LSU RTG

Hyperplane Arrangements

Talk 1: Introduction

For this talk we will live in $\mathbb{R}^n$

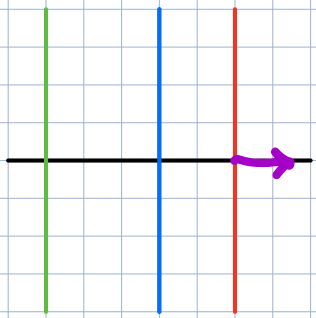
Dfn.

An $\overbrace{\underline{\text{affine hyperplane}}}^{\text{hyperplane}}$ H is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. In particular,

$$H = \{ v \in \mathbb{R}^n \mid \alpha \cdot v = c \}$$

normal of H

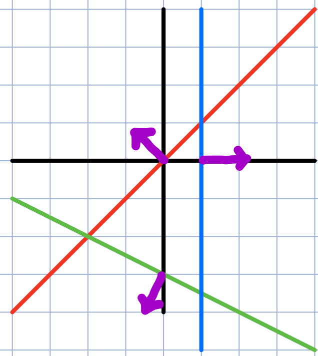
for some fixed, nonzero $\alpha \in \mathbb{R}^n$ and some $c \in \mathbb{R}$. If $c = 0$, we say that H is a <u>linear hyperplane</u> (note that lin. h.p. are also affine).



$H_1: (1,0)\cdot(x,y) = -3$

$H_2: (1,0)\cdot(x,y) = 0$

$H_3: (1,0)\cdot(x,y) = 2$



$H_1: (1,2)\cdot(x,y) = -3$

$H_2: (1,0)\cdot(x,y) = 1$

$H_3: (1,-1)\cdot(x,y) = 0$

B1

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normal of H

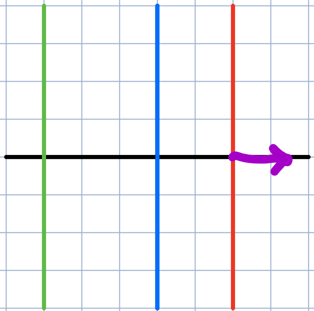
for some fixed, nonzero $\alpha \in \mathbb{R}^n$ and some $\overset{\text{nonzero}}{c} \in \mathbb{R}$. If $c = 0$, we say that H is a linear hyperplane (note that lin. h.p. are also affine).

A collection of finitely many hyperplanes $\mathcal{A} = \{H_i\}_i$ is called a hyperplane arrangement.

The dimension of $\mathcal{A}$, $\dim(\mathcal{A}) = n$.

The rank of $\mathcal{A}$, $\operatorname{rank}(\mathcal{A}) = \dim(\operatorname{span}\{\alpha_1, \dots, \alpha_k\})$

If $\operatorname{rank}(\mathcal{A}) = \dim(\mathcal{A})$, $\mathcal{A}$ is essential.



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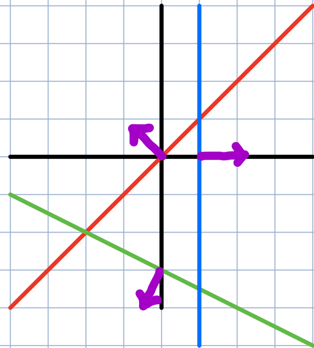
$H_3: (1,0)\cdot(x,y) = 2$

$\mathcal{A}_1$

$\dim(\mathcal{A}_1) = 2$, $\operatorname{rank}(\mathcal{A}) = 1$



$H_1: (1,2)\cdot(x,y) = -3$

$H_2: (1,0)\cdot(x,y) = 1$

$H_3: (1,1)\cdot(x,y) = 0$

$\mathcal{A}_2$

$\dim(\mathcal{A}_2) = 2$, $\operatorname{rank}(\mathcal{A}_2) = 2$,

B1

For this talk we will live in $\mathbb{R}^n$

Dfn.

An $\overbrace{\text{\underline{affine hyperplane}}}^{\text{hyperplane}}$ H is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. In particular,

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normal of H

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The <u>dimension</u> of $\mathcal{A}$, $\dim(\mathcal{A}) = n$.

