Structural Models of Utility Maximization

Tyler Ransom

University of Oklahoma, Dept. of Economics

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Outline

Today's plan

1. Describe static discrete choice models
2. How do they fit in with other data science models we've talked about in this class?
3. Derive logit/probit probabilities from intermediate microeconomic theory
4. Go through examples of how to estimate
5. How discrete choice models relate to sample selection bias

Note: These slides are based on the introductory lecture of a PhD course taught at Duke University by Peter Arcidiacono, and are used with permission. That course is based on Kenneth Train's book *Discrete Choice Methods with Simulation*, which is freely available here (PDF).

Outline

What are discrete choice models?

- Discrete choice models are one of the workhorses of structural economics
- Deeply tied to economic theory:
  - utility maximization
  - revealed preference
- Used to model "utility" (broadly defined), for example:
  - consumer product purchase decisions
  - firm market entry decisions
  - investment decisions

Why use discrete choice models?

- Provides link between human optimization behavior and economic theory
- Parameters of these models map directly to economic theory
- Parameter values can quantify a particular policy
- Can be used to form counterfactual predictions (e.g. by adjusting certain parameter values)
- Allows a research to quantify “tastes”

Why not use discrete choice models?

- They're not the best predictive models
  - Trade-off between out-of-sample prediction and counterfactual prediction
- You don't want to form counterfactual predictions, you just want to be able to predict handwritten digits
- You aren't interested in economic theory
- The math really scares you
- You don't like making assumptions
  - e.g. that decision-makers are rational

Example of a discrete choice model

- Cities in the Bay Area are interested in how the introduction of rideshare services will impact ridership on Bay Area Rapid Transit (BART)
- Questions that cities need to know the answers to:
  - Is rideshare a substitute for public transit or a complement?
  - How inelastic is demand for BART? Should fares be $\uparrow$ or $\downarrow$?
  - Should BART services be scaled up to compete with rideshares?
  - Will the influx of rideshare vehicles increase traffic congestion / pollution?
- Each of these questions requires making a counterfactual prediction
- In particular, need a way to make such a prediction confidently and in a way that is easy to understand

Properties of discrete choice models

1. Agents choose from among a **finite** set of alternatives (called the *choice set*)
2. Alternatives in choice set are **mutually exclusive**
3. Choice set is **exhaustive**

Example illustrating these properties

- In San Francisco, people can commute to work by the following (and *only* the following) methods:
  - Drive a personal vehicle (incl. motorcycle)
  - Carpool in a personal vehicle
  - Use taxi/rideshare service (incl. Uber, Lyft, UberPool, LyftLine, etc.)
  - BART (bus, train, or both)
  - Bicycle
  - Walk

Outline

1. Intro
2. Discrete choice
3. **Math**
4. Estimation
5. Sample selection
6. Dynamic Discrete Choice

Mathematically representing utility

Let d_i indicate the choice individual (or decision-maker) i makes where $d_i \in \{1, \cdots, J\}$. Individuals choose d to maximize their utility, U. U generally is written as:

$$U_{ij} = u_{ij} + \varepsilon_{ij} \tag{1}$$

where:

1. u_{ij} relates observed factors to the utility individual i receives from choosing option j
2. ε_{ij} are unobserved to the researcher but observed to the individual
3. $d_{ij} = 1$ if $u_{ij} + \varepsilon_{ij} > u_{ij'} + \varepsilon_{ij'}$ for all $j' \neq j$

Breakdown of the assumptions

- Examples of what's in ε
  - Person's mental state when making the decision
  - Choices of friends or relatives (maybe, depends on the data)
  - $\vdots$
  - Anything else about the person that is not in our data
- Reasonable to assume additive separability?
  - This is a big assumption: that there are no interactive effects between unobservable and observable factors
  - This results in linear separation regions and may be too restrictive
  - For now, go with it, and remember that there are no free lunches

