Variational principles for the classical water-wave problem
MTHM005 MMath Project Report

Author: Connor Watson

Supervisor: Hamid Alemi Ardakani

1 Abstract

By deriving Bernoulli's equation of an inviscid, incompressible, and irrotational fluid, we aim to take the first variation of the time and two-dimensional space volume integral of the Bernoulli pressure equation in cartesian coordinates to recover the governing equations of the classical water-wave problem. In considering where the fluid meets the outside atmosphere (the free-surface), the fluid bodies floor (rigid wall or bottom boundary), and the main body of the fluid (field equation for fluid motion), the following governing equations were recovered:

$$\begin{array}{cc} \text{Dynamic free-surface boundary condition} & \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy = 0 \\ \text{Kinematic free-surface boundary condition} & h_t + \phi_x h_x - \phi_y = 0 \\ \text{Bottom boundary condition} & \phi_y = 0 \\ \text{Field equation for fluid motion} & \Delta\phi = 0 \end{array} \tag{1.0.1}$$

Next, each utilised term was translated to a moving mesh. Here, the water waves translated to a rectangular shape with a constant level free-surface.

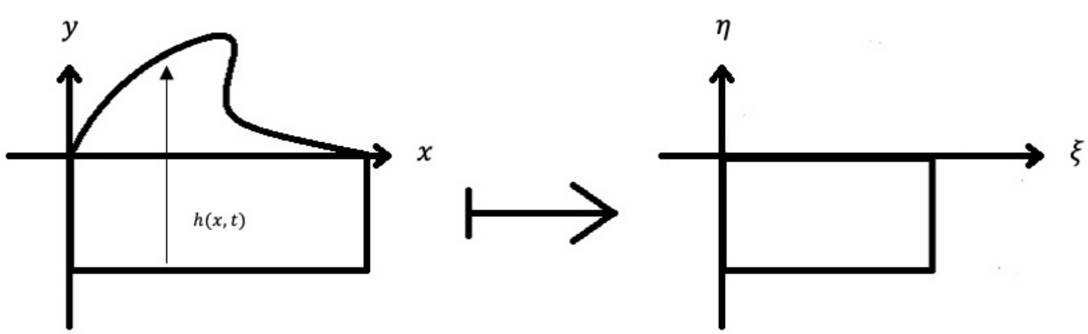

Figure 1: Mapping a time-dependent cartesian fluid to a fixed boundary moving mesh

This allowed us to take the first variation of the volume integral of the transformed Bernoulli's pressure equation in the curvilinear coordinate system to find governing equations:

$$
\begin{array}{cc}
\text{Dynamic free-surface boundary conditions} & \frac{1}{2}Y_\xi(U^2-V^2)-UVX_\xi-gYY_\xi=\Phi_\tau Y_\xi-\Phi_\xi Y_\tau \\
 & \frac{1}{2}X_\xi(U^2-V^2)+UVY_\xi+gYX_\xi=\Phi_\xi X_\tau-\Phi_\tau X_\xi \\
\text{Kinematic free-surface boundary condition} & J^{-1}g_{12}\phi_\xi\big|^{\eta=0}-J^{-1}g_{22}\phi_\eta\big|^{\eta=0}+X_\xi Y_\tau-Y_\xi X_\tau=0 \\
\text{Bottom boundary condition} & x_\xi\phi_\eta-\phi_\xi x_\eta=0 \\
\text{Field equation for fluid motion} & \Delta_g\phi=0
\end{array}
\tag{1.0.2}
$$

Finally, a rigid body was placed in the fluid:

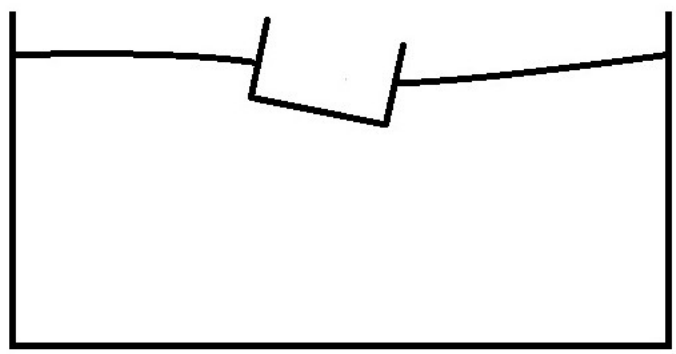

Figure 2: Rigid floating body in a sloshing fluid

Taking the variations of new terms that appeared, cartesian governing equations from (1.0.2) were recovered again as expected, as well as the contact boundary condition for a rigid body $\partial\phi/\partial\boldsymbol{n}=\boldsymbol{\nabla}\phi\cdot\boldsymbol{n}$ and boundary conditions for the rigid body:

$$
\begin{gathered}
-\mathbb{I}\ddot{\theta}+m\overline{y}\left(\ddot{q_1}\cos\theta+\ddot{q_2}\sin\theta\right)+m\overline{x}(\ddot{q_1}\sin\theta-\ddot{q_2}\cos\theta)+mg(\overline{y}\sin\theta-\overline{x}\cos\theta)+\int_s P(\dot{\boldsymbol{Q}}\boldsymbol{x}_s\cdot\boldsymbol{n})ds=0 \\
-m\overline{y}\dot{\theta}\left(\dot{\theta}\sin\theta+\ddot{\theta}\cos\theta\right)+m\overline{x}\dot{\theta}(\dot{\theta}\cos\theta+\ddot{\theta}\sin\theta)-m\ddot{q_1}+\int_s Pn_1\delta q_1 ds=0 \\
m\overline{y}\dot{\theta}\left(\dot{\theta}\cos\theta+\ddot{\theta}\sin\theta\right)+m\overline{x}\dot{\theta}(\dot{\theta}\sin\theta-\ddot{\theta}\cos\theta)-m\ddot{q_2}-mg+\int_s Pn_2\delta q_2 ds=0.
\end{gathered}
\tag{1.0.3}
$$

2 Introduction

By defining a function L as the Lagrangian that consists of kinetic energy subtracted by potential energy, we may write $L = Kinetic - Potential = K - P$ as [4]:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}m||\dot{\boldsymbol{q}}||^2 - P(\boldsymbol{q}). \tag{2.0.1}$$

For $\boldsymbol{q}$, a point mass position, in this example we have that $\boldsymbol{q} \in \mathbb{R}^2$, as the model is in two dimensions. Substituting this into the following Euler-Lagrange equation [4] gives:

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{\boldsymbol{q}}}\left(\frac{1}{2}m||\dot{\boldsymbol{q}}||^2 - P(\boldsymbol{q})\right)\right) - \frac{\partial}{\partial \boldsymbol{q}}\left(\frac{1}{2}m||\dot{\boldsymbol{q}}||^2 - P(\boldsymbol{q})\right) \\ &= \frac{d}{dt}(m\dot{\boldsymbol{q}}) + \frac{\partial P}{\partial \boldsymbol{q}} = m\ddot{\boldsymbol{q}} + \frac{\partial P}{\partial \boldsymbol{q}} \implies m\ddot{\boldsymbol{q}} = -\frac{\partial P}{\partial \boldsymbol{q}}. \end{aligned} \tag{2.0.2}$$

Define $\boldsymbol{q}_0(t)$ as a smooth curve, such that a and b are its end points. This curve can be deformed by adding some variable into the curve's equation. Define this variable as s, where $s \in (-\epsilon, \epsilon)$ with $\epsilon > 0$, thus it's deformation equation is defined $\boldsymbol{q}(t, s)$ where $\boldsymbol{q}(t, s)\big|_{s=0} = \boldsymbol{q}(t, 0) = \boldsymbol{q}_0(t)$. The amount this defined curve $\boldsymbol{q}$ 'varies' from its original path $\boldsymbol{q}_0$ is called its variation, and is clearly defined with derivative [4]:

$$\delta\boldsymbol{q}(t) = \frac{d}{ds}\bigg|_{s=0} \boldsymbol{q}(t, s). \tag{2.0.3}$$

If the function has fixed endpoints, i.e., where $\boldsymbol{q}_0(a) = \boldsymbol{q}(a, s)$ and $\boldsymbol{q}_0(b) = \boldsymbol{q}(b, s)$ $\forall s$, then clearly it's variation at those points is zero. If we have a functional $S(\boldsymbol{q}) = \int_a^b L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))dt$, we relate that $\delta S = \frac{d}{ds}\big|_{s=0} S(\boldsymbol{q}(t, s))$ also [4]. Taking this relation into consideration:

$$\begin{aligned} \delta S &= \frac{d}{ds}\bigg|_{s=0} S(\boldsymbol{q}(t, s)) = \frac{d}{ds}\bigg|_{s=0} \int_a^b L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))dt = \int_a^b \left(\frac{d}{ds}\bigg|_{s=0} L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))\right) dt \\ &= \int_a^b \left(\frac{\partial L}{\partial \boldsymbol{q}} \cdot \frac{d}{ds}\bigg|_{s=0} \boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d}{ds}\bigg|_{s=0} \dot{\boldsymbol{q}}\right) dt = \int_a^b \left(\frac{\partial L}{\partial \boldsymbol{q}} \cdot \delta\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta\dot{\boldsymbol{q}}\right) dt \\ &= \int_a^b \left(\frac{\partial L}{\partial \boldsymbol{q}} \cdot \delta\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d}{dt}\delta\boldsymbol{q}\right) dt. \end{aligned} \tag{2.0.4}$$

Considering the $\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d}{dt}\delta\boldsymbol{q}$ term, we have that $\int_a^b \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d}{dt}\delta\boldsymbol{q} dt = \int_a^b \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot d\delta\boldsymbol{q}$. Here we can use integration by parts, defining $u = \frac{\partial L}{\partial \dot{\boldsymbol{q}}},\ dv = d\delta\boldsymbol{q}$:

$$\int_a^b \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot d\delta\boldsymbol{q} = \left[\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta\boldsymbol{q}\right]_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \cdot \delta\boldsymbol{q}. \tag{2.0.5}$$

Due to endpoint conditions, we have $\delta\boldsymbol{q} = 0$ at $t = b$ and $t = a$, hence $\left[\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \delta\boldsymbol{q}\right]_a^b = 0$. Therefore we have that $\int_a^b \left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d}{dt}\delta\boldsymbol{q}\right) dt = -\int_a^b \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \cdot \delta\boldsymbol{q}$. Therefore:

$$\delta S = \int_a^b \left(\frac{\partial L}{\partial \boldsymbol{q}} \cdot \delta \boldsymbol{q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \cdot \delta \boldsymbol{q} \right) dt = \int_a^b \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\right) \cdot \delta \boldsymbol{q} dt = 0. \tag{2.0.6}$$

As we have previously defined the Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = 0$, therefore the above integral equals zero [4]. Hence proven for a functional S we have that $\delta S = 0$.

The aim is to reduce the Navier-Stokes equations of motion for an inviscid, incompressible, and irrotational flow. This allows us to define a Lagrangian for the classical water-wave problem, and use (2.0.6) to solve the boundary-value problem. Denote:

- Fluid velocity vector depending on spacial variables and time: $\boldsymbol{u}(x, y, t)$
- Lagrangian or Material derivative: $D/Dt = \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$
- Fluid density: ρ
- Kinematic fluid viscosity using Dynamic fluid viscosity μ: $\nu = \mu/\rho$
- Fluid pressure depending on spacial variables and time: $p(x, y, t)$
- Vertical gravitational constant vector: $\boldsymbol{g} = (0, -g)$

The Navier-Stokes equations for an incompressible fluid are [1]:

$$\frac{D\boldsymbol{u}}{Dt} - \nu \boldsymbol{\nabla}^2 \boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{g}. \tag{2.0.7}$$

For an inviscid fluid $\mu = \nu = 0$ [1], and for an incompressible fluid $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$ [1], this transforms the the Navier-Stokes equation to:

$$\frac{D\boldsymbol{u}}{Dt} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{g}. \tag{2.0.8}$$

Throughout, we will be referencing a two-dimensional model. We aim to study variational principles for the boundary-value problem for the classical water waves in two-dimensions. In two dimensions, the velocity field is $\boldsymbol{u} = \boldsymbol{u}(x, y, t) = (u, v, 0)$, where u and v are the velocity components in the x and y direction respectively. So, the material derivative in two-dimensions becomes:

$$\frac{D\boldsymbol{u}}{Dt} = \frac{Du}{Dt}\hat{x} + \frac{Dv}{Dt}\hat{y}. \tag{2.0.9}$$

Using (2.0.8) and (2.0.9) we separate horizontal and vertical components respectively:

$$u_t + uu_x + vu_y = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2.0.10}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}\frac{\partial p}{\partial v} - g. \tag{2.0.11}$$

For an irrotational fluid the vorticity $\boldsymbol{\omega}$ is zero [3], i.e., $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{\omega} = 0$. As the curl of the gradient of a scalar field is zero, this implies $\boldsymbol{u}$ can be expressed as the gradient of a scalar field; this scalar field is called the velocity potential $\phi = \phi(x, y, t)$ [2], where $\boldsymbol{u} = \boldsymbol{\nabla}\phi$:

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}. \tag{2.0.12}$$

3 The classical water-wave problem in two-dimensions

In this section, the boundary-value problem for the two-dimensional water waves is presented.

