

PHYS4B: Electromagnetism for Scientists and Engineers

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Winter 2025

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1. Electrostatics

Introduction to Electrostatics

Charge

Definition Much like inertia, charge is a fundamental property of matter that describes how an object interacts with electric fields. It is measured in Coulombs $[C]$. Practically, charge indicates if the object has an excess or deficiency of electrons.

Quantization of Charge Charge is quantized, meaning that it can only exist in discrete values. The smallest possible charge is the charge of an electron. Protons and electrons carry equal but opposite charges, represented by e .

$$q = \pm Ne \implies e = 1.6 \times 10^{-19} C$$

Where N is any integer. This means that the charge of an object is always a multiple of the charge of an electron.

Conservation of Charge Much like energy and matter, charge is conserved in a closed system. This means that the total charge in a system will remain constant.

$$\sum q_i = \sum q_f$$

Conductors and Insulators Electrical conductors are materials in which some of the electrons are not bound to atoms and can move relatively freely through the material. This allows for the easy transfer of charge. Metals are the most common conductors. Electrical insulators are materials in which all of the electrons are bound to atoms and cannot move freely. This impedes the transfer of charge. Glass, rubber, and plastic are common insulators.

Coloumb's Law

Definition Coulomb's law describes the fundimantal force between 2 charged objects. It is given by the equation:

$$F_e = k_e \frac{q_1 q_2}{r^2} \implies \vec{F}_{12} = k_e \frac{q_1 q_2}{r^2} \hat{r}_{12}$$

Where F_e is the electrostatic force in Newtons, k_e is Coulomb's constant, q_1 and q_2 are the charges of the objects in Coulombs, r is the distance between the objects in meters, and \hat{r}_{12} is a unit vector pointing from object 1 to object 2.

Notice the similarities between Coulomb's law and Newton's law of gravitation. These similarities are because both are fundamental forces of nature. Both are inverse square laws, meaning that the force between the objects decreases exponentially as the distance between them increases. Both forces are proportional to a property of matter (mass for gravity and charge for electrostatics) and a constant (G and k_e respectively).

The main difference is that electrostatic forces can be attractive or repulsive, while gravity is always attractive. This is because charge can be positive or negative, while it is (for our purposes) impossible to have negative mass.

Electric Fields

Definition The electric field is a vector field that describes the force experienced by a charge at any point in space. It is measured in Newtons per Coulomb [$\frac{N}{C}$]. It is given by the equation:

$$\vec{E} = \frac{\vec{F}_e}{q}$$

Where \vec{E} is the electric field in Newtons per Coulomb, \vec{F}_e is the electrostatic force experienced by the particle in Newtons, and q is the charge of the particle in Coulombs.

Electric Field Lines

Definition Electric field lines are a visual representation of the electric field. They are drawn such that the electric field is tangent to the line at any point. The electric field lines are drawn such that they point away from positive charges and towards negative charges.



The density of the lines leaving or terminating at a particle is proportional to the charge of the particle.

$$\frac{N_2}{N_1} = \left| \frac{q_2}{q_1} \right|$$

Where N is the number of field lines coming from a charge, and q is the charge of that particle.

The number of field lines leaving the positive charge equals the number terminating at the negative charge.



(a) Field lines between two equal and opposite charges

Two field lines leave $+2q$ for every one that terminates on $-q$.



(b) Field lines between two unequal and opposite charges

Charge Density

Definition Charge density is the amount of charge per unit length (λ), area (σ), or volume (ρ) depending on the geometry of the object.

$$\text{Linear Charge Density : } \lambda = \frac{Q}{\ell} \implies dq = \lambda d\ell$$

$$\text{Surface Charge Density : } \sigma = \frac{Q}{A} \implies dq = \sigma dA$$

$$\text{Volume Charge Density : } \rho = \frac{Q}{V} \implies dq = \rho dV$$

Where Q is the total charge, ℓ is the length, A is the area, and V is the volume of the object.

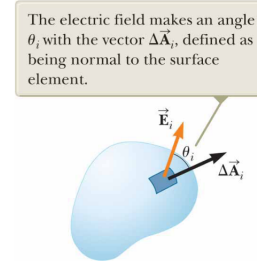
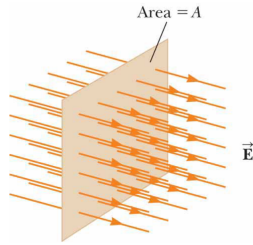
Electric Field Caused by Different Charged Geometry

Charged objects can have different geometries resulting in different electric fields. Here are some common geometries and their electric fields.

$$\begin{aligned}\text{Infinite Line} : E_{line} &= \frac{\lambda}{2\pi\epsilon_0 r} \\ \text{Infinite Plane} : E_{plane} &= \frac{\sigma}{2\epsilon_0} \\ \text{Parallel Plates} : E_{||,plate} &= \frac{\sigma}{\epsilon_0} \\ \text{Ring} : E_{ring} &= \frac{k_e Q x}{(x^2 + a^2)^{3/2}}\end{aligned}$$

Where λ is the linear charge density, σ is the surface charge density, k_e is Coulomb's constant, a is the radius of the ring, x is the distance from the ring, Q is the total charge of the object, and a is the radius of the ring.

Electric Flux



Definition Electric flux is defined as the number of electric field lines passing through a surface. It is measured in Volt meters $[Vm]$. It is given by the equation:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = EA \cos(\theta)$$

Where Φ_E is the electric flux in Newtons per Coulomb, \vec{E} is the electric field in Newtons per Coulomb, $d\vec{A}$ is the differential area vector, and θ is the angle between \vec{E} and $d\vec{A}$.

Gauss's Law

Definition Gauss's Law states that electric flux through a closed surface is equal to the charge enclosed by the surface divided by the permittivity of free space. This is to say that any flux generated by electric fields originating from charges outside the surface will cancel out, thus we only care about the enclosed charge.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

Where Φ_E is the electric flux in Vm , \vec{E} is the electric field in Newtons per Coulomb, $d\vec{A}$ is the differential area vector, Q_{enc} is the charge enclosed by the surface, and ϵ_0 is the permittivity of free space.

