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EXTENDING CLASSICAL GEOMETRY WITH AUTOMATIC THEOREM PROVING: APPLICATIONS TO CYCLIC POLYGONS

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ABSTRACT. The problem of determining the area of a polygon given only its side lengths has intrigued mathematicians for centuries, with solutions ranging from Heron's formula for triangles to Brahmagupta's formula for cyclic quadrilaterals. However, extending such methods to general cyclic polygons remains challenging due to the complex algebraic relationships between side lengths and geometric constraints. This paper explores the use of automatic theorem-proving (ATP) techniques, leveraging computational algebra tools like Macaulay2, to systematically address this problem.

By encoding polygonal area determination as a set of algebraic and geometric constraints, ATP tools not only verify existing formulas but also derive new ones. We apply this approach to compute areas for specific classes of cyclic polygons, including convex and non-convex configurations up to hexagons. The results highlight ATP's role in automating the discovery of higher-order geometric relationships. This work highlights the potential of computational methods to address long-standing mathematical challenges and underscores the value of ATP tools in advancing geometric problem-solving and symbolic computation.

1. INTRODUCTION

Heron's formula for triangles and Brahmagupta's formula for cyclic quadrilaterals elegantly connect geometry and algebra, but generalizing these results to polygons with more sides poses significant challenges. While strides have been made to generalize these area formulas, particularly for cyclic polygons, they tend to involve intricate algebraic relationships.

Previous research by Robbins explored polynomial formulations for cyclic pentagons and hexagons, relating areas to the discriminants of cubic polynomials [2]. Maley, Robbins, and Roskies further developed "generalized Heron polynomials" for heptagons, octagons, and semicyclic polygons [1]. These studies demonstrated the potential for algebraic methods to generalize classic geometric results.

This paper leverages Macaulay2, a computational algebra system, to explore area formulas for polygons beyond traditional analytical approaches. Using Macaulay2, we (1) confirm Heron's formula computationally, (2) derive and validate area expressions for both convex and non-convex cyclic quadrilaterals, and (3) construct algebraic ideals for cyclic pentagons and hexagons to efficiently compute areas.

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These results extend existing theoretical findings, showcasing the power of computational algebra to derive area formulas for complex polygons and highlighting the interplay between algebraic methods and geometric intuition.

2. Preliminaries

2.1. **Cyclic Polygons.** For polygons with more than three sides, we consider cyclic polygons (where all vertices lie on a circle) due to the lack of rigidity in non-cyclic polygons. For example, a general quadrilateral can have equal side lengths but different areas. Focusing on cyclic polygons resolves this ambiguity and allows for consistent equations for polygons with more than three sides.

2.2. **Convexity of a Polygon.** In cyclic polygons, convexity is defined as having no self-intersecting edges. This study primarily calculates areas of convex polygons, with some results extended to nonconvex cases.

3. Area of a Triangle

Given the side lengths a, b, c of a triangle, Heron's Formula gives us

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} \tag{3.1}$$

where

$$s = \frac{1}{2}(a+b+c) \tag{3.2}$$

We construct an ideal with the system of equations as described in Appendix **??**, and then eliminate the unnecessary symbols x_1, y_1 to get the resulting ideal $J = I \cap \mathbb{Q}[a, b, c, S]$ that has a single generator. This generator gives us an equation which must be equal to 0 in order for all the equations in **??** to be satisfied. By setting the equation equal to 0, and isolating the term S^2, we immediately see that we have

$$S^2 = \frac{1}{16}(a+b+c)(-a+b+c)(a-b+c)(a+b-c) \tag{3.3}$$

which is equal, up to sign, to Heron's formula in 3.1.

4. Area of a Cyclic Quadrilateral

Next we automatically prove Brahmagupta's Formula for the area of a cyclic convex quadrilateral, and find the formula for the area of the cyclic non-convex quadrilateral.

Brahmagupta's Formula, given the sides a, b, c, d of a quadrilateral, is

$$\text{Area} = \sqrt{(s-a)(s-b)(s-c)(s-d)} \tag{4.1}$$

where

$$s = \frac{1}{2}(a+b+c+d) \tag{4.2}$$

We construct an ideal with the system of equations as described in Appendix **??**, and then eliminate the unnecessary symbols $x_2, x_3, x_4, y_2, y_3, y_4, r$ to get the resulting ideal $J = I \cap \mathbb{Q}[a, b, c, d, S]$ that has a single generator. This generator gives us an equation which must be equal to 0 in order for all the equations in **??** to be satisfied. Factoring the generator yields separate equations for convex and non-convex quadrilaterals. By setting the equation for the convex equal to 0, and isolating the term S^2, we see that we have

$$S^2 = \frac{1}{16}(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d) \tag{4.3}$$

which is equal, up to sign, to Brahmagupta's formula in 4.1.

For the nonconvex solution, we solve for S^2 and obtain

$$S^2 = \frac{1}{16}(-a+b+c-d)(a-b+c-d)(a+b-c-d)(a+b+c+d) \tag{4.4}$$

5. Area of a Cyclic Pentagon

As the number of sides increases, the complexity of deriving area formulas grows significantly. Unlike the elegant closed-form expressions for triangles and quadrilaterals, cyclic pentagons require solving higher-degree polynomials that encode intricate relationships between the side lengths and the area. These complexities make deriving a simple, explicit formula impractical.

Using automatic theorem-proving techniques, we construct a principal ideal whose generator provides an equation in terms of the side lengths. This equation can be solved to determine the possible areas of cyclic pentagons. Among these, the cyclic convex pentagon has the largest area, which can be computed using the derived equation.

We construct the ideal I based on the system of equations in Appendix **??**, eliminate extraneous variables, and obtain $J = I \cap \mathbb{Q}[a, b, c, d, e, S]$. This ideal J has a single generator, enabling us to compute the roots for specific values of a, b, c, d, e. The degree-14 generator for S includes only even powers, resulting in seven possible areas for a cyclic pentagon, up to sign. This result aligns with Robbins's theoretical discoveries [2].

6. Area of a Cyclic Hexagon

For cyclic hexagons, the computational challenges increase further, as the degree of the polynomial grows, requiring more computational resources and sophisticated techniques. This complexity underscores the difficulty in deriving closed-form expressions for polygons with many sides.

Using the same method as for the pentagon, we construct the ideal I, eliminate unnecessary variables, and obtain $J = I \cap \mathbb{Q}[a, b, c, d, e, f, S]$. By substituting specific values for a, b, c, d, e, f, the generator reveals 14 possible areas of a cyclic hexagon. These results were confirmed through computations in Macaulay2 and again align with Robbins's theoretical findings [2].

7. Conclusion

This study demonstrates the effectiveness of automatic theorem-proving techniques in deriving and validating area formulas for cyclic polygons up to six sides. By leveraging algebraic tools like Macaulay2, we confirmed classical formulas for triangles and quadrilaterals, while deriving polynomial relationships for pentagons and hexagons. The results highlight the computational challenges that arise as the number of sides increases, particularly due to the exponential growth in the complexity of the underlying polynomials.

Despite these challenges, the methodology provides a robust framework for future exploration. Advancements in symbolic computation and optimization algorithms could extend this approach to cyclic polygons with more than six sides. This work underscores the potential of computational methods to bridge gaps in classical mathematics, offering new perspectives on long-standing geometric problems and enabling automated discovery of mathematical results.

References

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