Polytopes, Amoeba, and Tropical Curves

Connections in Tropical Geometry

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GT Directed Reading Program, Fall 2024

Table of Contents

Tropical Curves

Definition

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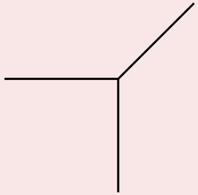

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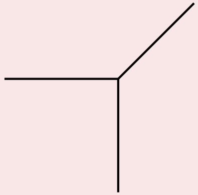

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This constitutes one construction of a *tropical curve*.

Table of Contents

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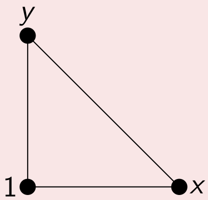
Here $x^{\mathbf{u}} = x_1^{u_1} x_2^{u_2} \ldots x_n^{u_n}$.

Polytopes

Example

The Newton Polytope newt(f) of $f = x_1 + x_2 + 1$ is the convex hull of

$$\{(1,0),(0,1),(0,0)\}:$$



Polytopes

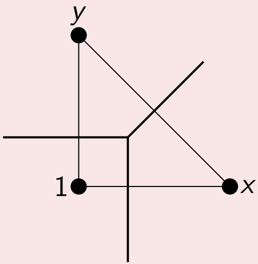
Definition

The *inner normal fan* of a polytope Σ is the polyhedral fan containing the normal cones to each face of Σ.

Polytopes

Example

The inner normal fan of newt(f), where $f = x_1 + x_2 + 1$, is



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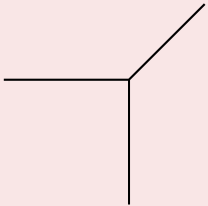

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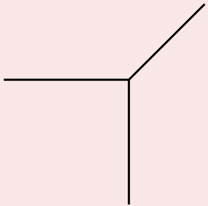

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Table of Contents

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Definition

The *amoeba* of a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ is the image of $\mathbb{V}(f)$ under the local logarithmic map

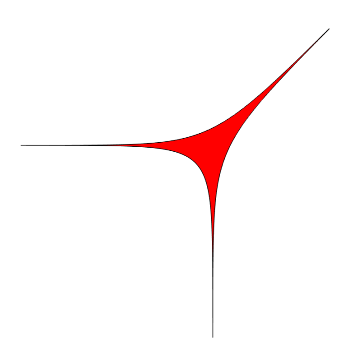
$$\text{Log} : (\mathbb{C}^*)^n \to \mathbb{R}^n,$$

$$\text{Log}(v_1, \ldots, v_n) = (\log|v_1|, \ldots, \log|v_n|)\,.$$

Amoebas

Example

The amoeba of $f = x_1 + x_2 + 1$ is the image of $\mathbb{V}(f)$ under Log:



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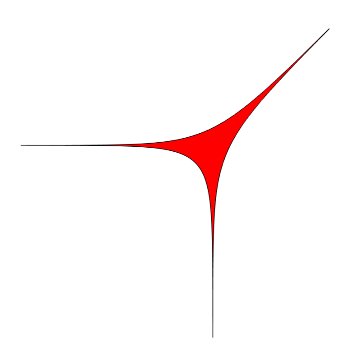

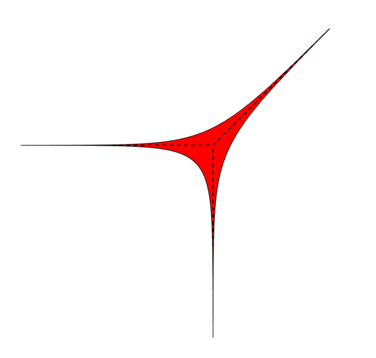

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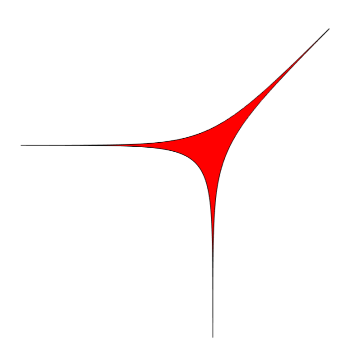

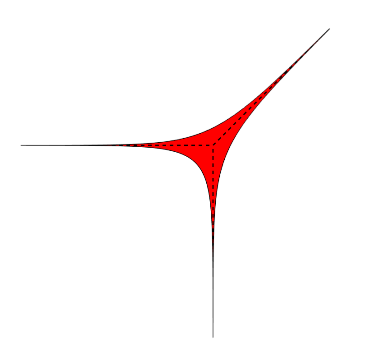

The *spine* of this amoeba is another construction of a *tropical curve*.

Table of Contents

Conclusion

Maybe you've noticed a pattern?

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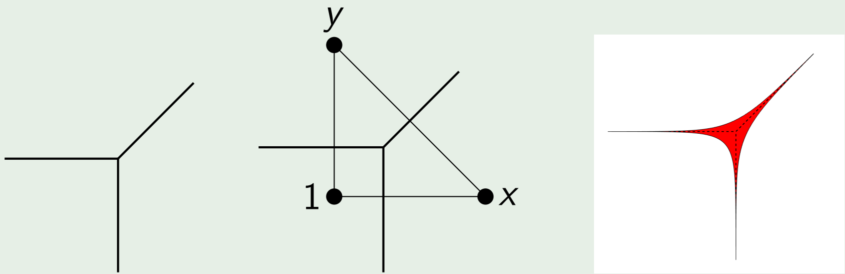
Idea!



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Keywords: Tropical Geometry, Maslov Dequantization, Log Semiring, Logarithmic Limit Set, Newton Polytope

Book: *Introduction to Tropical Geometry* - Diane Maclagan and Bernd Sturmfels