3.1 The kinematic free-surface boundary condition

If we assume that in the fluid, there is a free-surface at height $y = h(x,t)$, then:

$$\frac{Dh}{Dt} = \frac{\partial}{\partial t}h + (\boldsymbol{u} \cdot \boldsymbol{\nabla})h = h_t + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)h = h_t + u\frac{\partial h}{\partial x}. \tag{3.1.1}$$

Since $h = h(x,t)$. Substituting for u in terms of the velocity potential gives:

$$\frac{Dh}{Dt} = h_t + u\frac{\partial h}{\partial x} = h_t + \phi_x h_x = \frac{Dy}{Dt}. \tag{3.1.2}$$

Further, as $\frac{Dy}{Dt} = \frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x} + v\frac{\partial y}{\partial y}$, and as y is independent of x and t, then $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial t} = 0$, and $\frac{\partial y}{\partial y} = 1$, giving $\frac{Dy}{Dt} = v = \phi_y$, we may conclude that on boundary $y = h(x,t)$:

$$\frac{Dy}{Dt} = \frac{Dh}{Dt} \implies h_t + \phi_x h_x - \phi_y = 0, \tag{3.1.3}$$

which is the kinematic boundary condition [2] [5].

3.2 Rigid wall boundary conditions

On the floor surface of a body of fluid, i.e., where $y = 0$, we have particles of fluid that can not physically pass vertically, as there is an unmoving horizontal rigid wall. This means that we may set the vertical component of the velocity field to zero, hence on the boundary $y = 0$ we have:

$$v = \phi_y = 0, \tag{3.2.1}$$

which is the bottom boundary condition [2] [5].

3.3 Dynamic free-surface boundary condition

Starting with the momentum equations (2.0.10) and substituting for the velocity field $\boldsymbol{u}(x,y,t)$ in terms of the velocity potential (2.0.12) gives:

$$\begin{gathered}
\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\implies \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right) + \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) = 0.
\end{gathered} \tag{3.3.1}$$

Therefore we can derive the following equations for substitution into (3.3.1):

$$\begin{aligned}\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} &= \frac{1}{2}\frac{\partial}{\partial x}(\phi_x^2),\\ \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} &= \frac{1}{2}\frac{\partial}{\partial x}(\phi_y^2).\end{aligned} \tag{3.3.2}$$

$$\begin{gathered}\implies \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right) + \frac{\partial}{\partial x}\left(\frac{1}{2}(\phi_x^2+\phi_y^2)\right) + \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) = 0\\ \implies \frac{\partial}{\partial x}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\phi_x^2+\phi_y^2) + \frac{p}{\rho} + gy\right] = 0 \implies \frac{\partial \phi}{\partial t} + \frac{1}{2}(\phi_x^2+\phi_y^2) + \frac{p}{\rho} + gy = c(y).\end{gathered} \tag{3.3.3}$$

Similarly, recall equation (2.0.11) and substitute in (2.0.12):

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g. \tag{3.3.4}$$

This equation further reduces to:

$$\begin{aligned}\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial y} &= \frac{1}{2}\frac{\partial}{\partial y}(\phi_x^2),\\ \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial y^2} &= \frac{1}{2}\frac{\partial}{\partial y}(\phi_y^2).\end{aligned} \tag{3.3.5}$$

$$\begin{gathered}\implies \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) + \frac{\partial}{\partial y}\left(\frac{1}{2}(\phi_x^2+\phi_y^2)\right) + \frac{\partial}{\partial y}\left(\frac{p}{\rho}\right) + g = 0\\ \implies \frac{\partial}{\partial y}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\phi_x^2+\phi_y^2) + \frac{p}{\rho} + gy\right] = 0 \implies \frac{\partial \phi}{\partial t} + \frac{1}{2}(\phi_x^2+\phi_y^2) + \frac{p}{\rho} + gy = c(x).\end{gathered} \tag{3.3.6}$$

Therefore, from (3.3.3) and (3.3.6) we can conclude that:

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\phi_x^2+\phi_y^2) + \frac{p}{\rho} + gy = \textsf{constant}. \tag{3.3.7}$$

The constant is absorbed into the derivatives of the velocity potential; this gives:

$$\phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) + \frac{p}{\rho} + gy = 0 \implies -\rho\left(\phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) + gy\right) = p(x,y,t). \tag{3.3.8}$$

This is Bernoulli's equation for the pressure field [2]. On the free-surface $y = h(x,t)$ the pressure is equivalent to the atmospheric pressure [2], giving that $p = 0$. Hence on the boundary $y = h(x,t)$, this gives the dynamic free-surface boundary condition [2] [5]:

$$\phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) + gy = 0. \tag{3.3.9}$$

3.4 The field equation for the fluid motion

The incompressibility condition gives:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{in } 0 < y < h(x,t). \tag{3.4.1}$$

Substituting (2.0.12) into the incompressibility condition (3.4.1) gives the Laplace equation for fluid motion in $0 < y < h(x,t)$:

$$\begin{aligned} 0 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} &= \frac{\partial}{\partial x}(\phi_x) + \frac{\partial}{\partial y}(\phi_y) = \phi_{xx} + \phi_{yy} \\ &\implies \Delta\phi = \phi_{xx} + \phi_{yy} = 0. \end{aligned} \tag{3.4.2}$$

which is the field equation for fluid motion [2] [5].

3.5 Luke's variational principle

The aim in this section is to present a variational principle for the boundary-value problem (3.1.3), (3.2.1), (3.3.9), and (3.4.2). Define:

$$L(\phi, h) = \int_0^{h(x,t)} \left(\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy \right) dy. \tag{3.5.1}$$

Luke's variational principle takes the form:

$$\delta S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} L(\phi, h) dx dt = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_0^{h(x,t)} \left(\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy \right) dy dx dt = 0. \tag{3.5.2}$$

Luke's variational principle ($\delta S = 0$), is the variation of the double integral of the pressure fluid between fixed horizontal points $[x_1, x_2]$ in the fluid, with a horizontal bottom topography $y = 0$ and the free-surface $y = h(x,t)$, and fixed times $[t_1, t_2]$ [5].

$L(\phi, h)$ is the Lagrangian functional, we may use this to find boundary conditions as by (2.0.6) $\delta L = 0$ as a first variation of a functional. This is equation (3.3.8), or Bernoulli's equation, where $L(\phi, h) = \int_0^{h(x,t)} p(x,y,t) dy$. The coefficient $-\rho$ is neglected because it does not affect $\delta S = 0$. Noting that in two-dimensional space $\phi_x^2 + \phi_y^2 = \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi$, Luke's variational principle may be written as:

$$\delta \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_0^{h(x,t)} \left(\phi_t + \frac{1}{2}(\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi) + gy \right) dy dx dt. \tag{3.5.3}$$

Taking into account that the first variation is taken with respect to ϕ and $h(x,t)$, the variational principle $\delta S = 0$ becomes:

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} \delta \int_0^{h(x,t)} \left(\phi_t + \frac{1}{2}(\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi) + gy \right) dy dx dt = 0. \tag{3.5.4}$$

We may use the Leibniz integral rule with variable limits to get:

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{x_1}^{x_2} p(x,y,t)|^{y=h}\delta h dxdt + \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \delta\left(\phi_t + \frac{1}{2}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi) + gy\right) dydxdt = 0 \\
\implies &\int_{t_1}^{t_2}\int_{x_1}^{x_2} p(x,h,t)\delta h dxdt + \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \left(\delta\phi_t + \frac{1}{2}(\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\delta\phi) + gy\right) dydxdt = 0 \\
\implies &\int_{t_1}^{t_2}\int_{x_1}^{x_2} p(x,h,t)\delta h dxdt + \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \left(\delta\phi_t + (\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\phi) + gy\right) dydxdt = 0.
\end{aligned} \tag{3.5.5}$$

First, consider the term $\delta\phi_t$:

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \delta\phi_t dydxdt. \tag{3.5.6}$$

By product rule we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \delta\phi dydxdt &= \int_{t_1}^{t_2}\int_{x_1}^{x_2}\frac{\partial}{\partial t}\int_0^{h(x,t)} \delta\phi dydxdt \\
&= \int_{t_1}^{t_2}\int_{x_1}^{x_2} h_t\delta\phi|^{y=h} dxdt + \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \delta\phi_t dydxdt.
\end{aligned} \tag{3.5.7}$$

As $\int_a^b \frac{dF}{dx}dx = F(b) - F(a)$, (3.8.7) becomes:

$$\int_{t_1}^{t_2}\frac{\partial}{\partial t}\int_{x_1}^{x_2}\int_0^{h(x,t)} (\delta\phi) dydxdt = \int_{x_1}^{x_2}\int_0^{h(x,t)} [\delta\phi|^{t_2} - \delta\phi|_{t_1}] dydx = 0. \tag{3.5.8}$$

Using endpoint conditions $\delta\phi(t_2) = \delta\phi(t_1) = 0$. Therefore we are left with:

$$\begin{aligned}
0 &= \int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \delta\phi_t dydxdt + \int_{t_1}^{t_2}\int_{x_1}^{x_2} h_t\delta\phi|^{y=h} dxdt \\
\implies &\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)} \delta\phi_t dydxdt = -\int_{t_1}^{t_2}\int_{x_1}^{x_2} h_t\delta\phi|^{y=h} dxdt.
\end{aligned} \tag{3.5.9}$$

Next, consider the divergence theorem:

$$\iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{u} dV = \iint_{\partial V} \boldsymbol{u}\cdot\boldsymbol{n} dS, \tag{3.5.10}$$

which relates the volume integral of the divergence of a vector field $\boldsymbol{u}$ over the volume V to the surface integral of the vector field $\widehat{\boldsymbol{u}}$ over the closed surface, noting that $\boldsymbol{u}\cdot\boldsymbol{n}$ is the flux of $\boldsymbol{u}$ through the surface ∂V. In this case, we apply the divergence theorem to derive Green's first identity, which is useful in further simplifications in Luke's variational principle. If we replace $\boldsymbol{u}$ in the divergence theorem with $\boldsymbol{u} = f\boldsymbol{\nabla}g$, where f and g are differentiable scalar functions, then using the vector identity $\boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}g) = \boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + f\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}g$ gives Green's first identity [6]:

$$\iiint_V \boldsymbol{\nabla}\cdot\boldsymbol{u}dV = \iint_{\partial V}\boldsymbol{u}\cdot\boldsymbol{n}dS \implies \iiint_V \boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla}g)dV = \iint_{dV} f\boldsymbol{\nabla}g\cdot\boldsymbol{n}dS$$
$$\implies \iiint_V(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + f\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}g)dV = \iiint_V(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + f\boldsymbol{\nabla}^2 g)dV = \iint_{dV}(f\boldsymbol{\nabla}g)\cdot\boldsymbol{n}dS$$
$$\implies \iiint_V(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g + f\Delta g)dV = \iint_{dV}(f\boldsymbol{\nabla}g)\cdot\boldsymbol{n}dS, \tag{3.5.11}$$

which may also be written in the form:

$$\iiint_V(\boldsymbol{\nabla}f\cdot\boldsymbol{\nabla}g)dV = \iint_{dV}(f\boldsymbol{\nabla}g)\cdot\boldsymbol{n}dS - \iiint_V(f\Delta g)dV. \tag{3.5.12}$$

