, and d is the distance between the plates in meters.

Electric Field Within a Capacitor The electric field within a capacitor is given by the equation:

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Where E is the electric field in N/C, σ is the charge density in C/m^2, Q is the charge in Coulombs, ϵ_0 is the permittivity of free space, and A is the area of the plates in m^2.

Note that any 2 parallel objects with opposite charges can act as a capacitor, for example the sky and the ground during a stormy day.

Voltage Across a Capacitor The voltage across a capacitor is given by the equation:

$$\Delta V = Ed = \frac{Qd}{\epsilon_0 A}$$

Where ΔV is the voltage in volts, E is the electric field in N/C, d is the distance between the plates in meters, Q is the charge in Coulombs, ϵ_0 is the permittivity of free space, and A is the area of the plates in m^2.

Energy Stored in a Capacitor The electric potential energy stored in a capacitor is given by the equation:

$$U_e = \frac{1}{2}Q\Delta V = \frac{1}{2}C\Delta V^2 = \frac{Q^2}{2C}$$

Where U_e is the electric potential energy in Joules, Q is the charge in Coulombs, ΔV is the voltage in volts and C is the capacitance in farads

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Dielectrics

Up until this point, we have made the assumption that the conductors in the capacitor have been seperated by air, however this is not always the case. Dielectrics are insulating materials that are placed between the plates of a capacitor. They increase the capacitance of a capacitor by a factor of κ, or the dielectric constant of that material. κ is a direct property of a material that has been determined through experimentation, and thus this value will have to be given in the problem or retrieved from a table.

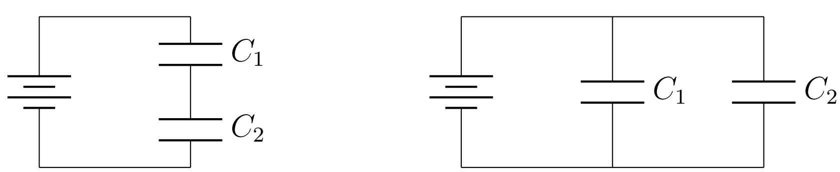
Capacitance and Electric Field with a Dielectric When a dielectric is placed between the plates of a capacitor, the capacitance and electric field change based off of the dielectric constant.

$$C_\kappa \propto \kappa$$
$$E_\kappa \propto \frac{1}{\kappa}$$

This changes the equations for capacitance, electric field, and voltage across a capacitor to the following:

$$C_\kappa = \kappa C_0 = \frac{\kappa Q}{\Delta V}$$
$$E_\kappa = \frac{\sigma}{\kappa \epsilon_0} = \frac{Q}{\kappa \epsilon_0 A}$$
$$\Delta V_\kappa = E_\kappa d = \frac{Qd}{\kappa \epsilon_0 A}$$

Capacitors in Series and Parallel



(a) Wired in series. (b) Wired in parallel.

Differences Capacitors wired in series and parallel behave differently. Capacitors wired in series carry the same charge, but different voltages.

$$q_{eq(s)} = q_1 = q_2 = \cdots = q_n$$

Capacitors wired in parallel behave in the opposite manner, carrying the same voltage, but different charges.

$$\Delta V_{eq(\|)} = \Delta V_1 = \Delta V_2 = \cdots = \Delta V_n$$

Equivilant Capacitance Multiple capacitors can be represented by a single capacitor with an equivilant capacitance. The equivilant capacitance of capacitors in series and in parallel is given by the following equations:

$$\frac{1}{C_{eq(s)}} = \sum_{i=1}^{n} \frac{1}{C_i}$$

$$C_{eq(\|)} = \sum_{i=1}^{n} C_i$$

Current and Resistance

Current

Definition Current is defined as the time rate of change of charge through an object. It is measured in amps.

$$I = \frac{dQ}{dt} \Rightarrow I = nqv_dA$$

Where I is the current in amps $[A] = [\frac{C}{s}]$, n is the free electron density in electrons/m^3, v_d is the drift velocity in m/s, and A is the cross-sectional area of the conductor in m^2.