Probabilistic choice

With the ε's unobserved, we must consider choices as probabilistic instead of certain. The Probability that i chooses alternative j is:

$$
\begin{aligned}
P_{ij} &= \Pr(u_{ij} + \varepsilon_{ij} > u_{ij'} + \varepsilon_{ij'} \; \forall \; j' \neq j) && (2) \\
&= \Pr(\varepsilon_{ij'} - \varepsilon_{ij} < u_{ij} - u_{ij'} \; \forall \; j' \neq j) && (3) \\
&= \int_{\varepsilon} I(\varepsilon_{ij'} - \varepsilon_{ij} < u_{ij} - u_{ij'} \; \forall \; j' \neq j) f(\varepsilon) d\varepsilon && (4)
\end{aligned}
$$

Transformations of utility

Note that, regardless of what distributional assumptions are made on the ε's, the probability of choosing a particular option does not change when we:

1. Add a constant to the utility of all options (utility is relative to one of the options, only differences in utility matter)
2. Multiply by a positive number (need to scale something, generally the variance of the ε's)

This is just like in consumer choice theory: utility is ordinal, and so is invariant to the above two transformations

Variables

Suppose we have:

$$u_{i1} = \alpha Male_i + \beta_1 X_i + \gamma Z_1$$
$$u_{i2} = \alpha Male_i + \beta_2 X_i + \gamma Z_2$$

Since only differences in utility matter:

$$u_{i1} - u_{i2} = (\beta_1 - \beta_2) X_i + \gamma (Z_1 - Z_2)$$

- Thus, we cannot tell whether men are happier than women, but can tell whether men have a preference for a particular option over another.
- We can only obtain **differenced** coefficient estimates on X's, and can obtain an estimate of a coefficient that is constant across choices only if the variable it is multiplying varies by choice.

Number of error terms

Similar to socio-demographic characteristics, there are restrictions on the number of error terms. Recall that he probability i will choose j is given by:

$$\begin{aligned} P_{ij} &= \Pr(u_{ij} + \varepsilon_{ij} > u_{ij'} + \varepsilon_{ij'} \ \forall \ j' \neq j) \\ &= \Pr(\varepsilon_{ij'} - \varepsilon_{ij} < u_{ij} - u_{ij'} \ \forall \ j' \neq j) \\ &= \int_{\varepsilon} I(\varepsilon_{ij'} - \varepsilon_{ij} < u_{ij} - u_{ij'} \ \forall \ j' \neq j) f(\varepsilon) d\varepsilon \end{aligned}$$

where the integral is J-dimensional.

Number of error terms (cont'd)

But we can rewrite the last line as $J-1$ dimensional integral over the differenced ε's:

$$P_{ij} = \int_{\tilde{\varepsilon}} I(\tilde{\varepsilon}_{ij'} < \tilde{u}_{ij'} \ \forall\ j' \neq j) g(\tilde{\varepsilon}) d\tilde{\varepsilon}$$

Note that this means one dimension of $f(\varepsilon)$ is not identified and must therefore be normalized.

Derivation of Logit Probability

Consider the case when the choice set is $\{1, 2\}$. The Type 1 extreme value cdf for ε_2 is:

$$F(\varepsilon_2) = e^{-e^{(-\varepsilon_2)}}$$

To get the probability of choosing 1, substitute in for ε_2 with $\varepsilon_1 + u_1 - u_2$:

$$Pr(d_1 = 1|\varepsilon_1) = e^{-e^{-(\varepsilon_1 + u_1 - u_2)}}$$

But ε_1 is unobserved so we need to integrate it out (see Appendix to these slides if you want the math steps)

Derivation of Logit Probability

In the end, we can show that, for any model where there are two choice alternatives and ε is drawn from the Type 1 extreme value distribution,

$$P_{i1} = \frac{\exp(u_{i1} - u_{i2})}{1 + \exp(u_{i1} - u_{i2})}, P_{i2} = \frac{1}{1 + \exp(u_{i1} - u_{i2})}$$

