Notes

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July 15, 2025

Abstract

These notes are taken from things I've read.

None of this work is my own – I have almost entirely copied directly from the given sources.

Small edits have been made in the way of phrasing, and I have attempted to make certain statements and/or arguments more transparent to my own mind.

Contents

Chapter 1

Mathematics

§ 1 Correlation bounds for fields and matroids

[HSW21] [Annotated copy]

This paper of Huh, Schröter, and Wang gives new bounds on the correlation for bases of matroids. It also proves a new bound on the correlation constant of fields. This paper leverages the Hodge theory for matroids developed by Adiprasito, Huh, and Katz.

1.1.1 Introduction

Motivated by Kirchoff's work on electrical networks and some conjectures by Mason, we seek to show that, for some matroid M and basis B, the correlation between events $i \in B$ and $j \in B$ is negative. More specifically, given some weight w_e on elements of a matroid's ground set and the supposition $\mathbb{P}(b = B) \propto \prod_{e \in b} w_e$, we wish to show that

$$\mathbb{P}(i \in B, j \in B)\mathbb{P}(i \notin B, j \notin B) \leq \mathbb{P}(i \in B, j \notin B)\mathbb{P}(i \notin B, j \in B).$$

We are able to show the following results:

Theorem 5

If M is a rank d matroid, then

$$\mathbb{P}(i \in B, j \in B)\mathbb{P}(i \notin B, j \notin B) \leq 2\left(1 - \frac{1}{d}\right)\mathbb{P}(i \in B, j \notin B)\mathbb{P}(i \notin B, j \in B).$$

Theorem 6

If M is a rank d matroid and i, j are free elements, then

$$\mathbb{P}(i \in B, j \in B)\mathbb{P}(i \notin B, j \notin B) \leq \left(1 - \frac{1}{d}\right)\mathbb{P}(i \in B, j \notin B)\mathbb{P}(i \notin B, j \in B).$$

Theorem 7

If $\mathbb{F}$ is a field, then the correlation constant $\alpha_{\mathbb{F}} = \sup\{\alpha(M)\}$ (where the supremum is taken over all matroids representable over $\mathbb{F}$) satisfies

$$8/7 \leq \alpha_{\mathbb{F}} \leq 2.$$

Theorems 5 and 6 follow from constructing a matrix whose determinant leverages the condition $\mathbb{P}(b = B) \propto \prod_{e \in b} w_e$ to show the appropriate bounds. Theorem 7's upper bound follows from Theorem 5, and the lower bound is given by construction.

These theorems prove a conjecture of Mason as a corollary, as well as yielding new bounds for the asymptotic entropy of I_m, the number of independent sets of size m.

1.1.2 Theorem 5

Before we begin, we summarize the proof of Theorem 5:

> We will construct an element y_e of the Chow ring that vanishes under products unless the es are basis elements. This will allow us to construct an element L_{ij} of the Chow ring whose image under deg is proportional to the probability of elements i and j being in a basis or not. This element L_{ij} has a natural description as being attached to a submodular function on $2^{\overline{E}}$, which will allow us to apply the Hodge-Riemann relations to a symmetric matrix whose determinant yields the result.

Let M be a matroid of rank d and $\overline{M}$ the matroid given by adding 0 as a coloop. We define

$$y_e \triangleq \sum_{0 \in \overline{F}, e \notin \overline{F}} x_{\overline{F}} \in A(\overline{M})$$

where our sum is over all proper flats $\overline{F}$ containing zero and not containing e. By the fact that this sum lives in the Chow ring we have that

$$y_e = \sum_{0 \in \overline{F}, e \notin \overline{F}} x_{\overline{F}} = \sum_{0 \notin \overline{F}, e \in \overline{F}} x_{\overline{F}}. \tag{1}$$

Further, we have that

$$x_{\overline{F}} \cdot y_e = 0 \tag{2}$$

for any $\overline{F}$ containing exactly one of 0 or e. In particular, $y_e^2 = 0$. We refer to (1) and (2) as the "xy relations." We first prove a number of results regarding these elements and their image under the deg isomorphism given in [AHK18].

Lemma 14

For any dependent set J,

$$\prod_{e \in J} y_e = 0.$$

Proof. Without loss of generality we suppose that J is a circuit. Let I be a maximal independent set in J, and pick some $f \in I$ and $g \in J \setminus I$. Then $(I \setminus f) \cup g$ is a maximal independent set of J as well. The set of flats containing $(I \setminus f) \cup 0$ and not containing f is equivalent to the set of flats containing $(I \setminus f) \cup 0$ and not containing g, as adding either would increase the rank of $(I \setminus f) \cup 0$. Therefore

$$\prod_{e \in I} y_e = y_f \prod_{e \in I \setminus f} y_e = y_g \prod_{e \in I \setminus f} y_e.$$

Therefore we may write

$$\prod_{e \in J} y_e = \prod_{e \in I} y_e \prod_{e \in J \setminus I} y_e = 0.$$

$\triangle$

Lemma 15

If B is a d-element set of M, then

$$\deg\left(\prod_{e \in B} y_e\right) = \begin{cases} 1 & \text{if } B \text{ is a basis,} \\ 0 & \text{if } B \text{ is not a basis.} \end{cases}$$

Proof. Note that the dependent case follows by Lemma 14. Now suppose B is a basis and without loss of generality write $B = \{1, \dots, d\}$. Let $\overline{F}_k$ be the smallest flat containing $0, 1, \dots, k-1$. Because B is a basis $\overline{F}_k$ is the only flat of M containing $0, 1, \dots, k-1$, not containing k, and comparable to $\overline{F}_{k-1}$. Then we have by the xy relations,

$$\begin{aligned} y_1 y_2 \dots y_{d-2} y_{d-1} y_d &= y_1 y_2 \dots y_{d-2} y_{d-1} x_{\overline{F}_d} \\ &= y_1 y_2 \dots y_{d-2} x_{\overline{F}_{d-1}} x_{\overline{F}_d} = \dots = x_{\overline{F}_1} \dots x_{\overline{F}_d}, \end{aligned}$$

and [AHK18] tells us that $\deg(x_{\overline{F}_1} \dots x_{\overline{F}_d}) = 1$. △

Now we will define an element

$$L_{ij} = L_{ij}(w) = \sum_{e \neq i, e \neq j} w_e y_e$$

and prove some nice properties about it. Let $\mathcal{B}^{ij}$ be the set of bases not containing i and j, $\mathcal{B}_j^i$ the set of bases containing j and not containing i, and $\mathcal{B}_{ij}$ the set of bases containing both i and j. Firstly,

$$\begin{aligned} \deg\left(L_{ij}^d\right) &= \deg\left(\left(\sum_{e \neq i, e \neq j} w_e y_e\right)^d\right) \\ &= d! \sum_{B \in \mathcal{B}^{ij}} \prod_{e \in B} w_e \end{aligned}$$

The last equality can be seen by noting that, in the product, any term whose corresponding collection of elements (i.e. $y_1 y_2$ corresponds to $\{1, 2\}$) is not a basis goes to zero by Lemma 15. Any term whose corresponding collection of elements does form a basis will be mapped to the product of the weights, and for each of these we will have $d!$ copies as there are $d!$ ways to order the basis elements, all of which will occur in the product.

We have the following equalities by similar arguments:

$$\deg\left(y_i L_{ij}^{d-1}\right) = (d-1)! \sum_{B \in \mathcal{B}_i^j} \prod_{e \in B} w_e,$$

$$\deg\left(y_i y_j L_{ij}^{d-2}\right) = (d-2)! \sum_{B \in \mathcal{B}_{ij}} \prod_{e \in B} w_e.$$

If $\mathcal{B}^{ij}$ or $\mathcal{B}_{ij}$ are empty, we have that $\mathbb{P}(i \in B, j \in B)\mathbb{P}(i \notin B, j \notin B) = 0$ and so Theorem 5 holds. Suppose now that both are nonempty.

Let L be any element of $A^1(\overline{M})$ (the first graded component of the Chow ring) associated to a strictly submodular function on $2^{\overline{E}}$. To L_{ij} we may associate a submodular function (see annotated text), and from this we see that we may apply the Hodge-Riemann relations to the element $L_{ij} + \varepsilon L$ for any $\varepsilon > 0$. These Hodge-Riemann relations imply that the matrix associated to the symmetric bilinear form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (\eta_1, \eta_2) \mapsto \deg\left(\eta_1 \eta_2 (L_{ij} + \varepsilon L)^{d-2}\right)$$

has exactly one positive eigenvalue. Then by continuity we also have that the matrix representing the symmetric bilinear form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (a_1, a_2) \mapsto \deg\left(a_1 a_2 L_{ij}^{d-2}\right)$$

also has at most one positive eigenvalue. Now consider the symmetric matrix

$$H_{ij} = \begin{bmatrix} 0 & \deg(y_i y_j L_{ij}^{d-2}) & \deg(y_i L_{ij} L_{ij}^{d-2}) \\ \deg(y_i y_j L_{ij}^{d-2}) & 0 & \deg(y_j L_{ij} L_{ij}^{d-2}) \\ \deg(y_i L_{ij} L_{ij}^{d-2} & \deg(y_j L_{ij} L_{ij}^{d-2} & \deg(L_{ij} L_{ij} L_{ij}^{d-2}) \end{bmatrix}.$$

By Cauchy's eigenvalue interlacing theorem we have that H_{ij} has at most one positive eigenvalue. However the fact that its lower-right diagonal entry is positive implies that it has at least one positive eigenvalue. Then the determinant is nonnegative and we find that it is a positive multiple of

$$2\left(1-\frac{1}{d}\right)\left(\sum_{B\in\mathcal{B}_i^j}\prod_{e\in B} w_e\right)\left(\sum_{B\in\mathcal{B}_j^i}\prod_{e\in B} w_e\right)-\left(\sum_{B\in\mathcal{B}^{ij}}\prod_{e\in B} w_e\right)\left(\sum_{B\in\mathcal{B}_{ij}}\prod_{e\in B} w_e\right)$$

which proves the claim.

