Contents

Question 1

The Society for Human Blood Consumption suggests that the equations

$$\dot{x} = \frac{1}{9}x(22 - 2x - y) \quad \text{and} \quad \dot{y} = \frac{1}{4}y(-1 + x - 2y)$$

best govern the coupling of human population density $x(t)$ to vampire population density $y(t)$. Restrict attention to

$$\Omega = \{(x, y) : x \geq 0, y \geq 0\}$$

(1a)

Use Maxima to find all equilibrium points in Ω.

The equilibrium points are at the coordinates

$$P_1(0, 0), \quad P_2(11, 0), \quad P_3(9, 4)$$

The point $(0, -\frac{1}{2})$ is also an equilibrium point, but we ignore it since it does not lie in Ω.

Refer to page (44) for the Maxima code.

(1b)

Use Maxima to compute the variational matrix at each equilibrium point.

Define the variational matrix $A(x, y)$ by

$$A(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix}$$

Then, using Maxima, we find that the variational matrix for each equilibrium point is:

$$A(P_1) = \begin{bmatrix} \frac{22}{9} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$$
$$A(P_2) = \begin{bmatrix} -\frac{22}{9} & -\frac{11}{9} \\ 0 & \frac{5}{2} \end{bmatrix}$$
$$A(P_3) = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$$

Refer to page (44) for the Maxima code.

(1c)

Use Maxima to find the eigenvalues associated with each equilibrium point.

The eigenvalues for each variational matrix are given by

Point	λ_1	λ_2
$P_1(0,0)$	$-\frac{1}{4}$	$\frac{22}{9}$
$P_2(11,0)$	$-\frac{22}{9}$	$\frac{5}{2}$
$P_3(9,4)$	$-2-i$	$-2+i$

Refer to page (44) for the Maxima code.

(1d)

Complete a table with each equilibrium, its eigenvalues, p and q, Δ and the type of equilibrium.

Calculations for P_1 and P_2 were done using Maxima, where $p = \lambda_1 + \lambda_2$, $q = \lambda_1\lambda_2$, $\Delta = p^2 - 4q$. See page (46).

For P_3, Maxima wasn't simplifying things as I wanted; however, we can compute:

$$\begin{aligned} p &= (-2-i) + (-2+i) = -2 - 2 + i - i \\ &= -4 \\ q &= (-2-i)(-2+i) = 4 + 2i - 2i - i^2 \\ &= 5 \\ \Delta &= (-4)^2 - 4(5) \\ &= -4 \end{aligned}$$

Equilibrium point	λ_1	λ_2	p	q	Δ	Type
$P_1(0,0)$	$-\frac{1}{4}$	$\frac{22}{9}$	$\frac{79}{36}$	$-\frac{11}{18}$	$\frac{9409}{1296}$	Saddle
$P_2(11,0)$	$-\frac{22}{9}$	$\frac{5}{2}$	$\frac{1}{18}$	$-\frac{55}{9}$	$\frac{7921}{324}$	Saddle
$P_3(9,4)$	$-2-i$	$-2+i$	-4	5	-4	Stable Spiral

(1e)

Determine the behavior of the system along the boundary of Ω.

The boundary of Ω is when $x = 0$ or $y = 0$.

First, consider when $x = 0$. Then the system is given by

$$\begin{aligned} & \dot{x} = \tfrac{1}{9}(0)(22 - 2(0) - y) & & \dot{y} = \tfrac{1}{4}y(-1 + (0) - 2y) \\ \Rightarrow \quad & \dot{x} = 0 & \Rightarrow \quad & \dot{y} = \tfrac{1}{4}y(-1 - 2y) \end{aligned}$$

Let $(0, y)$ be a solution starting on this boundary. $\dot{y}$ is negative for all starting points (assuming y is positive). Thus, y is decreasing along the boundary $x = 0$. Also, $\dot{x} = 0$, and so the solution stays on the boundary. We may conclude that solutions along this line approach $P_1(0, 0)$.

Now, suppose that $y = 0$. Then

$$\begin{aligned} & \dot{x} = \tfrac{1}{9}x(22 - 2x - (0)) & & \dot{y} = \tfrac{1}{4}(0)(-1 + x - 2(0)) \\ \Rightarrow \quad & \dot{x} = \tfrac{1}{9}x(22 - 2x) & \Rightarrow \quad & \dot{y} = 0 \end{aligned}$$

Suppose that $(x, 0)$ is a starting point of a solution on this boundary. First, consider the case when $x < 11$. Then $22 - 2x > 0$ and so $\dot{x} > 0$. Also, $\dot{y} = 0$, so the solution stays on this boundary. We can then conclude that x will approach the equilibrium point $P_2(11, 0)$.

If $x > 11$, then $22 - 2x < 0$ and so $\dot{x} < 0$. Thus, x is decreasing on this boundary line. Again, the solution will approach $P_2(11, 0)$.

(1f)

Use Maxima to construct (1) a direction field around each equilibrium point and (2) a complete phase portrait that includes all equilibrium points. Will vampires continue to prey on humans forever, or can we expect their activity to essentially cease in the long run?

The four direction fields are shown on the following two pages. Each one has three nearby solutions that demonstrate the behavior of the system.

Let $(x(t), y(t))$ be a solution to the system. Vampires will continue to prey on humans forever as long as $x(t) \not\to 0$ and $y(t) \not\to 0$. Thus, vampires will continue to prey on humans as long as $(x(t), y(t))$ does not approach the boundary of Ω. We already found that if a solution begins on the boundary of Ω, it stays on that boundary. Also, the direction fields show that a solution that does not start on the boundary of Ω will not approach the boundary of Ω. Thus, vampires will prey on humans forever as long as $x(0) \neq 0$ and $y(0) \neq 0$.

See page (47) for the Maxima code of the direction fields.