Electric Potential

Definition Electric potential is the amount of work needed to move a unit charge from a reference point to a specific point in space. It is measured in volts. The electric potential at a point in space is given by the following equation.

$$V = \frac{U_e}{q} = \frac{k_e q}{r}$$

Where V is the electric potential in volts, U_e is the electric potential energy in joules, and q is the charge in coulombs.

Electric Potential Energy Electric potential energy is the energy stored in a system of charges. It is measured in joules. The electric potential energy of a system of 2 charges is given by the following equation.

$$U_e = q_1 V = k_e \frac{q_1 q_2}{r}$$

Where U_e is the electric potential energy in joules, k_e is Coulomb's constant, q_1 and q_2 are the charges in coulombs, and r is the distance between the charges in meters.

Forces are Derivatives of Potentials and the electric force is no different. The electric force is the derivative of the electric potential energy with respect to the distance between the charges. We can derive this relationship as follows.

$$\begin{aligned} \vec{F}_e &= -\frac{dU_e}{d\vec{\ell}} \Rightarrow \vec{F}_e \cdot d\vec{\ell} = -dU_e \Rightarrow \int \vec{F}_e d\vec{\ell} = -\int dU_e \\ \Rightarrow \int \vec{F}_e d\vec{\ell} &= -\Delta U_e \Rightarrow W = -\Delta U_e \end{aligned}$$

Where \vec{F}_e is the electric force in newtons, $\vec{\ell}$ is radial distance from the charge in meters, ΔU_e is the change in electric potential energy in joules, and W is the work done in joules.

Voltage

Voltage (ΔV) is the electric potential difference between two points in space. It is measured in volts. The voltage between two points in space is given by the following equation.

$$\Delta V = V_b - V_a = \int \vec{E} \cdot d\vec{\ell}$$

Where ΔV is the voltage in volts, V_b and V_a are the electric potentials at points b and a respectively, \vec{E} is the electric field in newtons per coulomb, and $\vec{\ell}$ is the distance between the points in meters.

Electric Field as a function of Voltage The electric field is the derivative of the electric potential with respect to the distance between the charges. We can derive this relationship as follows.

$$\vec{E} = -\vec{\nabla}V \implies \begin{cases} E_x = -\frac{\partial V}{\partial x} \\ E_y = -\frac{\partial V}{\partial y} \\ E_z = -\frac{\partial V}{\partial z} \end{cases}$$

2. Circuits

Capacitors

Definition Capacitors are devices that can store charge. They are made of one or more pairs of conductors separated by an insulator. In circuit diagrams, they are denoted using the following symbol:



Capacitance

Definition Capacitance is the ability of a capacitor to store charge. It is measured in farads.

$$C = \frac{Q}{\Delta V} = \epsilon_0 \frac{A}{d}$$

Where C is capacitance in farads, Q is charge in Coulombs, ΔV is voltage in volts, ϵ_0 is the permittivity of free space, A is the area of the plates in m^2 , and d is the distance between the plates in meters.

Electric Field Within a Capacitor The electric field within a capacitor is given by the equation:

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Where E is the electric field in N/C , σ is the charge density in C/m^2 , Q is the charge in Coulombs, ϵ_0 is the permittivity of free space, and A is the area of the plates in m^2 .

Note that any 2 parallel objects with opposite charges can act as a capacitor, for example the sky and the ground during a stormy day.

Voltage Across a Capacitor The voltage across a capacitor is given by the equation:

$$\Delta V = Ed = \frac{Qd}{\epsilon_0 A}$$

Where ΔV is the voltage in volts, E is the electric field in N/C , d is the distance between the plates in meters, Q is the charge in Coulombs, ϵ_0 is the permittivity of free space, and A is the area of the plates in m^2 .

Energy Stored in a Capacitor The electric potential energy stored in a capacitor is given by the equation:

$$U_e = Q\Delta V = \frac{1}{2}C\Delta V^2 = \frac{Q^2}{2C}$$

Where U_e is the electric potential energy in Joules, Q is the charge in Coulombs, ΔV is the voltage in volts and C is the capacitance in farads

Dielectrics

Up until this point, we have made the assumption that the conductors in the capacitor have been separated by air, however this is not always the case. Dielectrics are insulating materials that are placed between the plates of a capacitor. They increase the capacitance of a capacitor by a factor of κ , or the dielectric constant of that material. κ is a direct property of a material that has been determined through experimentation, and thus this value will have to be given in the problem or retrieved from a table.

Capacitance and Electric Field with a Dielectric When a dielectric is placed between the plates of a capacitor, the capacitance and electric field change based off of the dielectric constant.

$$C_{\kappa} \propto \kappa$$

$$E_{\kappa} \propto \frac{1}{\kappa}$$

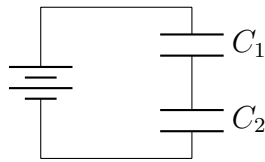
This changes the equations for capacitance, electric field, and voltage across a capacitor to the following:

$$C_{\kappa} = \kappa C_0 = \frac{\kappa Q}{\Delta V}$$

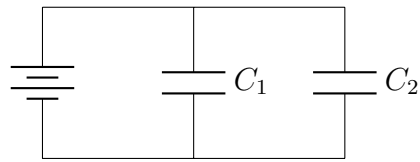
$$E_{\kappa} = \frac{\sigma}{\kappa \epsilon_0} = \frac{Q}{\kappa \epsilon_0 A}$$

$$\Delta V_{\kappa} = E_{\kappa} d = \frac{Qd}{\kappa \epsilon_0 A}$$

Capacitors in Series and Parallel



(a) Wired in series.



(b) Wired in parallel.