1.1.3 Theorem 6

Again, we summarize the proof:

> The methods used here are similar to those from Theorem 5. We begin by reducing the statement to an equivalent statement on the collections of m-independent sets in a deletion of our matroid. To prove this equivalent statement we construct an element α of the Chow ring and apply an extension of Lemma 15 to this element. Then we speedily proceed in constructing a matrix whose (nonnegative) determinant realizes the inequality.

Recall that an element $i \in M$ is *free* if it is not a coloop and if any circuit containing it is of cardinality $d+1$, where d is the rank of our matroid.

Let Y be a rank d matroid with free elements i and j. Let $Z \setminus \{i,j\}$. When $d=1$ no basis of Y contains both i and j, so the claim holds trivially. Suppose now that $d \geq 2$.

Denote by ℓ_m the collection of m-independent sets in M. Because i and j are free we have natural bijections

$$\ell_d(Y) \simeq \mathcal{B}^{ij}(Y), \quad \ell_{d-1}(Z) \simeq \mathcal{B}_i^j(Y) \simeq \mathcal{B}_i^j(Y), \quad \ell_{d-2}(Z) \simeq \mathcal{B}_{ij}(Y).$$

If the rank of Z is less than d, then we have that Theorem 6 holds as the left hand side of the inequaltiy is zero. If the rank of Z is d, then Theorem 6 for Y is equivalent to Proposition 17 for Z:

Proposition 17

For any matroid M of rank $d \geq 2$ and any set $w=(w_e)$ of positive weights,

$$\left(\sum_{I\in\ell_{d-1}}\prod_{e\in I} w_e\right) \geq \frac{d}{d-1}\left(\sum_{I\in\ell_{d-2}}\prod_{e\in I} w_e\right)\left(\sum_{I\in\ell_{d-1}}\prod_{e\in I} w_e\right).$$

The rest of this section will be occupied with proving Proposition 17. Define an element

$$\alpha = \sum_{0\in\overline{F}} x_{\overline{F}}.$$

The linear relations in $A(\overline{M})$ show that we may equivalently define α by summing flats containing any $e \in M$. Now we prove an extension of Lemma 15 critical to our proof.

Lemma 18

For any m-element subset I of E, we have

$$\deg\left(\alpha^{d-m}\prod_{e\in I} y_e\right) = \begin{cases} 1 & \text{if } I \text{ is independent in } M, \\ 0 & \text{if } I \text{ is dependent in } M. \end{cases}$$

Proof. We use descending induction on m. The case $m = d$ is Lemma 15, and if I is dependent the claim follows by Lemma 14. What remains is to show that the claim holds for independent I satisfying $|I| < d$. For the inductive step, suppose without loss of generality that $\{1, \ldots, d\}$ is a basis of M. It suffices to show that

$$(y_1 \ldots y_{m-1})y_m\alpha^{d-m} = (y_1 \ldots y_{m-1})\alpha^{d-m+1}.$$

Taking the difference of the right and left hand side we find

$$(y_1 \ldots y_{m-1})\alpha^{d-m+1} - (y_1 \ldots y_{m-1})y_m\alpha^{d-m} = (y_1 \ldots y_{m-1})(\alpha - y_m)\alpha^{d-m}.$$

Now note that we may write

$$\alpha - y_m = \sum_{0 \in \overline{F}} x_{\overline{F}} - \sum_{0 \in \overline{F}, m \notin \overline{F}} x_{\overline{F}} = \sum_{0, m \in \overline{F}} x_{\overline{F}}.$$

Recall that by the xy-relations we have that if $\overline{F}$ contains 0 and i, then $y_i x_{\overline{F}} = 0$. Now we have that

$$y_i(\alpha - y_m) = \sum_{0, m \in \overline{F}} y_i x_{\overline{F}} = \sum_{0, i, m \in G} x_G.$$

Therefore we may write

$$(y_1 \ldots y_{m-1})(\alpha - y_m)\alpha^{d-m} = \alpha^{d-m} \sum_{\overline{G}} x_{\overline{G}},$$

where the sum on the right hand side is over all proper flats $\overline{G}$ containing $\{1, \ldots, m\}$[1]. We claim that for any such $\overline{G}$ we have

$$x_{\overline{G}}\alpha^{d-m} = 0.$$

To see this to be true, use the linear relations in $A(\overline{M})$ to write

$$x_{\overline{G}}\alpha^{d-m} = x_{\overline{G}} \left(\sum_{\overline{F}_{m+1}} x_{\overline{F}_{m+1}} \right) \cdots \left(\sum_{\overline{F}_d} x_{\overline{F}_d} \right)$$

where each sum on the right hand side is taken over all flats containing $m + k$. Since $\{1, \ldots, d\}$ is a basis of M, no proper flat can contain all of them. Then by the quadratic relations in $A(\overline{M})$ this is indeed zero, as we claim. Therefore the right and left hand sides of our original statement are equivalent, proving the inductive step and therefore the lemma itself. △

We now prove Proposition 17. Define another element of the Chow ring

$$L_0 = L_0(w) = \sum_{e \in E} w_e y_e.$$

By Lemma 18, for any nonnegative integer $m \leq d$ we have

$$\deg(\alpha^{d-m} L_0^m) = m! \left(\sum_{I \in \ell_m} \prod_{e \in I} w_e \right).$$

Let L be any element of $A^1(\overline{M})$ attached to a strictly submodular function on $2^{\overline{E}}$. To L_0 we may associate a submodular function (see annotated text), and from this we see that we may apply the Hodge-Riemann relations to the element $L_0 + \varepsilon L$ for any $\varepsilon > 0$. These Hodge-Riemann relations imply that the matrix associated to the symmetric bilinear form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (\eta_1, \eta_2) \mapsto \deg(\eta_1 \eta_2 (L_0 + \varepsilon L)^{d-2})$$

[1] This claim is slightly different from the one in [HSW21]

must have exactly one positive eigenvalue. Then the matrix representing the symmetric form

$$A^1(\overline{M}) \times A^1(\overline{M}) \to \mathbb{R}, \quad (a_1, a_2) \mapsto \deg(a_1 a_2 L_0^{d-2})$$

has at most one positive eigenvalue. Now consider the symmetric matrix

$$H_0 = . \begin{bmatrix} \deg(\alpha\alpha L_0^{d-2}) & \deg(\alpha L_0 L_0^{d-2}) \\ \deg(\alpha L_0 L_0^{d-2}) & \deg(L_0 L_0 L_0^{d-2}) \end{bmatrix}.$$

By Cauchy's eigenvalue interlacing theorem we have that H_0 has at most one positive eigenvalue. However the fact that its lower-right diagonal entry is positive implies that it has at least one positive eigenvalue. Indeed, the determinant is a positive multiple of

$$\frac{d}{d-1}\left(\sum_{I\in\ell_d}\prod_{e\in I} w_e\right)\left(\sum_{I\in\ell_{d-2}}\prod_{e\in I} w_e\right) - \left(\sum_{I\in\ell_{d-1}}\prod_{e\in I} w_e\right)^2,$$

which is nonpositive by the conditions on the eigenvalues, proving the claim.

1.1.4 Theorem 7

§ 2 Hyperbolicity and stable polynomials in combinatorics and probability

[Pem12] [Annotated copy]

1.2.1 Definitions and properties

A (homogeneous) polynomial is *hyperbolic* in direction $\mathbf{x}$ if it satisfies $p(\mathbf{x} + i\mathbf{y}) \neq 0$ for all $\mathbf{y} \in \mathbb{R}^d$. We care mostly about the homogeneous case.

Related to this notion is stability. A polynomial is *stable* if it is nonzero on the upper half-plane. In fact, we have the following relationship between the two:

Proposition 1.3

A real homogeneous polynomial p is stable if and only if it is hyperbolic in direction ξ for all ξ in the positive orthant.

So stability is just hyperbolicity across the entire positive orthant.