Figure 1: Direction field around $P_1(0,0)$

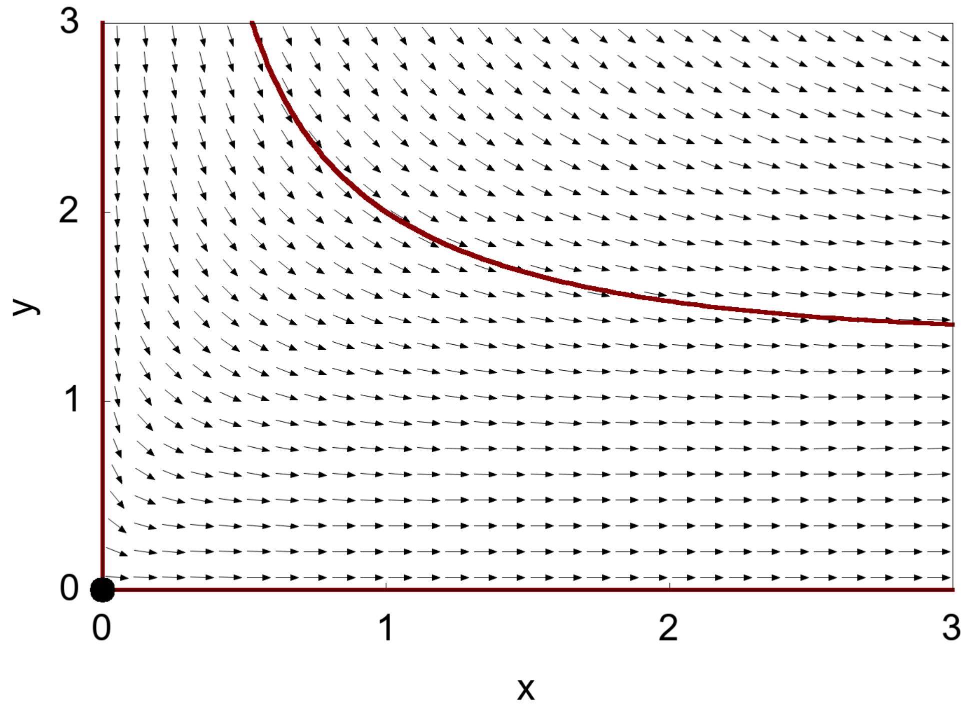


Figure 2: Direction field around $P_2(11,0)$

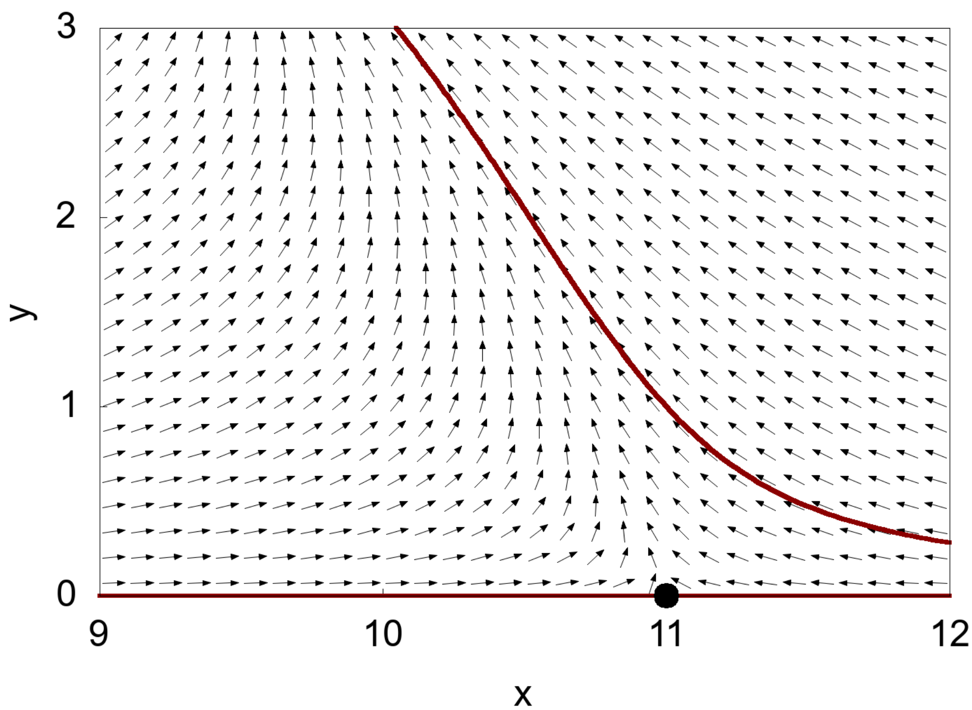


Figure 3: Direction field around $P_3(9,4)$

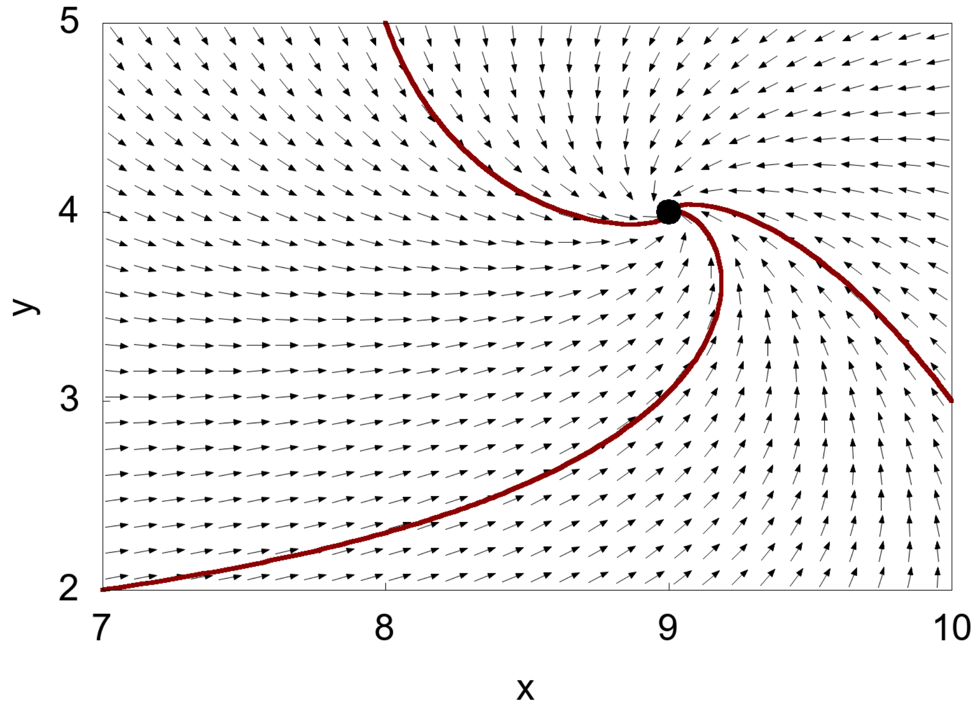
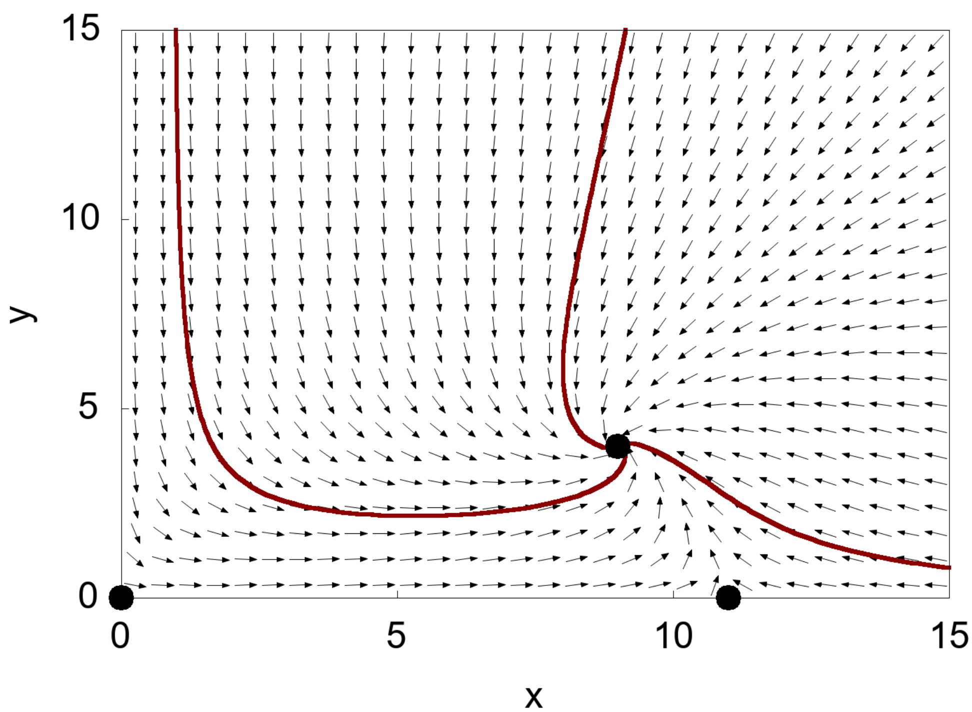


Figure 4: Complete direction field



Question 2

Hauntedness $X(t)$ of a forest and the density of ghouls $Y(t)$ inhabiting it obey the following system of nonlinear equations:

$$\begin{aligned} \dot{X} &= 4X(\ln Y - \ln B) \\ \dot{Y} &= Y(\ln A - \ln X) + 2aY(\ln B - \ln Y) + Y(\ln A - \ln X)^7 \end{aligned} \tag{1}$$

where a is the density of forgotten headstones, A is the amount of plant overgrowth, and B is the thickness of the evening fog. Restrict attention to

$$\Omega = \{(X, Y) : X > 0, Y > 0\}$$

(2a)

Make the change of variables $x = \ln\left(\frac{X}{A}\right)$ and $y = \ln\left(\frac{Y}{B}\right)$ to obtain an equivalent system

$$\dot{x} = 4y \quad \text{and} \quad \dot{y} = -x - 2ay - x^7 \tag{2}$$

Let $x = \ln(\frac{X}{A})$ and $y = \ln(\frac{Y}{B})$. Then

$$\begin{aligned} \dot{x} &= \frac{dx}{dX}\frac{dX}{dt} \\ &= \left(\frac{1}{A}\frac{1}{\frac{X}{A}}\right)(4X(\ln Y - \ln B)) \\ &= \frac{1}{X}4X(\ln Y - \ln B) \\ &= 4(\ln Y - \ln B) \\ &= 4\ln\left(\frac{Y}{B}\right) \\ &= 4y \end{aligned}$$

Also, we have

$$\begin{aligned}
\dot{y} &= \frac{dy}{dY}\frac{dY}{dt} \\
&= \left(\frac{1}{B}\frac{1}{\frac{Y}{B}}\right)\left(Y(\ln A - \ln X) + 2aY(\ln B - \ln Y) + Y(\ln A - \ln X)^7\right) \\
&= \frac{1}{Y}\left(Y(\ln A - \ln X) + 2aY(\ln B - \ln Y) + Y(\ln A - \ln X)^7\right) \\
&= (\ln A - \ln X) + 2a(\ln B - \ln Y) + (\ln A - \ln X)^7 \\
&= -(\ln X - \ln A) - 2a(\ln Y - \ln B) - (\ln X - \ln A)^7 \\
&= -\ln\left(\frac{X}{A}\right) - 2a\ln\left(\frac{Y}{B}\right) - \left(\ln\left(\frac{X}{A}\right)\right)^7 \\
&= -x - 2ay - x^7
\end{aligned}$$

as desired.

(2b)

Show that there exists a unique critical point P of (2).

We want to solve the system

$$\begin{aligned}
4y &= 0 \\
-x - 2ay - x^7 &= 0
\end{aligned}$$

The first equation tells us that $y = 0$, which we can plug into the second equation to find:

$$\begin{aligned}
& & -x - 2ay - x^7 &= 0 \\
&\Rightarrow & -x - 2a(0) - x^7 &= 0 \\
&\Rightarrow & -x - x^7 &= 0 \\
&\Rightarrow & -x(1 + x^6) &= 0
\end{aligned}$$

Hence, either $-x = 0$ or $(1 + x^6) = 0$. The first equation has a solution $x = 0$, while the latter equation has no solutions.

Thus, our only solution is $P^*(0, 0)$.

(2c)

Find a real number c such that $V(x, y) = x^2 + 4y^2 + cx^8$ is a non-strict Lyapunov function for P on $\Phi = \mathbb{R}^2$.

Let $c = \frac{1}{4}$ so that

$$V(x, y) = x^2 + 4y^2 + \frac{1}{4}x^8$$

I claim that this is a non-strict Lyapunov function. First, we check that it is positive definite. Observe that $V(0, 0) = 0$ (on our critical point). Next, suppose that $(x, y) \neq (0, 0)$. Notice that all terms of $V(x, y)$ are raised to an even power. This means that each term in $V(x, y)$ is either 0 or positive. Furthermore, at least one of x^2 or $4y^2$ is positive since we assume that $(x, y) \neq (0, 0)$. Thus, $V(x, y) > 0$ for $(x, y) \neq (0, 0)$, and so $V(x, y)$ is positive definite.

Next, we verify that $\dot{V}(x, y)$ is negative definite. Observe that

$$\begin{aligned} \dot{V}(x, y) &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\ &= (2x + 8\left(\tfrac{1}{4}\right)x^7)4y + 8y(-x - 2ay - x^7) \\ &= 8xy + 8x^7y - 8xy - 16ay^2 - 8x^7y \\ &= -16ay^2 \end{aligned}$$

Also, notice that

$$\begin{aligned} & y^2 \geq 0 \\ \Rightarrow \quad & ay^2 \geq 0 \\ \Rightarrow \quad & -16ay^2 \leq 0 \end{aligned}$$

Thus, $\dot{V}(x, y) \leq 0$ for all $(x, y) \in \Phi$, and so it is negative definite.

Finally, we verify that $V(x, y)$ is radially unbounded. Observe that:

$$\lim_{x \to \infty} x^2 + 4y^2 + \tfrac{1}{4}x^8 = \infty \qquad \lim_{y \to \infty} x^2 + 4y^2 + \tfrac{1}{4}x^8 = \infty$$
$$\lim_{x \to -\infty} x^2 + 4y^2 + \tfrac{1}{4}x^8 = \infty \qquad \lim_{y \to -\infty} x^2 + 4y^2 + \tfrac{1}{4}x^8 = \infty$$

Thus, $V(x, y)$ is radially unbounded because $V(x, y) \to \infty$ as $|x| \to \infty$ or as $|y| \to \infty$.

(2d)

Find the subset J of Φ° on which $\dot{V}(x, y) = 0$.

The subset J of Φ° is where $\dot{V}(x, y) = 0$. This is when

$$\begin{aligned} & -16ay^2 = 0 \\ \Rightarrow \quad & y^2 = 0 \\ \Rightarrow \quad & y = 0 \end{aligned}$$

However, the point $P^*(0, 0)$ is not an element of Φ°, so we ignore it. Thus, we can define J as

$$J = \{(x, 0) \in \mathbb{R}^2 : x \neq 0\}$$

(2e)

Show that any solution of (2) that starts on J immediately leaves J.

Suppose that $(x, 0) \in J$. Then $x \neq 0$. First, suppose that $x > 0$. Then $x + x^7$ is positive since $x > 0$ and $x^7 > 0$. Thus

$$\begin{aligned}\dot{y} &= -x - 2a(0) - x^7 \\ &= -x - x^7 \\ &= -(x + x^7) \\ &< 0\end{aligned}$$

and so a trajectory starting at $(x, 0)$ immediately leaves J because $\dot{y}$ is negative (and so y decreases).

Next, suppose that $x < 0$. Then $x + x^7 < 0$ because $x < 0$ and $x^7 < 0$. Therefore

$$\begin{aligned}\dot{y} &= -x - 2a(0) - x^7 \\ &= -x - x^7 \\ &= -(x + x^7) \\ &> 0\end{aligned}$$

and so a trajectory starting at $(x, 0)$ immediately leaves J since $\dot{y}$ is positive.

Thus, every solution that starts on J immediately leaves J.

(2f)

Identify the basin of attraction for P in (2). Then, interpret this result in terms of the original haunted forest represented in (1).

Because $V(x, y)$ is a non-strict Lyapunov function for $\Phi = \mathbb{R}^2$ and every solution in J immediately leaves J, we know that $\Phi \subseteq B(P^*)$. Since Φ is the entire plane, we conclude that P^* is a global attractor.

To go back to the original differential equations, recall that $x = \ln\left(\frac{X}{A}\right)$ and $y = \ln\left(\frac{Y}{B}\right)$. Then we have

$$\begin{aligned} x &= \ln\left(\frac{X}{A}\right) \\ \Rightarrow \quad e^x &= \frac{X}{A} \\ \Rightarrow \quad Ae^x &= X \end{aligned}$$

Similarly,

$$\begin{aligned} y &= \ln\left(\frac{Y}{B}\right) \\ \Rightarrow \quad e^y &= \frac{Y}{B} \\ \Rightarrow \quad Be^y &= Y \end{aligned}$$

Thus, the critical point $P^*(0, 0)$ will become

$$\begin{aligned} Q^*(X, Y) &= (Ae^x, Be^y) \\ &= (Ae^0, Be^0) \\ &= (A, B) \end{aligned}$$

Thus, we conclude that in our original system of differential equations, all solutions in Ω will approach the global attractor $Q^*(A, B)$.