Drift Velocity Electrons are bouncing around randomly. When an electric field is applied, the bouncing is directed in a direction, but it is still chaotic. This bouncing results in heat being generated. Heat is defined as the kinetic energy of a particle. The speed of the drift of the electrons is called the drift velocity (v_d).

$$v_d = \frac{I_{avg}}{nqA} = \frac{I}{nqA}$$

Where v_d is the drift current, n is the free electron density in $\frac{g}{m^2}$, q is the charge of the current carrier (usually an electron) in C, and A is the cross-sectional area of the conductor in m^2.

Current Density Current density is the current per unit area. It can be calculated using the following formula:

$$J = \frac{I}{A} = \sigma A = nqv_d$$

Where J is the current density in $Amps/m^2$, I is the current in amps, A is the cross-sectional area of the conductor in m^2, σ is the conductivity of the material, and n is the free electron density in electrons/m^3.

Voltage can be calculated as a function of current density and conductivity as follows:

$$\Delta V = E\ell = \frac{\ell J}{\sigma}$$

Where ΔV is the voltage in volts, E is the electric field in volts per meter, ℓ is the length of the conductor in meters,

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Resistance

Definition Resistance is defined as the ratio of voltage to current, also known as Ohm's law. It is measured in ohms.

$$R = \frac{\Delta V}{I}$$

Where R is the resistance in Ω, ΔV is the voltage in volts, and I is the current in A.

Resistivity Resistivity is the fundimental property of a material that determines how much it resists the flow of current. It is measured in ohm-meters $[\Omega m]$.

$$\rho = \frac{1}{\sigma} = \frac{RA}{\ell} \Longrightarrow R = \rho\frac{\ell}{A}$$

Where ρ is resistivity in Ωm, σ is conductivity in S/m, R is resistance in Ω, A is the cross-sectional area of the conductor in m^2, and ℓ is the length of the conductor in meters.

Conductivity Conductivity is the inverse of resistivity. It is measured in Siemens per meter $[\frac{S}{m}]$. $[S] = [\frac{1}{\Omega}]$

$$\sigma = \frac{1}{\rho} = \frac{\ell}{RA} \Longrightarrow R = \frac{\ell}{\sigma A}$$

Where ρ is resistivity, σ is conductivity, R is resistance in Ω, I is current in amps, and ΔV is voltage in volts.

Ohmic vs. Non-Ohmic devices Ohmic devices are devices that have a Voltage vs Current slope of $\frac{1}{R}$. Non-ohmic devices have a slope that changes with voltage or current.

Resistivity and Temperature The resistivity of a material changes with the temperature of the material. The equation for resistivity as a function of temperature is given by:

$$\rho_t = \rho_0[1 + \alpha\Delta T]$$
$$\alpha = \frac{\Delta\rho/\rho_0}{\Delta T}$$

Where ρ_t is the resistivity of the material at a given temperature in Ωm, ρ_0 is the resistivity of the material at a reference temperature in Ωm, α is the temperature coefficient in $°C^{-1}$ (a material constant, much like κ), $\Delta\rho$ is the change in resistivity in Ωm, and ΔT is the change in temperature in $°C$.

Electrical Power

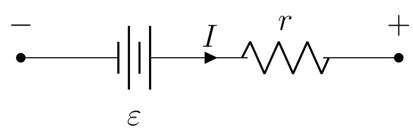
Definition Power is defined as the rate at which energy is transferred or converted. It is measured in watts.

$$P = \frac{dU_e}{dt} = I^2 R = \frac{(\Delta V)^2}{R}$$

Where P is power in watts, U_e is electric potential energy in Joules, ΔV is voltage in volts, and I is current in amps.