Homogeneous hyperbolic polynomials

Hyperbolicity is equivalent to the following "real-rootedness" property:

Proposition 2.2

The homogeneous polynomial p is hyperbolic in direction $\mathbf{x}$ if and only if for any $\mathbf{y} \in \mathbb{R}^d$ the univariate polynomial $t \mapsto p(\mathbf{y} + t\mathbf{x})$ has only real roots.

Proof. Homogeneity yields that for any nonzero λ, $p(\lambda\mathbf{z}) = 0$ if and only if $p(\mathbf{z}) = 0$. Letting $\lambda = is$ for real s we find that

$$\text{“}p(\mathbf{x} + i\mathbf{y}) \neq 0 \text{ for all } \mathbf{y} \in \mathbb{R}^d\text{”}$$

is equivalent to

$$\text{“}p(\mathbf{y} + is\mathbf{x}) \neq 0 \text{ for all } y \in \mathbb{R}^d \text{ and all nonzero real } s,\text{”}$$

as we may simply scale by λ and then rescale $\mathbf{y}$ appropriately. △

This reformulation, where we write $t \mapsto p(\mathbf{y}+t\mathbf{x})$, will prove useful in our study of hyperbolic polynomials. Indeed, let us see that this polynomial gives us a useful narrowing of our object of study.

Denoting by $t_k(\mathbf{x}, \mathbf{y})$ the roots of $t \mapsto p(\mathbf{y}+t\mathbf{x})$, we may write

$$p(\mathbf{y}+t\mathbf{x}) = p(\mathbf{x}) \prod_{k=1}^{m} (t - t_k(\mathbf{x}, \mathbf{y})).$$

Proposition 2.2 shows us that these roots must all be real if p is to be hyperbolic. Therefore $p(\mathbf{y}+t\mathbf{x})/p(\mathbf{x})$ is a real polynomial, and so we may restrict our discussion to polynomials with real coefficients without loss of generality.

Now we present two ways to construct new hyperbolic polynomials from old ones.

Proposition 2.5

(i) If q_1 and q_2 are hyperbolic in direction $\mathbf{x}$, then so is $q_1 q_2$.

(ii) Let p be a homogeneous polynomial of degree m that is hyperbolic in direction $\mathbf{x}$. Denote by $p_0, \ldots, p_m$ the coefficients of $t \mapsto p(\mathbf{y}+t\mathbf{x})$, as below:

$$p(\mathbf{y}+t\mathbf{x}) = \sum_{k=0}^{m} p_k(\mathbf{y}) t^k.$$

Then p_k is hyperbolic in direction $\mathbf{x}$.

Proof. The first claim follows by definition. For the second claim, first we will show that if p is hyperbolic in direction $\mathbf{x}$, then so is its directional derivative, $D_{\mathbf{x}}(p)$.

Note that if f has only real roots, then so does f'. Now we see that if $f(t) = p(\mathbf{y}+t\mathbf{x})$, then $f'(t)$ given by $t \mapsto D_{\mathbf{x}}(p)(\mathbf{y}+t\mathbf{x})$ has only real roots for all $\mathbf{y} \in \mathbb{R}^d$. Therefore $D_{\mathbf{x}}(p)$ is hyperbolic in direction $\mathbf{x}$.

Now recall the expansion of $p(\mathbf{y}+t\mathbf{x})$ into homogeneous parts,

$$p(\mathbf{y}+t\mathbf{x}) = \sum_{k=0}^{m} p_k(\mathbf{y}) t^k.$$

Each part is given by

$$p_k(\mathbf{y}) = \frac{1}{k!} \left(\frac{d}{dt} \right)^k \bigg|_{t=0} p(\mathbf{y}+t\mathbf{x}) = (D_{\mathbf{x}})^k \, (p)(\mathbf{y}).$$

As we already noted, the directional derivative preserves hyperbolicity, and so we find that each p_k is hyperbolic in direction $\mathbf{x}$. △

Cones of hyperbolicity for homogeneous polynomials

Hyperbolic polynomials are not just hyperbolic in one direction. In fact, by deleting the hyperplane on which they are zero we form a number of connected components of $\mathbb{R}^d$ on which the polynomial may or may not be hyperbolic.

Proposition 2.6

Suppose p is a homogeneous polynomial hyperbolic in direction ξ. Dividing by a real multiple of $p(\xi)$ we may suppose that p is real and that $p(\xi) = 1$. Let $K(p, \xi)$ denote the connected component of the set $\mathbb{R}^d \setminus \{\mathbf{x} \in \mathbb{R}^d \mid p(\mathbf{x}) = 0\}$ that contains ξ.

(i) p is hyperbolic in direction $\mathbf{x}$ for all $\mathbf{x} \in K(p, \xi)$.

(ii) The set $K(p, \xi)$ is an open convex cone (the *cone of hyperbolicity for* p)

(iii) $K(p, \xi)$ is equal to the set $\mathcal{K}$ of vectors $\mathbf{x}$ for which all roots of $t \mapsto p(\mathbf{x} + t\xi)$ are real and negative.

Proof. We first note that the condition

"the roots of $t \mapsto p(\mathbf{x} + t\xi)$ are all real and negative"

is equivalent to the condition

"no root of $t \mapsto p(\mathbf{x} + t\xi)$ is real and nonnegative."

This is because hyperbolicity of p in direction ξ gives us that the roots are always real (Proposition 2.2), so negativity is the only condition that can be violated.

We begin by proving (i). By continuity of a polynomial with respect to its coefficients, $\mathcal{K}$ must be open. We can also immediately see that $\xi \in \mathcal{K}$, as $p(\xi + t\xi) = (1 + t)^m p(\xi)$ where m is the degree of p.

Suppose that $\mathbf{x}$ is in the closure $\overline{\mathcal{K}}$. Then by the continuity of the roots of p we must have that $p(\mathbf{x} + t\xi) \neq 0$ for $t > 0$, as t here is both real and nonnegative. This implies that $p(\mathbf{x}) \neq 0$, and so $\mathbf{x} \in \mathcal{K}$. Therefore $\mathcal{K}$ is closed in $\mathbb{R}^d \setminus \{\mathbf{x} \mid p(\mathbf{x}) = 0\}$.

We can also note that if $\mathbf{x} \in \mathcal{K}$, then $\mathbf{x} + t\xi$ is also in $\mathcal{K}$. Now by homogeneity we have that $s\mathbf{x} + t\xi \in \mathcal{K}$ for all $s, t > 0$. Then sending s to zero we find that $\mathcal{K}$ is star convex at ξ, and therefore connected. Therefore $\mathcal{K}$ is a connected clopen component of $\mathbb{R}^d \setminus \{\mathbf{x} \mid p(\mathbf{x} = 0\}$ containing ξ, which tells us that it is equal to $K(p, \xi)$.

Now we prove (i) and (ii) by a somewhat roundabout argument. We will first show that for $\mathbf{v} \in \mathcal{K}$, $\mathbf{x}$ real, and s and t complex,

$$p(\mathbf{x} + t\mathbf{v} + s\xi) \neq 0 \text{ whenever } \mathfrak{Im}\,(s) \leq 0,\ \mathfrak{Im}\,(t) \leq 0,\ \mathfrak{Im}\,(s) + \mathfrak{Im}\,(t) < 0. \tag{2.5}$$

First, suppose $\mathfrak{Im}\,(s) < 0$. If $u \geq 1$, then the polynomial

$$t \mapsto p\left(\frac{\mathbf{x}}{u} + t\mathbf{v} + \left(\frac{s}{u} + i\left(\frac{1}{u} - 1\right)\right)\xi\right)$$

is nonzero for real t because here ξ has nonzero imaginary part. As $u \to \infty$, the number of roots in the above polynomial that are in the lower half-plane remains constant. The limit at $u = \infty$ is $t \mapsto p(t\mathbf{v} - i\xi)$, which has roots in the upper half-plane given by $-i/\mathbf{y}_k$ where $\mathbf{y}_k$ are the roots of $t \mapsto p(\mathbf{v} + t\xi)$, which are negative real by (iii).

Because the number of roots in the lower half-plane remains constant through the limit, and we have just shown that there are no roots in the lower half-plane in the limit, there must only be roots in the upper half-plane for all u. In particular, for $u = 1$ we have

$$p(\mathbf{x} + t\mathbf{v} + s\xi) \neq 0 \text{ whenever } \mathfrak{Im}\,(s) < 0,\ \mathfrak{Im}\,(t) \leq 0.$$

This proves part of (2.5). To see that it holds for $\mathfrak{Im}\,(s) = 0$ and $\mathfrak{Im}\,(t) < 0$, note that because $\mathcal{K}$ is open we have $\mathbf{v} - \varepsilon\xi \in \mathcal{K}$ for some $\varepsilon > 0$. Hence

$$p(\mathbf{x} + t\mathbf{v} + s\xi) = p(\mathbf{x} + t(\mathbf{v} - \varepsilon\xi) + (s + \varepsilon t)\xi),$$

which is nonzero by the above discussion. Therefore (2.5) holds.