Question 3

Let $x(t)$ be the density of good clowns and let $y(t)$ be the density of bad clowns. They obey:

$$\dot{x} = a - ex - bxy \quad \text{and} \quad \dot{y} = -cy + dxy$$

where a is the rate at which good clowns enter the population, e is the rate at which good clowns exit the population, b is the rate at which evil clowns attack good clowns, c is the rate at which evil clowns exit the population, and d is the rate at which bad clowns emerge. All constants are positive. Restrict attention to

$$\Omega = \{(x, y) : x \geq 0, y \geq 0\}$$

(3a)

Determine all equilibrium points.

We must solve the system

$$\begin{aligned} a - ex - bxy &= 0 \\ -cy + dxy &= 0 \end{aligned}$$

First, notice from the first equation that

$$\begin{aligned} -cy + dxy &= 0 \\ \Rightarrow \quad y(-c + dx) &= 0 \end{aligned}$$

Thus, either $y = 0$ or $-c + dx = 0$. In the latter case, we have

$$\begin{aligned} & -c + dx = 0 \\ \Rightarrow \quad & dx = c \\ \Rightarrow \quad & x = \frac{c}{d} \end{aligned}$$

Thus, the first equation tells us that either $y = 0$ or $x = \frac{c}{d}$. Plugging $y = 0$ into the first equation gives us:

$$\begin{aligned} & a - ex - bxy = 0 \\ \Rightarrow \quad & a - ex - bx(0) = 0 \\ \Rightarrow \quad & a - ex = 0 \\ \Rightarrow \quad & -ex = -a \\ \Rightarrow \quad & x = \frac{a}{e} \end{aligned}$$

Thus, $\left(\frac{a}{e}, 0\right)$ is a solution. If we plug $x = \frac{c}{d}$ into the first equation, we find:

$$\begin{aligned} & a - ex - bxy = 0 \\ \Rightarrow \quad & a - e\left(\frac{c}{d}\right) - b\left(\frac{c}{d}\right)y = 0 \\ \Rightarrow \quad & a - \frac{ec}{d} - \frac{bcy}{d} = 0 \\ \Rightarrow \quad & \frac{ad}{d} - \frac{ec}{d} - \frac{bcy}{d} = 0 \\ \Rightarrow \quad & ad - ec - bcy = 0 \\ \Rightarrow \quad & -bcy = -ad + ec \\ \Rightarrow \quad & y = \frac{ad - ec}{bc} \\ \Rightarrow \quad & y = \frac{ad - ce}{bc} \end{aligned}$$

Thus, a second solution is $\left(\frac{c}{d}, \frac{ad-ce}{bc}\right)$. Hence, we see that the two equilibrium points to our differential equation are

$$P_1^*\left(\frac{a}{e}, 0\right) \qquad \text{and} \qquad P_2^* = \left(\frac{c}{d}, \frac{ad-ce}{bc}\right)$$

(3b)

Find the parameter conditions under which each equilibrium point is feasible and stable. Summarize this information in a table.

Recall that our differential equations are:

$$\dot{x} = a - ex - bxy \quad \text{and} \quad \dot{y} = -cy + dxy$$

and our equilibrium points are $P_1^*\left(\frac{a}{e}, 0\right)$ and $P_2^* = \left(\frac{c}{d}, \frac{ad-ce}{bc}\right)$. Define $f(x, y) = \dot{x}$ and $g(x, y) = \dot{y}$. First, we will check the feasibility conditions.

Because the parameters are assumed to be positive, $\frac{a}{e}$ will always be positive, and so the equilibrium $P_1^*\left(\frac{a}{e}, 0\right)$ is always feasible. Similarly, $\frac{c}{d}$ will always be positive, so P_2^* will be feasible as long as $\frac{ad-ce}{bc} > 0$. Simplifying this inequality, we find:

$$\begin{aligned} \frac{ad-ce}{bc} &> 0 \\ \Rightarrow \quad ad - ce &> 0 \\ \Rightarrow \quad ad &> ce \end{aligned}$$

Thus, P_2^* is feasible as long as $ad > ce$.

Next, we must calculate the eigenvalues of each of these points. The variational matrix is given by:

$$\begin{aligned} A(x,y) &= \begin{bmatrix} \frac{\partial f}{\partial x}(x,y) & \frac{\partial f}{\partial y}(x,y) \\ \frac{\partial g}{\partial x}(x,y) & \frac{\partial g}{\partial y}(x,y) \end{bmatrix} \\ &= \begin{bmatrix} -e-by & -bx \\ dy & -c+dx \end{bmatrix} \end{aligned}$$

For P_1^*, the variational matrix is

$$\begin{aligned} A\left(\frac{a}{e},0\right) &= \begin{bmatrix} -e-b(0) & -b\left(\frac{a}{e}\right) \\ d(0) & -c+d\left(\frac{a}{e}\right) \end{bmatrix} \\ &= \begin{bmatrix} -e & -\frac{ab}{e} \\ 0 & -c+\frac{ad}{e} \end{bmatrix} \end{aligned}$$

Since this matrix is upper-triangular, we see that the eigenvalues of P_1^* are $\lambda_1 = -e$ and $\lambda_2 = -c + \frac{ad}{e}$. The first eigenvalue is always negative. The second eigenvalue is negative whenever $-c + \frac{ad}{e} < 0$. Furthermore, we have

$$\begin{aligned} & -c + \frac{ad}{e} < 0 \\ \Rightarrow \qquad & \frac{ad}{e} < c \\ \Rightarrow \qquad & ad < ce \end{aligned}$$

Thus, λ_2 is negative as long as $ad < ce$. Then P_1^* is stable when $ad < ce$. Recall that P_2^* is feasible if $ad > ce$. Therefore, we see that P_1^* is stable when P_2^* is infeasible.

The variational matrix for P_2^* is:

$$
\begin{aligned}
A\left(\frac{c}{d}, \frac{ad-ce}{bc}\right) &= \begin{bmatrix} -e-b\left(\frac{ad-ce}{bc}\right) & -b\left(\frac{c}{d}\right) \\ d\left(\frac{ad-ce}{bc}\right) & -c+d\left(\frac{c}{d}\right) \end{bmatrix} \\
&= \begin{bmatrix} -e-\frac{ad-ce}{c} & -\frac{bc}{d} \\ \frac{d(ad-ce)}{bc} & -c+c \end{bmatrix} \\
&= \begin{bmatrix} -e-\frac{ad}{c}+\frac{ce}{c} & -\frac{bc}{d} \\ \frac{d(ad-ce)}{bc} & 0 \end{bmatrix} \\
&= \begin{bmatrix} -e-\frac{ad}{c}+e & -\frac{bc}{d} \\ \frac{d(ad-ce)}{bc} & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{ad}{c} & -\frac{bc}{d} \\ \frac{d(ad-ce)}{bc} & 0 \end{bmatrix}
\end{aligned}
$$

Rather than computing the eigenvalues explicitly, we will find the trace p and determinant q. We see that the trace p is

$$p = -\frac{ad}{c} + 0 = -\frac{ad}{c}$$

and the determinant q is

$$
\begin{aligned}
q &= \left(-\frac{ad}{c}\right)(0) - \left(\frac{d(ad-ce)}{bc}\right)\left(-\frac{bc}{d}\right) \\
&= -\left(\frac{d(ad-ce)}{bc}\right)\left(-\frac{bc}{d}\right) \\
&= \frac{d(ad-ce)bc}{bcd} \\
&= ad-ce
\end{aligned}
$$

Clearly, $p < 0$. Also, recall that P_2^* is feasible if $ad > ce$. Thus, we see that $q > 0$. We can therefore conclude that P_2^* is stable when feasible.

Our findings are summarized in the following table:

Point	Feasible	Stable
$P_1^* \left(\frac{a}{e}, 0\right)$	Always	$ad < ce$
$P_2^* \left(\frac{c}{d}, \frac{ad-ce}{bc}\right)$	$ad > ce$	When feasible

(3c)

Show that the boundary equilibrium is either a stable node or a saddle point. Then, show that the positive equilibrium is a stable node when

$$\frac{c}{d} < \frac{a}{e} < \frac{c}{d} + \frac{1}{4}\left(\frac{a^2 d}{c^2 e}\right)$$

and a stable spiral when

$$\frac{a}{e} > \frac{c}{d} + \frac{1}{4}\left(\frac{a^2 d}{c^2 e}\right)$$

Our boundary equilibrium is $P_1^*\left(\frac{a}{e}, 0\right)$ since it lies on the x-axis. Recall that the eigenvalues of P_1^* are $\lambda_1 = -e$ and $\lambda_2 = -c + \frac{ad}{e}$. Compute the trace ($p$) and determinant ($q$) of our eigenvalues:

$$\begin{aligned} p &= -e + (-c + \frac{ad}{e}) \\ &= -e - c + \frac{ad}{e} \\ q &= (-e)\left(-c + \frac{ad}{e}\right) \\ &= ce - \frac{ade}{e} \\ &= ce - ad \end{aligned}$$

First, suppose that P_1^* is unstable. Our stability condition for P_1^* was that $-c + \frac{ad}{e} < 0$ or, equivalently, $ad < ce$. Since P_1^* is unstable, we then have $ad > ce$. Hence, $q < 0$, and so P_1^* is a saddle point.

Now, suppose that P_1^* is stable. Then we know that $-c + \frac{ad}{e} < 0$ and, equivalently, $ad < ce$. Thus, $p < 0$ since both $-e$ and $-c + \frac{ad}{e}$ are negative. Also, $q > 0$ because $ad < ce$ and so $q = ce - ad > 0$.

Consequently, P_1^* is either a stable spiral or a stable node. To show that P_1^* is a stable node, we must compute the determinant Δ.

We find:

$$\begin{aligned}
\Delta &= p^2 - 4q \\
&= \left(-e - c + \frac{ad}{e}\right)^2 - 4(ce - ad) \\
&= e^2 + ce - ad + ce + c^2 - \frac{acd}{e} - ad - \frac{acd}{e} + \left(\frac{ad}{e}\right)^2 \\
&\quad - 4(ce - ad) \\
&= e^2 + 2ce + c^2 - 2ad + \left(\frac{ad}{e}\right)^2 - \frac{2acd}{e} - 4ce + 4ad \\
&= e^2 - 2ce + c^2 + 2ad + \left(\frac{ad}{e}\right)^2 - \frac{2acd}{e} \\
&= e^2 - 2ce + c^2 + 2ad - \frac{2acd}{e} + \left(\frac{ad}{e}\right)^2 \\
&= e^2 - 2ce + c^2 + \frac{2ade}{e} - \frac{2acd}{e} + \left(\frac{ad}{e}\right)^2 \\
&= (e - c)^2 + \frac{2ad}{e}(e - c) + \left(\frac{ad}{e}\right)^2 \\
&= \left((e - c) + \frac{ad}{e}\right)^2
\end{aligned}$$

This determinant is always greater than or equal to zero. We ignore the borderline case where $\Delta = 0$, so we just have $\Delta > 0$. Thus, when P_1^* is stable, it must be a stable spiral because $p < 0, q > 0$, and $\Delta > 0$.

Hence, P_1^* is always a saddle point or a stable node.