Suppose we have a data set with N observations. The log likelihood function we maximize is then:

$$\ell(\beta, \gamma) = \sum_{i=1}^{N} (d_{i1} = 1)(u_{i1} - u_{i2}) - \ln\left(1 + \exp(u_{i1} - u_{i2})\right)$$

Derivation of Probit Probability

In the probit model, we assume that ε is Normally distributed. So for a binary choice we have:

$$P_{i1} = \Phi\left(u_{i1} - u_{i2}\right), P_{i2} = 1 - \Phi\left(u_{i1} - u_{i2}\right)$$

where $\Phi\left(\cdot\right)$ is the standard normal cdf

The log likelihood function we maximize is then:

$$\ell(\beta, \gamma) = \sum_{i=1}^{N} (d_{i1} = 1) \ln\left(\Phi\left(u_{i1} - u_{i2}\right)\right) + (d_{i2} = 1) \ln\left(1 - \Phi\left(u_{i1} - u_{i2}\right)\right)$$

Pros & Cons of Logit & Probit

Logit model:

- Has a much simpler objective function
- Is by far most popular
- ... but has more restrictive assumptions about how people substitute choices
- (this is known as the Independence of Irrelevant Alternatives or IIA assumption)

Probit model:

- Much more difficult to estimate
- ... but can accommodate more realistic choice patterns

Outline

Estimation in R

The R function `glm` is the easiest way to estimate a binomial logit or probit model:

```
library(mlogit)
data(Heating) # load data on residential heating choice in CA
levels(Heating$depvar) <- c("gas","gas","elec","elec","elec")
estim <- glm(depvar ~ income+agehed+rooms+region,
             family=binomial(link='logit'),data=Heating))
print(summary(estim))
```

Interpreting the coefficients

Estimated coefficients using the code in the previous slide:

```
Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  -0.599142   0.252032  -2.377   0.0174 *
income        0.015579   0.027905   0.558   0.5766
agehed       -0.006535   0.003342  -1.955   0.0505 .
rooms         0.024291   0.026916   0.902   0.3668
regionscostl -0.053096   0.126665  -0.419   0.6751
regionmountn  0.041827   0.169787   0.246   0.8054
regionncostl -0.219136   0.137692  -1.591   0.1115
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Interpreting the coefficients

- Positive coefficients ⇒ household more likely to choose the non-baseline alternative (in this case: electric)
  - Whatever the first `level` of the factor dependent variable is will be the "baseline" alternative
- Negative coefficients imply the reverse
- Coefficients **not** linked to changes in probability of choosing the alternative (since probability is a nonlinear function of X)

Forming predictions

To get predicted probabilities for each observation in the data:

```
Heating$predLogit <- predict(estim, newdata = Heating, type = "response")
print(summary(Heating$predLogit))
```

Estimating a probit model

For the probit model, we repeat the same code, except change the “link” function from “logit” to “probit”

```
estim2 <- glm(depvar ~ income+agehed+rooms+region,
             family=binomial(link='probit'),data=Heating))
print(summary(estim2))
Heating$predProbit <- predict(estim2, newdata = Heating, type = "response
    ")
print(summary(Heating$predProbit))
```

A simple counterfactual simulation

- We talked a lot about doing counterfactual comparisons, but how do we *actually* do it?
- Let's show how to do this on a previous example. Suppose that we introduce a policy that makes richer people more likely to use electric heating.
- Mathematically, what does this look like?
- It would correspond to an increase in the parameter in front of *income* in our regression

A simple counterfactual simulation

- Suppose the coefficient increased by a factor of 4. What is the new share of gas vs. electricity usage?