The upshot of this diversion can now be made apparent. Setting $s = 0$ in (2.5) we see that any root of $t \mapsto p(\mathbf{x} + t\mathbf{v})$ satisfies $\mathfrak{Im}\,(t) \geq 0$. Because p is real, we can conjugate the argument to find that $\mathfrak{Im}\,(t) \leq 0$ and therefore $\mathfrak{Im}\,(t) = 0$. Therefore $t \mapsto p(\mathbf{x} + t\mathbf{v})$ has only real roots, and therefore p is hyperbolic with respect to any $\mathbf{v} \in \mathcal{K}$. Further, it follows that $\mathcal{K}$ is star-convex with respect to every $\mathbf{v} \in \mathcal{K}$, implying that $\mathcal{K}$ itself is convex. △

1.2.2 Semi-continuity and Morse deformations

Localization

Given a function f analytic on some neighborhood of the origin, we define its *order of vanishing* to be the least total degree of a nonzero monomial in its Taylor expansion. The sum of all such terms is called the *homogeneous part of f at the origin* and is denoted $\text{ßhom}(f)$. For any $\mathbf{x}$ we define the *localization of f at $\mathbf{x}$* by

$$\text{ßhom}(f, \mathbf{x}) \triangleq \text{ßhom}(f(\mathbf{x} + \bullet)).$$

Proposition 3.1

Let p be any hyperbolic homogeneous polynomial of degree m. Fix $\mathbf{x}$ with $p(\mathbf{x}) = 0$ and let $\tilde{p} = \text{ßhom}(p, \mathbf{x})$. If p is hyperbolic in direction $\mathbf{u}$, then $\tilde{p}$ is also hyperbolic in direction $\mathbf{u}$. Consequently, any cone of hyperbolicity for p is contained in one for $\tilde{p}$.

Proof. Suppose Q is a polynomial satisfying $\deg(\text{ßhom}(Q)) = 0$. Then we have

$$\text{ßhom}(Q)(\mathbf{y}) = \lim_{\lambda \to \infty} \lambda^k Q(\lambda^{-1}\mathbf{y}).$$

This limit is uniform as $\mathbf{y}$ varies over compact sets. Applying this reasoning to $Q(\bullet) = p(\mathbf{x} + \bullet)$ with $\mathbf{y} + t\mathbf{u}$ instead of $\mathbf{y}$, we find that

$$\tilde{p}(\mathbf{y} + t\mathbf{u}) = \text{ßhom}(p, \mathbf{x})(\mathbf{y} + t\mathbf{u}) = \lim_{\lambda \to \infty} p(\mathbf{x} + \lambda^{-1}(\mathbf{y} + t\mathbf{u}))$$

uniformly as t varies over compact sub-intervals of $\mathbb{R}$. Because p is hyperbolic we have that for any fixed λ this polynomial has only real zeroes. Now Hurwitz' theorem on the continuity of zeroes tells us that, because this limit is uniformly bounded on intervals, has all real zeroes or vanishes. We know that $\tilde{p}$ does not vanish as it has degree $k \geq 1$, so it must have all real zeroes. Equivalently, $\tilde{p}$ is hyperbolic in direction $\mathbf{u}$. △

We now shift our focus to cones defined using these localizations. For p, a hyperbolic homogeneous polynomial with cone of hyperbolicity B, denote by $\tilde{B}$ the cone of hyperbolicity of $\text{ßhom}(p, \mathbf{x})$ containing B. We define

$$K^{p,B}(\mathbf{x}) \triangleq \begin{cases} \tilde{B} & \text{if } p(\mathbf{x}) = 0, \\ \mathbb{R}^d & \text{if } p(\mathbf{x}) \neq 0. \end{cases}$$

Suppose that $\mathbf{x}_n \to \mathbf{x}$. It is not in general true that $\text{ßhom}(f, \mathbf{x}_n) \to \text{ßhom}(f, \mathbf{x})$, but it is true that

$$\text{ßhom}(f, \mathbf{x}_n) = \text{ßhom}(\text{ßhom}(f, \mathbf{x}), \mathbf{x}_n)$$

for sufficiently large n. This implies that if p is homogeneous and hyperbolic with cone of hyperbolicity B, then $K^{p,B}$ is "semi-continuous" in $\mathbf{x}$:

$$K^{p,B}(\mathbf{x}) \subseteq \liminf K^{p,B}(\mathbf{x}_n) \text{ as } \mathbf{x}_n \to \mathbf{x}.$$

This is a result of Atiyah, Bott, and Gårding. We will want a version of this semi-continuity for inhomogeneous polynomials.

Amoeba boundaries

The *amoeba* of a polynomial is the image of its zero-set under a coordinatewise (real) logarithmic map:

$$ßamoeba(q) \triangleq \{(\log|z_1|, \dots, \log|z_d|) \mid q(\mathbf{z})\}.$$

The connected components of $\mathbb{R}^d \setminus ßamoeba(q)$ are convex and in 1-1 correspondence with Laurent series expansions of q[2]. We now give two results from [BP11] that demonstrate how amoebas relate to hyperbolicity.

Proposition 3.5 ([BP11] Proposition 2.12)

Let q be any polynomial and let B be a component of the complement of $ßamoeba(q)$. Denote the boundary of B by ∂B and fix some $\mathbf{x} \in \partial B$. Let $f = q \circ \exp$ so that f vanishes at some point $\mathbf{x} + i\mathbf{y}$. Let $f_{\mathbf{y}} \triangleq ßhom(f, \mathbf{x} + i\mathbf{y})$. Then each $f_{\mathbf{y}}$ is hyperbolic in at least one direction and one of its cones of hyperbolicity contains B. We denote this cone by $K^{q,B}(\mathbf{y})$.

Theorem 3.6 ([BP11] Corollary 2.15)

If B is a component of $\mathbb{R}^d \setminus ßamoeba(q)$ and $\mathbf{x} \in \partial B$, then the family of cones $K^{p,B}(\mathbf{y})$ defined in Proposition 3.5 satisfies

$$K^{q,B}(\mathbf{y}) \subseteq \liminf K^{q,B}(\mathbf{y}_n) \text{ whenever } \mathbf{y}_n \to \mathbf{y}.$$

What we have just seen is that the localization of *any* polynomial to a point on its amoeba's boundary is hyperbolic, and this semi-continuity condition persists for the localizations of inhomogeneous polynomials.

§ 3 Negative dependence and the geometry of polynomials

[BBL08] [Annotated copy]

Reference sheet

A polynomial is said to be *multi-affine* if it has degree at most one in each variable.

A measure μ is said to satisfy the *negative lattice condition* (NLC) if

$$\mu(S)\mu(T) \geq \mu(S \cup T)\mu(S \cap T),$$

and a (multi-affine) polynomial is said to be NLC if it is the generating polynomial of an NLC measure. In fact, there is a 1-1 correspondence between measures on the boolean algebra on n elements and multi-affine polynomials in n variables.

A measure is said to be *symmetric* if for all $\sigma \in \mathfrak{S}_n$ we have $\sigma(\mu) = \mu$. Equivalently, the generating polynomial g_μ is symmetric. A measure is *almost-symmetric* if g_μ is symmetric in all but at most one variable.

A collection of subsets $\mathcal{A} \subseteq 2^{[n]}$ is an *increasing event* if it is closed upwards under containment.

A measure is *pairwise negatively correlated* (p-NC) if $\mu(X_i)\mu(X_j) \geq \mu(X_iX_j)$ whenever $1 \leq i \neq j \leq n$. This is the weakest negative dependence property.

[2]This is my own interpretation of what I believe to be a typo in the original manuscript, informed by a quick reading of [GKZ94].

A measure satisfies the *hereditary negative lattice condition* (h-NLC) if every projection of that measure is NLC. The *strong hereditary negative lattice condition* (h-NLC+) is satisfied by a measure if imposing any external field results in an h-NLC measure.

A polynomial $f \in \mathcal{P}_n$ (the space of all multi-affine polynomials in n variables with nonnegative coefficients satisfying $f(\mathbf{1}) = 1$) is said to be *Rayleigh* if

$$\frac{\partial f}{\partial z_i}(x)\frac{\partial f}{\partial z_j}(x) \geq \frac{\partial^2 f}{\partial z_i \partial z_j}(x) f(x)$$

for all $1 \leq i, j \leq n$ and all $x \in \mathbb{R}_+^n$.

Proposition 2.2

A measure in $\mathfrak{B}_n$ or a polynomial in $\mathcal{P}_n$ is Rayleigh if and only if it is h-NLC+.

§ 4 Polynomials with the half-plane property and matroid theory

[Brä07] [Annotated copy]

This paper of Brändén proves a number of results that relate the support of multi-affine polynomials satisfying the half-plane property to matroid theory, resolving a conjecture of Choe, Oxley, Sokal, and Wagner.

1.4.1 Delta-matroids and jump systems

Delta-matroids, introduced by Bouchet, generalize the exchange axioms for independent sets. A *delta-matroid* is a pair $(\mathcal{F}, E)$ where $\mathcal{F}$ is a collection of subsets of a finite set E satisfying

(a) $\bigcup_{A \in \mathcal{F}} A = E$

(b) (Symmetric exchange axiom) For any $A, B \in \mathcal{F}$ and $x \in A \Delta B$, there exists some $y \in A \Delta B$ such that $A \Delta \{x, y\} \in \mathcal{F}$.

