Moduli spaces - A collection of interesting examples

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Contents

Abstract

In this survey we offer an example-based introduction to moduli spaces, spaces which parameterize objects up to some notion of isomorphism. We give particular attention to the moduli space of algebraic curves. Moduli spaces are natural objects of study for algebraic and differential geometers, as well as algebraic topologists, as they give a geometric structure to isomorphism classes upon which further study may be conducted.

§ 1 Introduction

Often it is useful to classify objects, such as groups or topological spaces, up to some notion of isomorphism. Moduli spaces are spaces that contain information on how these equivalence classes are related, and how one might "continuously vary" between them.

A moduli space is a space whose points are in bijection with an equivalence class of objects, and whose open sets parameterize some notion of similarity between two equivalence classes. Here, the term space can refer to a variety or scheme as the topic arises in algebraic geometry, but it can also be a topological space or manifold.

We want the geometry of the moduli space to reflect the similarity of objects we are parameterizing, so that the points of the space are in *natural* bijection with the classes of objects. This idea can be made more precise by considering a family of objects over a space S that correspond uniquely to a map from S to the moduli space.

For example, if the objects we wish to find the moduli space of had a complete set of metric invariants, then ideally the moduli space would also have a metric, and objects with close invariants would be nearby in the moduli space. Further, if we moved invariants continuously, we would get a continuous path in the moduli space, which would be a *continuous family* of objects.

Moduli spaces arise in many contexts, and for many classes of objects moduli spaces are quite natural. Moduli spaces arise in algebraic geometry, where varieties, schemes, vector bundles, and much more have natural descriptions as moduli spaces. Further, the language of algebraic stacks yields itself rather naturally to describing and extending concepts from the theory of moduli spaces. The curious reader can find an accessible introduction to algebraic stacks in [1].

In this paper, we will spend most of our time considering specific examples of moduli spaces. The most elementary of these is the space of lines in the plane. We then follow with a discussion of the moduli space of triangles, which illustrates why moduli spaces may need even further restrictions to be made precise. A treatment of the moduli space(s) of rational and elliptic curves is also given, before concluding with an introduction to the formalism of moduli spaces and applications.

§ 2 Lines in the plane

A simple example of a moduli space is the moduli space of lines in the plane. To construct this space we seek a parameterization of lines that associates each line to a point in our space.

A very familiar parameterization of (nonvertical) lines in $\mathbb{R}^2$ comes in the form

$$y = mx + b.$$

This is a two-parameter family of lines in the plane, where each line is uniquely identified with a ordered pair (m, b) (the slope and y-intercept). That is, the moduli space of (nonvertical) lines in $\mathbb{R}^2$ is just $\mathbb{R}^2$ itself.

2.1 Approach 1

Naturally, we wish to parameterize the vertical lines in the plane as well. This will require a radically different approach (given in [4]) that we present below.

If we regard each line in $\mathbb{R}^2$ as sitting on the plane $z = 1$ in $\mathbb{R}^3$, we can uniquely identify each line with a plane in $\mathbb{R}^3$ that passes through the origin and is not parallel to the plane $z = 1$.

For each one of these planes we may write it in the form $ax + by + cz = 0$. This gives a natural identification, where we identify each plane $ax+by+cz = 0$ with the point $(a, b, c) \in \mathbb{R}^3$.

Now note that each of these points can be identified with the unique line in $\mathbb{R}^3$ containing (a, b, c) and $(0, 0, 0)$, which itself can be identified with a point in $\mathbb{RP}^2$, the space of lines in $\mathbb{R}^3$ through the origin. These identifications allow us to draw one long, convoluted string between lines in the plane and points in $\mathbb{RP}^2$, the moduli space of lines in $\mathbb{R}^3$.

But recall that each plane that is *not parallel* to $z = 1$ can be identified with a line in $\mathbb{R}^2$. Therefore the plane identified with $(0, 0, 1)$ cannot be in our moduli space, and so the moduli space of lines in $\mathbb{R}^2$ is exactly $\mathbb{RP}^2$ minus the point $[0 : 0 : 1]$.