```
estim$coefficients["income"] <- 4*estim$coefficients["income"]
Heating$predLogitCfl <- predict(estim, newdata = Heating, type = "
    response")
print(summary(Heating$predLogitCfl))
```

This policy would increase electric usage by 7 percentage points (from 22% to 29%)

Outline

Discrete choice models and sample selection bias

- Discrete choice models are common tools used to evaluate sample selection bias
- Why? Because variables that are MNAR can be thought of as following a utility-maximizing process
- Examples:
  - Suppose you want to know what the returns to schooling are, but you only observe wages for those who currently hold jobs
  - As a result, your estimate of the returns to schooling might be invalidated by the non-randomness of the sample of people who are currently working
  - How to get around this? Use a discrete choice model (This was the problem we ran into in PS7, if you recall)

Heckman selection correction

The Heckman selection model specifies two equations:

$$u_i = \beta x_i + \nu_i$$
$$y_i = \gamma z_i + \varepsilon_i$$

- The first equation is a utility maximization problem, determining if the person is in the labor force. Can think of ν_i as "desire to work"
- x_i may include: number of children in the household
- The second equation is the log wage equation, where y_i is only observed for people who are in the labor force.
- To solve the model, one needs to use the so-called "Heckit" model, which involves adding a correction term in the wage equation which accounts for the fact that workers are not randomly selected.

Estimating Heckman selection in R

R has a package called `sampleSelection` which incorporates the Heckman selection model[1]

```
library(sampleSelection)
data('Mroz87')
Mroz87$kids <- (Mroz87$kids5 + Mroz87$kids618) > 0
# Comparison of linear regression and selection model
outcome1 <- lm(wage ~ exper, data = Mroz87)
summary(outcome1)
selection1 <- selection(selection = lfp ~ age + I(age^2) + faminc + kids
    + educ,
outcome = wage ~ exper, data = Mroz87, method = '2step')
summary(selection1)
```

[1] This code taken from Garrett Glasgow's website: http://www.polsci.ucsb.edu/faculty/glasgow/ps207/ps207_class6.r

Estimation output

Output from a regression of wage on experience:

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  1.30434    0.18937   6.888 1.20e-11 ***
exper        0.10067    0.01419   7.093 3.03e-12 ***
---
```

Estimation output

Output from the Heckman selection model: (edited for length)

```
Probit selection equation:
               Estimate Std. Error t value Pr(>|t|)
(Intercept) -4.157e+00  1.402e+00  -2.965 0.003126 **
kidsTRUE    -4.490e-01  1.309e-01  -3.430 0.000638 ***
Outcome equation:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)  7.12492    0.80425   8.859   <2e-16 ***
exper        0.02962    0.02059   1.439    0.151
Multiple R-Squared:0.0823,      Adjusted R-Squared:0.0779
Error terms:
                Estimate Std. Error t value Pr(>|t|)
invMillsRatio   -5.075      1.108  -4.581 5.42e-06 ***
sigma            4.977         NA      NA       NA
rho             -1.020         NA      NA       NA
```

Reading the output

- Because there are two equations, there are now more parameters
- Using just the regression on workers led us to believe the returns to experience were $\approx 10\%$
- Taking into account the selectivity of labor force participants leads us to conclude the returns to experience are much lower ($\approx 3\%$ and not statistically different from zero)
- Viability of the model depends on the assumption that's made: in this case, that having children only affects labor supply preferences and doesn't affect wages
  - Wage discrimination against mothers in the labor market would invalidate this assumption
  - Back to the idea that to get causal inference we have to impose more assumptions

Outline

The optimal stopping problem

- Much of life is concerned with knowing when to stop:
  - How many people to date before making/accepting a marriage proposal
  - How much to study for upcoming exams
  - How long to "hodl" an asset
- All of the above cases involve forming expectations about:
  1. The long-run value of making a particular choice
  2. ... relative to the long-run value of alternatives
- Expectations about the future imply that we need to think "dynamically" (i.e. think over the longterm)
- Today we'll go through the math on how to do this

Relation to reinforcement learning

- Reinforcement learning is based on the optimal stopping problem
- At each state X (e.g. game board configuration), observe reward y (e.g. win probability)
- In each period (i.e. gameplay turn), choose the decision that maximizes the (present value) expected reward
- With structural models, "reward" is utility