Now apply Green's first identity to Luke's principle. Denoting $\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}g$ and $\boldsymbol{\nabla}\delta\phi = \boldsymbol{\nabla}f$ follows from (3.5.12) that:

$$\iiint_V(\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\delta\phi)dV = \iiint_V(\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\phi)dV = \iint_{\partial V}(\delta\phi\boldsymbol{\nabla}\phi)\cdot\boldsymbol{n}dS - \iiint_V(\delta\phi\Delta\phi)dV. \tag{3.5.13}$$

The term $(\delta\phi\boldsymbol{\nabla}\phi)\cdot\boldsymbol{n}$ on the right hand side of (3.5.13), can be written as:

$$\iint_{dV}(\delta\phi\boldsymbol{\nabla}\phi)\cdot\boldsymbol{n}dS = \int_{t_1}^{t_2}\int_{dA}(\delta\phi\boldsymbol{\nabla}\phi)\cdot\boldsymbol{n}dS = \int_{t_1}^{t_2}\int_{dA}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{n})\delta\phi dS. \tag{3.5.14}$$

For our two-dimensional problem the fluid occupies the region $0 \le y \le h(x,t)$. Hence denoting the boundary ∂V of the volume V by $\Sigma = \{(x,y) : y = 0, x_1 < x < x_2\}$, which is the sea's bottom floor, and $\Gamma = \{(x,y) : y = h(x,t), x_1 < x < x_2\}$, which is the free-surface of the fluid, we have:

$$\iint_{dV}(\delta\phi\boldsymbol{\nabla}\phi)\cdot\boldsymbol{n}dS = \int_{t_1}^{t_2}\int_{\Sigma}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{n})\delta\phi dS + \int_{t_1}^{t_2}\int_{\Gamma}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{n})\delta\phi dS. \tag{3.5.15}$$

First considering the static surfaces. The static surface in question is the floor of the pool of liquid, i.e., when $y = 0$. This is an assumed level and smooth surface, hence the vector $\boldsymbol{n} = (0, -1)$. Therefore as $\boldsymbol{\nabla}\phi = (\phi_x, \phi_y)$:

$$\int_{t_1}^{t_2}\int_{\Sigma}(\boldsymbol{\nabla}\phi\cdot\boldsymbol{n})\delta\phi dt = \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left(\begin{bmatrix}\phi_x\\ \phi_y\end{bmatrix}\cdot\begin{bmatrix}0\\ -1\end{bmatrix}\right)\delta\phi\Big|_{y=0} dxdt = -\int_{t_1}^{t_2}\int_{x_1}^{x_2}\phi_y\Big|_{y=0}\delta\phi dxdt. \tag{3.5.16}$$

Next, consider the free-surface of waves. This is an assumed smooth but uneven surface, hence $\boldsymbol{n}$, the outward pointing normal to the surface of the fluid, is a function of x at any time t. At the free-surface, $y = h(x,t)$. Hence, $y - h(x,t) = 0$. In a geometric space, we can see this wave instead as a curve, denoted a function of x and y where $q(x,y) = y - h(x,t) = 0$. Therefore, for a curve, the normal vector:

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla}q}{||\boldsymbol{\nabla}q||} = \frac{\begin{bmatrix}-h_x\\ 1\end{bmatrix}}{\left|\left|\begin{bmatrix}-h_x\\ 1\end{bmatrix}\right|\right|} = \frac{\begin{bmatrix}-h_x\\ 1\end{bmatrix}}{\sqrt{1+h_x^2}} = \begin{bmatrix}-h_x\\ 1\end{bmatrix}\ell^{-1}, \tag{3.5.17}$$

where we denote $\ell = \sqrt{1 + h_x^2}$. Thus as $\boldsymbol{\nabla}\phi = (\phi_x, \phi_y)$:

$$\begin{aligned}\int_{t_1}^{t_2} \int_{\Gamma} (\boldsymbol{\nabla}\phi \cdot \boldsymbol{n})\delta\phi dt &= \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left(\begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} \cdot \begin{bmatrix} -h_x \\ 1 \end{bmatrix} \right) \ell^{-1} \delta\phi|^{y=h} dxdt \\ &= \int_{t_1}^{t_2} \int_{x_1}^{x_2} (\phi_x h_x + \phi_y)\, \ell^{-1} \delta\phi\Big|^{y=h} dxdt,\end{aligned} \tag{3.5.18}$$

where we evaluate the integrand at $y = h$ as Γ is the free-surface.

3.6 From Luke's variational principle to the boundary-value problem for the classical water-wave

Firstly, from the summation of (3.5.9) and (3.5.18), we see that we can recover the kinematic boundary condition at the free-surface $y = h(x, t)$:

$$\begin{aligned}\int_{t_1}^{t_2} \int_{x_1}^{x_2} \left(\begin{bmatrix} \phi_x \\ \phi_y \end{bmatrix} \cdot \begin{bmatrix} -h_x \\ 1 \end{bmatrix} \right) \ell^{-1} \delta\phi|^{y=h} dxdt &- \int_{t_1}^{t_2} \int_{x_1}^{x_2} h_t \delta\phi|^{y=h} dxdt = 0 \\ = \int_{t_1}^{t_2} \int_{x_1}^{x_2} &(\phi_y - \phi_x h_x - h_t)\delta\phi|^{y=h} dxdt \\ \implies h_t + \phi_x h_x - \phi_y = 0 &\qquad \text{at} \;\; y = h(x, t).\end{aligned} \tag{3.6.1}$$

Next, take (3.5.16). The invariance of L with respect to variation in the velocity potential ϕ along the bottom surface $y = 0$ recovers the bottom boundary condition:

$$-\int_{t_1}^{t_2} \int_{x_1}^{x_2} \phi_y \delta\phi dxdt = 0 \implies -\phi_y = 0 \implies \phi_y = v = 0 \qquad \text{at} \;\; y = 0. \tag{3.6.2}$$

Furthermore, take the right-hand term $\delta\phi\Delta\phi$ of (3.5.13). In $0 < y < h(x, t)$ we recover the field equation for the fluid motion:

$$\begin{aligned}-\iiint_V (\delta\phi\Delta\phi) dV = 0 &\implies -\Delta\phi = -\phi_{xx} - \phi_{yy} = 0 \\ \implies \Delta\phi = \phi_{xx} + \phi_{yy} = 0 &\qquad \text{in} \;\; 0 < y < h(x, t).\end{aligned} \tag{3.6.3}$$

Finally, from the first term in (3.5.5) we can conclude that the invariance of the Lagrangian with respect to a variation in the free-surface height δh yields the dynamic free-surface boundary condition:

$$\phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy = 0 \qquad \text{at} \;\; y = h(x, t). \tag{3.6.4}$$

4 Transforming an irrotational two-dimensional cartesian wave model to curvilinear coordinates

The aim of this section is to transform the boundary-value problem for the classical water-wave problem (3.6.1), (3.6.2), (3.6.3), and Luke's variational principle (3.5.2) to the viewpoint of a time-dependent moving mesh.

4.1 Denoting the coordinate system

Introduce an arbitrary time-dependent transformation of the form [2]

$$(\xi,\eta,\tau) \mapsto \begin{bmatrix} x(\xi,\eta,\tau) \\ y(\xi,\eta,\tau) \\ t(\xi,\eta,\tau) \end{bmatrix}, \tag{4.1.1}$$

where (ξ,η) are coordinates of a fixed, time-independent computational space $\Omega = \{\xi_1 \leq \xi \leq \xi_2, -\widetilde{h} \leq \eta \leq 0\}$. Hence:

$$\begin{bmatrix} \xi \\ \eta \\ \tau \end{bmatrix} = \begin{bmatrix} \xi(x,y,t) \\ \eta(x,y,t) \\ \tau(x,y,t) \end{bmatrix}, \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} x(\xi,\eta,\tau) \\ y(\xi,\eta,\tau) \\ t(\xi,\eta,\tau) \end{bmatrix}. \tag{4.1.2}$$

4.2 Transformed derivatives

Using (4.1.2) and the chain rule, we have derivatives:

$$\frac{\partial}{\partial \xi} = x_\xi \frac{\partial}{\partial x} + y_\xi \frac{\partial}{\partial y} + t_\xi \frac{\partial}{\partial t}, \tag{4.2.1}$$

$$\frac{\partial}{\partial \eta} = x_\eta \frac{\partial}{\partial x} + y_\eta \frac{\partial}{\partial y} + t_\eta \frac{\partial}{\partial t}, \tag{4.2.2}$$

$$\frac{\partial}{\partial \tau} = x_\tau \frac{\partial}{\partial x} + y_\tau \frac{\partial}{\partial y} + t_\tau \frac{\partial}{\partial t}, \tag{4.2.3}$$

which can be displayed in matrix form:

$$\begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \\ \partial/\partial\tau \end{bmatrix} = \begin{bmatrix} x_\xi & y_\xi & t_\xi \\ x_\eta & y_\eta & t_\eta \\ x_\tau & y_\tau & t_\tau \end{bmatrix} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial t \end{bmatrix} \implies \begin{bmatrix} x_\xi & y_\xi & t_\xi \\ x_\eta & y_\eta & t_\eta \\ x_\tau & y_\tau & t_\tau \end{bmatrix}^{-1} \begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \\ \partial/\partial\tau \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial t \end{bmatrix}. \tag{4.2.4}$$

Furthermore, for our water waves hydrodynamics problem assume that $t = t(\tau)$ and $\tau = \tau(t)$ instead. This also implies that $t = \tau$, and it follows that $t_\tau = 1, t_\xi = t_\eta = 0$. Hence the equation simplifies to

$$\begin{bmatrix} x_\xi & y_\xi & 0 \\ x_\eta & y_\eta & 0 \\ x_\tau & y_\tau & 1 \end{bmatrix}^{-1} \begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \\ \partial/\partial\tau \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial t \end{bmatrix}. \tag{4.2.5}$$

We find the determinant:

$$\begin{vmatrix} x_\xi & y_\xi & 0 \\ x_\eta & y_\eta & 0 \\ x_\tau & y_\tau & 1 \end{vmatrix} = x_\xi y_\eta - y_\xi x_\eta = J, \tag{4.2.6}$$

where J is the Jacobian. Denoting $(\cdot)_{MoC}$ as the matrix of cofactors, and $(\cdot)_{Adj}$ the adjoint matrix, we calculate the inverse:

$$\begin{aligned}
\begin{bmatrix} x_\xi & y_\xi & 0 \\ x_\eta & y_\eta & 0 \\ x_\tau & y_\tau & 1 \end{bmatrix}_{MoC} &= \begin{bmatrix} y_\eta & -x_\eta & x_\eta y_\tau - y_\eta x_\tau \\ -y_\xi & x_\xi & -x_\xi y_\tau + y_\xi x_\tau \\ 0 & 0 & x_\xi y_\eta - y_\xi x_\eta \end{bmatrix} = \begin{bmatrix} y_\eta & -x_\eta & x_\eta y_\tau - y_\eta x_\tau \\ -y_\xi & x_\xi & -x_\xi y_\tau + y_\xi x_\tau \\ 0 & 0 & J \end{bmatrix} \\
\implies \begin{bmatrix} x_\xi & y_\xi & 0 \\ x_\eta & y_\eta & 0 \\ x_\tau & y_\tau & 1 \end{bmatrix}_{Adj} &= \begin{bmatrix} y_\eta & -y_\xi & 0 \\ -x_\eta & x_\xi & 0 \\ x_\eta y_\tau - y_\eta x_\tau & -x_\xi y_\tau + y_\xi x_\tau & J \end{bmatrix} \\
\implies \begin{bmatrix} x_\xi & y_\xi & 0 \\ x_\eta & y_\eta & 0 \\ x_\tau & y_\tau & 1 \end{bmatrix}^{-1} &= J^{-1} \begin{bmatrix} y_\eta & -y_\xi & 0 \\ -x_\eta & x_\xi & 0 \\ x_\eta y_\tau - y_\eta x_\tau & -x_\xi y_\tau + y_\xi x_\tau & J \end{bmatrix}.
\end{aligned} \tag{4.2.7}$$