Above we write $A \Delta B = (A \cup B) \setminus (A \cap B)$ for the symmetric difference operator. Delta-matroids whose elements of $\mathcal{F}$ are hereditary (i.e. $A \in \mathcal{F}$, $B \subseteq A \implies B \in \mathcal{F}$) are independent sets of some matroid, and requiring that all elements of $\mathcal{F}$ be the same size results in a delta-matroid whose family $\mathcal{F}$ is a family of bases for some matroid.

Bouchet and Cunningham introduced jump systems as further generalizations of delta-matroids. Let $\alpha, \beta \in \mathbb{Z}^n$ and define

$$|\alpha| \triangleq \sum_{i=1}^{n} |\alpha_i|.$$

Also define the *set of steps from α to β* by

$$\mathrm{St}(\alpha, \beta) \triangleq \{\sigma \in \mathbb{Z}^n \mid |\sigma| = 1, |\alpha - \beta + \sigma| = |\alpha - \beta| - 1\}.$$

Jump systems are subsets of $\mathbb{Z}^n$ satisfying the *two-step axiom*:

If $\alpha, \beta \in \mathcal{F}$, $\sigma \in \mathrm{St}(\alpha, \beta)$, and $\alpha + \sigma \notin \mathcal{F}$ then there is $\tau \in \mathrm{St}(\alpha + \sigma, \beta)$ such that $\alpha + \sigma + \tau \in \mathcal{F}$.

As a quick note, delta-matroids are precisely the jump systems satisfying $\mathcal{F} \subseteq \{0, 1\}^n$. We may view the vectors here as indicator vectors.

1.4.2 The support of polynomials with the half-plane property

We say a polynomial has the *half-plane propety* if it is nonzero on some open half-plane in $\mathbb{C}$ whose boundary contains the origin. We also refer to these polynomials as being H-stable for some half-plane H. If a polynomial is H-stable on the right half-plane, we say it is *Hurwitz stable*.

Proposition 3.1

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be H-stable. Then either $\partial f/\partial z_1 = 0$ or $\partial f/\partial z_1$ is H-stable.

§ 5 Semimatroids and their Tutte polynomials

[Ard04] [Annotated copy]

1.5.1 Introduction

A *semimatroid* is a triple $(S, \mathcal{C}, r_{\mathcal{C}})$ consisting of a finite set S, a simplicial complex $\mathcal{C}$ on S, and a function $r_{\mathcal{C}} : \mathcal{C} \to \mathbb{N}$ satisfying

(**R1**) If $X \in \mathcal{C}$, then $0 \leq r_{\mathcal{C}}(X) \leq |X|$.

(**R2**) If $X, Y \in \mathcal{C}$ and $X \subseteq Y$, then $r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(Y)$.

(**R3**) If $X, Y \in \mathcal{C}$ and $X \cup Y \in \mathcal{C}$, then $r_{\mathcal{C}}(X) + r_{\mathcal{C}}(Y) \geq r_{\mathcal{C}}(X \cup Y) + r_{\mathcal{C}}(X \cap Y)$.

(**CR1**) If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) = r_{\mathcal{C}}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$.

(**CR2**) If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(Y)$, then there exists some element $y \in Y \setminus X$ such that $X \cup y \in \mathcal{C}$.

We refer to $\mathcal{C}$ as the *collection of central sets*. As with matroids, S is the *ground set* and $r_{\mathcal{C}}$ the *rank function*. Semimatroids satisfy stronger, "local" versions of R2, CR1, and CR2:

Proposition

If $(S, \mathcal{C}, r_{\mathcal{C}})$ is a semimatroid, then

(**R2'**) If $X \cup x \in \mathcal{C}$, then $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X) + 1$ or $r_{\mathcal{C}}(X \cup x) = r_{\mathcal{C}}(X)$.

(**CR1'**) If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) = r_{\mathcal{C}}(X \cap Y)$, then $X \cup Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup Y) = r_{\mathcal{C}}(Y)$.

(**CR2'**) If $X, Y \in \mathcal{C}$ and $r_{\mathcal{C}}(X) < r_{\mathcal{C}}(Y)$, then $X \cup y \in \mathcal{C}$ and $r_{\mathcal{C}}(X \cup y) = r_{\mathcal{C}}(X) + 1$ for some $y \in Y \setminus X$.

Proof. First we prove (R2'). By (R2) we have that $r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(X \cup x)$, and by (R3) we have that $r_{\mathcal{C}}(X) + r_{\mathcal{C}}(x) \geq r_{\mathcal{C}}(X \cup x) + r_{\mathcal{C}}(\emptyset)$, implying that $r_{\mathcal{C}}(X \cup x) - r_{\mathcal{C}}(X) \leq r_{\mathcal{C}}(x) - r_{\mathcal{C}}(\emptyset)$, and the right hand side of this inequality is either zero or one by (R1), and so $0 \leq r_{\mathcal{C}}(X \cup x) - r_{\mathcal{C}}(X) \leq 1$.

Now we prove (CR1'). Because $r_{\mathcal{C}}(X) = r_{\mathcal{C}}(X \cap Y)$ we have that $X \cup Y \in \mathcal{C}$. Then by (R2) we have that $r_{\mathcal{C}}(Y) \leq r_{\mathcal{C}}(X \cup Y)$ while (R3) gives that $r_{\mathcal{C}}(Y) \geq r_{\mathcal{C}}(X \cup Y)$.

Now a proof of (CR2'). By repeatedly applying (CR2) to the sets X and Y we find that $X \cup y_1 \cup \cdots \cup y_k \in \mathcal{C}$ must satisfy $r_{\mathcal{C}}(X \cup y_1 \cup \cdots \cup y_k) = r_{\mathcal{C}}(Y)$. Our claim is that $r_{\mathcal{C}}(X \cup y_i) = r_{\mathcal{C}}(X) + 1$ for some $y_i \in Y$.

Suppose for the sake of contradiction that there exists no such y_i. Then because $r_{\mathcal{C}}(X \cup y_1) = r_{\mathcal{C}}(X)$ we may apply (CR1') to both $X \cup y_1$ and $X \cup y_2$, finding that $X \cup y_1 \cup y_2 \in \mathcal{C}$. Further, $r_{\mathcal{C}}(X \cup y_1 \cup y_2) = r_{\mathcal{C}}(X \cup y_2 = r_{\mathcal{C}}(X)$. Then we may apply (CR1') again, finding that $r_{\mathcal{C}}(X \cup y_1 \cup y_2 \cup y_3) = r_{\mathcal{C}}(X)$ in the same fashion. This may continue until we have $r_{\mathcal{C}}(X \cup y_1 \cup \cdots \cup y_k) = r_{\mathcal{C}}(X) = r_{\mathcal{C}}(Y)$, but this contradicts the fact that $r_{\mathcal{C}}(X) < r_{\mathcal{C}}(Y)$. △

Now for the connection between semimatroids and affine arrangements.

Proposition 2.2

Let $\mathcal{A}$ be an affine hyperplane arrangement in $\Bbbk^n$. Let $\mathcal{C}_\mathcal{A}$ be the collection of central subarrangements of $\mathcal{A}$, and let $r_\mathcal{A}$ be the rank function on $\mathcal{A}$ (i.e. the codimension of a subspace, which is the rank function on $L(\mathcal{A})$, the intersection semilattice). Then $(\mathcal{A}, \mathcal{C}_\mathcal{A}, r_\mathcal{A})$ is a semimatroid.

Proof. Let $\mathcal{A} = \{H_1, \dots, H_n\}$ and let v_i be the normal vector defining H_i. Immediately we have that

$$\dim(\operatorname{span}(v_{i_1}, \dots, v_{i_k})) = r_\mathcal{A}(H_{i_1} \cap \dots \cap H_{i_k})$$

for any $\{H_{i_1}, \dots, H_{i_k}\} \in \mathcal{C}_\mathcal{A}$. Now we see that (R1), (R2), and (R3) are simply properties of elements of a vector space.

We now show (CR1). Assume that $X, Y \in \mathcal{C}_\mathcal{A}$ and that $r_\mathcal{A}(X) = r_\mathcal{A}(X \cap Y)$. Let $A = \bigcap_{H \in X} H$ and $B = \bigcap_{H \in Y} H$, so that A and B are themselves intersections of hyperplanes (elements of the intersection semilattice). Since $X \cap Y \subseteq X$ and $r_\mathcal{A}(X) = r_\mathcal{A}(X \cap Y)$, we must have that $\bigcap_{H \in X \cap Y} H = A$. This is because if $\bigcap_{H \in X \cap Y} H \supsetneq A$ then we would find a contradiction with the fact that $X \cap Y \subseteq X$, and if $A \supsetneq \bigcap_{H \in X \cap Y} H$ then X would have to be of higher rank, violating the fact that $r_\mathcal{A}(X) = r_\mathcal{A}(X \cap Y)$. Further, because $X \cap Y \subseteq Y$ we have that $\bigcap_{H \in X \cap Y} H \supseteq B$, by a similar argument to above. Therefore $A \supseteq B$ and so every hyperplane in $X \cup Y$ contains B, meaning that the arrangement is central and so $X \cup Y \in \mathcal{C}_\mathcal{A}$.