Differences Capacitors wired in series and parallel behave differently. Capacitors wired in series carry the same charge, but different voltages.

$$q_{eq(s)} = q_1 = q_2 = \cdots = q_n$$

Capacitors wired in parallel behave in the opposite manner, carrying the same voltage, but different charges.

$$\Delta V_{eq(||)} = \Delta V_1 = \Delta V_2 = \cdots = \Delta V_n$$

Equivilant Capacitance Multiple capacitors can be represented by a single capacitor with an equivilant capacitance. The equivilant capacitance of capacitors in series and in parallel is given by the following equations:

$$\frac{1}{C_{eq(s)}} = \sum_{i=1}^n \frac{1}{C_i}$$
$$C_{eq(\parallel)} = \sum_{i=1}^n C_i$$

Current and Resistance

Current

Definition Current is defined as the time rate of change of charge through an object. It is measured in amps.

$$I = \frac{dQ}{dt} \Rightarrow I = nqv_d A$$

Where I is the current in amps $[A] = [\frac{C}{s}]$, n is the free electron density in electrons/ m^3 , v_d is the drift velocity in m/s , and A is the cross-sectional area of the conductor in m^2 .

Drift Velocity Electrons are bouncing around randomly. When an electric field is applied, the bouncing is directed in a direction, but it is still chaotic. This bouncing results in heat being generated. Heat is defined as the kinetic energy of a particle. The speed of the drift of the electrons is called the drift velocity (v_d).

$$v_d = \frac{I_{avg}}{nqA} = \frac{I}{nqA}$$

Where v_d is the drift current, n is the free electron density in $\frac{g}{m^2}$, q is the charge of the current carrier (usually an electron) in C , and A is the cross-sectional area of the conductor in m^2 .

Current Density Current density is the current per unit area. It can be calculated using the following formula:

$$J = \frac{I}{A} = \sigma A = nqv_d$$

Where J is the current density in $Amps/m^2$, I is the current in amps, A is the cross-sectional area of the conductor in m^2 , σ is the conductivity of the material, and n is the free electron density in electrons/ m^3 .

Voltage can be calculated as a function of current density and conductivity as follows:

$$\Delta V = E\ell = \frac{\ell J}{\sigma}$$

Where ΔV is the voltage in volts, E is the electric field in volts per meter, ℓ is the length of the conductor in meters,

Resistance

Definition Resistance is defined as the ratio of voltage to current, also known as Ohm's law. It is measured in ohms.

$$R = \frac{\Delta V}{I}$$

Where R is the resistance in Ω , ΔV is the voltage in volts, and I is the current in A .

Resistivity Resistivity is the fundamental property of a material that determines how much it resists the flow of current. It is measured in ohm-meters [Ωm].

$$\rho = \frac{1}{\sigma} = \frac{RA}{\ell} \implies R = \rho \frac{\ell}{A}$$

Where ρ is resistivity in Ωm , σ is conductivity in S/m , R is resistance in Ω , A is the cross-sectional area of the conductor in m^2 , and ℓ is the length of the conductor in meters.

Conductivity Conductivity is the inverse of resistivity. It is measured in Siemens per meter [$\frac{S}{m}$]. $[S] = [\frac{1}{\Omega}]$

$$\sigma = \frac{1}{\rho} = \frac{\ell}{RA} \implies R = \frac{\ell}{\sigma A}$$

Where ρ is resistivity, σ is conductivity, R is resistance in Ω , and ℓ is the length of the conductor in meters.

Ohmic vs. Non-Ohmic devices Ohmic devices are devices that have a Voltage vs Current slope of $\frac{1}{R}$. Non-ohmic devices have a slope that changes with voltage or current.

Resistivity and Temperature The resistivity of a material changes with the temperature of the material. The equation for resistivity as a function of temperature is given by:

$$\rho_t = \rho_0[1 + \alpha\Delta T]$$
$$\alpha = \frac{\Delta\rho/\rho_0}{\Delta T}$$

Where ρ_t is the resistivity of the material at a given temperature in Ωm , ρ_0 is the resistivity of the material at a reference temperature in Ωm , α is the temperature coefficient in $^{\circ}C^{-1}$ (a material constant, much like κ), $\Delta\rho$ is the change in resistivity in Ωm , and ΔT is the change in temperature in $^{\circ}C$.

Electrical Power

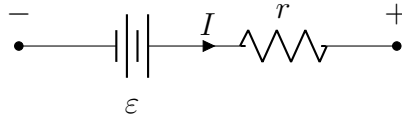
Definition Power is defined as the rate at which energy is transferred or converted. It is measured in watts.

$$P = \frac{dU_e}{dt} = I^2R = \frac{(\Delta V)^2}{R} = IV$$

Where P is power in watts, U_e is electric potential energy in Joules, ΔV is voltage in volts, and I is current in amps.

Direct Current Circuits

Electromotive Force



Circuit Diagram of a Simple Battery

Definition Electromotive force (emf) is the voltage produced by a battery or other voltage source. It is measured in volts. A simple battery consists of a source of emf and a resistor to represent the internal resistance of the battery. Unless the battery is ideal, $\Delta V_{terminal} \neq \varepsilon$. In this case, we can find the voltage at the terminals of the battery using the following equation.

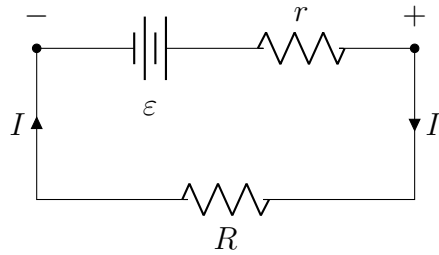
$$\begin{aligned}\Delta V_{terminal} &= \varepsilon - Ir \\ \varepsilon &= \Delta V_{terminal} + Ir\end{aligned}$$

Where $\Delta V_{terminal}$ is the voltage at the terminals of the battery, ε is the emf of the battery, I is the current through, and r is the internal resistance of the battery.

When a non-ideal battery is in use, we need to represent the current in terms of the emf and the resistors in the circuit. The current and emf in the circuit can be found using the following equations.

$$\begin{aligned}I &= \frac{\varepsilon}{R + r} \\ \varepsilon &= IR + Ir\end{aligned}$$

Where ε is the emf in volts, I is the current in amps, r is the internal resistance of the battery in ohms, and R is the resistance of the resistor in ohms.