Therefore, using this substitution into (4.2.5) we get:

$$J^{-1} \begin{bmatrix} y_\eta & -y_\xi & 0 \\ -x_\eta & x_\xi & 0 \\ x_\eta y_\tau - y_\eta x_\tau & -x_\xi y_\tau + y_\xi x_\tau & J \end{bmatrix} \begin{bmatrix} \partial/\partial\xi \\ \partial/\partial\eta \\ \partial/\partial\tau \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial t \end{bmatrix}, \tag{4.2.8}$$

which implies [2]:

$$\frac{\partial}{\partial x} = J^{-1}\left(y_\eta \frac{\partial}{\partial \xi} - y_\xi \frac{\partial}{\partial \eta}\right), \tag{4.2.9}$$

$$\frac{\partial}{\partial y} = J^{-1}\left(x_\xi \frac{\partial}{\partial \eta} - x_\eta \frac{\partial}{\partial \xi}\right), \tag{4.2.10}$$

$$\frac{\partial}{\partial t} = J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\frac{\partial}{\partial \xi} + (y_\xi y_\tau - x_\xi y_\tau)\frac{\partial}{\partial \eta} + J\frac{\partial}{\partial \tau}\right). \tag{4.2.11}$$

4.3 Geometric conservation law (GCL)

Denoting:

$$F_1 = x_\eta y_\tau - y_\eta x_\tau. \tag{4.3.1}$$

$$F_2 = x_\tau y_\xi - y_\tau x_\xi. \tag{4.3.2}$$

we may see that:

$$\begin{aligned}
\frac{\partial J}{\partial \tau} &= x_{\xi\tau} y_\eta + x_\xi y_{\eta\tau} - y_{\xi\tau} x_\eta - y_\xi x_{\eta\tau} \\
\frac{\partial F_1}{\partial \xi} &= x_{\eta\xi} y_\tau + x_\eta y_{\xi\tau} - y_{\xi\eta} x_\tau - y_\eta x_{\xi\tau} \\
\frac{\partial F_2}{\partial \eta} &= y_{\xi\eta} x_\tau + y_\xi x_{\eta\tau} - x_{\xi\eta} y_\tau - x_\xi y_{\eta\tau}.
\end{aligned} \tag{4.3.3}$$

Hence:

$$\begin{aligned}
&\frac{\partial J}{\partial \tau} + \frac{\partial F_1}{\partial \xi} + \frac{\partial F_2}{\partial \eta} \\
&= x_{\xi\tau} y_\eta + x_\xi y_{\eta\tau} - y_{\xi\tau} x_\eta - y_\xi x_{\eta\tau} + x_{\eta\xi} y_\tau + x_\eta y_{\xi\tau} \\
&\quad - y_{\xi\eta} x_\tau - y_\eta x_{\xi\tau} + y_{\xi\eta} x_\tau + y_\xi x_{\eta\tau} - x_{\xi\eta} y_\tau - x_\xi y_{\eta\tau} \\
&= (x_{\xi\tau} y_\eta - y_\eta x_{\xi\tau}) + (x_\xi y_{\eta\tau} - x_\xi y_{\eta\tau}) + (x_{\eta\xi} y_\tau - x_{\xi\eta} y_\tau) \\
&\quad + (x_\eta y_{\xi\tau} - y_{\xi\tau} x_\eta) + (y_{\xi\eta} x_\tau - y_{\xi\eta} x_\tau) + (y_\xi x_{\eta\tau} - y_\xi x_{\eta\tau}) = 0.
\end{aligned} \tag{4.3.4}$$

$$\implies \frac{\partial J}{\partial \tau} + \frac{\partial F_1}{\partial \xi} + \frac{\partial F_2}{\partial \eta} = 0. \tag{4.3.5}$$

This is called the geometric conservation law [2].

4.4 Transformed kinematic boundary condition

The fluid occupies a simply-connected subset of $\mathbb{R}^2$, with coordinates (x, y). In $\mathbb{R}^2$ the set $\Gamma(t)$, defined by $\Gamma(t) = \{(x, y) \in \mathbb{R}^2 : x = X(\xi, t) \text{ and } y = Y(\xi, t), \xi_1 \leq \xi \leq \xi_2\}$, is a parametric representation of a curve in the plane for each value of t, and this curve represents the free-surface of the fluid. Hence, $\lim_{\eta \to 0} x(\xi, \eta, \tau) = X(\xi, \tau)$ and $\lim_{\eta \to 0} y(\xi, \eta, \tau) = Y(\xi, \tau)$ at free-surface $\boldsymbol{X} = (X, Y)$.

At any time t, the unit tangent vector to the free-surface curve is $\boldsymbol{t} = (X_\xi, Y_\xi)/\sqrt{X_\xi^2 + Y_\xi^2}$. Hence the unit outward-pointing normal vector to the free-surface is:

$$\boldsymbol{n} = \ell^{-1} \begin{bmatrix} -Y_\xi \\ X_\xi \end{bmatrix}, \tag{4.4.1}$$

where $\ell = \sqrt{X_\xi^2 + Y_\xi^2}$. Along the boundary $\eta = 0$, the kinematic free-surface boundary condition requires that the normal component of the velocity to be continuous. This implies that the normal velocity of the surface needs to be equal to the normal velocity of the fluid at the surface [2], i.e.:

$$\begin{aligned} \boldsymbol{n}\cdot\boldsymbol{X}_t &= \boldsymbol{n}\cdot\boldsymbol{\nabla}\phi \qquad \text{for} \quad (x,y)\in\Gamma(t) \\ \implies \ell^{-1}(X_\xi Y_t - Y_\xi X_t) &= \ell^{-1}(-Y_\xi\phi_x + X_\xi\phi_y) \\ \implies X_\xi Y_t - Y_\xi X_t &= -Y_\xi\phi_x + X_\xi\phi_y. \end{aligned} \tag{4.4.2}$$

Using (4.2.9), (4.2.10), and (4.2.11) we can transform ϕ_x, ϕ_y, ϕ_t:

$$\frac{\partial\phi}{\partial x} = J^{-1}\left(y_\eta\frac{\partial}{\partial\xi} - y_\xi\frac{\partial}{\partial\eta}\right)\phi = J^{-1}\left(y_\eta\phi_\xi - y_\xi\phi_\eta\right). \tag{4.4.3}$$

$$\frac{\partial\phi}{\partial y} = J^{-1}\left(x_\xi\phi_\eta - x_\eta\phi_\xi\right). \tag{4.4.4}$$

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\frac{\partial}{\partial\xi} + (y_\xi y_\tau - x_\xi y_\tau)\frac{\partial}{\partial\eta} + J\frac{\partial}{\partial\tau}\right)\phi \\ &= J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\phi_\xi + (y_\xi y_\tau - x_\xi y_\tau)\phi_\eta + J\phi_\tau\right). \end{aligned} \tag{4.4.5}$$

Therefore for the non-special case, we find general boundary conditions for free-surface $\eta = 0$ using the above substitutions of (4.4.4) and (4.4.5) into (4.4.2)

$$-Y_\xi\phi_x + X_\xi\phi_y = -J^{-1}Y_\xi(y_\eta\phi_\xi - y_\xi\phi_\eta) + J^{-1}X_\xi(x_\xi\phi_\eta - x_\eta\phi_\xi), \tag{4.4.6}$$

$$\implies X_\xi Y_\tau - Y_\xi X_\tau = -J^{-1}Y_\xi(y_\eta\phi_\xi - y_\xi\phi_\eta) + J^{-1}X_\xi(x_\xi\phi_\eta - x_\eta\phi_\xi) \ : \ \eta = 0, \tag{4.4.7}$$

or introducing the surface variables (Φ, U, V) by $\lim_{\eta\to0}\phi(\xi,\eta,\tau) = \Phi(\xi,\tau)$, $\lim_{\eta\to0}u(\xi,\eta,\tau) = U(\xi,\tau)$, and $\lim_{\eta\to0}v(\xi,\eta,\tau) = V(\xi,\tau)$ [2], (4.4.7) takes the form:

$$X_\xi Y_\tau - Y_\xi X_\tau = -UY_\xi + VX_\xi \qquad \text{on} \quad \eta = 0. \tag{4.4.8}$$

4.5 Transformed rigid wall or bottom boundary condition

Using (3.2.1) and (4.4.4), and recalling that necessarily $J^{-1} \neq 0$, we recover the rigid wall boundary condition at $\eta = -\widetilde{h}$ in curvilinear coordinates [2]:

$$\phi_y = \frac{1}{J}(x_\xi\phi_\eta - \phi_\xi x_\eta) = 0 \implies x_\xi\phi_\eta - \phi_\xi x_\eta = 0. \tag{4.5.1}$$

4.6 Transformed field equation for the fluid motion

We may use (4.2.9), (4.2.10), (4.4.3), and (4.4.4) substituted into (3.4.2) to give:

$$\phi_{xx} = \frac{\partial}{\partial x}\left[J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)\right] = J^{-1}\left[y_\eta\frac{\partial}{\partial\xi}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) - y_\xi\frac{\partial}{\partial\eta}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta))\right]$$

$$\phi_{yy} = \frac{\partial}{\partial y}\left[J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)\right] = J^{-1}\left[x_\xi\frac{\partial}{\partial\eta}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) - x_\eta\frac{\partial}{\partial\xi}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi))\right]$$

$$\begin{aligned}\implies \phi_{xx} + \phi_{yy} = J^{-1}\Bigg[&y_\eta\frac{\partial}{\partial\xi}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) - x_\eta\frac{\partial}{\partial\xi}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) \\ &+x_\xi\frac{\partial}{\partial\eta}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) - y_\xi\frac{\partial}{\partial\eta}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta))\Bigg].\end{aligned} \tag{4.6.1}$$

Recognising that:

$$\begin{aligned}&y_\eta\frac{\partial}{\partial\xi}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) - y_\xi\frac{\partial}{\partial\eta}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) \\ &= y_\eta\frac{\partial}{\partial\xi}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) + y_{\xi\eta}J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta) \\ &-y_\xi\frac{\partial}{\partial\eta}(J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) - y_{\xi\eta}J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta) \\ = &\frac{\partial}{\partial\xi}(y_\eta J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) - \frac{\partial}{\partial\eta}(y_\xi J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta))\end{aligned} \tag{4.6.2}$$

$$\begin{aligned}&x_\xi\frac{\partial}{\partial\eta}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) - x_\eta\frac{\partial}{\partial\xi}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) \\ &= x_\xi\frac{\partial}{\partial\eta}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) + x_{\xi\eta}J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi) \\ &-x_\eta\frac{\partial}{\partial\xi}(J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) - x_{\xi\eta}J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi) \\ = &\frac{\partial}{\partial\eta}(x_\xi J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) - \frac{\partial}{\partial\xi}(x_\eta J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)),\end{aligned}$$

means we can substitute to give:

$$
\begin{aligned}
\phi_{xx}+\phi_{yy} &= J^{-1}\left[\frac{\partial}{\partial\xi}(y_\eta J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta)) - \frac{\partial}{\partial\xi}(x_\eta J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi))\right.\\
&\quad\left.+\frac{\partial}{\partial\eta}(x_\xi J^{-1}(x_\xi\phi_\eta - x_\eta\phi_\xi)) - \frac{\partial}{\partial\eta}(y_\xi J^{-1}(y_\eta\phi_\xi - y_\xi\phi_\eta))\right]\\
&= J^{-1}\left[\frac{\partial}{\partial\xi}(J^{-1}(y_\eta^2\phi_\xi - y_\xi y_\eta\phi_\eta) - J^{-1}(x_\xi x_\eta\phi_\eta - x_\eta^2\phi_\xi))\right.\\
&\quad\left.+\frac{\partial}{\partial\eta}(J^{-1}(x_\xi^2\phi_\eta - x_\eta x_\xi\phi_\xi) - J^{-1}(y_\eta y_\xi\phi_\xi - y_\xi^2\phi_\eta))\right]\\
&= J^{-1}\left[\frac{\partial}{\partial\xi}(J^{-1}(\phi_\xi(x_\eta^2+y_\eta^2) - \phi_\eta(x_\xi x_\eta + y_\xi y_\eta)))\right.\\
&\quad\left.+\frac{\partial}{\partial\eta}(J^{-1}(\phi_\eta(x_\xi^2+y_\xi^2) - \phi_\xi(x_\eta x_\xi + y_\eta y_\xi)))\right].
\end{aligned}
\tag{4.6.3}
$$