Now, consider the positive equilibrium point $P_2^*\left(\frac{c}{d}, \frac{ad-ce}{bc}\right)$. In part (b), we find that $p = -\frac{ad}{c}$ and $q = ad - ce$. Thus, the discriminant Δ is

$$\begin{aligned}\Delta &= \left(-\frac{ad}{c}\right)^2 - 4(ad-ce)\\ &= \left(\frac{ad}{c}\right)^2 - 4(ad-ce)\end{aligned}$$

Suppose that

$$\frac{c}{d} < \frac{a}{e} < \frac{c}{d} + \frac{1}{4}\left(\frac{a^2d}{c^2e}\right)$$

The first inequality tells us that $\frac{c}{d} < \frac{a}{e}$, which implies $ad > ce$. Thus, P_2^* is feasible and stable. To show that P_2^* is a stable node, we must show that $\Delta > 0$. Observe that

$$\begin{aligned}
& \frac{a}{e} < \frac{c}{d} + \frac{1}{4}\left(\frac{a^2d}{c^2e}\right)\\
\Rightarrow\quad & (4de)\frac{a}{e} < 4de\left(\frac{c}{d} + \frac{1}{4}\left(\frac{a^2d}{c^2e}\right)\right)\\
\Rightarrow\quad & 4ad < 4ce + \frac{a^2d^2}{c^2}\\
\Rightarrow\quad & 4ad < 4ce + \frac{(ad)^2}{c^2}\\
\Rightarrow\quad & 4ad < 4ce + \left(\frac{ad}{c}\right)^2\\
\Rightarrow\quad & 0 < 4ce - 4ad + \left(\frac{ad}{c}\right)^2\\
\Rightarrow\quad & 0 < \left(\frac{ad}{c}\right)^2 - 4(ad-ce)\\
\Rightarrow\quad & 0 < \Delta
\end{aligned}$$

Thus, since $p < 0$, $q > 0$ and $\Delta > 0$, P_2^* is a stable node.

Now, suppose that

$$\frac{a}{e} > \frac{c}{d} + \frac{1}{4}\left(\frac{a^2 d}{c^2 e}\right)$$

Then the exact same process as above (with the inequality going the other way) will show that $0 > \Delta$. Since $p < 0$, $q > 0$ and $\Delta < 0$, P_2^* is a stable spiral in this case.

(3d)

Construct all possible partial and complete flow diagrams. For the complete flow diagrams, be sure to include the behavior on the boundary.

To summarize our findings thus far, refer to the following table:

Condition	$P_1^*\left(\frac{a}{e}, 0\right)$	$P_2^*\left(\frac{c}{d}, \frac{ad-ce}{bc}\right)$
$\frac{c}{d} > \frac{a}{e}$	Stable Node	Unfeasible
$\frac{c}{d} < \frac{a}{e} < \frac{c}{d} + \frac{1}{4}\left(\frac{a^2d}{c^2}e\right)$	Saddle	Stable Node
$\frac{a}{e} > \frac{c}{d} + \frac{1}{4}\left(\frac{a^2d}{c^2e}\right)$	Saddle	Stable Spiral

I found that the partial flow diagrams for x and y were similar regardless of each condition.

For the partial flow diagram of x, the x-nullcline is the line

$$a - ex - bxy = 0$$

The partial flow diagram of y has a y-nullcline of

$$-cy + dxy = y(-c + dx) = 0$$

Thus, either $y = 0$ or $-c + dx = 0$. The latter equation can be solved to get $x = \frac{c}{d}$.

Both of these partial flow diagrams are shown on the next page. For both diagrams, I used the parameters $a = b = c = d = e = 1$. The Maxima code for all flow diagrams is on page (50).

Figure 5: Partial Flow Diagram of $a - ex - bxy = 0$

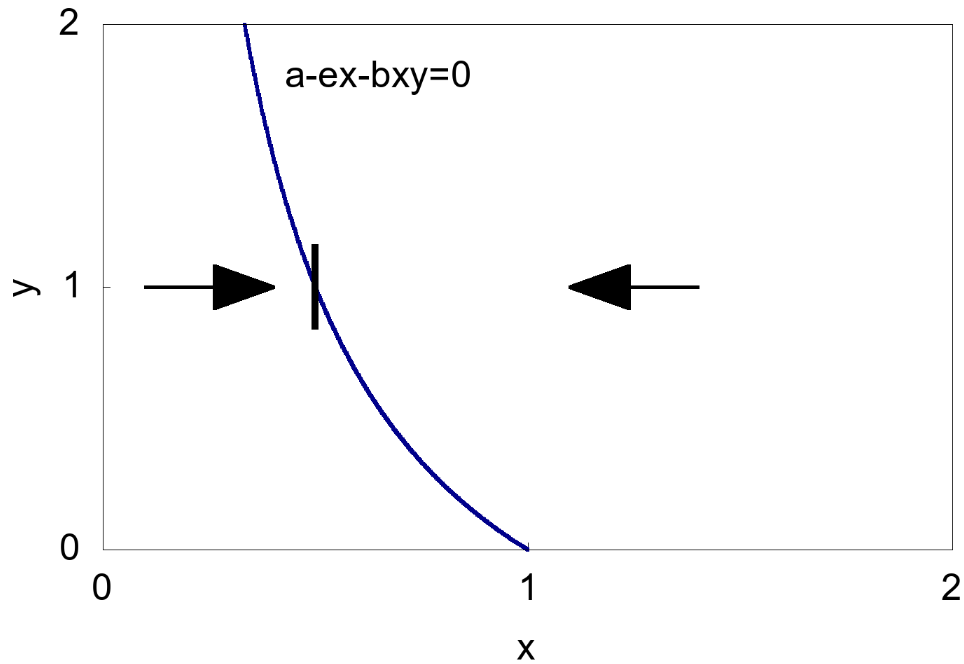
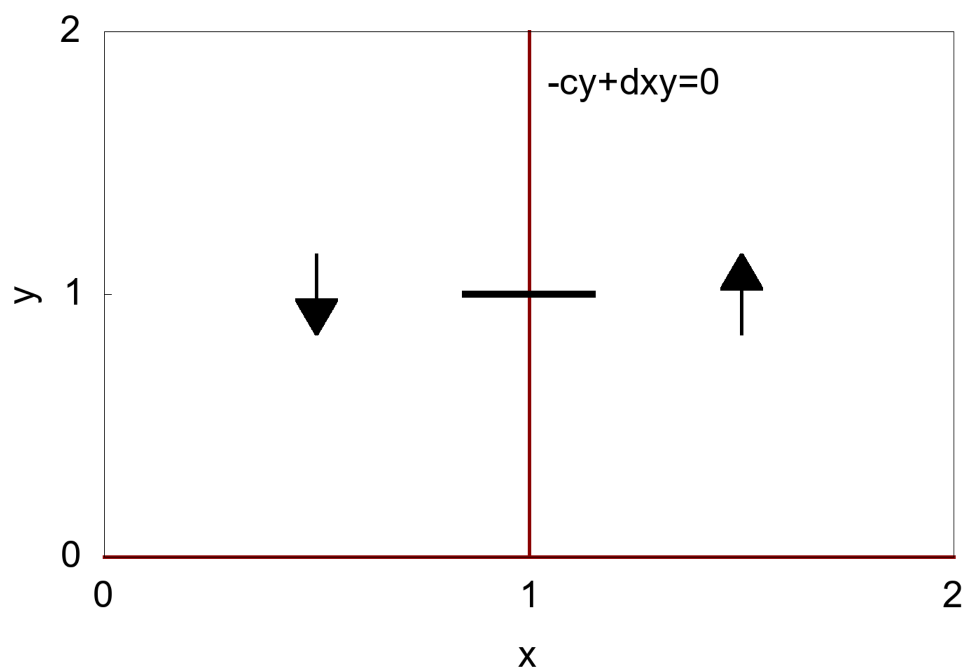


Figure 6: Partial Flow Diagram of $-cy + dxy = 0$



For the complete flow diagrams, there are two cases. If $\frac{c}{d} > \frac{a}{e}$, then P_1 is stable and P_2^* is unfeasible; if $\frac{c}{d} < \frac{a}{d}$, then P_1^* is a saddle point and P_2^* is stable. The flow diagrams look the same regardless of whether P_2^* is a stable node or stable spiral.

The flow diagram representing $\frac{c}{d} > \frac{a}{e}$ uses parameter values $a = b = d = e = 1$ and $c = 1.5$. The diagram representing $\frac{c}{d} < \frac{a}{e}$ uses $a = b = d = e = 1$ and $c = 0.5$. Again the code can be found on page (50).

(Note that the arrows that start on the y-axis are supposed to represent the behavior on the boundary $x = 0$)

Figure 7: Complete Flow Diagram when $\frac{c}{d} > \frac{a}{d}$

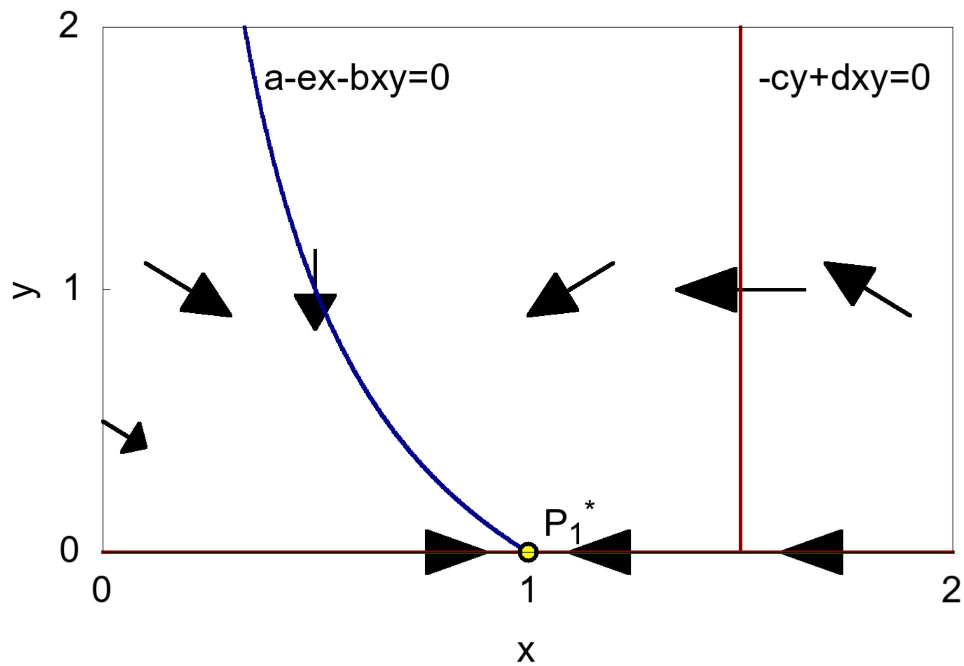
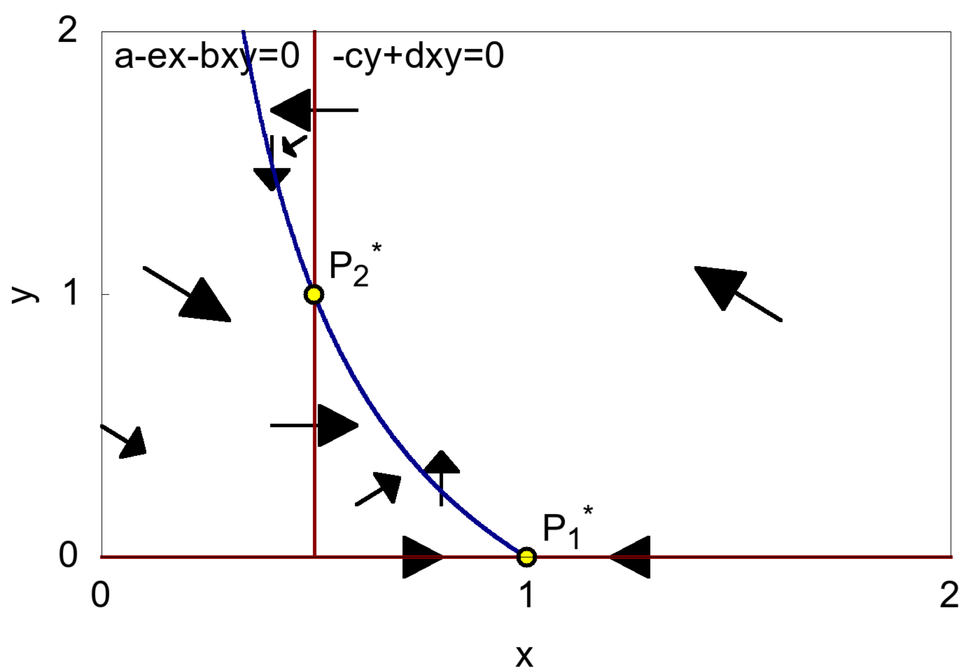


Figure 8: Complete Flow Diagram when $\frac{c}{d} < \frac{a}{d}$



(3e)

Rule out the existence of cycles.

