Area of Cyclic Polygons by Elimination of Variables

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Heron's Formula

Q: Can we calculate the area of a polygon given only its side lengths?

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Q: Can we calculate the area of a polygon given only its side lengths?

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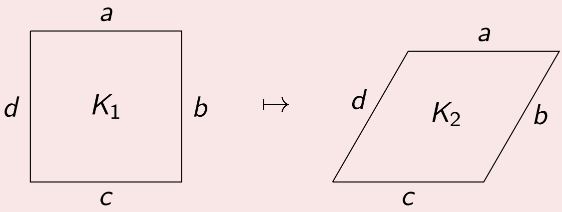
Q: When is "sometimes"?

A: When it is *rigid* (when its area is fixed uniquely by its side lengths).

Heron's Formula

Example

A triangle is rigid, but a quadrilateral is not.



$$K_1 \neq K_2$$

Heron's Formula

So, because all triangles are rigid, we can show

Heron's Formula (1st Century BC)

Given a triangle T with side lengths a, b, c, the area K of T is given by

$$K = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the *semiperimeter*, $s = \frac{1}{2}(a+b+c)$.

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Brahmagupta's Formula

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Brahmagupta's Formula describes the area of a cyclic quadrilateral.

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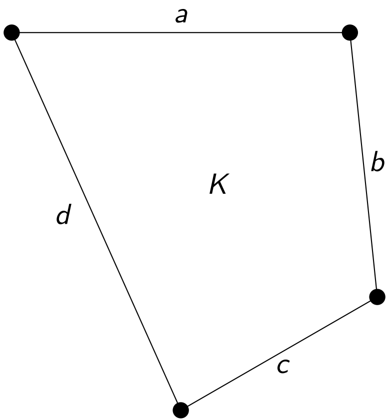
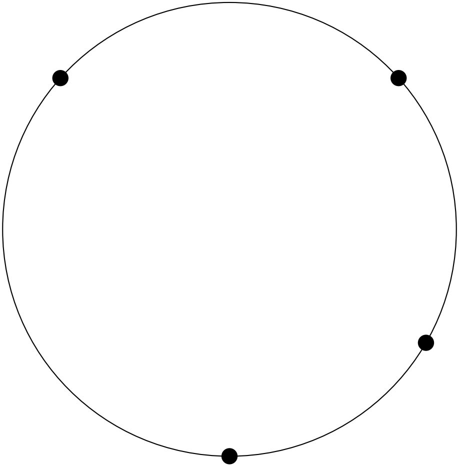


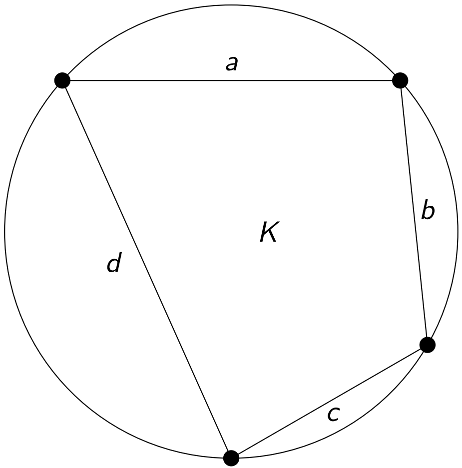
Brahmagupta's Formula

Create the circle described by these four points



Brahmagupta's Formula

Inscribe the quadrilateral



Brahmagupta's Formula

Brahmagupta's Formula (7th Century)

Given sidelengths a, b, c, d, the area K of the cyclic convex quadrilateral with these side lengths is given by

$$K = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where s is the *semiperimeter*, $s = \frac{1}{2}(a+b+c)$.

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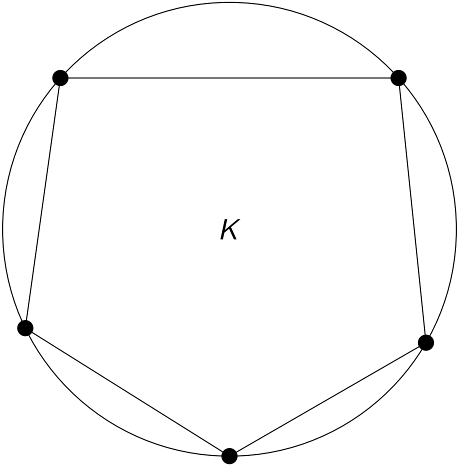
Gauss' Pentagon Formula



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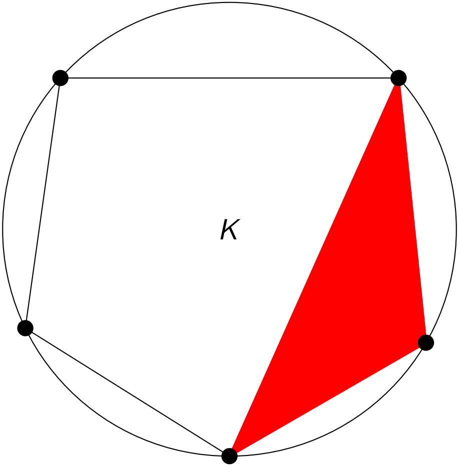

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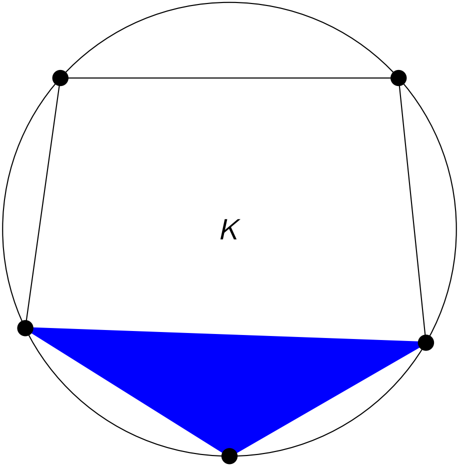