Considering the covariant metric tensor of transformation (4.1.1), for covariant metric tensor G [4] we have:

$$
g_{ab} = \begin{bmatrix} \left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial y}{\partial\xi}\right)^2 & \left(\frac{\partial y}{\partial\eta}\right)\left(\frac{\partial y}{\partial\xi}\right) + \left(\frac{\partial x}{\partial\eta}\right)\left(\frac{\partial x}{\partial\xi}\right) \\ \left(\frac{\partial y}{\partial\eta}\right)\left(\frac{\partial y}{\partial\xi}\right) + \left(\frac{\partial x}{\partial\eta}\right)\left(\frac{\partial x}{\partial\xi}\right) & \left(\frac{\partial x}{\partial\eta}\right)^2 + \left(\frac{\partial y}{\partial\eta}\right)^2 \end{bmatrix} = \begin{bmatrix} x_\xi^2+y_\xi^2 & y_\eta y_\xi + x_\xi x_\eta \\ y_\eta y_\xi + x_\xi x_\eta & x_\eta^2 + y_\eta^2 \end{bmatrix},
\tag{4.6.4}
$$

hence we can define [2]:

$$
g_{11} = x_\xi^2 + y_\xi^2,\ g_{12} = g_{21} = y_\eta y_\xi + x_\xi x_\eta,\ g_{22} = x_\eta^2 + y_\eta^2.
\tag{4.6.5}
$$

To convert a covariant metric tensor g_{ab} to a contravariant tensor g^{ab} we take the inverse of the covariant tensor:

$$
g^{ab} = g_{ab}^{-1} = \frac{1}{|g_{ab}|}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11}\end{bmatrix} = \frac{1}{|g_{ab}|}\begin{bmatrix} x_\eta^2+y_\eta^2 & -(y_\eta y_\xi + x_\xi x_\eta) \\ -(y_\eta y_\xi + x_\xi x_\eta) & x_\xi^2+y_\xi^2\end{bmatrix},
\tag{4.6.6}
$$

where:

$$
\begin{aligned}
|g_{ab}| &= (x_\xi^2+y_\xi^2)(x_\eta^2+y_\eta^2) - (y_\eta y_\xi + x_\xi x_\eta)^2\\
&= x_\xi^2x_\eta^2 + y_\xi^2y_\eta^2 + y_\xi^2x_\eta^2 + x_\xi^2y_\eta^2 - y_\eta^2y_\xi^2 - x_\xi^2x_\eta^2 - 2y_\eta y_\xi x_\xi x_\eta\\
&= x_\eta^2y_\xi^2 + y_\eta^2x_\xi^2 - 2y_\eta y_\xi x_\xi x_\eta = (x_\xi y_\eta - y_\xi x_\eta)^2 = J^2.
\end{aligned}
\tag{4.6.7}
$$

Hence, we have contravariant tensor matrix and values:

$$
\begin{aligned}
g^{ab} &= J^{-2}\begin{bmatrix} x_\eta^2+y_\eta^2 & -(y_\eta y_\xi + x_\xi x_\eta) \\ -(y_\eta y_\xi + x_\xi x_\eta) & x_\xi^2+y_\xi^2\end{bmatrix}\\
g^{11} &= J^{-2}(x_\eta^2+y_\eta^2),\ g^{12} = g^{21} = -J^{-2}(y_\eta y_\xi + x_\xi x_\eta),\ g^{22} = J^{-2}(x_\xi^2+y_\xi^2).
\end{aligned}
\tag{4.6.8}
$$

By substitution into (4.6.3) we see that:

$$
\Delta_g\phi = \phi_{xx}+\phi_{yy} = \frac{1}{J}\frac{\partial}{\partial\xi}\left[Jg^{11}\phi_\xi + Jg^{12}\phi_\eta\right] + \frac{1}{J}\frac{\partial}{\partial\eta}\left[Jg^{22}\phi_\eta + Jg^{12}\phi_\xi\right] = 0,
\tag{4.6.9}
$$

which is the Laplace-Beltrami operator acting on ϕ [2]. For $-\widetilde{h} < \eta < 0$ we have:

$$\frac{1}{J}\frac{\partial}{\partial \xi}\left[Jg^{11}\phi_\xi + Jg^{12}\phi_\eta\right] + \frac{1}{J}\frac{\partial}{\partial \eta}\left[Jg^{22}\phi_\eta + Jg^{12}\phi_\xi\right] = 0, \tag{4.6.10}$$

and as $J \neq 0$ we see that we must have:

$$\frac{\partial}{\partial \xi}\left[Jg^{11}\phi_\xi + Jg^{12}\phi_\eta\right] + \frac{\partial}{\partial \eta}\left[Jg^{22}\phi_\eta + Jg^{12}\phi_\xi\right] = 0, \tag{4.6.11}$$

which is the field equation for the fluid motion (at $-\widetilde{h} < \eta < 0$) in curvilinear coordinates [2].

4.7 Transformed Bernoulli's equation and the dynamic free-surface boundary condition

Using (4.4.3) and (4.4.4) we can derive:

$$\begin{aligned} \phi_x^2 &= J^{-2}\left(y_\eta\phi_\xi - y_\xi\phi_\eta\right)^2 = J^{-2}\left(y_\eta^2\phi_\xi^2 + y_\xi^2\phi_\eta^2 - 2y_\eta\phi_\xi y_\xi\phi_\eta\right) \\ \phi_y^2 &= J^{-2}\left(x_\xi\phi_\eta - x_\eta\phi_\xi\right)^2 = J^{-2}\left(x_\xi^2\phi_\eta^2 + x_\eta^2\phi_\xi^2 - 2x_\xi\phi_\eta x_\eta\phi_\xi\right). \end{aligned} \tag{4.7.1}$$

Hence substituting (4.7.1) and (4.4.5) into (3.3.8) gives:

$$\begin{aligned} \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy &= J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\phi_\xi + (y_\xi x_\tau - x_\xi y_\tau)\phi_\eta + J\phi_\tau\right) \\ &+ \frac{J^{-2}}{2}(y_\eta^2\phi_\xi^2 + y_\xi^2\phi_\eta^2 - 2y_\eta\phi_\xi y_\xi\phi_\eta + x_\xi^2\phi_\eta^2 + x_\eta^2\phi_\xi^2 - 2x_\xi\phi_\eta x_\eta\phi_\xi) + gy \\ &= \phi_\tau + J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\phi_\xi + (y_\xi x_\tau - x_\xi y_\tau)\phi_\eta\right) \\ &+ \frac{J^{-2}}{2}((x_\eta^2 + y_\eta^2)\phi_\xi^2 + (x_\xi^2 + y_\xi^2)\phi_\eta^2 + 2\phi_\xi\phi_\eta(-y_\eta y_\xi - x_\xi x_\eta)) + gy. \end{aligned} \tag{4.7.2}$$

Substituting (4.6.8) into (4.7.2) gives:

$$\begin{aligned} \phi_t + \frac{1}{2}(\phi_x^2 + \phi_y^2) + gy &= \phi_\tau + J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\phi_\xi + (y_\xi x_\tau - x_\xi y_\tau)\phi_\eta\right) \\ &+ \frac{J^{-2}}{2}(J^2g^{11}\phi_\xi^2 + J^2g^{22}\phi_\eta^2 + 2\phi_\xi\phi_\eta J^2g^{12}) + gy \\ &= \phi_\tau + J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\phi_\xi + (y_\xi x_\tau - x_\xi y_\tau)\phi_\eta\right) + \frac{1}{2}(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2) + gy, \end{aligned} \tag{4.7.3}$$

hence Bernoulli's equation becomes:

$$\begin{aligned} p(\phi, x, y) = -\rho\Bigg[&\phi_\tau + J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\phi_\xi + (y_\xi x_\tau - x_\xi y_\tau)\phi_\eta\right) \\ &+ \frac{1}{2}(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2) + gy\Bigg]. \end{aligned} \tag{4.7.4}$$

On the free-surface, pressure is equivalent to the atmospheric pressure, i.e., $p = 0$, therefore:

$$\phi_\tau + J^{-1}\left((x_\eta y_\tau - y_\eta x_\tau)\phi_\xi + (y_\xi x_\tau - x_\xi y_\tau)\phi_\eta\right) + \frac{1}{2}(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2) + gY = 0 \quad \text{at} \quad \eta = 0, \tag{4.7.5}$$

which is the dynamic free-surface boundary condition in curvilinear coordinates [2].

4.8 Transformed Luke's principle

Recall the properties of the wedge product for differential one forms da and db [4]:

$$\begin{cases} da \wedge db = -db \wedge da \\ da \wedge da = 0 \\ db \wedge db = 0 \end{cases} \tag{4.8.1}$$

As we have $x = x(\xi, \eta, \tau),\ \ y = y(\xi, \eta, \tau), t = \tau$ then:

$$\begin{aligned} dx &= x_\xi d\xi + x_\eta d\eta + x_\tau d\tau \\ dy &= y_\xi d\xi + y_\eta d\eta + y_\tau d\tau \\ dt &= d\tau. \end{aligned} \tag{4.8.2}$$

Hence by substitution and usage of (4.8.1) identities:

$$\begin{gathered} dx \wedge dy \wedge dt = (x_\xi d\xi + x_\eta d\eta + x_\tau d\tau) \wedge (y_\xi d\xi + y_\eta d\eta + y_\tau d\tau) \wedge d\tau \\ = (x_\xi y_\xi d\xi \wedge d\xi + x_\xi y_\eta d\xi \wedge d\eta + x_\xi y_\tau d\xi \wedge d\tau + x_\eta y_\xi d\eta \wedge d\xi + x_\eta y_\eta d\eta \wedge d\eta + x_\eta y_\tau d\eta \wedge d\tau \\ + x_\tau y_\xi d\tau \wedge d\xi + x_\tau y_\eta d\tau \wedge d\eta + x_\tau y_\tau d\tau \wedge d\tau) \wedge d\tau \\ = [(x_\xi y_\eta - y_\xi x_\eta) d\xi \wedge d\eta + (x_\xi y_\tau - y_\xi x_\tau) d\xi \wedge d\tau + (x_\eta y_\tau - y_\eta x_\tau) d\eta \wedge d\tau] \wedge d\tau \\ = [J d\xi \wedge d\eta - F_2 d\xi \wedge d\tau + F_1 d\eta \wedge d\tau] \wedge d\tau \\ = J d\xi \wedge d\eta \wedge d\tau - F_2 d\xi \wedge d\tau \wedge d\tau + F_1 d\eta \wedge d\tau \wedge d\tau = J d\xi \wedge d\eta \wedge d\tau \\ \implies dxdydt = J d\xi d\eta d\tau. \end{gathered} \tag{4.8.3}$$

Using a substitution of (4.7.5) and (4.8.3) into the integral S of (3.5.2) gives:

$$
\begin{aligned}
&\int_{t_1}^{t_2}\int_{x_1}^{x_2}\int_0^{h(x,t)}\left(\phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2)+\frac{p}{\rho}+gy\right)dydxdt\\
&=\int_{\tau_1}^{\tau_2}\int_0^{l}\int_{-\widetilde{h}}^{0}\Big[\phi_\tau+J^{-1}\left((x_\eta y_\tau-y_\eta x_\tau)\phi_\xi+(y_\xi x_\tau-x_\xi y_\tau)\phi_\eta\right)\\
&+\frac{J^{-2}}{2}((x_\eta^2+y_\eta^2)\phi_\xi^2+(x_\xi^2+y_\xi^2)\phi_\eta^2+2\phi_\xi\phi_\eta(-y_\eta y_\xi-x_\xi x_\eta))+gy\Big]Jd\xi d\eta d\tau \qquad (4.8.4)\\
&=\int_{\tau_1}^{\tau_2}\int_0^{l}\int_{-\widetilde{h}}^{0}\Big[J\phi_\tau+(x_\eta y_\tau-y_\eta x_\tau)\phi_\xi+(y_\xi x_\tau-x_\xi y_\tau)\phi_\eta\\
&+\frac{J^{-1}}{2}\left((x_\eta^2+y_\eta^2)\phi_\xi^2+(x_\xi^2+y_\xi^2)\phi_\eta^2+2\phi_\xi\phi_\eta(-y_\eta y_\xi-x_\xi x_\eta)\right)+Jgy\Big]d\xi d\eta d\tau.
\end{aligned}
$$