To recapitulate, we have constructed the following "chain of identifications"

$$\begin{aligned}\{\text{lines in } \mathbb{R}^2\} &\leftrightarrow \{\text{planes through origin in } \mathbb{R}^3 - \text{plane } z = 0\} \\ &\leftrightarrow \{\text{lines through origin in } \mathbb{R}^3 - \text{line through } (0, 0, 1)\} \\ &\leftrightarrow \{\mathbb{RP}^2 - \text{line given by } (0, 0, 1)\},\end{aligned}$$

which allows us to identify each line in $\mathbb{R}^2$ with a point in $\mathbb{RP}^2 \setminus \{[0 : 0 : 1]\}$, the moduli space of lines in $\mathbb{R}^2$. This space is the open Möbius strip.

2.2 Approach 2

Another way in which we can describe a line in $\mathbb{R}^2$ is by the angle $\theta \in [0,\pi)$ between the line and the x-axis and its distance $r \in \mathbb{R}$ to the origin along its perpendicular that goes through the origin, where each line has a notion of a vector perpendicular to it indicating positive distance. Then we have the parameter space $[0,\pi) \times \mathbb{R}$. The problem with this space is that lines that are described by an angle that is close to π are "almost" a line described by $\theta = 0$. We want the topology of this moduli space to reflect that by having these points be close together. One way to fix this is to consider the angle as defined by the space $[0,\pi]$ and then to identify $(0,r) \sim (\pi,-r)$. (Exercise: why is r identified with $-r$ here?). Again we get the open Möbius strip $([0,\pi]\times\mathbb{R})/\sim$ and points parameterizing vertical and almost vertical lines in this space are nearby.

2.3 Tangent lines

Now we will introduce the notion of a continuous family of objects in a moduli space. Consider all the lines tangent to the unit circle, so we have a continuous family $\mathcal{F}$ of lines which is parameterized by S^1. Then we can build a map from S^1 to the moduli space $M = ([0,\pi]\times\mathbb{R})/\sim$ where each point in S^1 maps to the point in the moduli space corresponding to that line. This map would be $f : S^1 \to M$ where

$$f(\theta) = \begin{cases} (\theta + \frac{\pi}{2}, -1) & \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ (\theta - \frac{\pi}{2}, 1) & \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}] \end{cases}.$$

Then the line given by $x \in S^1$ is the same as the line given by $f(x) \in M$, and so this corresponds to the line $\mathcal{F}_x$. We say that we have $\mathcal{F}$ over S^1, or $\mathcal{F}/S^1$. In general, every continuous family $\mathcal{F}/T$, where T is a topological space, gives rise to a continuous map $f : T \to M$, called the moduli map of $\mathcal{F}$.

Going in the other direction, say we have a space $[0,1]$ and a map from this space to the moduli space, lets say $\gamma : [0,1] \to M$ with

$$\gamma(t) = \begin{cases} (2\pi t, 1) & t \in [0, \frac{1}{2}] \\ (2\pi(t - \frac{1}{2}), -1) & t \in [\frac{1}{2}, 1] \end{cases}.$$

From this map we can construct a family $\mathcal{F}$ of lines over the space $[0,1]$ and we can call this the family of lines parameterized by $[0,1]$ by pullback. The line given by $t \in [0,1]$ is the one given by $\gamma(t) \in M$. Via the path γ we pull back the family $\mathcal{F}/M$ to obtain a family parameterized by $[0,1]$ denoted by $\gamma^*\mathcal{F}$ which is defined such that $(\gamma^*\mathcal{F})_t = \mathcal{F}_{\gamma(t)}$. For this example the family $\gamma^*\mathcal{F}/[0,1]$ is all lines of distance 1, so all lines tangent to the unit circle.

§ 3 Triangles

Another simple example is the moduli space of triangles. This example is worth looking into because it illustrates how the symmetries (or automorphisms) of the object can determine the type of moduli space, and how problems arise with having unique continuous families of objects when we have these symmetries.