Now we show (CR2). Suppose that $X, Y \in \mathcal{C}_\mathcal{A}$ and $r_\mathcal{A}(X) < r_\mathcal{A}(Y)$. Denote by L_X and L_Y the normal vectors to X and Y, respectively. Because $\operatorname{rank}(L_X) < \operatorname{rank}(L_Y)$ there must exist some vector $L \in L_Y$ corresponding to a hyperplane $y \in Y$ that is not in the span of L_X. Therefore y has nonempty intersection with A, implying that $X \cup y$ is a central subarrangement and therefore $X \cup y \in \mathcal{C}_\mathcal{A}$. △

We now present another formulation of semimatroids, via a closure operator. For a semimatroid $(S, \mathcal{C}, r_\mathcal{C})$ and a set $X \in \mathcal{C}$ we define the *closure of X in $\mathcal{C}$* to be

$$\operatorname{cl}_\mathcal{C}(X) = \{x \in S \mid X \cup x \in \mathcal{C},\, r_\mathcal{C}(X \cup x) = r_\mathcal{C}(X)\}.$$

Of course when it is clear from context we will simply write $\operatorname{cl}(X)$ instead of $\operatorname{cl}_\mathcal{C}(X)$.

Proposition 2.4

The closure operator of a semimatroid satisfies the following for all $X, Y \in \mathcal{C}$ and all $x, y \in S$.

(**CLR1**) $\operatorname{cl}(X) \in \mathcal{C}$ and $r_\mathcal{C}(\operatorname{cl}(X)) = r_\mathcal{C}(X)$.

(**CL1**) $X \subseteq \operatorname{cl}(X)$.

(**CL2**) If $X \subseteq Y$, then $\operatorname{cl}(X) \subseteq \operatorname{cl}(Y)$.

(**CL3**) $\operatorname{cl}(\operatorname{cl}(X)) = \operatorname{cl}(X)$.

(**CL4**) If $X \cup x \in \mathcal{C}$ and $y \in \operatorname{cl}(X \cup x)$, then $X \cup y \in \mathcal{C}$ and $x \in \operatorname{cl}(X \cup y)$.

Proof. To check (CLR1) we repeat the proof of (CR2'). Let $\operatorname{cl}(X) = \{x_1, \dots, x_k\}$. Since $r_\mathcal{C}(X \cup x_1) = r_\mathcal{C}(X)$, we may apply (CR1') to find that $r_\mathcal{C}(X \cup x_1 \cup x_2) = r_\mathcal{C}(X)$. Continuing in this way we find that $X \cup x_1 \cup \dots \cup x_k \in \mathcal{C}$ and $r_\mathcal{C}(X \cup x_1 \cup \dots \cup x_k) = r_\mathcal{C}(X)$.

Note that (CL1) follows by definition.

For (CL2), let $x \in \operatorname{cl}(X)$. Then $X \cup x \in \mathcal{C}$ and $r_\mathcal{C}(X \cup x) = r_\mathcal{C}(X)$. By applying (CR1') to $X \cup x$ and Y we find that $Y \cup x \in \mathcal{C}$ and $r_\mathcal{C}(Y \cup x) = r_\mathcal{C}(Y)$. Therefore $x \in \operatorname{cl}(Y)$. △

§ 6 A pithy look at the polytope algebra

[Cas21] [Annotated copy]

1.6.1 Valuations

For $S \subseteq \mathbb{R}^d$ we define the indicator function of S to be

$$[S](p) = \begin{cases} 1 & \text{if } p \in S, \\ 0 & \text{if } p \notin S. \end{cases}.$$

A function $f : \mathbb{R}^d \to \mathbb{Z}$ is *simply polytopal* if it can be written as

$$f = \alpha_1[Q_1] + \cdots + \alpha_k[Q_k]$$

for some integer $\alpha_1, \ldots, \alpha_k$ and polytopes $Q_1, \ldots, Q_k$. We write $\mathcal{SP}_d$ for the set of all simply polytopal functions. By definition $\mathcal{SP}_d$ is spanned by indicator functions of $\mathcal{P}_d$, the space of d-polytopes, but $\mathcal{P}_d$ does not form a basis.

A *valuation* is a group homomorphism $\phi : \mathcal{SP}_d \to G$ for G and abelian group. One of the more canonical valuations in the theory of polytopes is the Euler characteristic:

Theorem 3.4

There exists a unique valuation $\chi : \mathcal{SP}_d \to \mathbb{Z}$ satisfying

(i) $\chi([\emptyset]) = 0$

(ii) $\chi([P]) = \begin{cases} 1 & \text{if } P \in \hat{\mathcal{P}}_d = \mathcal{P}_d \setminus \{\emptyset\}, \\ 0 & \text{otherwise.} \end{cases}$

Even though $\mathcal{SP}_d$ is spanned by indicator functions of $\mathcal{P}_d$, it contains indicators for non-polytopal sets as well. As an example, the Euler formula is equivalent to the relation

$$[P^\circ] = \sum_{F \subseteq P} (-1)^{\dim(P) - \dim(F)}[F],$$

and $\chi([P^\circ]) = (-1)^{\dim(P)}$.

We endow $\mathcal{SP}_d$ with the structure of a commutative ring by the multiplication $[P] * [Q] = [P + Q]$. The unit here is $[\mathbf{0}]$, the indicator function at the origin.

Proposition 3.7

Let $P \in \hat{\mathcal{P}}_d$. Then $[P]$ is invertible, and its inverse is given by

$$[P]^{-1} = (-1)^{\dim(P)}[-\mathrm{relint}(P)].$$

Proposition 3.8

If $P = \mathrm{conv}(v_1, \ldots, v_m)$ is a polytope, then

$$([P] - [v_1]) * \cdots * ([P] - [v_m]) = 0.$$

Sketch of proof. We prove the claim for the $m-1$-simplex in $\mathbb{R}^m$, then handwave away the generalization.

Let $e_I = \sum_{i\in I} e_i$ for any $I \subseteq [m]$. We must show that

$$\sum_{I\subseteq[m]} (-1)^{|I|}[(m-|I|)\triangle_{m-1} + e_I] = 0.$$

For some $q \in \mathbb{R}^m$, how do we decide if $q \in (m-|I|)\triangle_{m-1} + e_I$? This is equivalent to $q - e_I \in (m-|I|)\triangle_{m-1}$, which is satisfied if and only if

$$q - e_I \geq 0, \quad \sum q_i = m.$$

These conditions follow by the nonnegativity and scaling of the simplex. Now define $I_0 = \{i \in [m] \mid q_i \geq 1\}$. For a fixed q satisfying the above conditions we have that

$$\begin{aligned}\left(\sum_{I\subseteq[m]} (-1)^{|I|}[(m-|I|)\triangle_{m-1} + e_I]\right)(q) &= \sum_{I\subseteq I_0} (-1)^{|I|} \\ &= \sum_{j=0}^{|I_0|} \binom{|I_0|}{j}(-1)^j \\ &= (1-1)^{|I_0|} = \begin{cases} 1 & \text{if } I_0 = \emptyset, \\ 0 & \text{if } I_0 \neq \emptyset. \end{cases}\end{aligned}$$

The first equality follows by the fact that if $I \not\subseteq I_0$ then $q - e_I < 0$ and so the corresponding summand dies by the indicator function. Notice now that we are left to conclude that, by the suppositon $\sum q_i = m$, that $I_0 \neq \emptyset$. This concludes the proof for the simplex.

For general polytopes, one can argue that writing $P = \pi(\triangle_{m-1})$ for $\pi : \mathbb{R}^m \to \mathbb{R}^d$ and $e_i \mapsto v_i$ that there is an induced map $\pi_* : \mathcal{SP}_m \to \mathcal{SP}_d$ that preserves linear relations, allowing the result to carry over. $\triangle$

1.6.2 The polytope algebra

We begin by defining the *translation ideal* of $\mathcal{SP}_d$,

$$\mathcal{T} = \mathbb{Z}\{[P+t] - [P] \mid P \in \mathcal{P}_d, t \in \mathbb{R}^d\}.$$

Then the *polytope algebra* Π^d is defined by $\Pi^d = \mathcal{SP}_d/\mathcal{T}$, and we denote the class of $[P]$ in Π^d by $[\![P]\!]$. Again the multiplicative identity is $[\![\mathbf{0}]\!]$.

We also define the *dilation maps* in terms of generators by $D_\lambda[\![P]\!] = [\![\lambda P]\!]$. These are ring endomorphisms. We begin by describing a relevant decomposition. The rest of this section will be dedicated to proving the following theorem:

Theorem 4.4

The polytope algebra is a graded ring, generated in degree one by $\Pi^d = \Pi_0 \oplus \cdots \oplus \Pi_d$. Furthermore,

(i) $\Pi_0 \cong \mathbb{Z}$.