Recall that $f(x,y) = a - ex - bxy = 0$ and $g(x,y) = -cy + dxy$. Let $B(x,y) = \frac{1}{y}$. I claim that $B(x,y)$ is a Dulac function. Compute:

$$\begin{aligned}
(Bf)_x + (Bg)_y &= \left(\frac{1}{y}(a - ex - bxy)\right)_x + \left(\frac{1}{y}(-cy + dxy)\right)_y \\
&= \left(\frac{a}{y} - \frac{ex}{y} - bx\right)_x + (-c + dx)_y \\
&= \left(-\frac{e}{y} - b\right) + 0 \\
&= -\frac{e}{y} - b
\end{aligned}$$

Since b and e are positive, we see that $(Bf)_x + (Bg)_y$ is negative for all $x, y \in \Omega$. Thus, $B(x,y)$ is a valid Dulac function, and our system of differential equations has no cycles.

(3f)

Form a conjecture about the basin of attraction for each equilibrium point. Then, use the results in part (d) to summarize clown population dynamics.

I conjecture that when $\frac{c}{d} > \frac{a}{d}$, the point $P_1^*\left(\frac{a}{e}, 0\right)$ is a global attractor on Ω. If $\frac{c}{d} < \frac{a}{e}$, then $P_1^*\left(\frac{a}{e}, 0\right)$ attracts all solutions on the positive x-axis, while $P_2^*\left(\frac{c}{d}, \frac{ad-ce}{bc}\right)$ attracts all other solutions on Ω. These conjectures are based on the flow diagrams in part (d).

These conjectures are summarized in the following table:

Condition	$B(P_1)$	$B(P_2^*)$
$\frac{c}{d} > \frac{a}{e}$	Ω	$\{(x, 0) \in \Omega\}$
$\frac{c}{d} < \frac{a}{e}$	Not feasible	$\{(x, y) \in \Omega : y > 0\}$

If $\frac{c}{d} > \frac{a}{e}$, then the population density of good clowns approaches $\frac{a}{e}$, while the bad clowns approach extinction.

If $\frac{c}{d} < \frac{a}{e}$, then good clowns approach a population density of $\frac{c}{d}$ and bad clowns approach a population density of $\frac{ad-ce}{bc}$, as long as we start with some bad clowns.

If $\frac{c}{d} < \frac{a}{e}$ and there are initially no bad clowns, the population of good clowns approaches $\frac{a}{e}$ and the population of bad clowns stays at 0.

Question 4

We search for cycles in the nonlinear system

$$\dot{x} = -x + ay + x^2y \quad \text{and} \quad \dot{y} = b - ay - x^2y$$

where a and b are positive constants. $x(t)$ is the concentration of ADP and $y(t)$ is the concentration of F6P for $t \geq 0$. Restrict attention to

$$\Omega = \{(x, y) : x \geq 0, y \geq 0\}$$

(4a)

Construct all possible partial and complete flow diagrams on Ω.

I found that the flow diagrams look similar regardless of the initial conditions a and b. The diagrams appear on the following two pages. I used the parameters $a = b = 1$.

Figure 9: Partial Flow Diagram for $\dot{x} = -x + ay + x^2y$

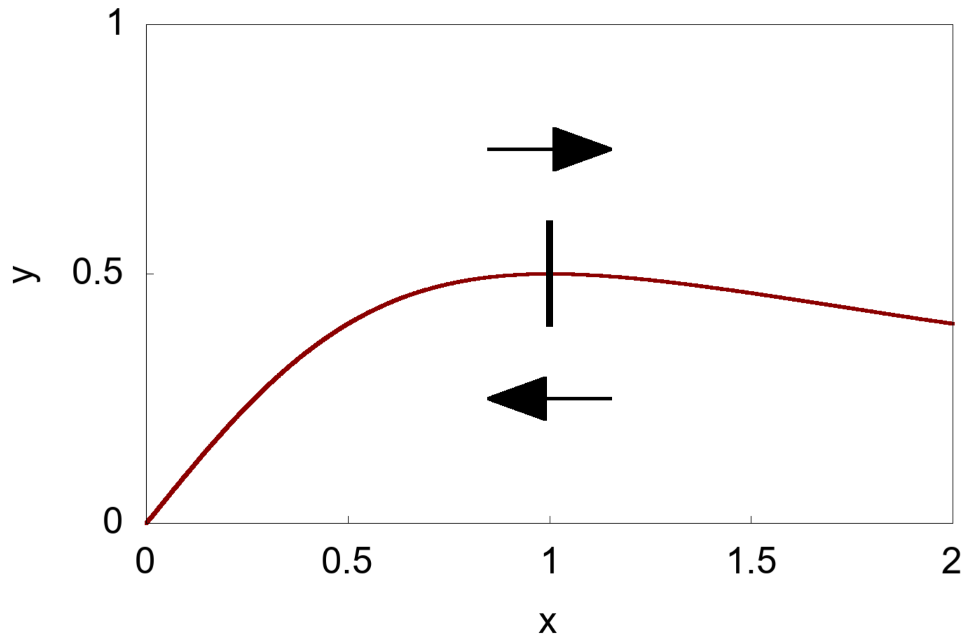


Figure 10: Partial Flow Diagram for $\dot{y} = b - ay - x^2y$

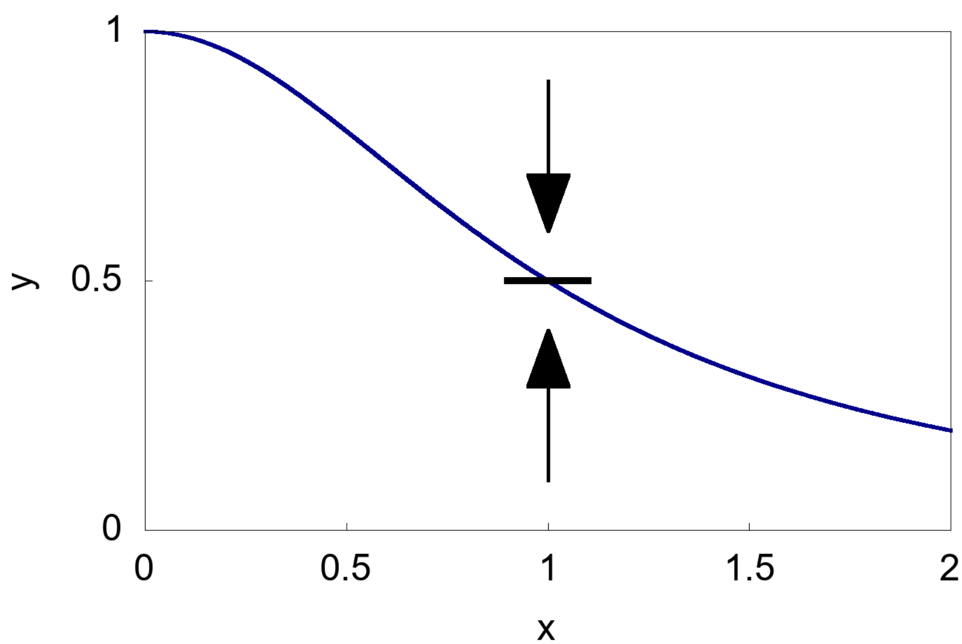
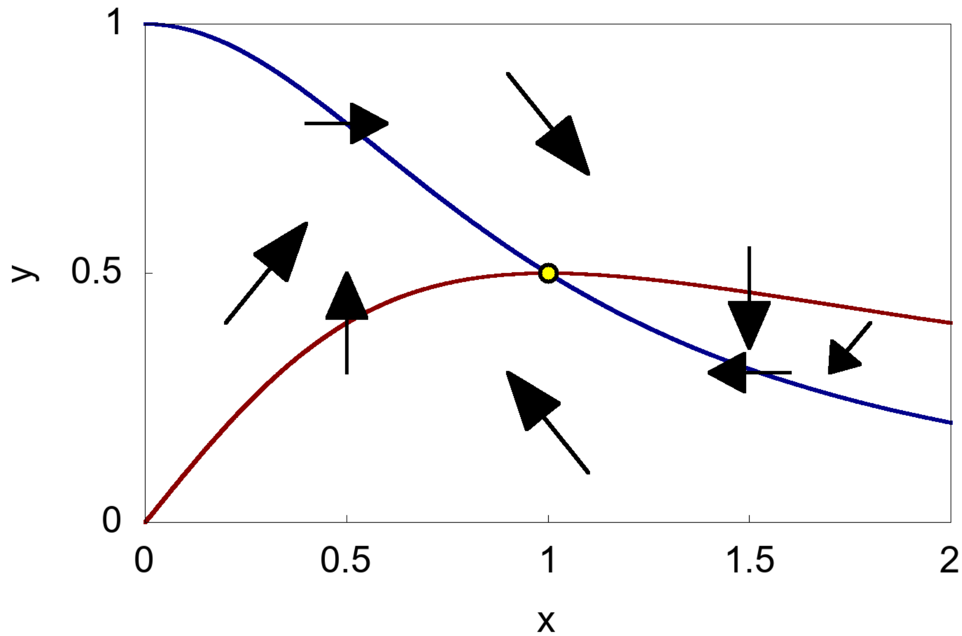


Figure 11: Complete Flow Diagram for $\dot{x}$ and $\dot{y}$



See page (55) for the Maxima code.

(4b)

Show that there is a single critical point $P(x^*, y^*)$ in Ω and compute its coordinates.

Looking at the complete flow diagram in part (a), we see that there is one critical point (the intersection of the nullclines). To find its coordinates, we must solve the system

$$\begin{aligned} -x + ay + x^2y &= 0 \\ b - ay - x^2y &= 0 \end{aligned}$$

Adding the two equations, we get

$$-x + b = 0$$

and so $x = b$. Then, plugging this into the first equation, we get

$$\begin{aligned} & -(b) + ay + (b)^2 y = 0 \\ \Rightarrow \quad & -b + y(a + b^2) = 0 \\ \Rightarrow \quad & y(a + b^2) = b \\ \Rightarrow \quad & y = \frac{b}{a + b^2} \end{aligned}$$

Thus, our equilibrium point is $P(x^*, y^*) = \left(b, \frac{b}{a+b^2}\right)$.

(4c)

Under what conditions is P a stable attractor and when is it a repeller? Use Maxima to plot a curve that divides the ab-plane into two regions: one where P is a stable attractor and one where P is a repeller.