We consider the moduli space of triangles up to similarity. Since scaling yields similar triangles, we can scale all these triangles to have perimeter 2, and equivalently consider the moduli space of triangles with perimeter 2 up to congruence. Since any triangle of a given perimeter is determined by the length of its sides, we can label the sides a, b, c with $a \leq b \leq c$, and letting M denote the moduli space we have

$$M = \{(a,b,c) \in \mathbb{R}^3_+ \mid a \leq b \leq c,\ c < a + b,\ a + b + c = 2\}.$$

This space is itself a right triangle in $\mathbb{R}^3$, where the edge corresponding to degenerate triangles (line segments) is not included. The hypotenuse of this triangle corresponds to isosceles triangles with short bases, and the other edge corresponds to isosceles triangles with wide bases. The vertex where these edges meet represents the equilateral triangle.

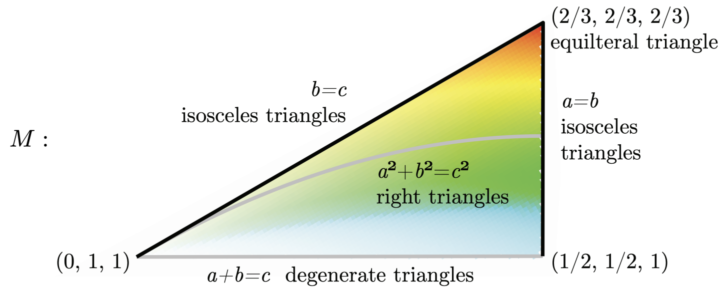

Figure 1: Moduli space of Triangles. Adapted from [1].

Letting T be a topological space, one might be interested in the family of triangles parameterized in T by the map $f : T \to M$. One would hope that this family is determined uniquely by the choice of map, as then we would have a well defined mapping between families and spaces. This is not the case, however, as there exist non-isomorphic triangle families that yield the same map $T \to M$.

An example of this is the constant map $f : T \to M : t \mapsto (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ that maps every point in the topological space to the point representing an equilateral triangle. To see the danger in this construction, form $T = S^1$ by gluing together the ends of the interval $[0, 1]$. Then the given map can parameterize multiple families. One possible family is given by gluing the two triangles at the endpoints by matching the labeled sides, while yet another is given by gluing these triangles after having been rotated sixty degrees.

So because here are six symmetries in this triangle, there are six families corresponding to the map f.

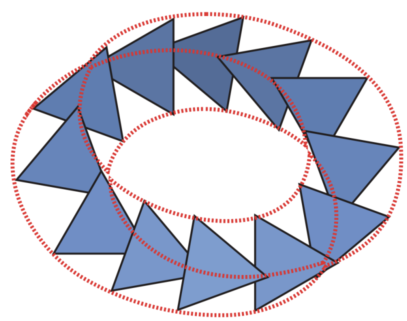

Figure 2: Family of triangles parameterized by S^1 obtained by gluing the triangles at the endpoints with a sixty degree rotation. Adapted from [1].

The fact that multiple families may correspond to the same map means that we are in fact not parameterizing families up to isomorphism, but up to some looser condition. When constructing moduli spaces this is important to keep in mind.

§ 4 Rational curves and the Riemann sphere

The examples we have shown thus far have been "toy" examples used to illustrate the essence of what a moduli space *does*.

In what follows we will construct the moduli space of genus zero (rational) curves, following the example of [9]. Unless otherwise specified we are always speaking of smooth complex algebraic curves.

Our discussion of curves will be largely informal, as the notions of genus are beyond the scope of this paper. To this end, the reader is not assumed to have any background in this area, but they may find [5] interesting.

The central object of study in this section is the *complex rational curve*. As noted, these curves are (in some sense) of genus 0 and therefore are all equivalent (in some sense) to $\mathbb{CP}^1$, the Riemann sphere. In this paper it is sufficient to view this equivalence as the ability to "identify" any rational curve with $\mathbb{CP}^1$.