(ii) Π_i is an $\mathbb{R}$-vector space for $i > 0$.

(iii) $\Pi_d \cong \mathbb{R}$.

That is, Π^d is *almost* an $\mathbb{R}$-algebra, which should be surprising given that $\mathcal{SP}_d$ contains only integer combinations of indicators.

Proposition 4.5

The map $\Pi^d \to \mathbb{Z}$ induced by χ gives a decomposition $\Pi^d = \mathbb{Z} \oplus \Pi_+$, where $\Pi_+ = \ker \chi$.

Proof. Begin by writing $x = \sum \alpha_i [\![P_i]\!]$. Then we may write

$$x = \sum \alpha_i \chi(P_i) \cdot [\![\mathbf{0}]\!] + \sum \alpha_i([\![P_i]\!] - 1) = \chi(x) \cdot [\![\mathbf{0}]\!] + (x - \chi(x) \cdot [\![\mathbf{0}]\!]).$$

This gives the desired decomposition, and in addition demonstrates that $\Pi_+ = \mathbb{Z}\{[\![P]\!] - 1 \mid P \in \hat{\mathcal{P}}_d\}$. △

Proposition 3.8 can now tell us that the generators of Π_+ are nilpotent:

Corollary 4.6

For $P \in \hat{\mathcal{P}}_d$ we have $([\![P]\!] - 1)^r = 0$ in Π^d for $r > d$.

Proof. We know from Proposition 3.8 that $([\![P]\!] - 1)^{f_0(P)} = 0$, we must now argue that we may lower the exponent. Begin by writing

$$[\![nP]\!] = [\![P]\!]^n = \sum_{i=0}^{n} \binom{n}{i} ([\![P]\!] - 1)^i.$$

Now triangulate P such that

$$[\![nP]\!] = \sum \alpha_i [\![nT_i]\!]$$

for some simplices T_i. We may expand the right hand side of the above equation similarly to how we expanded $[\![nP]\!]$, finding that $[\![nP]\!]$ is a polynomial in n of degree d (because each T_i is a simplex of dimension at most d), which implies that, in the right hand side's expansion, $([\![P]\!] - 1)^r$ vanishes for $r > d$. △

Now we seek another intermediate result:

Theorem 4.7

Π_+ is a $\mathbb{Q}$-vector space.

This result follows from the following pair of lemmas.

Lemma 4.8

The abelian group Π_+ is divisible.

Proof. It suffices to show the result true for m prime. Suppose $x \in \Pi_+$. We seek $h \in \Pi_+$ such that $m \cdot h = x$. Let $N = m^e$ with e such that $N > d + 1$. Recall from the proof of Proposition 4.5 that $\Pi_+ = \mathbb{Z}\{[\![P]\!] - 1 \mid P \in \hat{\mathcal{P}}_d\}$. Therefore we may write

$$x = [\![P]\!] - 1 = \left[\!\!\left[\frac{1}{N}P\right]\!\!\right]^N - 1 = \sum_{i=0}^{d} \binom{N}{i} \left(\left[\!\!\left[\frac{1}{N}P\right]\!\!\right] - 1\right)^i - 1 = \sum_{i=1}^{d} \binom{N}{i} \left(\left[\!\!\left[\frac{1}{N}P\right]\!\!\right] - 1\right)^i.$$

However now we have that m divides the binomial coefficients, and so m divides x (i.e. $h = \sum_{i=1}^{n} \frac{1}{m}\binom{N}{i}([\![P]\!] - 1)^i \in \Pi_+$). △

Lemma 4.12

The abelian group Π_+ is torsion-free

Proof. Consider the filtration $\Pi_+ = Z_1 \supset Z_2 \supset \ldots Z_{d+1}$ where Z_r is generated by elements of the form $([\![P]\!] - 1)^j$. The proof of the lemma follows from two observations:

The first observation is that $D_\lambda(\llbracket P\rrbracket - 1)^r = (\llbracket \lambda P\rrbracket - 1)^r$, as D_λ is a ring endomorphism, and so it commutes with multiplication. This implies that $D_\lambda Z_r \subset Z_r$ and $Z_r \subset D_{\lambda^{-1}} Z_r$.

The second observation is that if $x \in Z_r$, then

$$D_n x - n^r x \in Z_{r+1},$$

for n a natural number. We check this fact on the generators.

$$D_n(\llbracket P\rrbracket - 1) = (\llbracket nP\rrbracket - 1) = \sum_{i=1}^{n} \binom{n}{i} (\llbracket P\rrbracket - 1)^i.$$

Raising this entire expression to the r power we find that $D_n(\llbracket P\rrbracket - 1)^r - n^r(\llbracket P\rrbracket - 1)^r \in Z_{r+1}$. Therefore the equation holds for the generators and it follows that it holds for all $x \in Z_r$.

Now suppose $x \in Z_r$ is a torsion element of order n. Then $D_n x = D_n x - n^r x$ since $nx = 0$ and hence $D_n x \in Z_{r+1}$ by observation two. Also, $x \in D_{n^{-1}} Z_{r+1} \subset Z_{r+1}$. This implies that $x \in Z_j$ for $j >> 0$, but since they are eventually zero, x must be zero. $\triangle$

Lemmas 4.8 and 4.12 prove Theorem 4.7. Now we define the exponential and logarithm.

$$\exp(z) \triangleq \sum_{k\geq 0} \frac{1}{k!} z^k, \qquad \log(1+z) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} z^k.$$

The usual identities hold here. With this in mind we define

$$\log(\llbracket P\rrbracket) = \log(1 + \llbracket P\rrbracket - 1) = \sum_{k=1}^{d} \frac{(-1)^{k-1}}{k} (\llbracket P\rrbracket - 1)^k.$$

Now we construct our grading on Π^d.

Theorem 4.13

For $P \in \mathcal{P}_d$, let $p = \log(\llbracket P\rrbracket)$. Then

$$\llbracket P\rrbracket^n = 1 + pn + \frac{1}{2}p^2 n^2 + \cdots + \frac{1}{d!} p^d n^d.$$

Proof.

$$\llbracket P\rrbracket^n = \exp(\log(\llbracket P\rrbracket^n)) = \exp(np) = \sum_{k=0}^{d} \frac{1}{k!} p^k n^k$$

$\triangle$

Naturally, we see that our grading takes the form

$$\llbracket P\rrbracket = 1 + p + \frac{1}{2}p^2 + \cdots + \frac{1}{d!} p^d.$$

Let us prove that this is indeed the grading we want. Let $\Pi_k = \mathbb{Z}\{\frac{1}{k!}\log(\llbracket P\rrbracket)^k \mid P \in \mathcal{P}_d\}$. We have immediately that $\Pi^d = \Pi_0 + \Pi_1 + \ldots$. We want this sum to be direct, which will require the following observations.

First, note that

$$D_r(\log(\llbracket P\rrbracket)) = \log(D_r\llbracket P\rrbracket) = \log(\llbracket P\rrbracket^r) = r\log(\llbracket P\rrbracket).$$

Also note that

$$x \in \Pi_k \iff D_r x = r^k x, \quad r \in \mathbb{Q}_{>0}.$$

Proposition 4.15

The sum is direct: $\Pi^d = \bigoplus_{i=0}^d \Pi_d$.

Proof. Suppose that there exist $x_i \in \Pi_i$ such that $x_0 + \cdots + x_d = 0$. Then $D_N(x_0 + \cdots + x_d) = x_0 + Nx_1 + \cdots + N^d x_d = 0$ for all N, which implies that $x_i = 0$. △

Proposition 4.16

The decomposition $\Pi^d = \bigoplus_{i=0}^d \Pi_i$ gives a standard grading.

Proof. What must be shown is that for $x \in \Pi_i$ and $y \in \Pi_j$ that $x * y \in \Pi_{i+j}$. This follows by the fact that D_r is a ring homomorphism:

$$D_r(x * y) = D_r(x) * D_r(y) = r^i x * r^j y = r^{i+j}(x * y).$$

△

1.6.3 The weight algebra

For $P \in \mathcal{P}_d$ and $c \in \mathbb{R}^d$, we denote by P^c the face of P in direction c. Explicitly, $P^c = \{x \in P \mid c^t x \geq c^t y \text{ for all } y \in P\}$.

A $(d-k)$-*frame* is a $(d-k)$-tuple of vectors $U = (\mathbf{u}_1, \ldots, \mathbf{u}_{d-k})$ in $\mathbb{R}^d$ such that $\mathbf{u}_i^t \mathbf{u}_j = \delta_{ij}$. Given a $(d-k)$-frame U we define the face $P^U \triangleq (\ldots((P^{\mathbf{u}_1})^{\mathbf{u}_2}))\ldots)^{\mathbf{u}_{d-k}}$. By orthogonality, the dimension must reduce each step, which implies that $\dim(P^U) \leq k$. We also define so-called *frame functionals* by $V_U(P) \triangleq \mathrm{Vol}_k(P^U)$.