First, we must compute the variational matrix for our equilibrium point. Let $f(x, y) = \dot{x} = -x + ay + x^2 y$ and $g(x, y) = \dot{y} = b - ay - x^2 y$. Then

$$\begin{aligned} A(x, y) &= \begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix} \\ &= \begin{bmatrix} -1 + 2xy & a + x^2 \\ -2xy & -a - x^2 \end{bmatrix} \\ A\left(b, \frac{b}{a + b^2}\right) &= \begin{bmatrix} -1 + 2(b)\left(\frac{b}{a+b^2}\right) & a + b^2 \\ -2(b)\left(\frac{b}{a+b^2}\right) & -a - b^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 + \frac{2b^2}{a+b^2} & a + b^2 \\ -\frac{2b^2}{a+b^2} & -a - b^2 \end{bmatrix} \end{aligned}$$

Now, we will compute the trace (p) and determinant (q) of this matrix:

$$\begin{aligned} p &= \left(-1 + \frac{2b^2}{a+b^2}\right) + (-a - b^2) \\ &= -1 + \frac{2b^2}{a+b^2} - a - b^2 \end{aligned}$$

$$\begin{aligned} q &= \left(-1 + \frac{2b^2}{a+b^2}\right)(-a - b^2) - \left(-\frac{2b^2}{a+b^2}\right)(a+b^2) \\ &= \left(1 - \frac{2b^2}{a+b^2}\right)(a+b^2) + \left(\frac{2b^2}{a+b^2}\right)(a+b^2) \\ &= \left(\frac{a+b^2}{a+b^2} - \frac{2b^2}{a+b^2}\right)(a+b^2) + \left(\frac{2b^2}{a+b^2}\right)(a+b^2) \\ &= a + b^2 - 2b^2 + 2b^2 \\ &= a + b^2 \end{aligned}$$

Notice that q is always positive since a and b are positive. Thus, P is never a saddle point.

P will be a stable attractor whenever $p < 0$. That is, P is a stable attractor if

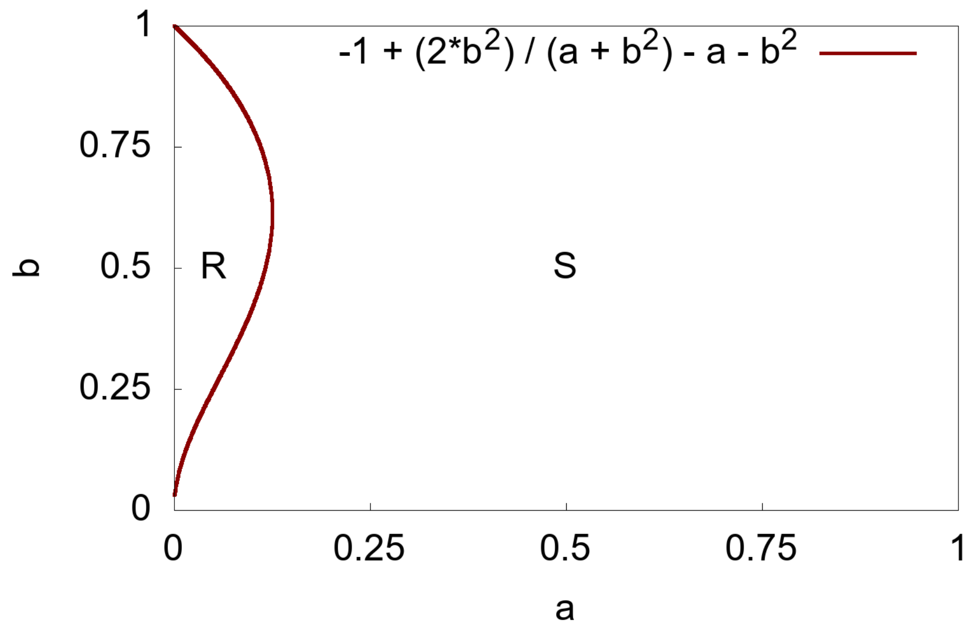
$$-1 + \frac{2b^2}{a+b^2} - a - b^2 < 0$$

Also, P is a repeller if

$$-1 + \frac{2b^2}{a+b^2} - a - b^2 > 0$$

Below is a diagram of the ab-plane demonstrating where P is stable and where P is a repeller. The region denoted by S is stable and the region marked by R is a repeller.

Figure 12: Effect of a, b on stability of P



See page (58) for the Maxima code.

(4d)

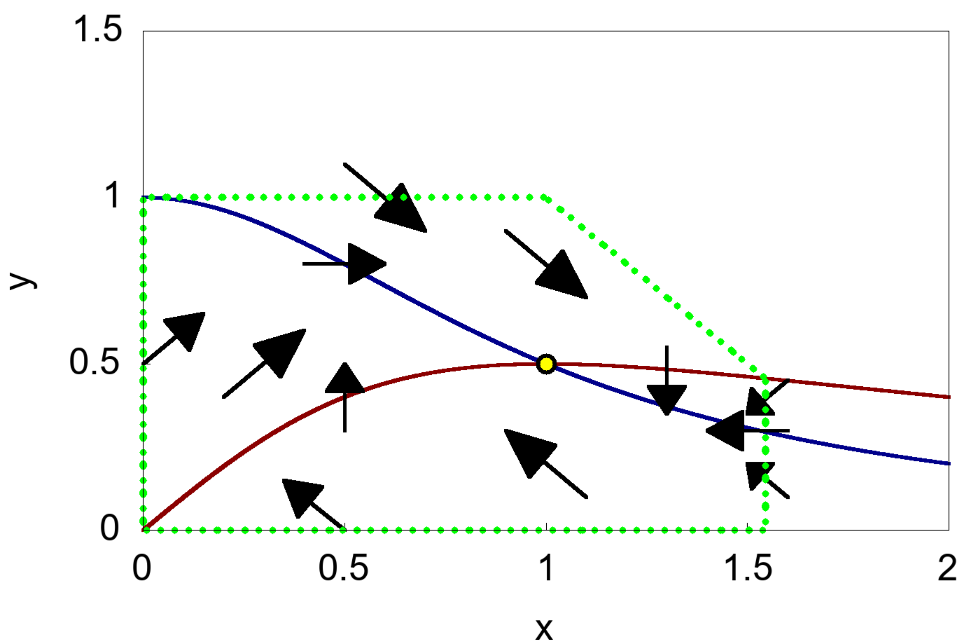
Consider the region Γ whose boundary consists of these five edges:

l_1: The line segment from $O(0,0)$ to $A(0,b/a)$.
l_2: The line segment from $A(0,b/a)$ to $B(b,b/a)$.
l_3: The line segment with slope -1 from $B(b,b/a)$ to $C(r,s)$ on the x-nu
l_4: The line segment from $C(r,s)$ to $D(r,0)$.
l_5: The line segment from $D(r,0)$ to $O(0,0)$.

Include arrows on the boundary of Γ to indicate the approximate direction of forward motion along it.

Below is the flow diagram from part (a), which now includes the region Γ as well as some additional arrows along the boundary of Γ.

Figure 13: Flow Diagram on the Boundary Γ (except l_3)



See page (59) for the Maxima code.

From this figure, we see that along the line segments l_1, l_2, l_4, and l_5, trajectories move from outside Γ to the inside of Γ. No trajectories can exit Γ along any of those line segments. Note that the arrows beginning at $(0, 0.5)$ and $(0.5, 0)$ on the diagram represent the behavior along the x and y axis, respectively.

Now, consider the segment l_3. This boundary is a straight line between $B\left(b, \frac{b}{a}\right)$ and $C(r, s)$, where C lies on the x-nullcline (the red curve). Compute:

$$\dot{x} + \dot{y} = -x + ay + x^2y + b - ay - x^2y = -x + b$$

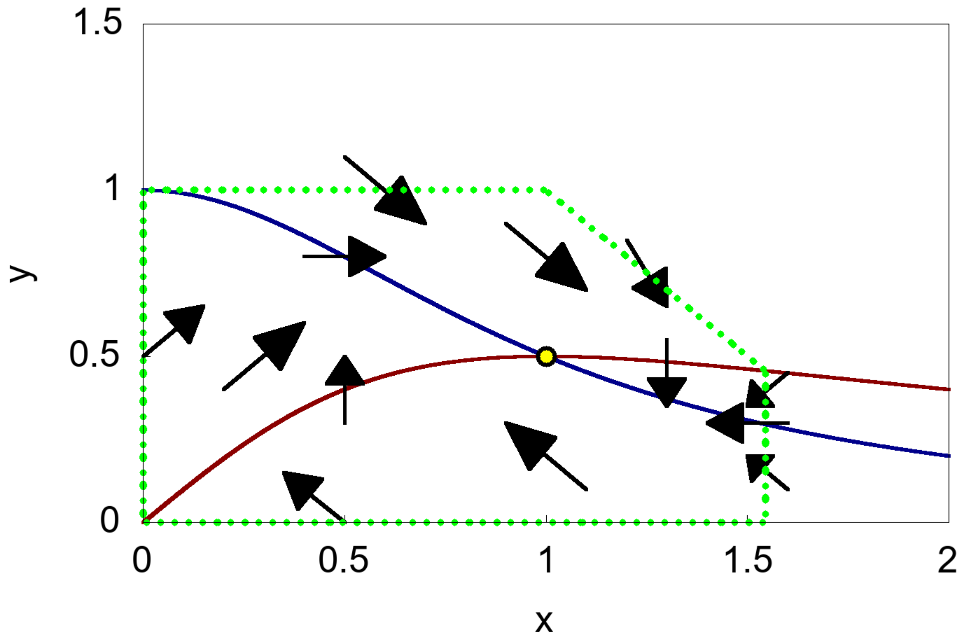
Because $x > b$ on the line segment l_3, we have that $\dot{x} + \dot{y} = -x + b < 0$. Thus,

$$\begin{aligned} & \dot{x} + \dot{y} < 0 \\ \Rightarrow \quad & \dot{x} < -\dot{y} \\ \Rightarrow \quad & 1 < \frac{-\dot{y}}{\dot{x}} \qquad (\dot{x} \text{ is positive on } l_3) \\ \Rightarrow \quad & -1 > \frac{\dot{y}}{\dot{x}} \\ \Rightarrow \quad & \frac{dy}{dx} < -1 \end{aligned}$$

Thus, near l_3, $\frac{dy}{dx} < -1$. Any trajectories near l_3 will have a slope that is more negative than the slope of l_3. Consequently, trajectories inside Γ will never intersect l_3. Any trajectory that intersects l_3 must come from outside of Γ.

With this insight, we can update our diagram with an arrow passing through l_3:

Figure 14: Flow diagram on the Boundary Γ (including l_3)



Thus, like we saw with the other line segments, any trajectory inside Γ cannot leave Γ by passing through l_3.

Refer to page (59) for the Maxima code.

(4e)

Use the result in part (d) to decide whether Γ is a forward invariant set.

Γ is a forward invarient set because any trajectory in Γ will stay in Γ. The line segments l_1, l_2, l_3, l_4, and l_5 completely encase Γ, but we observed in part (d) that no trajectory inside Γ can leave Γ through any of those segments.

(4f)

Under what conditions does Γ (and therefore Ω) definitely contain a cycle?

Recall from part (b) that the system has one equilibrium point $P\left(b, \frac{b}{a+b^2}\right)$. By our construction of Γ, P is always an element of Γ. This is because P shares the same x-coordinate as B, and the y-coordinate of P is always less than the y-coordinate of B ($\frac{b}{a+b^2} < \frac{b}{a}$).

Γ is also a non-empty, compact and forward invariant set. Thus, by the Poincare-Bendixson Theorem, we know that

1. Γ contains a critical point, a cycle, or both.
2. If all critical points in Γ are repellers, then Γ contains a cycle.

If P is stable, then we cannot be completely sure if Γ contains a cycle.

In part (c), we saw that P is a repeller if

$$-1 + \frac{2b^2}{a+b^2} - a - b^2 > 0$$

Thus, whenever this condition holds, the Poincare-Bendixson theorem guarentees that Γ contains a cycle.