Here we are concerned not just with curves themselves, but also with some number of distinct specified points on the curve. We will call these points *marked points* and denote a curve C with n marked points $x_1, \ldots, x_n$ by $(C, x_1, \ldots, x_n)$. Now we may give the set-theoretic definition of the moduli space(s) of algebraic curves.

Definition 4.1. *For fixed numbers $g, n \geq 0$, with $2g - 2 + n < 0$, the* moduli space of smooth complex algebraic curves of genus g with n marked points *is denoted by $\mathcal{M}_{g,n}$. For each configuration of n marked points $x_1, \ldots, x_n$, we have that*

1. *No two points can coincide so $x_i \neq x_j$ for $i \neq j$.*
2. *Two configurations of points $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ if there exists an automorphism ϕ of the curve such that $\phi(x_i) = x'_i$.*

This definition implies that the points of $\mathcal{M}_{g,n}$ correspond to isomorphism classes of these curves.

As discussed, the purpose of a moduli space is to give some characterization of solutions to classification problems. Naturally, the next question we must answer in the pursuit of a moduli space of rational curves is, "how do we classify rational curves?"

The answer lies in the automorphism group, $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{kI \mid k \in \mathbb{C}\}$.

Proposition 4.2. *The automorphism group of $\mathbb{CP}^1$ is $\mathrm{PSL}(2, \mathbb{C})$, acting by the Möbius transformation*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}.$$

Now we wish to note an important fact: For any three distinct points $x_1, x_2, x_3 \in \mathbb{CP}^1$ there exists a unique element of $\mathrm{PSL}(2, \mathbb{C})$ taking x_1 to 0, x_2 to 1, and x_3 to ∞. This identification is given by the linear system

$$b = -ax_1, \quad a(x_2 - x_1) = c(x_2 - x_3), \quad d = -cx_3.$$

This system has a unique solution across all of $\mathrm{PSL}(2, \mathbb{C})$. Therefore we have our first result in the study of moduli spaces

Proposition 4.3. *The moduli space of rational curves with three marked points is a point:*

$$\mathcal{M}_{0,3} \cong \left\{(\mathbb{CP}^1, 0, 1, \infty)\right\}.$$

That is, our unique identification of *any* rational curve (C, x_1, x_2, x_3) with $(\mathbb{CP}^1, 0, 1, \infty)$ yields a single isomorphism class in the moduli space. Naturally, we wish to answer this question for $n > 3$ as well.

In the case $n = 4$, we have curves (C, x_1, x_2, x_3, x_4). We may identify any three of these points with 0, 1, and ∞ as before, but the fourth point remains as a free variable. This implies that

Proposition 4.4.

$$\mathcal{M}_{0,4} \cong \mathbb{CP}^1 \setminus \{0, 1, \infty\}.$$

as once we have fixed three points by 0, 1, and ∞, the fourth point is allowed to take on any distinct value in $\mathbb{CP}^1$.

Perhaps at this point one might guess that the trend will continue. Indeed,

Proposition 4.5.

$$\mathcal{M}_{0,n} \cong \{(x_1, \ldots, x_{n-3} \in \left(\mathbb{CP}^1\right)^{n-3} \mid \ x_i \notin \{0, 1, \infty\} \text{ and } x_i \neq x_j \text{ for any } i \neq j\}.$$

Proof. Let $(C, x_1, \ldots, x_n)$ be a rational curve. Then without loss of generality we have an identification $(C, x_1, \ldots, x_n) \sim (\mathbb{CP}^1, 0, 1, \infty, y_1, \ldots, y_{n-3})$ given by elements of $\mathrm{PSL}(2, \mathbb{C})$ as above.

Note, however, that all additional marked points are free variables in the linear system determined by the identifications $x_1 \sim 0$, $x_2 \sim 1$, $x_3 \sim \infty$. Therefore there is precisely one isomorphism class of curves for each choice of our $n-3$ parameters, yielding a moduli space isomorphic to all possible parameter choices:

$$\{(x_1, \ldots, x_{n-3} \in \mathbb{CP}^{n-3} \mid \ x_i \notin \{0, 1, \infty\} \text{ and } x_i \neq x_j \text{ for any } i \neq j\}.$$

□

§ 5 Elliptic curves and lattices

An elliptic curve over the field K is a non-singular curve defined by an equation of the form

$$y^2 z = Ax^3 + Bx^2 z + Cxz^2 + Dz^3$$

with coefficients in K living in the projective plane $\mathbb{P}^2_K$.