Direct Current Circuits

Electromotive Force



Circuit Diagram of a Simple Battery

Definition Electromotive force (emf) is the voltage produced by a battery or other voltage source. It is measured in volts. A simple battery consists of a source of emf and a resistor to represent the internal resistance of the battery. Unless the battery is ideal, $\Delta V_{terminal} \neq \varepsilon$. In this case, we can find the voltage at the terminals of the battery using the following equation.

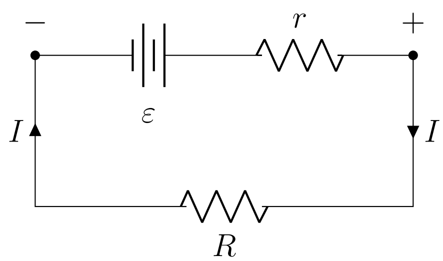
$$\Delta V_{terminal} = \varepsilon - Ir$$
$$\varepsilon = \Delta V_{terminal} + Ir$$

Where $\Delta V_{terminal}$ is the voltage at the terminals of the battery, ε is the emf of the battery, I is the current through, and r is the internal resistance of the battery.

When a non-ideal battery is in use, we need to represent the current in terms of the emf and the resistors in the circuit. The current and emf in the circuit can be found using the following equations.

$$I = \frac{\varepsilon}{R + r}$$
$$\varepsilon = IR + Ir$$

Where ε is the emf in volts, I is the current in amps, r is the internal resistance of the battery in ohms, and R is the resistance of the resistor in ohms.



A Simple Battery wired to a resistor

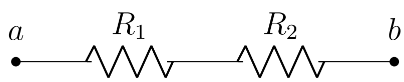
Power The internal resistance also affects power delivery. The power delevered to each of the resistors in the above circuit is given by the following equations.

$$
\begin{aligned}
P &= P_R + P_r \\
P_R &= I^2 R \\
P_r &= I^2 r
\end{aligned}
$$

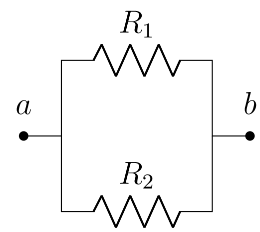
Note that the specific current value used will depend on how the resistors in the circuit are connected.

Equivilant Resistance

Definition Much like capacitors, multiple resistors can be represented by a single resistor with an equivilant resistance. How the resistors are connected will determine how the equivilant resistance is calculated and how the current and voltage change across them.



Two resistors in series



Two resistors in parallel

In Series Resistors in series can be combined into a single resistor with a resistance equal to the sum of the individual resistances. The current through each resistor in series is the same, but the voltage across each resistor is different.

$$R_{eq(s)} = \sum_{i=1}^{n} R_i$$
$$I_{eq(s)} = I_1 = I_2 = \cdots = I_n$$
$$\Delta V_{eq(s)} = \sum_{i=1}^{n} \Delta V_i$$

In Parallel Resistors in parallel can be combined into a single resistor with a resistance equal to the reciprocal of the sum of the reciprocals of the individual resistances. The voltage across each resistor in parallel is the same, but the current through each resistor is different.

$$\frac{1}{R_{eq(\|)}} = \sum_{i=1}^{n} \frac{1}{R_i}$$
$$I_{eq(\|)} = \sum_{i=1}^{n} I_i$$
$$\Delta V_{eq(\|)} = \Delta V_1 = \Delta V_2 = \cdots = \Delta V_n$$

Kirchhoff's Laws

Kirchhoff's Laws are fundamental tools used to analyze electrical circuits. They consist of two rules: the Junction Rule and the Loop Rule. These laws help us determine the current and voltage at different points in a circuit.

Junction Rule The Junction Rule states that at any junction (or node) in a circuit, the sum of the currents entering the junction must equal the sum of the currents leaving the junction. This is a consequence of the conservation of charge.

$$\sum_{junction} I = 0 \quad \Longrightarrow \quad \sum_{junction} I_{in} = \sum_{junction} I_{out}$$

To apply the Junction Rule:

1. Identify all junctions in the circuit.
2. Write an equation for each junction, setting the sum of currents entering the junction equal to the sum of currents leaving the junction.
3. Solve the system of equations to find the unknown currents.