The <u>rank</u> of $\mathcal{A}$, $\operatorname{rank}(\mathcal{A}) = \dim(\operatorname{span}\{\alpha_1, \dots, \alpha_k\})$

If $\operatorname{rank}(\mathcal{A}) = \dim(\mathcal{A})$, $\mathcal{A}$ is <u>essential.</u>

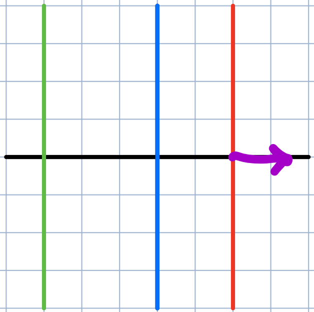
Let $\mathcal{R}(\mathcal{A})$ be the set of connected components of

$$X = \mathbb{R}^n \setminus \bigcup_{H \in \mathcal{A}} H$$

set of regions

and $r(\mathcal{A}) = |\mathcal{R}(\mathcal{A})|$.

B1



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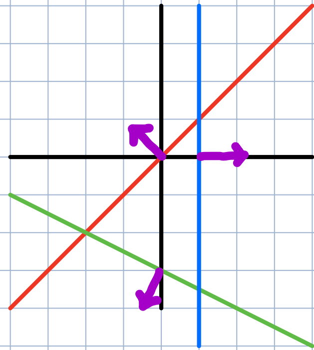
$H_3 : (1,0)\cdot(x,y) = 2$

$\mathcal{A}_1$

$\dim(\mathcal{A}_1) = 2$, $\operatorname{rank}(\mathcal{A}) = 1$

$r(\mathcal{A}_1) = 4.$



$H_1 : (1,2)\cdot(x,y) = -3$

$H_2 : (1,0)\cdot(x,y) = 1$

$H_3 : (1,1)\cdot(x,y) = 0$

$\mathcal{A}_2$

$\dim(\mathcal{A}_2) = 2$, $\operatorname{rank}(\mathcal{A}_2) = 2$,

$r(\mathcal{A}_2) = 7$

B1

Q: How to "make" an arrangement essential?

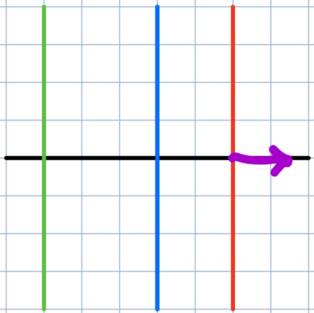
A: Essentialization.

① Let $\mathcal{A}$ be a h.p. arr. in $\mathbb{R}^n$ w/ h.p.s $H_1, \ldots, H_k$.

② Form $W = \text{span}\{\alpha_1, \ldots, \alpha_k\}$ for α_i normal of H_i.

③ Let $\text{ess}(\mathcal{A}) = \{ H_1 \cap W, \ldots, H_k \cap W \}$

Now $\text{ess}(\mathcal{A})$ is an essential hyperplane arrangement!



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$\mathcal{A}_1$

$\dim(\mathcal{A}_1) = 2$, $\operatorname{rank}(\mathcal{A}) = 1$

$r(\mathcal{A}) = 4$

$X =$ x- axis $\quad \downarrow$ ess

$H_1 : x = 3$

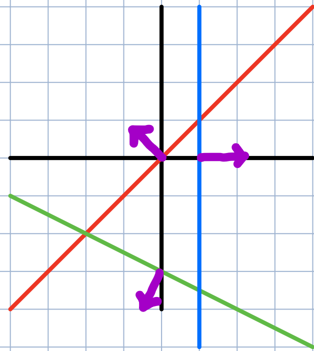
$H_2 : x = 0$

$H_3 : x = 2$

$\dim(\operatorname{ess}(\mathcal{A}_1)) = 1$,

$\operatorname{rank}(\operatorname{ess}(\mathcal{A}_1)) = 1$,

$r(\mathcal{A}) = 4$



$H_1 : (1,2)\cdot(x,y) = -3$

$H_2 : (1,0)\cdot(x,y) = 1$

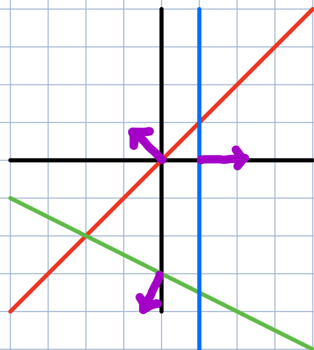
$H_3 : (1,1)\cdot(x,y) = 0$

$\mathcal{A}_2$

$\dim(\mathcal{A}_2) = 2$, $\operatorname{rank}(\mathcal{A}_2) = 2$,

$r(\mathcal{A}_2) = 7$

$X = \mathbb{R}^2 \quad \downarrow$ ess



B1

LSU RTG

Hyperplane Arrangements

Talk 7: Generic Translations and Face Counting

Defn.