A Simple Battery wired to a resistor

Power The internal resistance also affects power delivery. The power delivered to each of the resistors in the above circuit is given by the following equations.

$$P = P_R + P_r$$

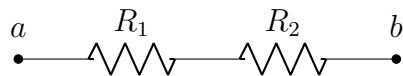
$$P_R = I^2 R$$

$$P_r = I^2 r$$

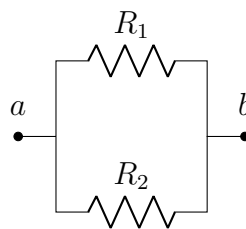
Note that the specific current value used will depend on how the resistors in the circuit are connected.

Equivalent Resistance

Definition Much like capacitors, multiple resistors can be represented by a single resistor with an equivalent resistance. How the resistors are connected will determine how the equivalent resistance is calculated and how the current and voltage change across them.



Two resistors in series



Two resistors in parallel

In Series Resistors in series can be combined into a single resistor with a resistance equal to the sum of the individual resistances. The current through each resistor in series is the same, but the voltage across each resistor is different.

$$R_{eq(s)} = \sum_{i=1}^n R_i$$

$$I_{eq(s)} = I_1 = I_2 = \cdots = I_n$$

$$\Delta V_{eq(s)} = \sum_{i=1}^n \Delta V_i$$

In Parallel Resistors in parallel can be combined into a single resistor with a resistance equal to the reciprocal of the sum of the reciprocals of the individual resistances. The voltage across each resistor in parallel is the same, but the current through each resistor is different.

$$\frac{1}{R_{eq(\parallel)}} = \sum_{i=1}^n \frac{1}{R_i}$$

$$I_{eq(\parallel)} = \sum_{i=1}^n I_i$$

$$\Delta V_{eq(\parallel)} = \Delta V_1 = \Delta V_2 = \cdots = \Delta V_n$$

Kirchhoff's Laws

Kirchhoff's Laws are fundamental tools used to analyze electrical circuits. They consist of two rules: the Junction Rule and the Loop Rule. These laws help us determine the current and voltage at different points in a circuit.

Junction Rule The Junction Rule states that at any junction (or node) in a circuit, the sum of the currents entering the junction must equal the sum of the currents leaving the junction. This is a consequence of the conservation of charge.

$$\sum_{junction} I = 0 \implies \sum_{junction} I_{in} = \sum_{junction} I_{out}$$

To apply the Junction Rule:

1. Identify all junctions in the circuit.
2. Write an equation for each junction, setting the sum of currents entering the junction equal to the sum of currents leaving the junction.
3. Solve the system of equations to find the unknown currents.

Loop Rule The Loop Rule states that the sum of the voltages around any closed loop in a circuit must equal zero. This is a consequence of the conservation of energy.

$$\sum_{closedloop} \Delta V = 0$$

To apply the Loop Rule:

1. Identify all independent loops in the circuit.
2. Choose a direction to traverse each loop (clockwise or counterclockwise).
3. Write an equation for each loop, summing the voltage drops (negative) and voltage rises (positive) around the loop and setting the sum equal to zero.
4. Solve the system of equations to find the unknown voltages and currents.

These rules can be used together to solve for the currents and voltages at different points in a circuit.

Steps to Solve a Circuit with Kirchhoff's Laws:

1. Label all currents and voltages: Assign a variable to each unknown current and voltage in the circuit.
2. Apply the Junction Rule: Write equations for each junction in the circuit.
3. Apply the Loop Rule: Write equations for each independent loop in the circuit.
4. Solve the system of equations: Use algebraic methods to solve the system of equations obtained from the Junction and Loop Rules.

Sign Conventions When applying Kirchhoff's Laws, it is important to follow these sign conventions:

- If the current enters the positive terminal of a resistor, the voltage drop across the resistor is positive.
- If the current enters the negative terminal of a resistor, the voltage drop across the resistor is negative.
- If the current enters the positive terminal of a battery, the voltage rise across the battery is negative.
- If the current enters the negative terminal of a battery, the voltage rise across the battery is positive.

RC Circuits

Definition An RC circuit is a circuit that contains a resistor and a capacitor. The capacitor stores energy in the form of an electric field, while the resistor dissipates energy in the form of heat. The time constant τ of an RC circuit is given by the product of the total resistance and capacitance in the circuit.

$$\tau = RC$$

Where τ is the time constant in seconds, R is the resistance in ohms, and C is the capacitance in farads.

Charging a Capacitor First, we must update our max charge equation to account for emf.

$$Q_{max} = C\varepsilon$$

Where Q_{max} is the max charge the capacitor can hold in Coulombs, C is the capacitance of the capacitor in Farads, and ε is the electromotive force in volts.

We can then find the charge and current of the capacitor as a function of time.

$$\begin{aligned} q(t) &= C\varepsilon(1 - e^{-t/\tau}) = Q_{max}(1 - e^{-t/\tau}) \\ I(t) &= \frac{dq}{dt} = \frac{\varepsilon}{R}e^{-t/\tau} = \frac{q(t) - Q_{max}}{\tau} \end{aligned}$$

Where $q(t)$ is the charge on the capacitor at time t in Coulombs, $I(t)$ is the current through the capacitor at time t in Amps, C is the capacitance of the capacitor in Farads, ε is the electromotive force in volts, R is the resistance in ohms, and τ is the time constant of the circuit in seconds.

Discharging a Capacitor We can also find the charge and current of a capacitor as it discharges as a function of time.

$$q(t) = Q_i e^{-t/\tau}$$
$$I(t) = -\frac{Q_i}{\tau} e^{-t/\tau}$$

Where Q_i is the initial charge on the capacitor in Coulombs, and the rest of the variables having the same definition as above.