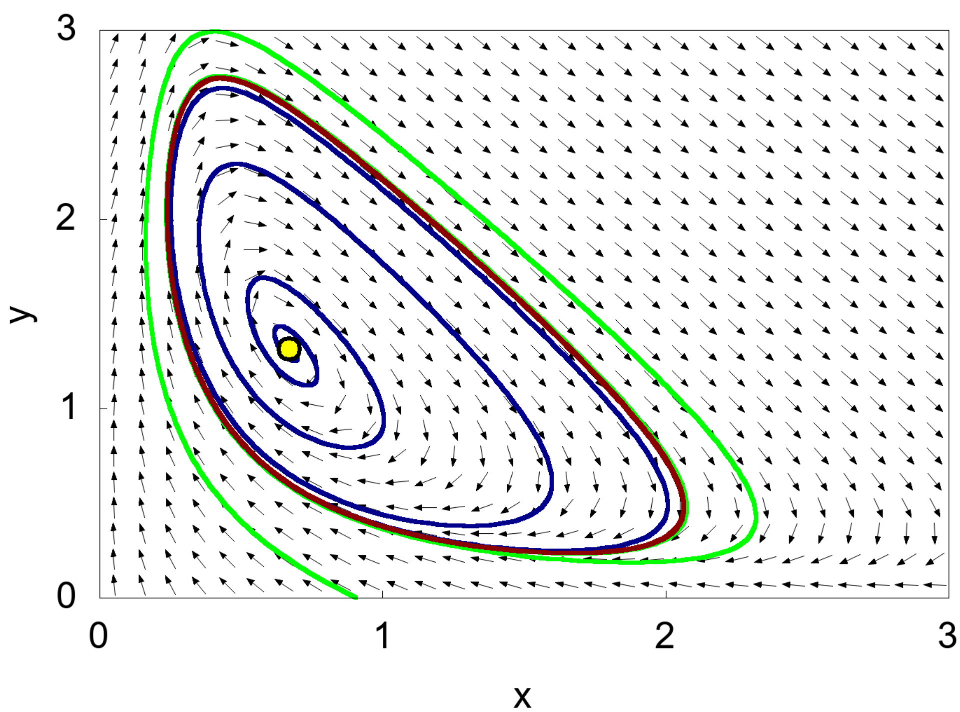
(Extra Credit)

Use Maxima to plot a cycle C in the xy-plane for the pair $(a, b) = (0.06, 0.67)$. In this figure, also include one non-constant trajectory enclosed by C and one outside C.

Below, I have plotted three trajectories. The red trajectory starts at $(0.41195, 1)$ and is approximately the cycle C. The blue trajectory starts at $(1, 1)$ and is enclosed by C. The green trajectory starts at $(0.25, 1)$ and is outside of C. The equilibrium point lies at approximately

$$P = \left(b, \frac{b}{a + b^2}\right) = \left(0.67, \frac{0.67}{0.06 + (0.67)^2}\right) \approx (0.67, 1.32)$$

Figure 15: Cycle C with two nearby trajectories



Refer to page (62) for the Maxima code.

Appendix

Below is the code for problems (1a), (1b), and (1c) starting on page (3).

```
(%i1) f(x, y) := 1/9 * x * (22 -2 * x - y)$
(%i2) g(x, y) := 1/4 * y * (-1 + x - 2 * y)$
(%i3) solve([f(x, y) = 0, g(x, y) = 0]);
                                                     1
(%o3) [[y = 0, x = 0], [y = 0, x = 11], [y = - -, x = 0], [y = 4, x = 9]]
                                                     2
(%i4) A(a, b) := matrix(
        [at(diff(f(x, y), x), [x = a, y = b]),
        at(diff(f(x, y), y), [x = a, y = b])],
        [at(diff(g(x, y), x), [x = a, y = b]),
        at(diff(g(x, y), y), [x = a, y = b])]);
(%o4) A(a, b) :=
 [ at(diff(f(x, y), x), [x = a, y = b])  at(diff(f(x, y), y), [x = a, y = b
 [
]
 [ at(diff(g(x, y), x), [x = a, y = b])  at(diff(g(x, y), y), [x = a, y = b
(%i5) A(0, 0);
                                [ 22       ]
                                [ --    0  ]
                                [ 9        ]
(%o5)                           [          ]
                                [        1 ]
                                [ 0    - - ]
                                [        4 ]
(%i6) eigenvalues(%);
                              22     1
(%o6)                       [[--, - -], [1, 1]]
                              9      4
(%i7) A(11, 0);
                              [    22     11 ]
                              [ - --    - -- ]
                              [    9      9  ]
(%o7)                         [              ]
                              [          5   ]
                              [   0      -   ]
                              [          2   ]
(%i8) eigenvalues(%);
                                22   5
(%o8)                       [[- --, -], [1, 1]]
                                9    2
```

```
(%i9) A(9, 4);
                                  [ - 2   - 1 ]
(%o9)                             [           ]
                                  [  1    - 2 ]
(%i10) eigenvalues(%);
(%o10)                 [[(- %i) - 2, %i - 2], [1, 1]]
```

Below is the code for the computations needed for problem (1d) on page (5).

```
(%i1) p(x, y) := x + y$
(%i2) q(x, y) := x * y$
(%i3) D(x, y) := p(x, y)^2 - 4 * q(x, y)$
(%i4) p(-1 / 4, 22 / 9);
                                                 79
(%o4)                                            --
                                                 36
(%i5) q(-1 / 4, 22 / 9);
                                                  11
(%o5)                                           - --
                                                  18
(%i6) D(-1 / 4, 22 / 9);
                                                9409
(%o6)                                           ----
                                                1296
(%i7) p(-22 / 9, 5 / 2);
                                                 1
(%o7)                                           --
                                                18
(%i8) q(-22 / 9, 5 / 2);
                                                 55
(%o8)                                          - --
                                                 9
(%i9) D(-22 / 9, 5 / 2);
                                                7921
(%o9)                                          ----
                                                324
```

Below is the code for the direction fields in problem (1f) on page (8).

```
(%i1) load(drawdf)$
(%i2) drawdf(
        [1/9 * x * (22 - 2 * x - y), 1/4 * y * (-1 + x - 2 * y)],
        [x,0,15],
        [y,0,15],
        xtics = [0, 5, 15],
        ytics = [0, 5, 15],
        font = "Arial",
        font_size = 40,
        xlabel = "x",
        ylabel = "y",
        line_width = 6,
        point_size = 3,
        color = darkred,
        solns_at([14, 1], [9, 14], [1, 14]),
        color = black,
        point_size = 5,
        points_at([0, 0], [11, 0], [9, 4])
)$
(%i3) draw_file (
        terminal = png,
        file_name = "Direction_Complete",
        dimensions = [1500, 1050]
)$
(%i4) drawdf(
        [1/9 * x * (22 - 2 * x - y), 1/4 * y * (-1 + x - 2 * y)],
        [x,0,3],
        [y,0,3],
        xtics = [0, 1, 3],
        ytics = [0, 1, 3],
        font = "Arial",
        font_size = 40,
        xlabel = "x",
        ylabel = "y",
        line_width = 6,
        point_size = 3,
        color = darkred,
        solns_at([0, 2], [1, 2], [1, 0]),
        color = black,
        point_size = 5,
        points_at([0, 0])
)$
(%i5) draw_file (
```

```
        terminal = png,
        file_name = "Direction_0_0",
        dimensions = [1500, 1050]
)$
(%i6) drawdf(
        [1/9 * x * (22 - 2 * x - y), 1/4 * y * (-1 + x - 2 * y)],
        [x,9,12],
        [y,0,3],
        xtics = [9, 1, 12],
        ytics = [0, 1, 3],
        font = "Arial",
        font_size = 40,
        xlabel = "x",
        ylabel = "y",
        line_width = 6,
        point_size = 3,
        color = darkred,
        solns_at([10, 0], [12, 0], [11, 1]),
        color = black,
        point_size = 5,
        points_at([11, 0])
)$
(%i7) draw_file (
        terminal = png,
        file_name = "Direction_11_0",
        dimensions = [1500, 1050]
)$
(%i8) drawdf(
        [1/9 * x * (22 - 2 * x - y), 1/4 * y * (-1 + x - 2 * y)],
        [x,7,10],
        [y,2,5],
        xtics = [7, 1, 10],
        ytics = [2, 1, 5],
        font = "Arial",
        font_size = 40,
        xlabel = "x",
        ylabel = "y",
        line_width = 6,
        point_size = 3,
        color = darkred,
        solns_at([7, 2], [8, 5], [10, 3]),
        color = black,
        point_size = 5,
        points_at([9, 4])
)$
(%i9) draw_file (
```

```
        terminal = png,
        file_name = "Direction_9_4",
        dimensions = [1500, 1050]
)$
```

Below is the code for the flow diagrams for question (3d) on page (27).

```
(%i1) load(drawdf)$
(%i2) set_draw_defaults(
        user_preamble = "set size ratio 0.618",
        font = "Arial",
        font_size = 40,

        xaxis = true,
        yaxis = true,

        xaxis_type = solid,
        yaxis_type = solid,

        xlabel = "x",
        ylabel = "y",

        line_type = solid,
        line_width = 5,

        point_size = 4,
        point_type = filled_circle
)$
(%i3) a : 1$
(%i4) b : 1$
(%i5) c : 1$
(%i6) d : 1$
(%i7) e : 1$
(%i8) draw2d(
        xrange = [0, 2],
        yrange = [0, 2],

        xtics = [0, 1, 2],
        ytics = [0, 1, 2],

        ip_grid = [500, 500],

        color = black,
        label(["a-ex-bxy=0", 0.65, 1.8]),

        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([1.4, 1], [-0.3, 0]),
        vector([0.1, 1], [0.3, 0]),
```

```
        color = darkblue,
        implicit(a - e*x - b*x*y=0, x, 0, 5, y, 0, 5),

        color = black,
        line_width = 10,
        head_angle = 0.01,
        vector([0.5, 0.85], [0, 0.3])
)$
(%i9) draw_file (
        terminal = png,
        file_name = "X_Flow",
        dimensions = [1500, 1050]
)$
(%i10) draw2d(
        xrange = [0, 2],
        yrange = [0, 2],

        xtics = [0, 1, 2],
        ytics = [0, 1, 2],

        ip_grid = [500, 500],

        color = black,
        label(["-cy+dxy=0", 1.25, 1.8]),

        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([1.5, 0.85], [0, 0.3]),
        vector([0.5, 1.15], [0, -0.3]),

        color = darkred,
        implicit(-c*y+d*x*y=0, x, 0, 5, y, 0, 5),
        explicit(0, x, 0, 2),

        color = black,
        line_width = 10,
        head_angle = 0.01,
        vector([0.85, 1], [0.3, 0])
)$
(%i11) draw_file (
        terminal = png,
        file_name = "Y_Flow",
        dimensions = [1500, 1500]
)$
```

```
(%i12) c : 0.5$
(%i13) draw2d(
        xrange = [0, 2],
        yrange = [0, 2],

        xtics = [0, 1, 2],
        ytics = [0, 1, 2],

        ip_grid = [500, 500],

        color = black,
        label(["a-ex-bxy=0", 0.6, 1.8], ["-cy+dxy=0", 1.75, 1.8],
        ["P_1^*", 1.1, 0.1]),

        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([1.9, 0.9], [-0.2, 0.2]),
        vector([1.2, 1.1], [-0.2, -0.2]),
        vector([0.1, 1.1], [0.2, -0.2]),
        vector([1.65, 1], [-0.3, 0]),
        vector([0.5, 1.15], [0, -0.3]),
        vector([-0.1, 0.6], [0.2, -0.2]),
        vector([0.6, 0], [0.3, 0]),
        vector([1.4, 0], [-0.3, 0]),
        vector([1.9, 0], [-0.3, 0]),

        color = darkblue,
        implicit(a - e*x - b*x*y=0, x, 0, 5, y, 0, 5),

        color = darkred,
        implicit(-c*y+d*x*y=0, x, 0, 5, y, 0, 5),
        explicit(0, x, 0, 2),

        color = black,

        point_size = 5,
        points([[a/e, 0]]),
        point_size = 3,
        color = yellow,
        points([[a/e, 0]])
)$
rat: replaced -1.5 by -3/2 = -1.5
(%i14) draw_file (
        terminal = png,
        file_name = "Complete_Flow_1",
```

```
        dimensions = [1500, 1500]
)$
(%i15) c : 0.5$
(%i16) draw2d(
        xrange = [0, 2],
        yrange = [0, 2],

        xtics = [0, 1, 2],
        ytics = [0, 1, 2],

        ip_grid = [500, 500],

        color = black,
        label(["a-ex-bxy=0", 0.25, 1.9], ["-cy+dxy=0", 0.75, 1.9],
        ["P_1^*", 1.1, 0.1], ["P_2^*", 0.6, 1.1]),

        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([1.6, 0.9], [-0.2, 0.2]),
        vector([0.48, 1.6], [-0.05, -0.05]),
        vector([0.1, 1.1], [0.2, -0.2]),
        vector([0.6, 1.7], [-0.2, 0]),
        vector([0.4, 1.6], [0, -0.2]),
        vector([-0.1, 0.6], [0.2, -0.2]),
        vector([0.6, 0], [0.2, 0]),
        vector([1.4, 0], [-0.2, 0]),
        vector([0.4, 0.5], [0.2, 0]),
        vector([0.6, 0.2], [0.1, 0.1]),
        vector([0.8, 0.2], [0, 0.2]),

        color = darkblue,
        implicit(a - e*x - b*x*y=0, x, 0, 5, y, 0, 5),

        color = darkred,
        implicit(-c*y+d*x*y=0, x, 0, 5, y, 0, 5),
        explicit(0, x, 0, 2),

        color = black,

        point_size = 5,
        points([[a/e, 0], [c/d, (a*d-c*e)/(b*c)]]),
        point_size = 3,
        color = yellow,
        points([[a/e, 0], [c/d, (a*d-c*e)/(b*c)]])
)$
```

```
rat: replaced -0.5 by -1/2 = -0.5
(%i17) draw_file (
        terminal = png,
        file_name = "Complete_Flow_2",
        dimensions = [1500, 1500]
)$
```