For $\mathcal{A}$ an arrangement of M hyperplanes given by

$$H_i = \{ v \in V \mid \alpha_i \cdot v = c_i \},$$

we say $c_1, \dots, c_M$ are <u>generic</u> if

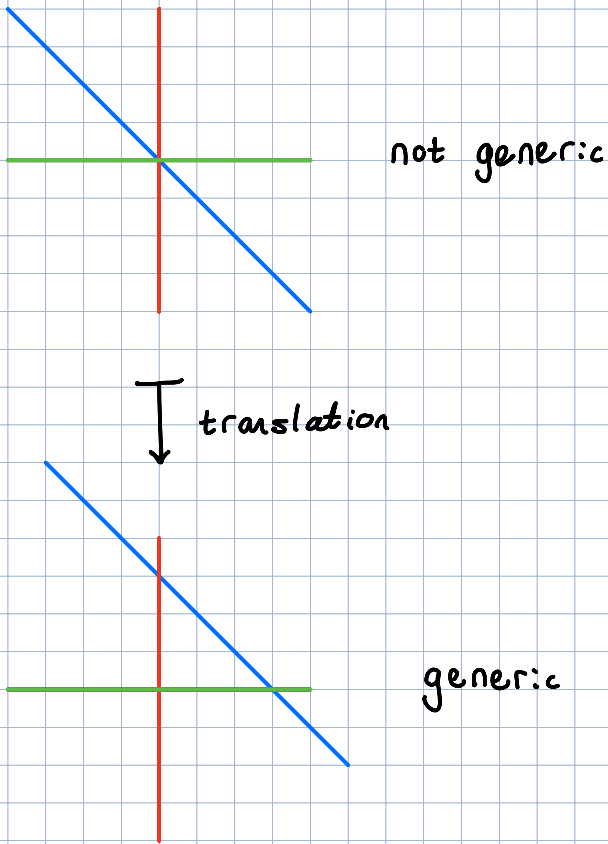
$$\alpha_{i_1}, \dots, \alpha_{i_k} \text{ are linearly independent iff } H_{i_1} \cap \cdots \cap H_{i_k} \neq \emptyset.$$

This is a description of our "c_i"s, of which there are many choices for an arbitrary set of "α_i"s.

Recall that if $c_i \neq 0$, H_i is a translate of some linear hyperplane, so to fulfill this condition is to have translated some hyperplanes in a linear arrangement.

This allows us to speak of "generic arrangements"

An arrangement $\mathcal{A} = \{H_1, \dots, H_M\}$ with H_i as above is <u>generic</u> if $c_1, \dots, c_M$ are generic.



B2

Def.

For $\mathcal{A}$ an arrangement of M hyperplanes given by

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we say $c_1, \dots, c_M$ are <u>generic</u> if

$\alpha_1, \dots, \alpha_K$ are linearly independent iff $H_1 \cap \cdots \cap H_K \neq \emptyset$.

This is a description of our "c_i"s, of which there are many choices for an arbitrary set of "α_i"s.

Recall that if $c_i \neq 0$, H_i is a translate of some linear hyperplane, so to fulfill this condition is to have translated some hyperplanes in a linear arrangement.

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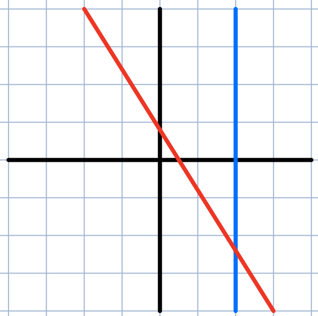
An arrangement $\mathcal{A} = \{ H_1, \dots, H_M \}$ with H_i as above is <u>generic</u> if $c_1, \dots, c_M$ are generic.