3. Magnetism

Magnetic Fields

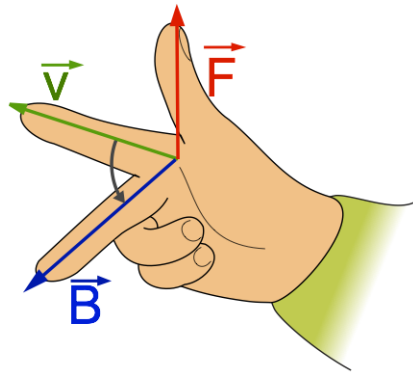
Definition A magnetic field is a vector field that describes the magnetic influence on moving electric charges, electric currents, and magnetic materials. They are generated by moving charges, or current. It is represented by the symbol \vec{B} and is measured in teslas (T).

Unlike electric fields, which can be generated by a single source charge, magnetic poles always come in pairs. There are no magnetic monopoles. The two types of magnetic poles are called north and south.

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

Where \vec{F}_B is the magnetic force in newtons, q is the charge in coulombs, \vec{v} is the velocity of the charged in meters per second, and \vec{B} is the magnetic field in teslas.

Note that the magnetic force is perpendicular to both the velocity of the charge and the magnetic field. This is different from the electric force, which is parallel to the electric field. The direction of the magnetic force can be determined using the right hand rule.



By convention, \odot represents a vector going into the page, and \otimes represents one coming out of the page.

Charged Particle in a Uniform Magnetic Field When a charged particle moves in a uniform magnetic field, it experiences a constant magnetic force that is directed radially inward. This causes the particle to move in uniform circular motion. We can revise the UCM equations in terms of the magnetic field and charge of the particle in UCM.

$$\begin{aligned} F_B &= qvB = \frac{mv^2}{r} \\ r &= \frac{mv}{qB} \\ \omega &= \frac{v}{r} = \frac{qv}{m} \\ T &= \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \end{aligned}$$

Where F_B is the magnetic force in newtons, q is the charge of the particle in coulombs, v is the velocity of the particle in meters per second, B is the magnetic field in teslas, m is the mass of the particle in kilograms, r is the distance of the particle from the axis of rotation in meters, ω is the angular velocity in radians per second, and T is the period in seconds.

Magnetic Force Acting on a Current-Carrying Wire When a current-carrying wire is placed in a magnetic field, it experiences a magnetic force. This force is given by the following equation.

$$\vec{F}_B = I\vec{\ell} \times \vec{B}$$

Where \vec{F}_B is the magnetic force in newtons, I is the current in amperes, $\vec{\ell}$ is the length of the wire in meters, and \vec{B} is the magnetic field in teslas. The direction of the force can be determined using the right hand rule.

Torque on a Current Loop in a Magnetic Field When a current loop is placed in a magnetic field, it experiences a torque. This torque is given by the following equation.

$$\begin{aligned}\vec{\tau}_{mag,dipole} &= \vec{\mu} \times \vec{B} \\ \vec{\mu} &= I\vec{A} \\ U_B &= -\vec{\mu} \cdot \vec{B}\end{aligned}$$

Where $\vec{\tau}_{mag,dipole}$ is the torque in newton-meters, $\vec{\mu}$ is the magnetic dipole moment in ampere-square meters, B is the magnetic field in teslas, I is the current in amperes, and A is the area of the loop in square meters.

Hall Effect The Hall effect is the production of a voltage difference (the Hall voltage) across an electrical conductor, when a magnetic field is applied perpendicular to the current. The Hall voltage is given by the following equation.

$$\begin{aligned}E_H &= v_d B = \frac{IB}{nq} \\ \Delta V_H &= E_H d = \frac{IBd}{nqA} = \frac{IB}{nqt}\end{aligned}$$

Where E_H is the Hall electric field in volts per meter, v_d is the drift velocity in meters per second, B is the magnetic field in teslas, n is the number of charge carriers per unit volume in per cubic meter, q is the charge of the particle in coulombs, I is the current in amperes, d is the thickness of the conductor in meters, A is the cross-sectional area of the conductor in square meters, and t is the time in seconds.

Sources of Magnetic Fields

Biot-Savart Law The Biot-Savart law describes the magnetic field produced by a current-carrying wire. It is given by the following equation.

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{\ell} \times \hat{r}}{r^2}$$
$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\ell} \times \hat{r}}{r^2}$$

Where $d\vec{B}$ is the magnetic field in teslas, μ_0 is the permeability of free space in henries per meter (a constant), I is the current in amperes, $d\vec{\ell}$ represents an infinitesimal segment of the wire's length, \hat{r} is the unit vector pointing from the wire to the point where the magnetic field is being measured, and r is the distance from the wire to the point where the magnetic field is being measured in meters.

Much like the electric field, the magnetic field generated by an object varies with its geometry. The two most common geometries are a straight wire and a loop of wire.

$$\vec{B}_{straight} = \frac{\mu_0 I}{2\pi r}$$
$$\vec{B}_{loop} = \frac{\mu_0 I}{2R}$$
$$\frac{\vec{F}_{B\parallel}}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi d}$$

Where $\vec{B}_{straight}$ is the magnetic field of a straight wire in teslas, \vec{B}_{loop} is the magnetic field of a loop of wire in teslas, μ_0 is the permeability of free space in henries per meter, I is the current in amperes, r is the distance from the wire to the point where the magnetic field is being measured in meters, and R is the radius of the loop in meters.

Ampere's Law Ampere's Law is similar to Gauss's Law, but for magnetic fields. Unlike Gauss's Law, Ampere's Law is a line integral rather than an area integral. It is given by the following equation.

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

Where \vec{B} is the magnetic field in Teslas, $d\vec{\ell}$ represents an infinitesimal segment of the wire's length, μ_0 is the permeability of free space in henries per meter, and I_{enc} is the current enclosed by the loop in amperes.

Toroids and Solenoids

Definitions A solenoid is a coil of wire that generates a magnetic field when a current is passed through it. A toroid is a solenoid bent into a circle.

Magnetic Field of a Solenoid The magnetic field generated by a solenoid is given by the following equation.

$$\vec{B} = \mu_0 n I$$
$$n = \frac{N}{L}$$

Where \vec{B} is the magnetic field in teslas, μ_0 is the permeability of free space in henries per meter, n is the number of turns per unit length in turns per meter, I is the current in amperes, N is the number of turns in the solenoid, and L is the length of the solenoid in meters.