The limits of integration have also changed. In this coordinate system, instead of fixed time $[t_1, t_2]$ and fixed horizontal $[x_1, x_2]$ boundaries, we have fixed time boundaries $[\tau_1, \tau_2]$ and fixed horizontal boundaries $[0, l]$, all boundaries of which are constants. The varying height $h = h(x, t)$ instead becomes a constant when we move from a wave free-surface to a fixed boundary free-surface at height 0. The bottom rigid wall boundary also changes from 0 to $-\widetilde{h}$ which is a constant, hence changing $[0, h]$ to $[-\widetilde{h}, 0]$. This can also be expressed as a transformation due to the new physical model [2]:

$$
\begin{cases} t_1 \ to \ t_2 \\ x_1 \ to \ x_2 \\ 0 \ to \ h(x,t) \end{cases} \mapsto \begin{cases} \tau_1 \ to \ \tau_2 \\ 0 \ to \ l \\ -\widetilde{h} \ to \ 0 \end{cases}. \qquad (4.8.5)
$$

(4.8.4) can also be expressed as:

$$
\widetilde{S}=\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\widetilde{L}(\phi,x,y)d\eta d\tau, \qquad (4.8.6)
$$

$$
\widetilde{L}=\int_0^l\left(J\phi_\tau+F_1\phi_\xi+F_2\phi_\eta+\frac{J}{2}\left(g^{11}\phi_\xi^2+2g^{12}\phi_\xi\phi_\eta+g^{22}\phi_\eta^2\right)+Jgy\right)d\xi. \qquad (4.8.7)
$$

Therefore Luke's variational principle in transformed coordinates becomes:

$$
\delta\widetilde{S}=\delta\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l\left(J\phi_\tau+F_1\phi_\xi+F_2\phi_\eta+\frac{J}{2}\left(g^{11}\phi_\xi^2+2g^{12}\phi_\xi\phi_\eta+g^{22}\phi_\eta^2\right)+Jgy\right)d\xi d\eta d\tau=0. \qquad (4.8.8)
$$

4.9 Simplification of the transformed integral

We have:

$$\delta \int_{\tau_1}^{\tau_2} \int_{-\tilde{h}}^{0} \int_{0}^{l} \left[J\phi_\tau + F_1\phi_\xi + F_2\phi_\eta + \frac{J}{2}\left(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2\right) + Jgy \right] d\xi d\eta d\tau = 0. \tag{4.9.1}$$

First we take the cartesian time derivative term $J\phi_\tau + F_1\phi_\xi + F_2\phi_\eta$. Using product rule and the GCL (4.3.5) we see that:

$$\begin{gathered} J\phi_\tau + F_1\phi_\xi + F_2\phi_\eta = (J\phi)_\tau - \phi J_\tau + (F_1\phi)_\xi - \phi(F_1)_\xi + (F_2\phi)_\eta - \phi(F_2)_\eta \\ = (J\phi)_\tau + (F_1\phi)_\xi + (F_2\phi)_\eta - \phi(J_\tau + (F_1)_\xi + (F_2)_\eta) = (J\phi)_\tau + (F_1\phi)_\xi + (F_2\phi)_\eta, \end{gathered} \tag{4.9.2}$$

$$\int_{\tau_1}^{\tau_2} \int_{-\tilde{h}}^{0} \int_{0}^{l} [J\phi_\tau + F_1\phi_\xi + \phi_\eta] \, d\xi d\eta d\tau = \int_{\tau_1}^{\tau_2} \int_{-\tilde{h}}^{0} \int_{0}^{l} [(J\phi)_\tau + (F_1\phi)_\xi + (F_2\phi)_\eta] \, d\xi d\eta d\tau. \tag{4.9.3}$$

Considering divergence theorem (3.5.10), but instead using the gradient $(\partial/\partial\xi, \partial/\partial\eta, \partial/\partial\tau)$ instead of $\boldsymbol{\nabla}$, and the vector $(F_1\phi, F_2\phi, J\phi)$ instead of $\boldsymbol{u}$ and the unit normal $\boldsymbol{N}$ on each boundary of the computational space:

$$\int_{\tau_1}^{\tau_2} \int_{-\tilde{h}}^{0} \int_{0}^{l} [(J\phi)_\tau + (F_1\phi)_\xi + (F_2\phi)_\eta] \, d\xi d\eta d\tau = \iint_S \left[\phi \begin{bmatrix} F_1 \\ F_2 \\ J \end{bmatrix} \cdot \boldsymbol{N} \right] dS. \tag{4.9.4}$$

Due to the periodic boundary conditions, the only time where this integrand is non-zero is when $\tau = \tau_2,\ \tau = \tau_1$, and the dynamic free-surface boundary condition when $\eta = 0$. Hence for surface normals $\boldsymbol{N} = (0,0,1),\ (0,0,-1),\ (0,1,0)$ respectively [2], we have the integral equations from (4.9.4):

$$\int_{-\tilde{h}}^{0} \int_{0}^{l} \phi J \Big|_{\tau=\tau_2} d\xi d\eta - \int_{-\tilde{h}}^{0} \int_{0}^{l} \phi J \Big|_{\tau=\tau_1} d\xi d\eta + \int_{\tau_1}^{\tau_2} \int_{0}^{l} \phi F_2 \Big|_{\eta=0} d\xi d\tau. \tag{4.9.5}$$

Recall that $F_2 = y_\xi x_\tau - x_\xi y_\tau$, and that at $\eta = 0$ we have $x = X,\ y = Y$, hence:

$$\int_{\tau_1}^{\tau_2} \int_{0}^{l} \phi F_2 \Big|_{\eta=0} d\eta d\tau = \int_{\tau_1}^{\tau_2} \int_{0}^{l} \Phi(Y_\xi X_\tau - X_\xi Y_\tau) d\xi d\tau$$

$$\begin{gathered} \implies \int_{\tau_1}^{\tau_2} \int_{-\tilde{h}}^{0} \int_{0}^{l} [J\phi_\tau + F_1\phi_\xi + F_2\phi_\eta] \, d\xi d\eta d\tau \\ = \int_{-\tilde{h}}^{0} \int_{0}^{l} \phi J \Big|_{\tau=\tau_2} d\xi d\eta - \int_{-\tilde{h}}^{0} \int_{0}^{l} \phi J \Big|_{\tau=\tau_1} d\xi d\eta + \int_{\tau_1}^{\tau_2} \int_{0}^{l} \Phi(Y_\xi X_\tau - X_\xi Y_\tau) d\xi d\tau. \end{gathered} \tag{4.9.6}$$

Next we may consider the potential energy term Jgy:

$$\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l gy(x_\xi y_\eta - x_\eta y_\xi) d\xi d\eta d\tau. \tag{4.9.7}$$

Using product rule, we may rearrange the integrand to:

$$\begin{aligned} g(yx_\xi y_\eta - yx_\eta y_\xi) &= g\left[yx_\xi y_\eta + \frac{1}{2}x_{\xi\eta}y^2 - \left(yx_\eta y_\xi + \frac{1}{2}x_{\xi\eta}y^2\right)\right] \\ = \frac{g}{2}\left[2yx_\xi y_\eta + x_{\xi\eta}y^2 - \left(2yx_\eta y_\xi + x_{\xi\eta}y^2\right)\right] &= \frac{g}{2}\left[(x_\xi y^2)_\eta - (x_\eta y^2)_\xi\right]. \end{aligned} \tag{4.9.8}$$

Hence the integral (4.9.7) becomes:

$$\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l gyJ d\xi d\eta d\tau = \int_{\tau_1}^{\tau_2}\frac{g}{2}\int_{-\widetilde{h}}^{0}\int_0^l \left[(x_\xi y^2)_\eta - (x_\eta y^2)_\xi\right] d\xi d\eta d\tau. \tag{4.9.9}$$

The integral will only recover boundary condition terms on boundaries $\eta = 0$ and $\eta = -\widetilde{h}$, hence reduces to:

$$\int_{\tau_1}^{\tau_2}\int_0^l \frac{g}{2}\int_{-\widetilde{h}}^{0}(x_\xi y^2)_\eta d\eta d\xi d\tau = \int_{\tau_1}^{\tau_2}\int_0^l \frac{g}{2}\left[x_\xi y^2\Big|_{\eta=0} - x_\xi y^2\Big|_{\eta=-\widetilde{h}}\right] d\xi d\tau. \tag{4.9.10}$$

Recognising that at $\eta = -\widetilde{h}$, we also have $y = h(x,t)$ and considering the second term:

$$\begin{aligned} \int_{\tau_1}^{\tau_2}\int_0^l \frac{g}{2}x_\xi y^2\Big|_{\eta=-\widetilde{h}} d\xi d\tau &= \int_{\tau_1}^{\tau_2}\int_0^l \frac{g}{2}x_\xi h^2 d\xi d\tau = \int_{\tau_1}^{\tau_2}\frac{g}{2}[x]_0^l h^2 d\tau \\ &= \int_{\tau_1}^{\tau_2}\frac{g}{2}lh^2 d\tau = \frac{1}{2}gh^2 l(\tau_2 - \tau_1). \end{aligned} \tag{4.9.11}$$

This term can be neglected as does not have contribute to the Euler-Lagrange equation, and hence the integral reduces to:

$$\int_{\tau_1}^{\tau_2}\int_0^l \frac{g}{2}x_\xi y^2\Big|_{\eta=0} d\xi d\tau = \int_{\tau_1}^{\tau_2}\int_0^l \frac{1}{2}gY^2 X_\xi d\xi d\tau. \tag{4.9.12}$$

Hence using (4.9.1), (4.9.6), and (4.9.12), the Luke variational principle in curvilinear coordinates takes the form:

$$\begin{aligned} \delta\widetilde{S} = &\int_{-\widetilde{h}}^{0}\int_0^l \phi J\Big|_{\tau=\tau_2} d\xi d\eta - \int_{-\widetilde{h}}^{0}\int_0^l \phi J\Big|_{\tau=\tau_1} d\xi d\eta + \int_{\tau_1}^{\tau_2}\left[\int_0^l \Phi(Y_\xi X_\tau - X_\xi Y_\tau) d\xi\right. \\ &\left. + \int_{-\widetilde{h}}^{0}\int_0^l \frac{J}{2}\left(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2\right) d\xi d\eta + \int_0^l \frac{1}{2}gY^2 X_\xi d\xi\right] d\tau = 0. \end{aligned} \tag{4.9.13}$$

5 Lagrangian variation of Bernoulli's equation in curvilinear coordinates

5.1 Definitions

We have:

$$\widetilde{S}(\phi, x, y) = \int_{-\widetilde{h}}^{0}\int_{0}^{l} \phi J\Big|_{\tau=\tau_2} d\xi d\eta - \int_{-\widetilde{h}}^{0}\int_{0}^{l} \phi J\Big|_{\tau=\tau_1} d\xi d\eta + \int_{\tau_1}^{\tau_2}\Bigg[\int_{0}^{l} \Phi(Y_\xi X_\tau - X_\xi Y_\tau) d\xi \\ + \int_{-\widetilde{h}}^{0}\int_{0}^{l} \frac{J}{2}\left(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2\right) d\xi d\eta + \int_{0}^{l}\frac{1}{2} g Y^2 X_\xi d\xi\Bigg] d\tau. \tag{5.1.1}$$

Taking the variations of the Lagrangian with fixed endpoints at $t = \tau_1$ and $t = \tau_2$, with contributions from the periodic boundary conditions $\delta\phi(\tau_1) = \delta\phi(\tau_2) = 0$, the boundary terms at $t = \tau_1$ and $t = \tau_2$ will vanish [2]. We may take the variation, giving the equation:

$$\delta\widetilde{S} = \int_{\tau_1}^{\tau_2}\Bigg[\delta\int_{0}^{l}\frac{1}{2} g Y^2 X_\xi d\xi + \delta\int_{0}^{l} \Phi(Y_\xi X_\tau - X_\xi Y_\tau) d\xi \\ + \delta\int_{-\widetilde{h}}^{0}\int_{0}^{l} \frac{J}{2}\left(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2\right) d\xi d\eta\Bigg] d\tau. \tag{5.1.2}$$

Define the kinetic energy term K:

$$K = \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_{0}^{l} \frac{J}{2}\left(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2\right) d\xi d\eta d\tau. \tag{5.1.3}$$

Define the potential energy term P and the boundary problem term B:

$$P = \int_{\tau_1}^{\tau_2}\int_{0}^{l}\frac{1}{2} g Y^2 X_\xi d\xi d\tau, \\ B = \int_{\tau_1}^{\tau_2}\int_{0}^{l} \Phi(Y_\xi X_\tau - X_\xi Y_\tau) d\xi d\tau. \tag{5.1.4}$$

Due to lengthy calculations, we take the first variation of each term separately.