Now some implications

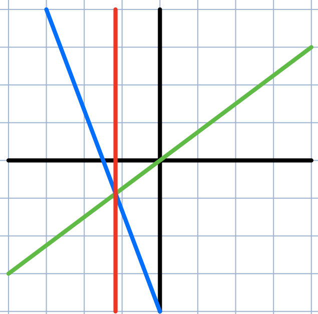
$\mathcal{A}$ is central and $m \leq n$

$\Rightarrow$ $\mathcal{A}$ is generic.



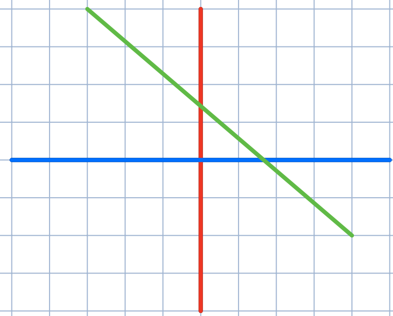
$\mathcal{A}$ is central and $m > n$

$\Rightarrow$ $\mathcal{A}$ is not generic.



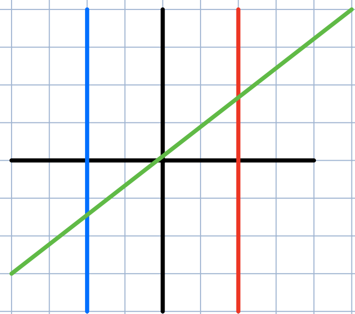
$\mathcal{A}$ is in gen. pos.

$\Rightarrow$ $\mathcal{A}$ is generic.



both

So generic is a weaker (but analogous) version of gen. pos.



not in gen. pos.

In particular, generic $\Rightarrow$ $\dim(H_1 \cap \cdots \cap H_k)$ is zero or $n-k$.

B2

Defn.

For $\mathcal{A}$ an arrangement of M hyperplanes given by

$$H_i = \{ v \in V \mid \alpha_i \cdot v = c_i \},$$

we say $c_1, \ldots, c_M$ are <u>generic</u> if

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This allows us to speak of "generic arrangements"

An arrangement $\mathcal{A} = \{H_1, \ldots, H_M\}$ with H_i as above is <u>generic</u> if $c_1, \ldots, c_M$ are generic.

Fact: If $\mathcal{A}$ is generic, $[\hat{0}, x] \cong B_K$, the boolean algebra of rank $K = \dim(x)$.

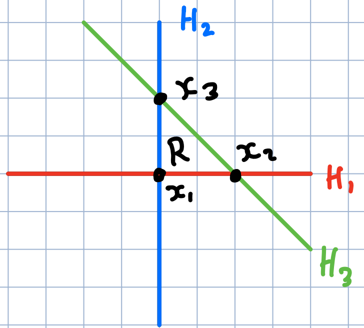
b.c. $x = H_1 \cap \cdots \cap H_K$ all nonparallel $\Rightarrow H_1, \ldots, H_K$ in gen pos.

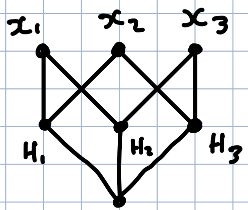
$$\Rightarrow \chi_{\mathcal{A}}(t) = \sum_{\substack{\mathcal{B} \\ \text{central}}} (-1)^{|\mathcal{B}|} t^{n-|\mathcal{B}|}$$

Dfn.

The closure of a region $R \in \mathcal{R}(\mathcal{A})$, denoted $\bar{R}$, is the union of R and the portions of the hyperplanes forming its boundary.

A <u>k-face</u> of $\mathcal{A}$ is a set $F = X \cap \bar{R}$ for some $R \in \mathcal{R}(\mathcal{A})$ and $X \in L(\mathcal{A})$ s.t. $\dim(\operatorname{span}(F)) = k$.





$R =$, $\quad \bar{R} = \triangle$

$\mathbb{R}^2 \cap \bar{R} = \bar{R}$ a 2d face

$H_1 \cap \bar{R} =$ a 1d face

$H_2 \cap \bar{R} =$ . . . .

$H_3 \cap \bar{R} =$ . . . .

$x_1 \cap \bar{R} = x_1$ a 0d face

$x_2 \cap \bar{R} = x_2$ . . . . . .

$x_3 \cap \bar{R} = x_3$ . . . .