Magnetic Field of a Toroid The magnetic field generated by a toroid is given by the following equation.

$$\vec{B} = \frac{\mu_0 N I}{2\pi r}$$

Where \vec{B} is the magnetic field in teslas, μ_0 is the permeability of free space in henries per meter, N is the number of turns in the toroid,

Magnetic Flux

Definition Magnetic flux is a measure of the number of magnetic field lines that pass through a given area. It is given by the following equation.

$$\Phi_B = \int \vec{B} \cdot d\vec{A} = BA \cos(\theta)$$

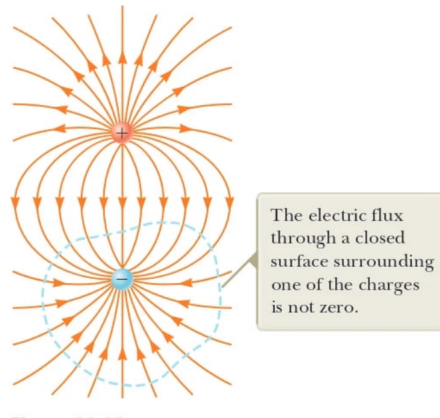
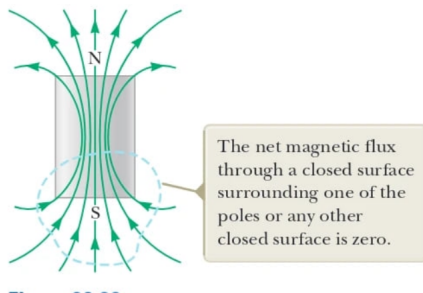
Where Φ_B is the magnetic flux in webers, \vec{B} is the magnetic field in teslas, and \vec{A} is the area in square meters.

Notice the similarity of this equation to the definition of electric flux, just with the magnetic field in place of the electric field.

Gauss's Law for Magnetism Gauss's Law for Magnetism is similar to Gauss's Law for Electricity, but for magnetic fields. It states that the magnetic flux through any surface surrounding a magnetic dipole is zero. This is given by the following equation.

$$\Phi_B = \oint \vec{B} \cdot d\vec{A} = 0$$

Where Φ_B is the magnetic flux in webers, \vec{B} is the magnetic field in teslas, and \vec{A} is the area in square meters.



Induction

Definition Induction is the process of generating an electromotive force in a closed circuit by changing the magnetic field around the circuit. It is the basis of many electrical devices, including transformers and generators. Induction is governed by Faraday's Law of Induction, which states that the induced electromotive force in a circuit is equal to the negative rate of change of the magnetic flux through the circuit. This is given by the following equations:

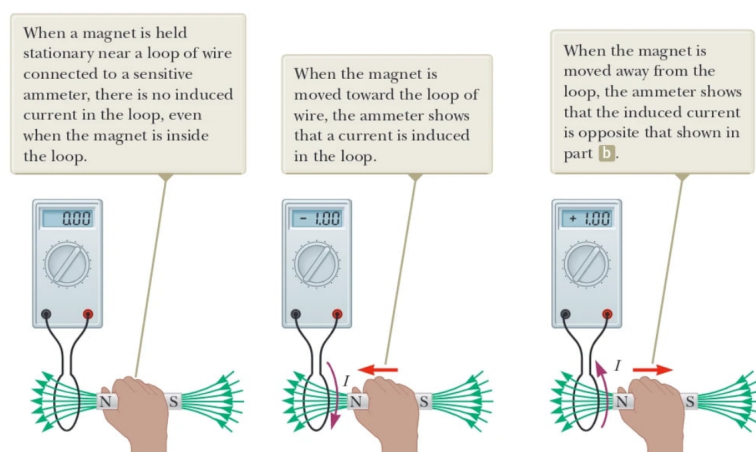
$$\begin{aligned}\varepsilon &= -\frac{d\Phi_B}{dt} = -\frac{\Delta\Phi_B}{\Delta t} \\ \varepsilon &= -N\frac{d\Phi_B}{dt} = -N\frac{\Delta\Phi_B}{\Delta t} \\ \varepsilon &= \frac{d}{dt}(BA\cos(\theta)) = -BA\sin(\theta) \\ \Rightarrow \Delta\Phi_B &= \Delta BA\cos(\theta) = B\Delta A\cos(\theta) = BA(\cos(\theta_f) - \cos(\theta_i))\end{aligned}$$

Where ε is the electromotive force in volts, N is the number of turns in the coil, Φ_B is the magnetic flux in webers, B is the magnetic field in teslas, A is the area of the coil in square meters, and θ is the angle between the magnetic field and the normal to the coil.

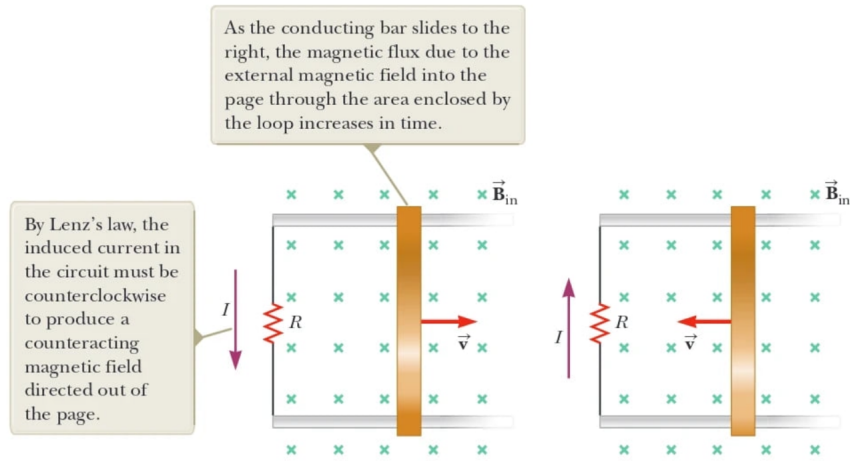
Motional EMF Motional electromotive force is the electromotive force generated by a moving conductor in a magnetic field. It is modeled by the following equations.

$$\begin{aligned}\varepsilon &= B\ell v \\ \Phi_B &= \int \vec{B} \cdot d\vec{A} = BA = B\ell w \\ \varepsilon &= \frac{B^2\ell^2v}{R}\end{aligned}$$

Where ε is the electromotive force in volts, Φ_B is the magnetic flux in webers, B is the magnetic field in teslas, ℓ is the length of the conductor in meters, w is the width of the conductor in meters, v is the velocity of the conductor in meters per second, and R is the resistance of the conductor in ohms.