5.2 Variation in curvilinear coordinates with respect to ϕ and Φ

First, we start with the kinetic energy and take the variation with respect to ϕ, with x and y fixed:

$$\begin{aligned}\delta\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l Kd\xi d\eta &= \frac{1}{2}\int_{-\widetilde{h}}^{0}\int_0^l \delta J(g^{11}\phi_\xi^2+2g^{12}\phi_\xi\phi_\eta+g^{22}\phi_\eta^2)d\xi d\eta d\tau \\ =\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l J(g^{11}\phi_\xi\delta\phi_\xi &+ g^{12}\phi_\eta\delta\phi_\xi+g^{12}\phi_\xi\delta\phi_\eta+g^{22}\phi_\eta\delta\phi_\eta)d\xi d\eta d\tau.\end{aligned} \tag{5.2.1}$$

Integrating this equation by parts gives:

$$\begin{aligned}\delta\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l Kd\xi d\eta = \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \Bigg(&\frac{\partial}{\partial\xi}((Jg^{11}\phi_\xi+Jg^{12}\phi_\eta)\delta\phi)+\frac{\partial}{\partial\eta}((Jg^{12}\phi_\xi+Jg^{22}\phi_\eta)\delta\phi) \\ &-\delta\phi\frac{\partial}{\partial\xi}(Jg^{11}\phi_\xi+Jg^{12}\phi_\eta)-\delta\phi\frac{\partial}{\partial\eta}(Jg^{12}\phi_\xi+Jg^{22}\phi_\eta)\Bigg)d\xi d\eta.\end{aligned} \tag{5.2.2}$$

If we denote:

$$\begin{aligned}A_1 &= (Jg^{11}\phi_\xi+Jg^{12}\phi_\eta)\delta\phi, \\ A_2 &= (Jg^{12}\phi_\xi+Jg^{22}\phi_\eta)\delta\phi.\end{aligned} \tag{5.2.3}$$

$$\Longrightarrow = \int_{-\widetilde{h}}^{0}\int_0^l\left(\frac{\partial A_1}{\partial\xi}+\frac{\partial A_2}{\partial\eta}\right)d\xi d\eta-\int_{-\widetilde{h}}^{0}\int_0^l J\Delta_g\phi\delta\phi d\xi d\eta. \tag{5.2.4}$$

Using Green's Theorem, the left hand integral reduces to:

$$\begin{aligned}\int_{-\widetilde{h}}^{0}\int_0^l\frac{\partial A_1}{\partial\xi}d\xi d\eta+\int_0^l\int_{-\widetilde{h}}^{0}\frac{\partial A_2}{\partial\eta}d\eta d\xi &= \int_{-\widetilde{h}}^{0}A_1(l,\eta,\tau)-A_1(0,\eta,\tau)d\eta \\ +\int_0^l A_2(\xi,0,\tau)-A_2(\xi,-\widetilde{h},\tau)d\xi &= \int_0^l A_2(\xi,0,\tau)d\xi-\int_0^l A_2(\xi,-\widetilde{h},\tau)d\xi,\end{aligned} \tag{5.2.5}$$

where $A_1(l,\eta,\tau)=A_1(0,\eta,\tau)$, due to periodic boundary conditions for ϕ. Now the variation of the fluid Lagrangian with respect to ϕ, with x and y fixed, becomes:

$$\begin{aligned}\delta\widetilde{S} = \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}-\int_l^0\Delta_g\phi J\delta\phi d\xi d\eta d\tau-\int_{\tau_1}^{\tau_2}\int_l^0(Jg^{12}\phi_\xi+Jg^{22}\phi_\eta)\delta\phi\big|_{-\widetilde{h}}^{0}d\xi d\tau \\ +\int_{\tau_1}^{\tau_2}\int_l^0(X_\xi Y_\tau-Y_\xi X_\tau)\delta\Phi d\xi d\tau=0.\end{aligned} \tag{5.2.6}$$

From the first term on the right-hand side it can be concluded that the invariance of the Lagrangian with respect to a variation in the velocity potential $\delta\phi$ recovers the fluid equation (4.6.9) in the transformed coordinates [2], i.e.,

$$\Delta_g\phi=0 \qquad \text{in } 0<\xi<l \text{ and } -\widetilde{h}<\eta<0. \tag{5.2.7}$$

The second and third integrals on the right-hand side give:

$$-\int_{\tau_1}^{\tau_2}\int_l^0 (Jg^{12}\phi_\xi + Jg^{22}\phi_\eta)\big|^{\eta=0}\delta\Phi d\xi d\tau + \int_{\tau_1}^{\tau_2}\int_l^0 (X_\xi Y_\tau - Y_\xi X_\tau)\delta\Phi d\xi d\tau \\ + \int_{\tau_1}^{\tau_2}\int_l^0 (Jg^{12}\phi_\xi + Jg^{22}\phi_\eta)\delta\phi\big|_{\eta=-\widetilde{h}} d\xi d\tau. \tag{5.2.8}$$

Now the invariance of the Lagrangian with respect to a variation in the velocity potential at the free-surface $\delta\Phi$ gives:

$$-(Jg^{12}\phi_\xi + Jg^{22}\phi_\eta)\big|^{\eta=0} + (X_\xi Y_\tau - Y_\xi X_\tau) = 0. \tag{5.2.9}$$

But from (4.6.5) and (4.6.8) we see that $Jg^{12} = -J^{-1}g_{12}$ and $Jg^{22} = J^{-1}g_{11}$. Hence: $\delta\Phi$ gives:

$$J^{-1}g_{12}\phi_\xi\big|^{\eta=0} - J^{-1}g_{22}\phi_\eta\big|^{\eta=0} + X_\xi Y_\tau - Y_\xi X_\tau = 0, \tag{5.2.10}$$

so after substituting for g_{12} and g_{11} evaluated at $\eta = 0$ recovers the kinematic free-surface boundary condition (4.4.7) [2]. Also, the invariance of the Lagrangian with respect to a variation in the velocity potential at the bottom boundary $\eta = -\widetilde{h}$ gives:

$$Jg^{12}\phi_\xi - Jg^{22}\phi_\eta = 0 \qquad \text{at } \eta = -\widetilde{h}, \tag{5.2.11}$$

or

$$-J^{-1}g_{12}\phi_\xi + J^{-1}g^{11}\phi_\eta = 0 \qquad \text{at } \eta = -\widetilde{h}, \tag{5.2.12}$$

or noting that $J \neq 0$

$$-g_{12}\phi_\xi + g^{11}\phi_\eta = 0 \implies (x_\xi^2 + y_\xi^2)\phi_\eta - (x_\xi x_\eta + y_\xi y_\eta)\phi_\xi \qquad \text{at } \eta = -\widetilde{h}. \tag{5.2.13}$$

But along the boundary line $\eta = -\widetilde{h}$ we have $y_\eta = 0$ and hence:

$$x_\xi^2\phi_\eta - x_\xi x_\eta\phi_\xi = 0 \implies x_\xi(x_\xi\phi_\eta - x_\eta\phi_\xi) = 0 \qquad \text{at } \eta = -\widetilde{h}. \tag{5.2.14}$$

Since $x_\xi \neq 0$ one concludes that:

$$x_\xi\phi_\eta - x_\eta\phi_\xi = 0 \implies \qquad \text{at } \eta = -\widetilde{h}. \tag{5.2.15}$$

This recovers the bottom boundary condition (4.5.1) in the transformed coordinates [2]. Now, it is left to recover the dynamic free-surface boundary condition (4.7.5) in the transformed coordinates.

5.3 Variation of kinetic energy in curvilinear coordinates with respect to x, y, X, and Y

First note that the kinetic energy in curvilinear coordinates can be written as:

$$\begin{aligned} K &= \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \frac{J}{2}\left(g^{11}\phi_\xi^2 + 2g^{12}\phi_\xi\phi_\eta + g^{22}\phi_\eta^2\right) d\xi d\eta d\tau \\ &= \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \left(\frac{1}{2}J^{-1}g_{22}\phi_\xi^2 - J^{-1}g_{12}\phi_\xi\phi_\eta + \frac{1}{2}J^{-1}g_{22}\phi_\eta^2\right) d\xi d\eta d\tau, \end{aligned} \tag{5.3.1}$$

using definitions from (4.6.5) and (4.6.8). Now take the variations of the kinetic energy with respect to x and y with ϕ fixed. This gives:

$$\delta K = \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \left(\frac{1}{2}\delta[J^{-1}(x_\eta^2+y_\eta^2)]\phi_\xi^2 - \delta[J^{-1}(x_\xi x_\eta + y_\xi y_\eta)]\phi_\xi\phi_\eta + \frac{1}{2}\delta[J^{-1}(x_\xi^2+y_\xi^2)]\phi_\eta^2\right) d\xi d\eta d\tau. \tag{5.3.2}$$

Now take the variation of each term separately. The first term of the integrand becomes:

$$\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \frac{1}{2}\left[(x_\eta^2+y_\eta^2)\delta J^{-1} + 2J^{-1}(x_\eta\delta x_\eta + y_\eta\delta y_\eta)\right]\phi_\xi^2 d\xi d\eta d\tau, \tag{5.3.3}$$

but

$$\delta J^{-1} = \delta\left(\frac{1}{J}\right) = -\frac{\delta J}{J^2} = -J^{-2}\delta(x_\xi y_\eta - x_\eta y_\xi) = -J^{-2}(x_\xi\delta y_\eta + y_\eta\delta x_\xi - x_\eta\delta y_\xi - y_\xi\delta x_\eta), \tag{5.3.4}$$

and so using integration by parts, the integral simplifies to:

$$\begin{aligned} &\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \left(-\frac{1}{2}J^{-2}(x_\eta^2+y_\eta^2)(x_\xi\delta y_\eta + y_\eta\delta x_\xi - x_\eta\delta y_\xi - y_\xi\delta x_\eta) + J^{-1}(x_\eta\delta x_\eta + y_\eta\delta y_\eta)\right)\phi_\xi^2 d\xi d\eta d\tau \\ &= \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \left(\frac{1}{2}\frac{\partial}{\partial\xi}[J^{-2}(x_\eta^2+y_\eta^2)y_\eta\phi_\xi^2]\delta x - \frac{1}{2}\frac{\partial}{\partial\xi}[J^{-2}(x_\eta^2+y_\eta^2)x_\eta\phi_\xi^2]\delta y\right. \\ &\quad + \frac{1}{2}\frac{\partial}{\partial\eta}[J^{-2}(x_\eta^2+y_\eta^2)x_\xi\phi_\xi^2]\delta y - \frac{1}{2}\frac{\partial}{\partial\eta}[J^{-2}(x_\eta^2+y_\eta^2)y_\xi\phi_\xi^2]\delta x \\ &\quad \left.-\frac{\partial}{\partial\eta}[J^{-1}x_\eta\phi_\xi^2]\delta x - \frac{\partial}{\partial\eta}[J^{-1}y_\eta\phi_\xi^2]\delta y\right) d\xi d\eta d\tau + \int_{\tau_1}^{\tau_2}\int_0^l J^{-1}x_\eta\phi_\xi^2\delta x\Big|_{\eta=-\widetilde{h}}^{\eta=0} d\xi d\tau \\ &\quad + \int_{\tau_1}^{\tau_2}\int_0^l J^{-1}y_\eta\phi_\xi^2\delta y\Big|^{\eta=0} d\xi d\tau - \int_{\tau_1}^{\tau_2}\int_0^l \frac{1}{2}J^{-2}(x_\eta^2+y_\eta^2)x_\xi\phi_\xi^2\Big|^{\eta=0}\delta Y d\xi d\tau \\ &\quad + \int_{\tau_1}^{\tau_2}\int_0^l \frac{1}{2}J^{-2}(x_\eta^2+y_\eta^2)y_\xi\phi_\xi^2\Big|^{\eta=0}\delta X d\xi d\tau, \end{aligned} \tag{5.3.5}$$

where we have periodicity in the ξ-direction and $y_\xi = 0,\ \delta y = 0$ at $\eta = -\widetilde{h}$ [2].