Below is the code for the flow diagrams in problem (4a) on page (33).

```
(%i1) load(draw)$
(%i2) set_draw_defaults(
        user_preamble = "set size ratio 0.618",
        font = "Arial",
        font_size = 40,

        xaxis = true,
        yaxis = true,

        xaxis_type = solid,
        yaxis_type = solid,

        xlabel = "x",
        ylabel = "y",

        line_type = solid,
        line_width = 5,

        point_size = 4,
        point_type = filled_circle
)$
(%i3) a : 1$
(%i4) b : 1$
(%i5) draw2d(
        xrange = [0, 2],
        yrange = [0, 1],

        xtics = [0, 0.5, 2],
        ytics = [0, 0.5, 1],

        color = black,
        ip_grid = [500, 500],
        color = darkred,
        implicit(-x + a*y + x^2*y = 0, x, 0, 2, y, 0, 1),
        color = black,
        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([0.85, 0.75], [0.3, 0]),
        vector([1.15, 0.25], [-0.3, 0]),

        line_width = 10,
        head_angle = 0.01,
```

```
        vector([1, 0.4], [0, 0.2])
)$
(%i6) draw_file(
        terminal = png,
        file_name = "4a_Partial_1",
        dimensions = [1500, 1050]
)$
(%i7) draw2d(
        xrange = [0, 2],
        yrange = [0, 1],

        xtics = [0, 0.5, 2],
        ytics = [0, 0.5, 1],

        color = black,
        ip_grid = [500, 500],
        color = darkblue,
        implicit(b - a*y -x^2*y = 0, x, 0, 2, y, 0, 1),
        color = black,
        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([1, 0.9], [0, -0.3]),
        vector([1, 0.1], [0, 0.3]),

        line_width = 10,
        head_angle = 0.01,
        vector([0.9, 0.5], [0.2, 0])
)$
(%i8) draw_file(
        terminal = png,
        file_name = "4a_Partial_2",
        dimensions = [1500, 1050]
)$
(%i9) draw2d(
        xrange = [0, 2],
        yrange = [0, 1],

        xtics = [0, 0.5, 2],
        ytics = [0, 0.5, 1],

        color = black,
        ip_grid = [500, 500],
        color = darkred,
        implicit(-x + a*y + x^2*y = 0, x, 0, 2, y, 0, 1),
        color = darkblue,
```

```
        implicit(b - a*y -x^2*y = 0, x, 0, 2, y, 0, 1),
        color = black,
        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([0.9, 0.9], [0.2, -0.2]),
        vector([1.1, 0.1], [-0.2, 0.2]),
        vector([0.2, 0.4], [0.2, 0.2]),
        vector([1.8, 0.4], [-0.1, -0.1]),
        vector([0.5, 0.3], [0, 0.2]),
        vector([1.5, 0.55], [0, -0.2]),
        vector([0.4, 0.8], [0.2, -0]),
        vector([1.6, 0.3], [-0.2, -0]),

        point_size = 5,
        points([[1, 0.5]]),
        point_size = 3,
        color = yellow,
        points([[1, 0.5]])
)$
(%i10) draw_file(
        terminal = png,
        file_name = "4a_Complete",
        dimensions = [1500, 1050]
)$
```

Below is the code for the *ab*-plane for problem (4c) on page (37).

```
(%i1) load(draw)$
(%i2) set_draw_defaults(
        user_preamble = "set size ratio 0.618",
        font = "Arial",
        font_size = 40,

        xaxis = true,
        yaxis = true,

        xaxis_type = solid,
        yaxis_type = solid,

        xlabel = "a",
        ylabel = "b",

        line_type = solid,
        line_width = 5,

        point_size = 4,
        point_type = filled_circle
)$
(%i3) draw2d(
        xrange = [0, 1],
        yrange = [0, 1],
        xtics = [0, 0.25, 1],
        ytics = [0, 0.25, 1],
        ip_grid = [500, 500],
        key_pos = top_right,
        color = black,
        label(["R", 0.05, 0.5], ["S", 0.5, 0.5]),
        color = darkred,
        key = "-1 + (2*b^2) / (a + b^2) - a - b^2",
        implicit(-1 + (2*b^2) / (a + b^2) - a - b^2 = 0, a, 0, 1, b, 0, 1)
)$
(%i4) draw_file(
        terminal = png,
        file_name = "ab_plane",
        dimensions = [1500, 1050]
)$
```

Below is the code for the flow diagrams with the boundary Γ used for problem (4d) on page (38).

```
(%i1) load(draw)$
(%i2)  set_draw_defaults(
        user_preamble = "set size ratio 0.618",
        font = "Arial",
        font_size = 40,

        xaxis = true,
        yaxis = true,

        xaxis_type = solid,
        yaxis_type = solid,

        xlabel = "x",
        ylabel = "y",

        line_type = solid,
        line_width = 5,

        point_size = 4,
        point_type = filled_circle
)$
(%i3) a : 1$
(%i4) b : 1$
(%i5) draw2d(
        xrange = [0, 2],
        yrange = [0, 1.5],

        xtics = [0, 0.5, 2],
        ytics = [0, 0.5, 1.5],

        color = black,
        ip_grid = [500, 500],
        color = darkred,
        implicit(-x + a*y + x^2*y = 0, x, 0, 2, y, 0, 1),
        color = darkblue,
        implicit(b - a*y -x^2*y = 0, x, 0, 2, y, 0, 1),
        color = black,
        line_width = 5,
        head_length = 0.15,
        head_angle = 20,
        vector([0.9, 0.9], [0.2, -0.2]),
        vector([0.5, 1.1], [0.2, -0.2]),
        vector([1.1, 0.1], [-0.2, 0.2]),
```

```
        vector([0.2, 0.4], [0.2, 0.2]),
        vector([1.6, 0.45], [-0.1, -0.1]),
        vector([1.6, 0.1], [-0.1, 0.1]),
        vector([0.5, 0.3], [0, 0.2]),
        vector([1.3, 0.55], [0, -0.2]),
        vector([0.4, 0.8], [0.2, -0]),
        vector([1.6, 0.3], [-0.2, -0]),
        vector([0, 0.5], [0.15, 0.15]),
        vector([0.5, 0], [-0.15, 0.15]),

        point_size = 5,
        points([[1, 0.5]]),
        point_size = 3,
        color = yellow,
        points([[1, 0.5]]),

        color = green,
        line_type = dots,
        implicit(x=0, x, 0, 2, y, 0, 1),
        explicit(1, x, 0, 1),
        explicit(2-x, x, 1, 1.544),
        implicit(x=1.544, x, 0, 2, y, 0, 0.456),
        explicit(0, x, 0, 1.544)
)$
rat: replaced -1.544 by -193/125 = -1.544
(%i6) draw_file(
        terminal = png,
        file_name = "Gamma_1",
        dimensions = [1500, 1050]
)$
(%i7) draw2d(
        xrange = [0, 2],
        yrange = [0, 1.5],

        xtics = [0, 0.5, 2],
        ytics = [0, 0.5, 1.5],

        color = black,
        ip_grid = [500, 500],
        color = darkred,
        implicit(-x + a*y + x^2*y = 0, x, 0, 2, y, 0, 1),
        color = darkblue,
        implicit(b - a*y -x^2*y = 0, x, 0, 2, y, 0, 1),
        color = black,
        line_width = 5,
        head_length = 0.15,
```

```
        head_angle = 20,
        vector([0.9, 0.9], [0.2, -0.2]),
        vector([0.5, 1.1], [0.2, -0.2]),
        vector([1.1, 0.1], [-0.2, 0.2]),
        vector([0.2, 0.4], [0.2, 0.2]),
        vector([1.6, 0.45], [-0.1, -0.1]),
        vector([1.6, 0.1], [-0.1, 0.1]),
        vector([0.5, 0.3], [0, 0.2]),
        vector([1.3, 0.55], [0, -0.2]),
        vector([0.4, 0.8], [0.2, -0]),
        vector([1.6, 0.3], [-0.2, -0]),
        vector([0, 0.5], [0.15, 0.15]),
        vector([0.5, 0], [-0.15, 0.15]),
        vector([1.2, 0.85], [0.1, -0.2]),

        point_size = 5,
        points([[1, 0.5]]),
        point_size = 3,
        color = yellow,
        points([[1, 0.5]]),

        color = green,
        line_type = dots,
        implicit(x=0, x, 0, 2, y, 0, 1),
        explicit(1, x, 0, 1),
        explicit(2-x, x, 1, 1.544),
        implicit(x=1.544, x, 0, 2, y, 0, 0.456),
        explicit(0, x, 0, 1.544)
)$
rat: replaced -1.544 by -193/125 = -1.544
(%i8) draw_file(
        terminal = png,
        file_name = "Gamma_2",
        dimensions = [1500, 1050]
)$
```

Below is the code used for the extra credit for problem 4 on page (43).

```
(%i1) load(drawdf)$
(%i2) a : 0.06$
(%i3) b : 0.67$
(%i4) drawdf(
        [-x + a*y + x^2*y, b - a*y - x^2*y],
        [x,0,3],
        [y,0,3],
        xtics = [0, 1, 3],
        ytics = [0, 1, 3],
        font = "Arial",
        font_size = 40,
        xlabel = "x",
        ylabel = "y",
        line_width = 6,
        point_size = 5,
        duration = 20,
        color = darkblue,
        solns_at([1, 1]),
        color = green,
        solns_at([0.25, 1]),
        color = black,
        points_at([b, b/(a + b^2)]),
        color = darkred,
        solns_at([0.41195, 1]),
        point_size = 3,
        color = yellow,
        points_at([b, b/(a + b^2)])
)$
(%i5) draw_file(
        terminal = png,
        file_name = "Cycle",
        dimensions = [1500, 1050]
)$
```