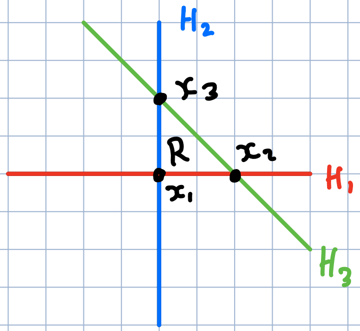
B2

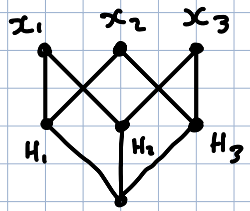
Dfn.

The closure of a region $R \in \mathcal{R}(\mathcal{A})$, denoted $\bar{R}$, is the union of R and the portions of the hyperplanes forming its boundary.

A <u>k-face</u> of $\mathcal{A}$ is a set $F = X \cap \bar{R}$ for some $R \in \mathcal{R}(\mathcal{A})$ and $X \in L(\mathcal{A})$ s.t. $\dim(\mathrm{span}(F)) = k$.

Fact: Every k-face of $\mathcal{A}$ corresponds uniquely to a region in $\mathcal{A}^X$ for some $X \in L(\mathcal{A})$ satisfying $\dim(X) = k$.





$R =$, $\overline{R} =$

$\mathbb{R}^2 \cap \overline{R} = \overline{R}$ a 2-face

$H_1 \cap \overline{R} =$ a 1-face

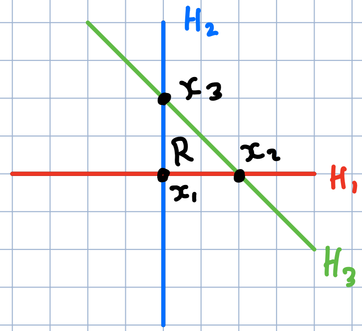
$H_2 \cap \overline{R} =$ . . . .

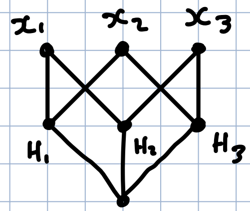
$H_3 \cap \overline{R} =$ . . . .

$x_1 \cap \overline{R} = x_1$ a 0-face

$x_2 \cap \overline{R} = x_2$ . . . . . .

$x_3 \cap \overline{R} = x_3$ . . . .





Note that 2-faces are just regions in $\mathcal{A}^{\mathbb{R}^2}$.

1-faces are regions in H_i

0-faces are regions in x_i

B2

Dfn.

The closure of a region $R \in \mathcal{R}(\mathcal{A})$, denoted $\bar{R}$, is the union of R and the portions of the hyperplanes forming its boundary.

A <u>k-face</u> of $\mathcal{A}$ is a set $F = x \cap \bar{R}$ for some $R \in \mathcal{R}(\mathcal{A})$ and $x \in L(\mathcal{A})$ s.t. $\dim(\operatorname{span}(F)) = k$.

<u>Fact:</u> Every k-face of $\mathcal{A}$ corresponds uniquely to a region in $\mathcal{A}^x$ for some $x \in L(\mathcal{A})$ satisfying $\dim(x) = k$.

Thm. 2.6.

Let $f_k(\mathcal{A}) = \#$ k-faces of $\mathcal{A}$. Then

$$f_k(\mathcal{A}) = \sum_{\dim(x)=k} \sum_{x \le y} (-1)^{\dim(x) - \dim(y)} \mu(x,y).$$

Dfn.

The closure of a region $R \in \mathcal{R}(\mathcal{A})$, denoted $\bar{R}$, is the union of R and the portions of the hyperplanes forming its boundary.

A <u>k-face</u> of $\mathcal{A}$ is a set $F = x \cap \bar{R}$ for some $R \in \mathcal{R}(\mathcal{A})$ and $x \in L(\mathcal{A})$ s.t. $\dim(\mathrm{span}(F)) = k$.

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$$f_k(\mathcal{A}) = \sum_{\dim(x)=k} \sum_{x \le y} (-1)^{\dim(x)-\dim(y)} \mu(x,y).$$

pf. By our Fact,

$$\#\ k\text{-faces} = \sum_{\dim(x)=k} r(\mathcal{A}^x) = \sum_{\dim(x)=k} \sum_{x \le y} (-1)^{\dim(x)-\dim(y)} \mu(x,y). \quad \triangle$$

LSU RTG

Hyperplane Arrangements

Talk 12: Geometric Lattices and Matroids Corresponding to Arrangements

Notation: I will write $\overline{X}$ for the closure of $X \subseteq S$, i.e.

$$\overline{X} := \{ y \in M \mid rk(X \cup y) = rk(X) \}$$

Writing $x \in M$ is equiv. to $x \in S$ for $M = (S, \mathcal{I})$.