Dynamic discrete choice models

With *dynamic* models, need a way to quantify present value of utility
Individual i's **flow utility** for option j at time t is:

$$\begin{aligned} U_{ijt} &= u_{ijt} + \varepsilon_{ijt} \\ &= X_{it}\alpha_j + \varepsilon_{ijt} \end{aligned}$$

Individual chooses d_{it} to maximize **expected lifetime utility**

$$\max_{d_{it}} V = E\left\{\sum_{\tau=t}^{T}\sum_{j}\beta^{\tau-t}(d_{it}=j)U_{ijt}\right\}$$

- V is the *value function*
- $\beta \in (0,1)$ is the *discount factor*
- T is the *time horizon*

Expectations

- Expectations taken over future states (X's) **and** errors
- ε's are iid over time
- Future states are not affected by ε's except through current and past choices:
$$E\left(X_{t+1}|d_t, ..., d_1, \varepsilon_t, ..., \varepsilon_1\right) = E\left(X_{t+1}|d_t, ..., d_1\right)$$

Human behavior vs. reinforcement learning

- In reinforcement learning, we typically don't have ε, unless we want to allow for "curiosity"
- Transitions in X much more dominant factor (e.g. if I move here, opponent will move there, ...)
- Real-life example of uncertainty in ε's:
  - "My significant other might take a job in another city next year, so if I want to move with him/her, I may not want to take this job offer today."
- Real-life example of uncertainty in X's:
  - "I might get laid off next year, which will influence my ability to pay off my car loan, so I might want not want to buy this Mercedes today, since my (expected) permanent income might be lower than my current income."

Dynamic programming & the Bellman equation

- We want to maximize the value function V
- It's helpful to write the value function as a recursive expression, where we separate out today's decision from all future decisions (this is called the *Bellman equation*, or the *dynamic programming problem*)
- The payoff from choosing alternative j today is the *flow utility* $= u_{ijt}$ from earlier in these slides
- The payoff from choosing alternative j in the future is the expected future utility conditional on choosing j today

How do we solve the Bellman equation?

- Requires solving backwards, just like in a dynamic game (cf. subgame perfect Nash equilibrium)

Two Period Example

Consider the utility of choice j in the last period:

$$\begin{aligned} U_{ijT} &= u_{ijT} + \varepsilon_{ijT} \\ &= X_{iT}\alpha_j + \varepsilon_{ijT} \end{aligned}$$

Define the **conditional valuation function** for choice j as the flow utility of j minus the associated ε plus the expected value of future utility conditional on j:

$$v_{ijT-1} = u_{ijT-1} + \beta E \max_{k \in J} \left\{ u_{ikT} + \varepsilon_{ikT} \middle| d_{iT-1} = j \right\}$$

where β is the discount factor.

Suppose X_{iT} was deterministic given X_{iT-1} and d_{T-1} and the ε's are Type 1 extreme value. What would the E max expression be? [$\ln \sum_k \exp\left(u_{ikT}\right)$]

Two Period Example (cont'd)

For $J = 2$ the log likelihood would then look like:

$$L(\alpha) = \sum_{i=1}^{N} \sum_{t=1}^{T} (d_{i1t} = 1)(v_{i1t} - v_{i2t}) - \ln\left(1 + \exp(v_{i1t} - v_{i2t})\right)$$

where

$$v_{ijt} = u_{ijt} + \beta E \max_{k \in J} \left\{ v_{ikt+1} + \varepsilon_{ikt+1} \middle| d_{it} = j \right\}$$

and where

$$u_{ijt} = X_{it}\alpha_j$$

Note: if $T = 2$ then $v_{ikt+1} = u_{ikT}$

Estimating a dynamic discrete choice model in R

- Because we have to loop backwards through time, we can't use a canned function like `lm()`
- Requires us to write a custom likelihood function
- This is because the flow utility parameters (α_j) appear in the flow utility function in *each* period
  - Side note: We don't typically estimate the discount factor (β) but instead assume a fixed value (most common: 0.90 or 0.95)
- To do this, write down an objective function (i.e. log likelihood function) and use `nloptr` to estimate the α's
- Once you have the α's you can do counterfactual simulations
- These simulations are likely to be more realistic because the model has incorporated forward-looking behavior

