What do I do?

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Note

This document is a work in progress, and I am currently soliciting feedback to improve readability and comprehension.

The goal is for this to be a relatively quick read that will give a serviceable introduction to what I am currently studying.

If you have criticism (constructive or otherwise), please let me know!

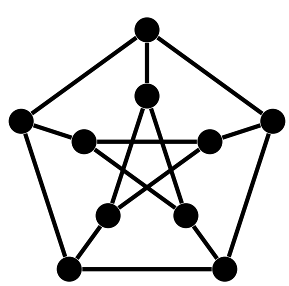
§ 1 Introduction

Borrowing words from June Huh, a famous mathematician, the questions I like to think about involve connecting discrete and continuous objects. In particular, I think it is really cool when we can answer hard questions about one by examining the other.

1.1 Discrete objects

When I say an object is discrete, what I am trying to say is that these objects are made up of small, separate parts. An example might be a train track. The slats in the middle of the track form the "parts" of my discrete object, and the track itself is the broader whole. These tracks might be infinite, or they might be finite, but they can be decomposed into little bits. The interesting properties of discrete objects arise from the way these little bits interact.

A mathematical example of a discrete object is a *graph*. A graph is a bunch of little dots, called *vertices*, and a bunch of lines that go between these vertices, called *edges*. Below is a special graph, called the Petersen Graph.



Here, the little bits we can decompose our object into are the vertices. A bunch of little dots are uninteresting by themselves, but it turns out that the structural phenomena we gain access to by adding edges between our vertices are large in number and very interesting.

Of course, there are many many more objects that might be called discrete. One important note is that discrete objects are **not** necessarily finite. That is, a object might have an infinite number of little bits, but as long as these little bits are separable we still say our object is discrete.

1.2 Continuous objects

Continuous objects are exact opposites of discrete objects. These objects cannot be decomposed into "little bits," as we say. Instead, we might imagine something like looking into space through a telescope. No matter how far we zoom in, there is always an infinite collection of "stuff" looking back at us.

A more mathematical example of this is the number line. If I pick two points on the number line, there is always another point (in fact an infinite number of points) between my two points. No matter how far I "zoom in," there is always more to see.

§ 2 Combinatorics

Combinatorics is basically the study of discrete objects. Wikipedia classifies questions in combinatorics into four categories:

1. (Enumeration) How many types of a discrete object exist? Can we classify them? If we can, how many are in each class? How many ways can we combine or rearrange discrete objects?

2. (Existence) What kind of discrete structures exist? What properties do discrete structures satisfy? If I want a discrete structure that looks a certain way, is it even possible to find it?

3. (Construction) How can we build discrete structures? What if I want to build a discrete structure that looks a certain way? Can I build up big structures from small ones? How can I think about big structures in terms of small structures?

4. (Optimization) If I have a discrete object, does it solve certain problems "best?" What kind of problems can be solved by what kinds of objects? How good is my solution to a certain question?

I am interested in a answering questions that concern a broad class of combinatorial objects called *matroids*, which themselves serve as a sort of abstraction of other combinatorial objects.

In particular, we might imagine that some combinatorial objects have properties that persist when we zoom in on them, so looking at smaller collections of their building blocks means we still see some of the same patterns. Matroids abstract this notion.

More specifically, a matroid is a pair E and $\mathcal{I}$ ("cal-I"), where E is some collection of objects, and $\mathcal{I}$ is a collection of subsets of E. We call the elements of $\mathcal{I}$ "independent sets," and stipulate that they must have two properties:

(i) (Hereditary Property) All subsets of an independent set are themselves independent sets.

(ii) (Replacement Property) If I have an independent set, I am always able to remove an element and find a new element that I "replace" it with. This replacement does not change the fact that it is independent.

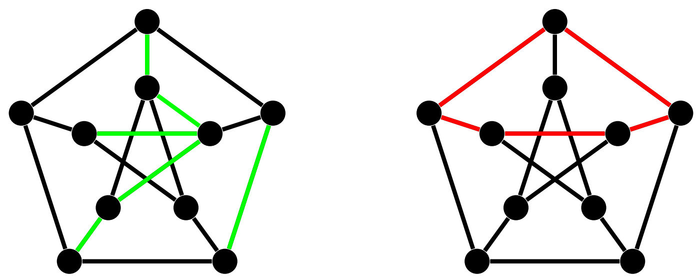
Our first property encapsulates the notion that whatever property I use to define my independent sets, it must be shared by all of the subsets (children). The second property is due to some other motivating examples of "hereditary" objects that are beyond the scope of this paper.

Example

One interesting class of matroids is the class of *graphical matroids*. A matroid is graphical if it is given by the following construction.