__Recall.__ A matroid $M = (S, \mathcal{I})$ is __simple__ if it is __loopless__ and has __no parallel points.__

No rank zero singletons

$rk(x) > 0 \quad \forall x \in M$

No rank one 2-subsets

$rk(\{x, y\}) > 1 \quad \forall \{x, y\} \in 2^S$

> Q: Can we have $\overline{X}$ contain enough ind. ys s.t. $rk(\overline{X}) > rk(X)$?
>
> A: No. Supp. B a basis of X. Then $\forall x \in \overline{X} \setminus X$ we have
>
> $$rk(B \cup x) \leq rk(X \cup x) = rk(X) = |B| = r(B) \leq r(B \cup x)$$
>
> So now $r(B \cup x) = r(B) \Rightarrow B \cup x$ is dependent $\Rightarrow B$ a basis for $\overline{X}$.

__Note.__ Setting $\widehat{M} = \{ \overline{x} \mid x \in M,\ x \notin \overline{\emptyset} \}$ (not a loop), we may construct a matroid $\widehat{M}$ s.t.

$$\{ \overline{x}_1, \ldots, \overline{x}_k \} \in \mathcal{I}(\widehat{M}) \text{ iff } \{ x_1, \ldots, x_k \} \in \mathcal{I}(M).$$

So $\widehat{M}$ is simple and $L(M) \cong L(\widehat{M})$

This means that, as far as lattices go, we only care about simple matroids.

M given by
$$\begin{array}{c}\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array}\end{array}$$

$$\mathcal{I}(M) = \{\emptyset, v_1, v_2, v_3, v_4, v_5, v_{13}, v_{14}, v_{15}, v_{23}, v_{24}, v_{25}, v_{34}, v_{35}, v_{45}, v_{134}, v_{135}, v_{145}, v_{234}, v_{235}, v_{345}\}$$

$$\varphi: v_1 \sim v_2 \longmapsto v_1 \quad v_3 \longmapsto v_3, \ v_4 \longmapsto v_4, \ v_5 \longmapsto v_5$$

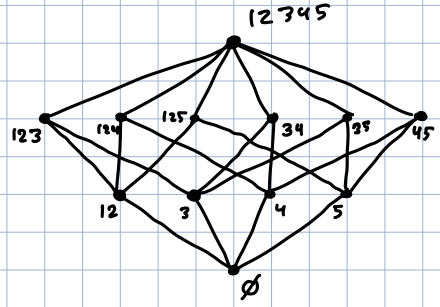
$\Longrightarrow \widehat{M} = \{v_1, v_3, v_4, v_5\}$ with

$$\mathcal{I}(\widehat{M}) = \{\emptyset, v_1, v_3, v_4, v_5, v_{13}, v_{14}, v_{15}, v_{34}, v_{35}, v_{45}, v_{134}, v_{135}, v_{145}, v_{345}\}$$

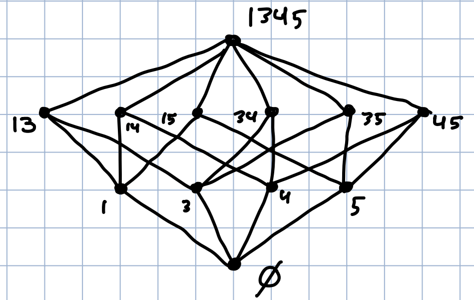
Note that $\widehat{M} \ncong M$!

"I know that if $rk(\{x,y\}) = 1$, x and y are always in flats together, so just make them one guy"

$L(M)$



$L(\widehat{M})$



Notation: I will write $\overline{X}$ for the closure of $X \subseteq S$, i.e.

$$\overline{X} := \{ y \in M \mid \mathrm{rk}(X \cup y) = \mathrm{rk}(X) \}$$

Recall. A matroid $M = (S, \mathcal{I})$ is simple if it is loopless and has no parallel points.