Objective function

```
objfun <- function(alpha,Choice,age) {
    J <- 2
    a  <- alpha[3]*(1-diag(J))

    u1 <- matrix(0, N, T)
    u2 <- matrix(0, N, T)
    for (t in 1:T) {
        u1[ ,t] <- 0*age[ ,t]
        u2[ ,t] <- alpha[1] + alpha[2]*age[ ,t]
    }
```

(continued on next slide)

```
    Like <- 0
    for (t in T:1) {
        for (j in 1:J) {
            # Generate FV
            dem <- exp(u1[ ,t] + a[1,j]+fv[ ,1,t+1])+
                   exp(u2[ ,t] + a[2,j]+fv[ ,2,t+1])
            fv[ ,j,t] <- beta*(log(dem)-digamma(1))
            p1 <- exp(u1[ ,t] + a[1,j] + fv[ ,1,t+1])/dem
            p2 <- exp(u2[ ,t] + a[2,j] + fv[ ,2,t+1])/dem
            Like <- Like - (LY[ ,t]==j)*((Choice[ ,t]==1)*log(p1)+(Choice
                [ ,2]==2)*log(p2))
        }
    }
    return ( sum(Like) )
}
```

Calling nloptr

```
## initial values
theta0 <- runif(3) #start at uniform random numbers equal to number of
    coefficients

## Algorithm parameters
options <- list("algorithm"="NLOPT_LN_NELDERMEAD","xtol_rel"=1.0e-6,"
    maxeval"=1e4)

## Optimize!
result <- nloptr( x0=theta0,eval_f=objfun,opts=options,Choice=Choice,age=
    age)
print(result)
```

Derivation of Logit Probability

$$
\begin{aligned}
Pr(d_1 = 1) &= \int_{-\infty}^{\infty} \left(e^{-e^{-(\varepsilon_1 + u_1 - u_2)}} \right) f(\varepsilon_1) d\varepsilon_1 \\
&= \int_{-\infty}^{\infty} \left(e^{-e^{-(\varepsilon_1 + u_1 - u_2)}} \right) e^{-\varepsilon_1} e^{-e^{-\varepsilon_1}} d\varepsilon_1 \\
&= \int_{-\infty}^{\infty} \exp\left(-e^{-\varepsilon_1} - e^{-(\varepsilon_1 + u_1 - u_2)} \right) e^{-\varepsilon_1} d\varepsilon_1 \\
&= \int_{-\infty}^{\infty} \exp\left(-e^{-\varepsilon_1} \left[1 + e^{u_2 - u_1} \right] \right) e^{-\varepsilon_1} d\varepsilon_1
\end{aligned}
$$

Derivation of Logit Probability

Now need to do the substitution rule where $t = \exp(-\varepsilon_1)$ and $dt = -\exp(-\varepsilon_1)d\varepsilon_1$.

Note that we need to do the same transformation of the bounds as we do to ε_1 to get t. Namely, $\exp(-\infty) = 0$ and $\exp(\infty) = \infty$.

Derivation of Logit Probability

Substituting in then yields:

$$
\begin{aligned}
Pr(d_1 = 1) &= \int_{\infty}^{0} \exp\left(-t\left[1 + e^{(u_2 - u_1)}\right]\right)(-dt) \\
&= \int_{0}^{\infty} \exp\left(-t\left[1 + e^{(u_2 - u_1)}\right]\right) dt \\
&= \left.\frac{\exp\left(-t\left[1+e^{(u_2-u_1)}\right]\right)}{-\left[1+e^{(u_2-u_1)}\right]}\right|_0^{\infty} \\
&= 0 - \frac{1}{-\left[1 + e^{(u_2 - u_1)}\right]} = \frac{\exp(u_1)}{\exp(u_1) + \exp(u_2)}
\end{aligned}
$$