For the purposes of this paper we will consider curves that can be simplified to the Weierstrass form of

$$y^2 = x^3 + Ax + B$$

over the field $K = \mathbb{C}$, and identify one marked point on them. More details can be found in [9].

Just as in the case of rational curves, we have a complex-analytic structure with which we identify elliptic curves – lattices.

Definition 5.1. *A lattice Λ is a discrete subgroup of $\mathbb{C}$ which contains an $\mathbb{R}$-basis for $\mathbb{C}$. If ω_1 and ω_2 are complex numbers that are linearly independent when considered as elements of $\mathbb{R}^2$, then the lattice is of the form $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and we write $\Lambda = [\omega_1, \omega_2]$.*

The space $\mathbb{C}/\Lambda$ is, topologically, a torus. We will now show that there exists a one-to-one correspondence between lattices and elliptic curves.

Definition 5.2. *The Weierstrass $\wp$-function relative to Λ is defined by the series*

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda, \omega \neq 0} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}.$$

The Eisenstein series of weight $k \in \mathbb{Z}_{>2}$ is

$$G_k(\Lambda) = \sum_{w \in \Lambda^*} w^{-k}.$$

The $\wp$-function is meromorphic on $\mathbb{C}$, meaning that is is analytic (holomorphic) everywhere except for a set of isolated points where it has poles.

We will use the $\wp$ function relative to Λ to associate with each lattice an elliptic curve. To do so, we first note that for any lattice there exists a natural elliptic curve whose variables are given by $\wp$:

An important result in complex analysis states that the Eisenstein series and $\wp$ functions are intimately related.

Proposition 5.3. *The Weierstrass $\wp$ function relative to Λ satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z)^3 - 60G_4(\Lambda)\wp(z) - 140G_6(\Lambda).$$

Note now that this differential equation identifies with each lattice an elliptic curve E_Λ of the form above, where the variables are Weierstrass $\wp$ functions. Further, the content of the theorem below proves that this identification is unique and exists for all elliptic curves.

Proposition 5.4 (Uniformization Theorem)**.** *For every elliptic curve E there exists a lattice Λ such that $E = E_\Lambda$.*

This identification provides us with the first step towards a moduli space of elliptic curves, as we have now identified elliptic curves with lattices over $\mathbb{C}$.

The next step in our identification will be to give some description of which lattices yield isomorphic elliptic curves, and how we may parameterize that space.

With the convention that the basis $[\omega_1, \omega_2]$ is oriented so that $\omega_1/\omega_2 \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, we may write that every lattice is homothetic (i.e. a scalar multiple of) some lattice of the form $\Lambda = [1, \tau]$, where $\tau \in \mathbb{H}$, the upper half plane.

Further, we have that $[\omega_1, \omega_2]$ and $[\omega_1', \omega_2']$ are two bases for Λ if and only if

$$A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} \text{ for some } A \in SL(2, \mathbb{Z}).$$

Therefore two lattices represented by $[1, \tau_1]$ and $[1, \tau_2]$ are homothetic if and only if there exists some $A \in \mathrm{SL}(2, \mathbb{Z})$ that maps τ_1 to τ_2. This allows us to conclude that the

Proposition 5.5. *The space of homothetic lattices is isomorphic to $\mathbb{H}/SL(2, \mathbb{Z})$.*

More details can be found in [8].

The last step to make in our construction now is to identify together homothetic lattices Λ_1 and Λ_2 with isomorphic elliptic curves, E_{Λ_1} and E_{Λ_2}.