No rank zero singletons $\quad$ No rank one 2-subsets

$\mathrm{rk}(x) > 0 \ \forall x \in M$ $\qquad$ $\mathrm{rk}(\{x,y\}) > 1 \ \forall \{x,y\} \in 2^S$

Note. Setting $\widehat{M} = \{ \overline{x} \mid x \in M, \ \mathrm{rk}(x) > 0 \}$, we may construct an isomorphism

$$\{ \overline{x}_1, \dots, \overline{x}_k \} \in \mathcal{I}(\widehat{M}) \text{ iff } \{x_1, \dots, x_k\} \in \mathcal{I}(M).$$

So $\widehat{M}$ is simple and $L(M) \cong L(\widehat{M})$

This means that, as far as lattices go, we only care about simple matroids.

Fact. M being simple $\Longrightarrow$ $L(M)$ uniquely determines it.

How? Identify $(x_1, \dots, x_k) \in \mathcal{I} \Longleftrightarrow \mathrm{rk}(x_1 \vee \cdots \vee x_k) = k$ in $L(M)$.

Simple $\Longrightarrow$ rank in $L(M)$ works exactly like rank in M.

Prop. 3.6

Maybe clarify this?

If $\mathcal{A}$ is central, then the linear Matroid whose ground set is given by the set of normals in $\mathcal{A}$ is simple and satisfies

$$L(M) \cong L(\mathcal{A}).$$

pf. (sketch) For simplicity note that $\mathcal{A}$ central $\Longrightarrow$ no parallel hyperplanes, and no loops b.c. nonzero normals.

For lattice isomorphism, we have a bijection given by

$$X = \bigcap_{H \in \mathcal{B}} H \longmapsto \{\alpha_H \mid \underbrace{\alpha_H \in \operatorname{span}\{\alpha_H \mid H \in \mathcal{B}\}}_{\text{normal is in span of normals of } \mathcal{B}}\} = \overline{\mathcal{B}}$$

and you may check that this preserves order.

B2

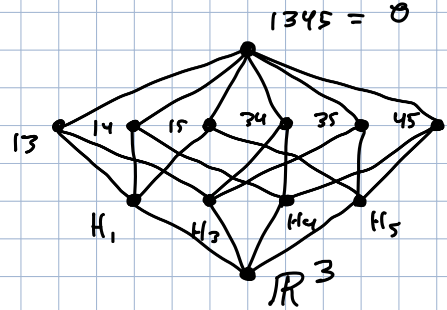
M given by $\begin{array}{ccccc} \left[\begin{array}{ccccc} 1 & \blacksquare & 0 & 0 & 1 \\ 0 & \blacksquare & 1 & 0 & 1 \\ 0 & \blacksquare & 0 & 1 & 1 \end{array}\right] \\ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \end{array}$

$\mathcal{A} = \{ H_i \mid H_i = \{ v \in \mathbb{R}^3 \mid v_i \cdot v = 0 \} \}$

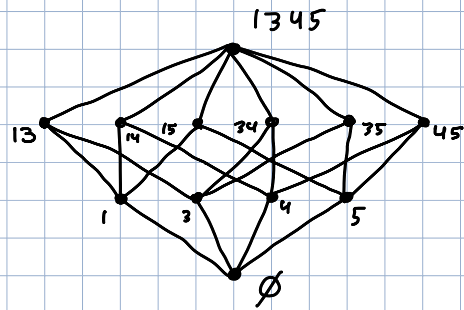
$\Rightarrow M = \{ v_1, v_3, v_4, v_5 \}$ with

$\mathcal{I}(M) = \{ \emptyset, v_1, v_3, v_4, v_5, v_{13}, v_{14}, v_{15}, v_{34}, v_{35}, v_{45}, v_{134}, v_{135}, v_{145}, v_{345} \}$

$L(\mathcal{A})$



$L(M)$



B3

Prop. 3.6.

Maybe clarify this?

If $\mathcal{A}$ is central, then the linear Matroid whose ground set is given by the set of normals in $\mathcal{A}$ is simple and satisfies

$$L(M) \cong L(\mathcal{A}).$$

pf. (sketch) For simplicity note that $\mathcal{A}$ central $\Longrightarrow$ no parallel hyperplanes, and no loops b.c. nonzero normals.

For lattice isomorphism, we have a bijection given by

$$X = \bigcap_{H \in \mathcal{B}} H \longmapsto \{\alpha_H \mid \underbrace{\alpha_H \in \operatorname{span}\{\alpha_H \mid H \in \mathcal{B}\}}_{\text{normal is in span of normals of } \mathcal{B}}\}$$

and you may check that this preserves order.