Lenz's Law Lenz's Law states that the direction of the induced current in a circuit is such that it opposes the change in magnetic flux that caused it. Practically, this means that when the magnetic flux increases, an opposing magnetic field is generated to counteract the increase. Conversely, if the magnetic flux decreases, the generated magnetic field will be in the same direction as the original field, opposing the decrease. The direction of this opposing field determines the direction of the induced current.



General Form of Faraday's Law

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$$

Which allows us to find the Electric Field generated by a changing magnetic field.

$$\vec{E} = -\frac{r}{2} \left(\frac{d\vec{B}}{dt} \right)$$

Where \vec{E} is the electric field in volts per meter, $d\vec{\ell}$ is an infinitesimal segment of the circuit in meters, Φ_B is the magnetic flux in webers, r is the distance from the wire in meters, and $\frac{d\vec{B}}{dt}$ is the rate of change of the magnetic field in teslas per second.

Inductance

Self-Inductance Self-inductance is the property of a circuit that causes it to generate an electromotive force in response to a change in current. It is measured in henries $[H] = [V \cdot \frac{s}{A}]$. The electromotive force generated by self-inductance is given by the following equation.

$$\begin{aligned}\varepsilon_L &= -L \frac{dI}{dt} \\ L &= \frac{\varepsilon_L}{dI/dt} = \frac{N\Phi_B}{I}\end{aligned}$$

Where ε_L is the electromotive force generated by self inductance in volts, L is the self-inductance in henries, I is the current in amperes, N is the number of turns in the coil, and Φ_B is the magnetic flux in webers.

RL Circuits

Definition An RL circuit is a circuit that contains a resistor and an inductor. When a current is passed through the circuit, the inductor generates an electromotive force that opposes the change in current. This causes the current to increase more slowly than it would in a circuit with only a resistor. The current in an RL circuit is given by the following equation.

$$\begin{aligned}I(t) &= \frac{\varepsilon}{R}(1 - e^{-t/\tau}) \\ \tau &= \frac{L}{R} \\ I_{max} &= \frac{\varepsilon}{R}\end{aligned}$$

Where $I(t)$ is the current in amperes, ε is the electromotive force in volts, R is the resistance in ohms, t is the time in seconds, τ is the time constant in seconds, L is the inductance in henries, and I_{max} is the maximum current in amperes.

Energy in a Magnetic Field The energy stored in a magnetic field is given by the following equation.

$$U_B = \frac{1}{2}LI^2$$

Where U_B is the energy stored in the magnetic field in joules, L is the inductance in henries, and I is the current in amperes.

Energy in a Solinoid in a Magnetic Field

$$u_B = \frac{U_B}{V} = \frac{B^2V}{2\mu_0}$$

Where u_B is the energy density in the magnetic field in joules per cubic meter, U_B is the energy stored in the magnetic field in joules, V is the volume of the solenoid in cubic meters, B is the magnetic field in teslas, and μ_0 is the permeability of free space in henries per meter.

Mutual Inductance Mutual inductance is the property of a circuit that causes it to generate an electromotive force in response to a change in current in a neighboring circuit. It is measured in henries.

$$\begin{aligned} M_{12} &= \frac{N_2\Phi_{12}}{I_1} & M_{21} &= \frac{N_1\Phi_{21}}{I_2} \\ \varepsilon_1 &= -M_{21}\frac{dI_2}{dt} & \varepsilon_2 &= -M_{12}\frac{dI_1}{dt} \end{aligned}$$

Where M_{12} is the mutual inductance from circuit 1 to circuit 2 in henries, N_2 is the number of turns in circuit 2, Φ_{12} is the magnetic flux from circuit 1 to circuit 2 in webers, I_1 is the current in circuit 1 in amperes, M_{21} is the mutual inductance from circuit 2 to circuit 1 in henries N_1 is the number of turns in circuit 1, Φ_{21} is the magnetic flux from circuit 2 to circuit 1 in webers, I_2 is the current in circuit 2 in amperes, ε_1 is the electromotive force generated in circuit 1 in volts, ε_2 is the electromotive force generated in circuit 2 in volts, and M_{12} is the mutual inductance from circuit 1 to circuit 2 in henries.

Oscillations in an LC Circuit An LC circuit is a circuit that contains a capacitor and an inductor. When a charge is passed through the circuit, the capacitor stores energy in the form of an electric field. When the capacitor is fully charged, the inductor generates an electromotive force that causes the charge to flow in the opposite direction. This causes the capacitor to discharge, and the cycle repeats. Note that this cycle is between electric potential energy and magnetic potential energy. The charge in an LC circuit is given by the following equation.

$$\begin{aligned}
 q(t) &= Q_{max} \cos(\omega t + \phi) \\
 \omega &= \frac{1}{\sqrt{LC}} \\
 I &= \frac{dq}{dt} = -\omega Q_{max} \sin(\omega t + \phi) = -I_{max} \sin(\omega t + \phi)
 \end{aligned}$$

Where $q(t)$ is the charge in coulombs, q_{max} is the maximum charge in coulombs, ω is the angular frequency in radians per second, L is the inductance in henries, C is the capacitance in farads, and t is the time in seconds.