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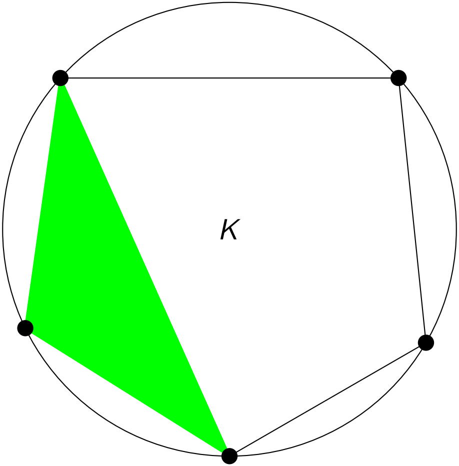

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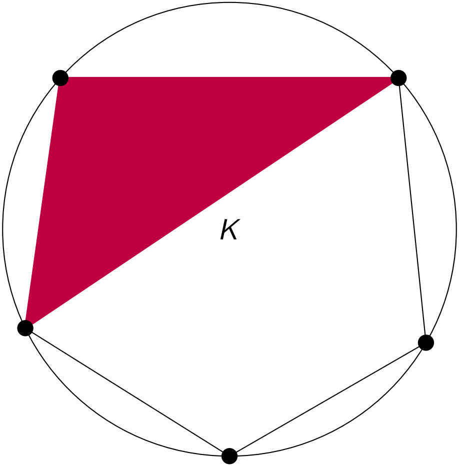

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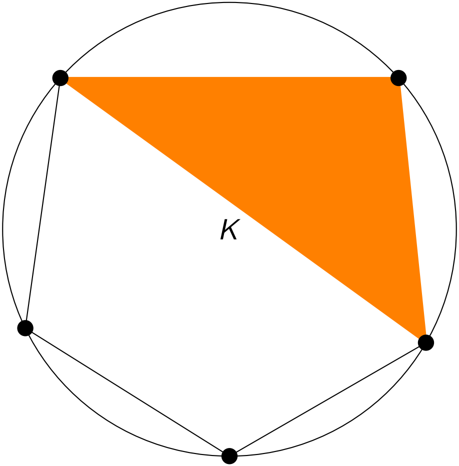

Gauss' Pentagon Formula

Gauss' Pentagon Formula (1823)

Given sidelengths a, b, c, d, e, the area K of the cyclic pentagon with these sidelengths is satisfies

$$K^2 - K(b_0 + b_1 + b_2 + b_3 + b_4) + b_0 b_1 + b_1 b_2 + b_3 b_4 + b_4 b_0 = 0,$$

where each b_i is the area of some distinct triangle formed by three consecutive vertices.

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Automatic Theorem Proving

The goal of this project is to prove Heron-type formulas for the area of cyclic polygons by automatic means.

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(a) Side lengths: $a_i^2 - (x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = 0$

(b) Circle: $x_i^2 + y_i^2 - r^2 = 0$

(c) Area: $K = \frac{1}{2}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \cdots + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right)$

then eliminating all variables except for the area and the side lengths.

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In all the proofs of the given theorems, edges are placed to form a *convex* polygon. This information is not encapsulated in the ideal.

Q: How else might these relations be interpreted?

A: These edges may form nonconvex polygons.

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There are two “combinatorial configurations” of a quadrilateral:

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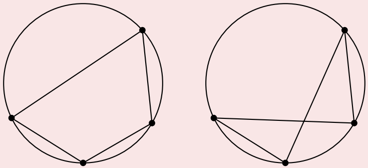

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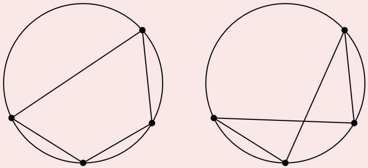

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When factored, the principal generator of the corresponding ideal yields two polynomials: one that agrees with Brahmagupta’s Formula and one that does not.

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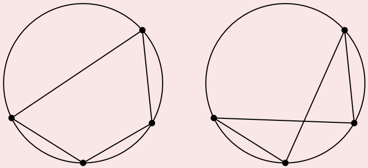

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When factored, the principal generator of the corresponding ideal yields two polynomials: one that agrees with Brahmagupta's Formula and one that does not.

These correspond to the area of the convex and nonconvex quadrilaterals, respectively.

Automatic Theorem Proving

Without restrictions on the order in which vertices are connected, there exist

$$\Delta_n = \frac{1}{2}\left((2(n-3)+1)\binom{2(n-3)}{n-3} - 2^{2(n-3)}\right)$$

combinatorial configurations of cyclic n-gons, where $n \geq 5$.

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The first few terms of this sequence are

$$\Delta_5 = 7, \Delta_6 = 38, \Delta_7 = 187, \Delta_8 = 874, \ldots$$

And so computational requirements grow quickly.

Automatic Theorem Proving

With this procedure we were able to verify all three stated formulas, as well as derive formulas in agreement with the findings of Robbins for the area of nonconvex 4- and 5-gons.

References

1. F. Miller Maley, David P. Robbins, and Julie Roskies, *On the areas of cyclic and semicyclic polygons*, Advances in Applied Mathematics **34** (2005), no. 4, 669-689 (en).
2. D.P. Robbins, *Areas of Polygons Inscribed in a Circle*, Discrete and Computational Geometry **12** (1994), no. 2, 223-236.