For $P, Q \in \mathcal{P}_d$ we say that Q is a *Minkowski summand* of P and write $Q \leq P$ if there exists $R \in \mathcal{P}_d$ such that $Q + R = P$. Similarly, Q is a *weak Minkowski summand* of P (denoted $Q \preceq P$) if there exists some $\lambda \in \mathbb{R}_{>0}$ and $R \in \mathcal{P}_d$ such that $Q + R = \lambda P$.

Fixing $P \in \mathcal{P}_d$, we define $\Pi(P) \triangleq \mathbb{Z}\{[\![Q]\!] \mid Q \preceq P\}$.

Proposition 5.3

The following hold.

(i) $\Pi(P)$ is a finitely generated graded subalgebra of Π^d

(ii) $Q \preceq P \implies \Pi(Q) \subset \Pi(P)$

(iii) $\Pi(P)$ is generated by Minkowski summands of P

(iv) $\Pi(P) + \Pi(Q) \subset \Pi(P + Q)$

Proof. $\Pi(P)$ is a subalgebra because the Minkowski sum of two weak summands is itself a weak summand. Gradation follows from the fact that $Q \in \Pi(P)$ implies that $\log(Q) \in \Pi(P)$. The other conditions are relatively easy to check, and finite generation will be dealt with following the proof of Theorem 5.8. △

There is a criteria for determining when Q is a summand of P:

Theorem 5.4 (Shephard)

The polytope Q is a Minkowski summand of P if and only if the following conditions are satisfied:

(i) $\dim(P^c) \geq \dim(Q^c)$ for any $c \in \mathbb{R}^d$

(ii) If for some $c \in \mathbb{R}^d$ we have $\dim(P^c) = 1$, then $\mathrm{vol}_1(P^c) \geq \mathrm{vol}_1(Q^c)$

Corollary 5.5

The polytope Q is a weak Minkowski summand of P if and only if $\dim(P^c) \geq \dim(Q^c)$ for all $c \in \mathbb{R}^d$.

Note that the condition above is equivalent to saying that $Q \preceq P$ if and only if the normal fan of P refines the normal fan of Q. Therefore we have that $P \preceq P'$ for some simple polytope P' for all $P \in \mathcal{P}_d$. Then $\Pi(P) \subset \Pi(P')$, which allows us to assume that P is simple.

Fix some simple polytope P together with frame functionals $U(F)$ for $F \in \mathcal{F}(P)$, the face lattice of P, such that $P^{U(F)} = F$. We define the *Minkowksi map* ϕ by

$$\phi : \Pi(P) \to \bigoplus_{i=0}^{d} \mathbb{R}^{f_i(P)},$$

where $[\![Q]\!] \mapsto \big(V_{U(F)}(Q)\big)_{F \in \mathcal{F}(P)}$. This map is injective, but surjectivity fails due to the Minkowski relations:

Theorem 5.8 (Minkowski relations)

Given a polytope P with unit facet normals $\mathbf{u}_1, \ldots, \mathbf{u}_n$, the following linear equation holds:

$$\sum_{i=0}^{n} \mathbf{u}_i \mathrm{Vol}_{d-1}(P^{\mathbf{u}_i}) = 0.$$

Proof. Choose some generic $\mathbf{u} \in \mathbb{R}^d$ to project the polytope onto $\mathbf{u}^\perp$ by the projection map π. The volume of the image of each facet $P^{\mathbf{u}_i}$ is proportional to $|\langle \mathbf{u}, \mathbf{u}_i \rangle| \mathrm{Vol}_{d-1}(P^{\mathbf{u}_i}$. Note that the projection of the lower facets (those with $\langle \mathbf{u}, \mathbf{u}_i \rangle < 0$) and the upper facets (those with $\langle \mathbf{u}, \mathbf{u}_i \rangle > 0$) both cover $\pi(P)$, and so they have the same volume. Therefore we may write

$$\sum_{\{i | \langle \mathbf{u}, \mathbf{u}_i \rangle < 0\}} \langle \mathbf{u}, \mathbf{u}_i \rangle \mathrm{Vol}_{d-1}(P^{\mathbf{u}_i}) = \sum_{\{i | \langle \mathbf{u}, \mathbf{u}_i \rangle < 0\}} \langle \mathbf{u}, \mathbf{u}_i \rangle \mathrm{Vol}_{d-1}(P^{\mathbf{u}_i}).$$

This implies that $\langle \mathbf{u}, \sum_{i=1}^{n} \mathbf{u}_i \mathrm{Vol}_{d-1}(P^{\mathbf{u}_i}) \rangle = 0$ for all $\mathbf{u}$, which proves the claim. △

These Minkowski relations inform the weight algebra's definition. We define a *k-balanced weight* on P to be a function $\omega_k : \mathcal{F}_k(P) \to \mathbb{R}$ such that for every $F \in \mathcal{F}_{k+1}(P)$ we have the following equality in $\mathrm{lin}(F)$:

$$\sum_{G \subset F \text{ a facet}} \omega_k(G) \cdot \mathbf{u}_{G/F} = 0,$$

where $\mathbf{u}_{G/F}$ is the unit outer normal of G in $\mathrm{lin}(F)$. We denote the set of all k-weights by $\Omega_k(P)$, and write $\Omega(P) = \bigoplus_k \Omega_k(P)$.

We can identify the set of weak Minkowski summands with the set of positive 1-weights, which is given as

$$\left\{ y \in \mathbb{R}^{f_1(P)} \;\middle|\; \sum_{i=1}^{m} y(E_i) \cdot \mathbf{E}_i = 0 \text{ if } E_1, \ldots, E_m \text{ form a 2-face, } y(E) \geq 0 \right\}.$$

This is a pointed cone in $\mathbb{R}^{f_1(P)}$, so it has finitely many rays. These correspond to the finite generating set of $\Pi(P)$ referenced in Theorem 5.3.

Theorem 5.11

The Minkowski map ϕ induces a graded isomorphism $\Pi(P) \cong \Omega(P)$ as graded vector spaces.

1.6.4 The structure of $\Omega(P)$

This section was taken from [McM96] to supplement the above paper.

Lemma 6.1

Weights $a \in \Omega_r(P)$ and $b \in \Omega_s(Q)$ induce a weight $a \times b \in \Omega_{r+s}(P \times Q)$.

Proof. We adopt the convention that if a is an r-weight on P and $F \in \mathcal{F}_k(P)$ with $k \neq r$, then $a(F) = 0$. We define $a \times b$ by

$$a \times b(F \times G) \triangleq a(F)b(G).$$

Now note that the Minkowski relation need only be checked on two kinds of face pairs: Either F is a facet of an $r+1$-face and G a facet of an s-face, or F a facet of an r-face and G a facet of an $s+1$-face. If this were not the case, then the facets we consider would be living in $\Omega_{r+s+1}(P \times Q)$ or some lower weight space.

Without loss of generality we assume the former case. Suppose $F' \in \Omega_{r+1}(P)$ and $G \in \mathcal{F}_s(Q)$. Then the unit outer normal vector of $F' \times G$ at a facet F is simply $(\mathbf{u}_F, \mathbf{0})$. The Minkowski relation then takes the form

$$\sum_F (a \times b)(F \times G)(\mathbf{u}_F \times G, \mathbf{0}) = \sum_F a(F)b(G)(\mathbf{u}_F, \mathbf{0})$$

which is zero by the Minkowski relation for a. That is, $a \times b \in \Omega_{r+s}(P \times Q)$. △

Theorem 6.2

If $a \in \Omega_r(P)$ and $b \in \Omega_s(Q)$, then there is an associative and commutative product $ab \in \Omega_{r+s}(P + Q)$.

Bibliography

[AHK18] Karim Adiprasito, June Huh, and Eric Katz. Hodge Theory for Combinatorial Geometries, May 2018.

[Ard04] Federico Ardila. Semimatroids and their Tutte polynomials, August 2004.

[BBL08] Julius Borcea, Petter Brändén, and Thomas M. Liggett. Negative dependence and the geometry of polynomials, July 2008.

[BP11] Yuliy Baryshnikov and Robin Pemantle. Asymptotics of multivariate sequences, part III: Quadratic points, August 2011.

[Brä07] Petter Brändén. Polynomials with the half-plane property and matroid theory. *Advances in Mathematics*, 216(1):302–320, December 2007.

[Cas21] Federico Castillo. A pithy look at the Polytope Algebra, November 2021.

[GKZ94] Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelevinsky. *Discriminants, Resultants, and Multidimensional Determinants.* Birkhäuser Boston, Boston, MA, 1994.

[HSW21] June Huh, Benjamin Schröter, and Botong Wang. Correlation bounds for fields and matroids. *Journal of the European Mathematical Society*, 24(4):1335–1351, June 2021.

[McM96] P. McMullen. Weights on polytopes. *Discrete & Computational Geometry*, 15(4):363–388, April 1996.

[Pem12] Robin Pemantle. Hyperbolicity and stable polynomials in combinatorics and probability, October 2012.