Take your favorite graph. We will let E, our "ground set," be the of edges of our graph. We will let the independent sets of our graph be the collections of edges that **do not** contain a cycle (i.e. a path that starts and ends at the same place without backtracking).

Below is an example of an independent set (green) and a non-independent set (red) on the Petersen graph.



So our matroid here has indepdent sets that are acyclic subgraphs (i.e. collections of edges inside of our graph that contain no cycles).

§ 3 Algebraic geometry

Algebraic geometry is a rich subject with an incredibly long history. Classical algebraic geometry seeks to understand two objects via one another.

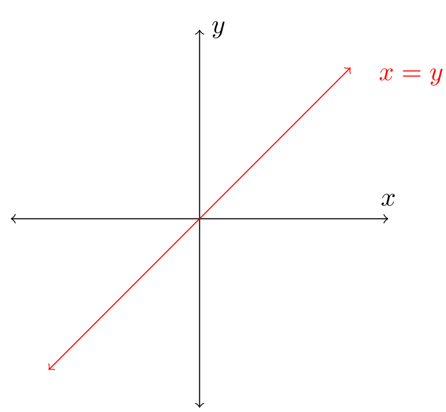
The first objects are polynomials (the "algebra"). Polynomials are functions that take in many inputs (variables) and output some number. Polynomials are defined by allowing our variables to be combined only in two ways: addition and multiplication. Some examples of polynomials are

$$\begin{aligned}
&x^2 + x + 1, \\
&x^2 - y^2 - r^2, \\
&x - 2, \\
&x^18 + y^3 - z^12 + w^34 + 10.
\end{aligned}$$

The *roots* of a polynomial are the values that, when plugged into our polynomial, give us zero. For instance, the polynomial $x - 1$ has one root, 1. This is because when I replace "x" with "1" I see that $1 - 1 = 0$.

These roots can be thought of as points in space, where the dimension of the space is the number of variables. So if I have two variables, let's say x and y, I can think of any polynomial that is made up of x and y as giving me some points in 2D space, namely the roots.

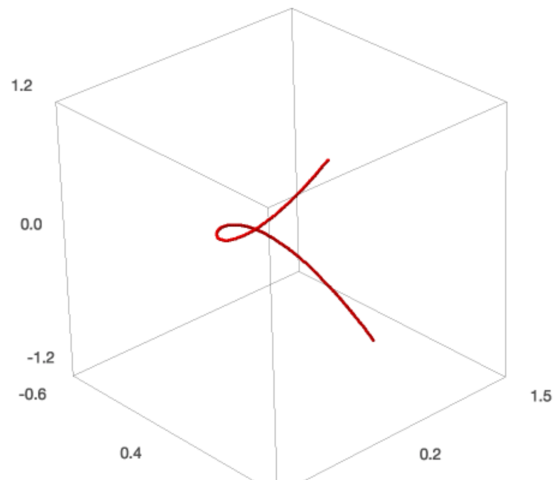
For instance, the polynomial $x - y$ is zero only when $x = y$. The roots here draw a line, which is below. Note that at any point on the line, the x and y coordinates are the same, just like the roots require.



Extending this notion to arbitrary numbers of variables and considering roots of many polynomials all together, we can draw very complicated shapes. In classical algebraic geometry a very important object is the *twisted cubic*, which is a curve drawn by the points that satisfy the three equations

$$\begin{aligned} xz - y^2 &= 0, \\ yw - z^2 &= 0, \\ xw - yz &= 0. \end{aligned}$$

The twisted cubic has four variables (x, y, z, and w), and draws a shape in 3D space (below).



Maybe at this point you can guess that the second object algebraic geometers study are shapes (the "geometry"). Many shapes can be drawn by carefully picking polynomials, and the connection between the two allows us to use one to understand the other.

In general, the shapes that are studied in algebraic geometry are continuous. Oftentimes it is the case that shapes that are important from an algebro-geometric point of view are important in other ways as well.

§ 4 Combinatorial Algebraic Geometry

The work that I enjoy studying occurs at the intersection of combinatorics and algebraic geometry (or other "theories" of geometry, but algebraic geometry is the best example I think). In particular, I study two particular ways of relating these fields.

The first way of relating these objects involves taking a continuous object and trying to reduce it to its simplest features. We want to do this in such a way that we still have *some* information about the original object, and hopefully that's enough to understand the important stuff. In particular, we hope that we can encapsulate these features in a discrete object that is easier to study.

The second way of relating these two objects is opposed to the first: Instead of taking a continuous object and trying to construct a discrete object, we will take a discrete object and try to construct a continuous one. One way of doing this (which we will demonstrate) is to try and use a matroid to divide a "big" continuous object into a bunch of smaller continuous objects.

4.1 Tropical geometry

4.2 Hyperplane arrangements