RLC Circuits

An RLC circuit is a circuit that contains a resistor, an inductor, and a capacitor. When a charge is passed through the circuit, the capacitor stores energy in the form of an electric field. When the capacitor is fully charged, the inductor generates an electromotive force that causes the charge to flow in the opposite direction. This causes the capacitor to discharge, and the cycle repeats. The energy model for this circuit is given by the following equation.

$$\begin{aligned}
 \Delta U_E + \Delta U_B + \Delta \varepsilon_{int} &= 0 \\
 L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} &= 0
 \end{aligned}$$

Where ΔU_E is the change in electric potential energy, ΔU_B is the change in magnetic potential energy, and $\Delta \varepsilon_{int}$ is the change in internal energy.

The charge in an RLC circuit is given by the following equation.

$$q(t) = q_{max} e^{-\frac{R}{2L}t} \cos(\omega_d t + \phi)$$
$$\omega_d = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

Where $q(t)$ is the charge in coulombs, q_{max} is the maximum charge in coulombs, R is the resistance in ohms, L is the inductance in henries, C is the capacitance in farads, ω_d is the damped angular frequency in radians per second, and t is the time in seconds.

Electromagnetic Waves

Definition Electromagnetic waves are waves of electric and magnetic fields that propagate through space. They are generated by accelerating charges and are characterized by their frequency and wavelength. Electromagnetic waves are governed by Maxwell's equations, which describe how electric and magnetic fields interact with each other.

Maxwell's Equations

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0} \quad \text{Gauss's Law}$$

$$\oint \vec{B} \cdot d\vec{A} = 0 \quad \text{Gauss's Law in Magnetism}$$

$$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt} \quad \text{Faraday's Law}$$

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt} \quad \text{Ampere-Maxwell Law}$$

Where \vec{E} is the electric field in volts per meter, \vec{B} is the magnetic field in teslas, $d\vec{A}$ is an infinitesimal segment of the circuit in meters, Q_{enc} is the charge enclosed by the loop in coulombs, ε_0 is the permittivity of free space in farads per meter, $d\vec{\ell}$ represents an infinitesimal segment of the wire's length, Φ_B is the magnetic flux in webers, μ_0 is the permeability of free space in henries per meter, I is the current in amperes, and Φ_E is the electric flux in webers.

Note that if there is no current, $\mu_0 I = 0$, and the Ampere-Maxwell Law simplifies to:

$$\oint \vec{B} \cdot d\vec{\ell} = \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

Wait a second ... How does an EM wave exist in the absence of charge or current? This can be explained using Maxwell's third and fourth equations. Note that I have rewritten them using divergence ($\nabla \cdot$) and curl ($\nabla \times$) notation. Divergence is a measure of how much a vector field is spreading out from a center point, and curl is a measure of how much a vector field is rotating around a point.

$$\begin{aligned}\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}\end{aligned}$$

Try to think of the left side of the equation as the effect of what the right side of the equation causes. Faraday's Law states that a change in the magnetic field causes a change in the electric field, causing it to bend. The Ampere-Maxwell Law states that a change in the electric field causes a change in the magnetic field, causing it to bend. This means that once an electromagnetic wave is generated, it can propagate through space without the need for a charge or current to sustain it, as the constant changes in both fields will keep the wave moving.

Electromagnetic Wave Equation

The electromagnetic wave equation is a second-order partial differential equation that describes how electromagnetic waves propagate through space. Assuming that the wave is propagating in the x direction, the wave equation is given by the following system of equations.

$$\begin{aligned}\frac{\partial^2 \vec{E}}{\partial x^2} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\ \frac{\partial^2 \vec{B}}{\partial x^2} &= \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \\ \frac{\partial^2 y}{\partial x^2} &= \frac{1}{v^2} \left(\frac{\partial^2 y}{\partial t^2} \right)\end{aligned}$$

Where \vec{E} is the electric field in volts per meter, \vec{B} is the magnetic field in teslas, x is the position in meters, t is the time in seconds, μ_0 is the permeability of free space in henries per meter, y is the wave function, x is the

position in meters and ε_0 is the permittivity of free space in farads per meter.

The Speed of Light The speed of light is the speed at which electromagnetic waves propagate through space. It is given by the following equation when the wave is propagating through a vacuum.

$$c_{vac} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3.00 \times 10^8 \text{ m/s}$$

The speed of light can also be calculated in terms of wavelength and frequency using the following equation.

$$c = \lambda f \quad \Rightarrow \quad f = \frac{c}{\lambda} \quad \Rightarrow \quad \lambda = \frac{c}{f}$$

Where c is the speed of light in meters per second, λ is the wavelength in meters, and f is the frequency in hertz.

Solution to the Wave Equation

$$c = \frac{E_{max}}{B_{max}} \quad \Rightarrow \quad E_{max} = c B_{max} \quad \Rightarrow \quad B_{max} = \frac{E_{max}}{c}$$

Poynting Vector The Poynting vector is a vector that describes the direction and magnitude of the energy flow in an electromagnetic wave. It is given by the following equation.

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

Where \vec{S} is the Poynting vector in watts per square meter, \vec{E} is the electric field in volts per meter, \vec{B} is the magnetic field in teslas, and μ_0 is the permeability of free space in henries per meter.

Energy Carried by Electromagnetic Waves The energy carried by an electromagnetic wave is given by the following equation.

$$U_E = \frac{1}{2}\epsilon_0 E_{max}^2$$

$$U_B = \frac{1}{2\mu_0} B_{max}^2$$

$$I = S_{avg} = \frac{cB_{max}^2}{2\mu_0} = \frac{c}{U_{avg}}$$

Momentum and Radiation Pressure Electromagnetic waves carry momentum and exert a pressure on objects they interact with. The momentum carried by an electromagnetic wave is given by the following equations.

$$P = \frac{S}{c} \quad \text{(Complete Absorption)}$$

$$P = \frac{2S}{c} \quad \text{(Complete Reflection)}$$

$$P = (1 + f)\frac{S}{c} \quad \text{(Partial Absorption)}$$

Where P is the momentum in newton-seconds, S is the Poynting vector in watts per square meter, c is the speed of light in meters per second, and f is the fraction of the wave that is absorbed (check this one for accuracy).