Definition 5.6. *For an elliptic curve $E : y^2 = x^3 + Ax + B$, we define the discriminant and j-invariant as below*

$$\Delta(E) = -16(4A^3 + 27B^2), \quad j(E) = -1728\frac{(4A)^3}{\Delta(E)}.$$

An important note now is that two elliptic curves are isomorphic if and only if they have the same j-invariant, and further that two lattices are homothetic if and only if $E_{\Lambda_1} \cong E_{\Lambda_2}$. Now our final result follows immediately.

Proposition 5.7.

$$\mathcal{M}_{1,1} \cong \mathbb{H}/SL(2, \mathbb{Z}).$$

§ 6 Generalizations

So far in this survey we have constructed moduli spaces for explicit objects: Lines, triangles, and curves. These spaces, however, seem as though they may be useful in other contexts. A natural question is to ask whether there exists a general theory of moduli spaces.

The answer, of course, is yes. Moduli spaces can be reformulated as *stacks*, category-theoretic objects that encapsulate exactly the properties we care about. What follows is a short, high level description of the ideas behind stacks, but the reader is **strongly** encouraged to consult [1] for a proper introduction.

6.1 Stacks

What is common in all of our descriptions of moduli spaces thus far is the following:

- (Objects) A class of objects we wish to "parameterize,"
- (Underlying space) A space by which to realize said "parameterization," and
- (Moduli map) A map between these two objects.

In the above discussions of curves, we have considered maps from our isomorphism classes of curves into topological spaces. In our discussion of triangles, we noted it was important that our moduli maps also preserve the "local similarity" between isomorphism classes. In other words, we care about the images of *objects* as well as the images of *morphisms* between them.

What this reveals is that we really care about *functors* from the category of whatever objects we wish to consider into the category of topological spaces. *Stacks* are objects which generalize these notions.

A stack is, loosely, a category of objects $\mathfrak{X}$ along with a functor to the category of topological spaces $\mathfrak{X} \to \mathrm{Top}$. This functor needs to satisfy a number of conditions for us to call it a stack, but the two most important are

(i) Isomorphic objects have the same image under our functor.

(ii) If two objects have a morphism between them, their images in Top do as well.

In modern algebraic geometry we replace Top with a *sheaf* (in which case we use the term *algebraic stack*). Stacks are wondrous objects, and [1, §2.6] gives a treatment of the moduli space of elliptic curves in this language.

§ 7 Further Reading

7.1 Tropical geometry

We have thus far seen two examples of moduli space of some algebro-geometric objects (i.e. curves). Tropical geometry is a field that attempts to study algebro-geometric objects by employing a degeneration technique known as *tropicalization*.

Many strong invariants of algebro-geometric objects are preserved under tropicalization. Given some class of algebraic curves (such as those above), the moduli space of abstract tropical curves can be used to construct results concerning the moduli space of the original class. More details can be found in [3].

7.2 Enumerative geometry

Enumerative geometry seeks to determine the number of geometric objects (usually complex curves in a smooth algebraic manifold) that satisfy a given set of conditions.

A classical problem in this field is to determine the number N_d of rational curves of degree d curves that pass through $3d-1$ points in the complex plane. This was solved by Kontsevich, who considered the moduli space of *stable maps*. This moduli space has a special recursive structure that can be used to compute the number of curves that pass through $3d-1$ points. More details can be found in [6].

7.3 Geometric invariant theory

Often we wish to "quotient" by a group action on a space, perhaps to remove symmetries (an example of this is found in [9]'s discussion of $\mathcal{M}_{2,0}$). Geometric invariant theory provides a number of techniques for constructing these quotient spaces, and one of its primary motivations was to construct moduli spaces of curves.

Recalling the triangle example from §3, one might hope there is some group that might act on our moduli space to remove the "symmetries," such as those demonstrated when considering maps $S^1 \to M$. If such a group action exists, then geometric invariant theory provides the tools to construct the moduli space explicitly. More details can be found in [7].

7.4 Deformation theory

Deformation theory studies the local geometry of spaces by applying some small perturbation to conditions on the space. Studying how geometric objects change with small perturbations gives the local structure of a moduli space. In fact, a useful characterization of the dimension of moduli spaces of algebraic curves in terms of cohomology can be shown using deformation theory. More details can be found in [2].

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