It may be worth noting that ind^t sets are ind^t subsets of normals, and what is here are flats.

Dfn.

A finite lattice L satisfying

$$rk(x) + rk(y) \geq rk(x \vee y) + rk(x \wedge y)$$

is said to be semimodular. A finite lattice L is atomic if every $x \in L$ is the join of some set of atoms.

A finite lattice that is both semimodular and atomic is geometric.

Thm. 3.8.

For L a finite lattice the following are equivalent,

1) L is geometric

2) $L \cong L(M)$ for some (unique, simple) matroid M.

pf. (sketch) $\Rightarrow$) Let A be the set of atoms and

$$\mathcal{I} = \{ T \subseteq A \mid rk(\vee T) = \# T \}.$$

Semimodularity of L yields inclusion property b.c. for any $T \subseteq A$ and $x \in A$,

$rk((\vee T) \vee x) \leq rk(\vee T) + 1.$ i.e. if $T \cup x \in \mathcal{I} \Rightarrow rk((\vee T) \vee x) = \# T + 1$
$\Rightarrow rk(\vee T) = \# T$ b.c. $(\vee T) \vee x$ covers $\vee T$ by P3.7.

For our property concerning sizes of maximal ind^t sets, an argument by contradiction follows b.c. L is atomic and so

$\Leftarrow$) Atomic because each flat is the join of some singletons (atoms).

Semimodularity follows for flats straightforwardly and

Now we have that results for simple matroids and geometric lattices can be applied to our study of central arrangements.

LSU RTG

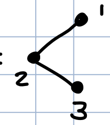
Hyperplane Arrangements

Capstone Part V.:

Graphic Arrangements

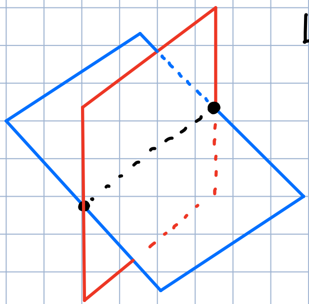
Graphic Arrangements: For G a graph with $V=\{1,\dots,n\}$ we define

$$\mathcal{A}_G = \{ H_{ij} \mid ij \in E \}, \quad H_{ij} = \{(v_1,\dots,v_n) \in \mathbb{R}^n \mid v_i = v_j\}.$$

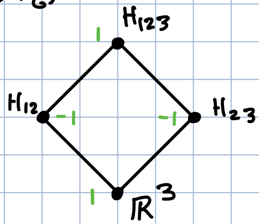
$G =$ 

$H_{12}: x_1 = x_2$

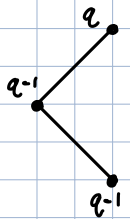
$H_{23}: x_2 = x_3$



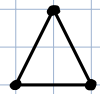
$L(\mathcal{A}_G)$:

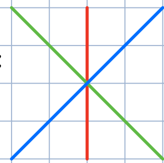


$$\chi_{\mathcal{A}_G}(t) = t^3 - 2t^2 + t$$

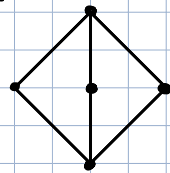


$$\chi_G(q) = q(q-1)^2$$
$$= q^3 - 2q^2 + q$$

$G =$ 

$\mathcal{A}_G \approx$  $= \mathcal{A}_2$

$L(\mathcal{A}_2)$



Now recall:

$$\chi_{\mathcal{A}_2}(t) = t^2 - 3t + 2$$

but we claim that

$\mathcal{A}_2$ is just $\mathcal{A}_G$ projected onto a plane

$$\Longrightarrow \chi_{\mathcal{A}_G}(t) = t^3 - 3t^2 + 2t$$
$$\chi_G(t) = t^3 - 3t^2 + 2t.$$

Graphic Arrangements: For G a graph with $V = \{1, \ldots, n\}$ we define

$$\mathcal{A}_G = \{ H_{ij} \mid ij \in E \}, \quad H_{ij} = \{ (v_1, \ldots, v_n) \in \mathbb{R}^n \mid v_i = v_j \}.$$

Punchline: $\chi_G(t) = \chi_{\mathcal{A}_G}(t)$

for any graph G.