Loop Rule The Loop Rule states that the sum of the voltages around any closed loop in a circuit must equal zero. This is a consequence of the conservation of energy.

$$\sum_{closedloop} \Delta V = 0$$

To apply the Loop Rule:

1. Identify all independent loops in the circuit.
2. Choose a direction to traverse each loop (clockwise or counterclockwise).
3. Write an equation for each loop, summing the voltage drops (negative) and voltage rises (positive) around the loop and setting the sum equal to zero.
4. Solve the system of equations to find the unknown voltages and currents.

These rules can be used together to solve for the currents and voltages at different points in a circuit.

Steps to Solve a Circuit with Kirchhoff's Laws:

1. Label all currents and voltages: Assign a variable to each unknown current and voltage in the circuit.
2. Apply the Junction Rule: Write equations for each junction in the circuit.
3. Apply the Loop Rule: Write equations for each independent loop in the circuit.
4. Solve the system of equations: Use algebraic methods to solve the system of equations obtained from the Junction and Loop Rules.

Sign Conventions When applying Kirchhoff's Laws, it is important to follow these sign conventions:

- If the current enters the positive terminal of a resistor, the voltage drop across the resistor is positive.
- If the current enters the negative terminal of a resistor, the voltage drop across the resistor is negative.
- If the current enters the positive terminal of a battery, the voltage rise across the battery is negative.
- If the current enters the negative terminal of a battery, the voltage rise across the battery is positive.

RC Circuits

Definition An RC circuit is a circuit that contains a resistor and a capacitor. The capacitor stores energy in the form of an electric field, while the resistor dissipates energy in the form of heat. The time constant τ of an RC circuit is given by the product of the total resistance and capacitance in the circuit.

$$\tau = RC$$

Where τ is the time constant in seconds, R is the resistance in ohms, and C is the capacitance in farads.

Charging a Capacitor First, we must update our max charge equation to account for emf.

$$Q_{max} = C\varepsilon$$

Where Q_{max} is the max charge the capacitor can hold in Coulombs, C is the capacitance of the capacitor in Farads, and ε is the electromotive force in volts.

We can then find the charge and current of the capacitor as a function of time.

$$
\begin{aligned}
q(t) &= C\varepsilon(1 - e^{-t/\tau}) &&= Q_{max}(1 - e^{-t/\tau}) \\
I(t) &= \frac{dq}{dt} = \frac{\varepsilon}{R}e^{-t/\tau} &&= \frac{q(t) - Q_{max}}{\tau}
\end{aligned}
$$

Where $q(t)$ is the charge on the capacitor at time t in Coulombs, $I(t)$ is the current through the capacitor at time t in Amps, C is the capacitance of the capacitor in Farads, ε is the electromotive force in volts, R is the resistance in ohms, and τ is the time constant of the circuit in seconds.

Discharging a Capacitor We can also find the charge and current of a capacitor as it discharges as a function of time.

$$
\begin{aligned}
q(t) &= Q_i e^{-t/\tau} \\
I(t) &= -\frac{Q_i}{\tau}e^{-t/\tau}
\end{aligned}
$$

Where Q_i is the initial charge on the capacitor in Coulombs, and the rest of teh variables having the same definition as above.

3. Magnetism

Magnetic Fields

Definition A magnetic field is a vector field that describes the magnetic influence on moving electric charges, electric currents, and magnetic materials. They are generated by moving charges, or current. It is represented by the symbol $\vec{B}$ and is measured in teslas (T).

Unlike electric fields, which can be generated by a single source charge, magnetic poles always come in pairs. There are no magnetic monopoles. The two types of magnetic poles are called north and south.

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

Where $\vec{F}_B$ is the magnetic force in newtons, q is the charge in coulombs, $\vec{v}$ is the velocity of the charge in meters per second, and $\vec{B}$ is the magnetic field in teslas. [Insert right hand rule stuff]