Similarly, for the second term of (5.3.2) we have

$$\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \Big(J^{-2}(x_\eta x_\xi + y_\eta y_\xi)(x_\xi \delta y_\eta + y_\eta \delta x_\xi - x_\eta \delta y_\xi - y_\xi \delta x_\eta)$$
$$-J^{-1}(x_\xi \delta x_\eta + x_\eta \delta x_\xi + y_\xi \delta y_\eta + y_\eta \delta y_\xi)\Big)\phi_\xi\phi_\eta d\xi d\eta d\tau$$
$$= \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \Big(-\frac{\partial}{\partial\xi}[J^{-2}(x_\eta x_\xi + y_\eta y_\xi) y_\eta \phi_\xi\phi_\eta]\delta x + \frac{\partial}{\partial\xi}[J^{-2}(x_\eta x_\xi + y_\eta y_\xi) x_\eta \phi_\xi\phi_\eta]\delta y$$
$$-\frac{\partial}{\partial\eta}[J^{-2}(x_\eta x_\xi + y_\eta y_\xi) x_\xi \phi_\xi\phi_\eta]\delta y + \frac{\partial}{\partial\eta}[J^{-2}(x_\eta x_\xi + y_\eta y_\xi) y_\xi \phi_\xi\phi_\eta]\delta x$$
$$+\frac{\partial}{\partial\eta}[J^{-1}x_\xi\phi_\xi\phi_\eta]\delta x + \frac{\partial}{\partial\eta}[J^{-1}y_\xi\phi_\xi\phi_\eta]\delta y + \frac{\partial}{\partial\xi}[J^{-1}x_\eta\phi_\xi\phi_\eta]\delta x + \frac{\partial}{\partial\xi}[J^{-1}y_\eta\phi_\xi\phi_\eta]\delta y\Big) d\xi d\eta d\tau$$
$$-\int_{\tau_1}^{\tau_2}\int_0^l J^{-1}x_\xi\phi_\xi\phi_\eta\delta x\Big|_{\eta=-\widetilde{h}}^{\eta=0} d\xi d\tau - \int_{\tau_1}^{\tau_2}\int_0^l J^{-1}y_\xi\phi_\xi\phi_\eta\delta y\Big|^{\eta=0} d\xi d\tau$$
$$+\int_{\tau_1}^{\tau_2}\int_0^l J^{-2}(x_\eta x_\xi + y_\eta y_\xi)x_\xi\phi_\xi\phi_\eta\Big|^{\eta=0}\delta Y d\xi d\tau - \int_{\tau_1}^{\tau_2}\int_0^l J^{-2}(x_\eta x_\xi + y_\eta y_\xi)y_\xi\phi_\xi\phi_\eta\Big|^{\eta=0}\delta X d\xi d\tau, \tag{5.3.6}$$

and for the third term

$$\int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \left(-\frac{1}{2}J^{-2}(x_\xi^2 + y_\xi^2)(x_\xi\delta y_\eta + y_\eta\delta x_\xi - x_\eta\delta y_\xi - y_\xi\delta x_\eta) + J^{-1}(x_\xi\delta x_\xi + y_\xi\delta y_\xi)\right)\phi_\eta^2 d\xi d\eta d\tau$$
$$= \int_{\tau_1}^{\tau_2}\int_{-\widetilde{h}}^{0}\int_0^l \Big(\frac{1}{2}\frac{\partial}{\partial\xi}[J^{-2}(x_\xi^2 + y_\xi^2)y_\eta\phi_\eta^2]\delta x - \frac{1}{2}\frac{\partial}{\partial\eta}[J^{-2}(x_\xi^2 + y_\xi^2)x_\eta\phi_\eta^2]\delta y$$
$$+\frac{1}{2}\frac{\partial}{\partial\eta}[J^{-2}(x_\xi^2 + y_\xi^2)x_\xi\phi_\eta^2]\delta y - \frac{1}{2}\frac{\partial}{\partial\eta}[J^{-2}(x_\xi^2 + y_\xi^2)y_\xi\phi_\eta^2]\delta x$$
$$-\frac{\partial}{\partial\xi}[J^{-1}x_\xi\phi_\eta^2]\delta x - \frac{\partial}{\partial\xi}[J^{-1}y_\xi\phi_\eta^2]\delta y\Big) d\xi d\eta d\tau$$
$$-\int_{\tau_1}^{\tau_2}\int_0^l \frac{1}{2}J^{-2}(x_\xi^2 + y_\xi^2)x_\xi\phi_\eta^2\Big|^{\eta=0}\delta Y d\xi d\tau + \int_{\tau_1}^{\tau_2}\int_0^l \frac{1}{2}J^{-2}(x_\xi^2 + y_\xi^2)y_\xi\phi_\eta^2\Big|^{\eta=0}\delta X d\xi d\tau. \tag{5.3.7}$$

From (5.3.5), (5.3.6), and (5.3.7) if we now take all of the δX and δY components:

$$\begin{aligned}
\delta X: \quad & J^{-1}X_\eta\Phi_\xi^2 - J^{-1}X_\xi\Phi_\xi\Phi_\eta + \tfrac{1}{2}J^{-2}(X_\xi^2 + Y_\xi^2)Y_\xi\Phi_\eta^2 + \tfrac{1}{2}J^{-2}(X_\eta^2 + Y_\eta^2)Y_\xi\Phi_\xi^2 \\
& -J^{-2}(X_\eta X_\xi + Y_\eta Y_\xi)Y_\xi\Phi_\xi\Phi_\eta = J^{-1}(X_\eta\Phi_\xi - X_\xi\Phi_\eta)\Phi_\xi + \tfrac{1}{2}Y_\xi\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi, \\
\delta Y: \quad & J^{-1}Y_\eta\Phi_\xi^2 - J^{-1}Y_\xi\Phi_\xi\Phi_\eta - \tfrac{1}{2}J^{-2}(X_\xi^2 + Y_\xi^2)X_\xi\Phi_\eta^2 - \tfrac{1}{2}J^{-2}(X_\eta^2 + Y_\eta^2)X_\xi\Phi_\xi^2 \\
& +J^{-2}(X_\eta X_\xi + Y_\eta Y_\xi)X_\xi\Phi_\xi\Phi_\eta = J^{-1}(Y_\eta\Phi_\xi - Y_\xi\Phi_\eta)\Phi_\xi - \tfrac{1}{2}X_\xi\boldsymbol{\nabla}\Phi\cdot\boldsymbol{\nabla}\Phi,
\end{aligned} \tag{5.3.8}$$

where J^{-1} is evaluated at $\eta = 0$.

It is worth noting that δx terms at $-\widetilde{h}$ disappear, as when considering these terms from (5.3.5) and (5.3.6):

$$\int_{\tau_1}^{\tau_2}\int_0^l J^{-1}x_\eta\phi_\xi^2\delta x\bigg|_{\eta=-\widetilde{h}} d\xi d\tau - \int_{\tau_1}^{\tau_2}\int_0^l J^{-1}x_\xi\phi_\xi\phi_\eta\delta x\bigg|_{\eta=-\widetilde{h}} d\xi d\tau$$
$$= \int_{\tau_1}^{\tau_2}\int_0^l J^{-1}\phi_\xi\left(\frac{x_\eta}{J}\phi_\xi - \frac{x_\xi}{J}\phi_\eta\right)\delta x\bigg|_{\eta=-\widetilde{h}} d\xi d\tau = \int_{\tau_1}^{\tau_2}\int_0^l J^{-1}\phi_\xi\phi_y\delta x\bigg|_{\eta=-\widetilde{h}} d\xi d\tau = 0, \tag{5.3.9}$$

by the bottom boundary condition.

5.4 Variation of the potential and boundary terms with respect to X

Starting with P, and using integration by parts:

$$\begin{aligned}\delta\int_0^l P d\xi &= \frac{1}{2}g\int_0^l Y^2\frac{\partial}{\partial\xi}\delta X d\xi \\ &= \frac{1}{2}g[Y^2\delta X]_0^l - g\int_0^l YY_\xi\delta X d\xi = \int_0^l -gYY_\xi\delta X d\xi.\end{aligned} \tag{5.4.1}$$

Next, we consider B, also integrated by parts:

$$\begin{aligned}\delta\int_{\tau_1}^{\tau_2}\int_0^l(\Phi Y_\xi X_\tau - \Phi X_\xi Y_\tau)d\xi d\tau &= \int_0^l\int_{\tau_1}^{\tau_2}(\Phi Y_\xi\delta X_\tau)d\tau d\xi \\ \int_{\tau_1}^{\tau_2}\int_0^l(-\Phi Y_\tau\delta X_\xi)d\xi d\tau &= \int_0^l\left([Y_\xi\Phi\delta X]_{t_1}^{t_2} - \int_{t_1}^{t_2}(Y_{\xi\tau}\Phi + Y_\xi\Phi_\tau)\delta X d\tau\right) \\ +\int_{\tau_1}^{\tau_2}\left(-[Y_\tau\Phi\delta X]_0^l + \int_0^l(Y_{\xi\tau}\Phi + Y_\tau\Phi_\xi)\delta X\right)d\xi &= \int_{\tau_1}^{\tau_2}\int_0^l(\Phi_\xi Y_\tau - \Phi_\tau Y_\xi)\delta X d\xi d\tau.\end{aligned} \tag{5.4.2}$$

5.5 Variation of the potential and boundary terms with respect to Y

Starting with P, and using integration by parts:

$$\delta\int_0^l P d\xi = \frac{1}{2}g\int_0^l X_\xi\delta Y^2 d\xi = \int_0^l gYX_\xi\delta Y d\xi. \tag{5.5.1}$$

Next, we consider B, also integrated by parts:

$$\begin{aligned}\delta\int_{\tau_1}^{\tau_2}\int_0^l(\Phi Y_\xi X_\tau - \Phi X_\xi Y_\tau)d\xi d\tau &= \int_{\tau_1}^{\tau_2}\int_0^l(\Phi X_\tau\delta Y_\xi)d\xi d\tau \\ \int_0^l\int_{\tau_1}^{\tau_2}(-\Phi X_\xi\delta Y_\tau)d\tau d\xi &= \int_0^l\left(-[X_\xi\Phi\delta Y]_{t_1}^{t_2} + \int_{t_1}^{t_2}(X_{\xi\tau}\Phi + X_\xi\Phi_\tau)\delta Y d\tau\right) \\ +\int_{\tau_1}^{\tau_2}\left([X_\tau\Phi\delta Y]_0^l - \int_0^l(X_{\xi\tau}\Phi + X_\tau\Phi_\xi)\delta Y\right)d\xi &= \int_{\tau_1}^{\tau_2}\int_0^l(\Phi_\tau X_\xi - \Phi_\xi X_\tau)\delta Y d\xi d\tau.\end{aligned} \tag{5.5.2}$$

5.6 Recovering the dynamic free-surface boundary condition in curvilinear coordinates

First we take the δX terms from (5.3.8), adding potential energy variation (5.4.1) and boundary term variation (5.4.2):

$$J^{-1}(X_\eta \Phi_\xi - X_\xi \Phi_\eta)\Phi_\xi + \frac{1}{2} Y_\xi \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Phi - gYY_\xi + \Phi_\xi Y_\tau - \Phi_\tau Y_\xi. \tag{5.6.1}$$

Next we take the δY terms from (5.3.8), adding potential energy variation (5.5.1) and boundary term variation (5.5.2):

$$J^{-1}(Y_\eta \Phi_\xi - Y_\xi \Phi_\eta)\Phi_\xi - \frac{1}{2} X_\xi \boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Phi + gYX_\xi + \Phi_\tau X_\xi - \Phi_\xi X_\tau. \tag{5.6.2}$$

Recall that $\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi = \phi_x^2 + \phi_y^2 = U^2 + V^2$, this relation can be used to recover the dynamic free-surface boundary condition. Firstly for (5.6.1):

$$\begin{aligned} &-V\Phi_\xi + \frac{1}{2} Y_\xi (U^2 + V^2) - gYY_\xi + \Phi_\xi Y_\tau - \Phi_\tau Y_\xi \\ &= -V(UX_\xi + VY_\xi) + V^2 Y_\xi + \frac{1}{2} Y_\xi (U^2 - V^2) - gYY_\xi + \Phi_\xi Y_\tau - \Phi_\tau Y_\xi \\ &= -UVX_\xi + \frac{1}{2} Y_\xi (U^2 - V^2) - gYY_\xi + \Phi_\xi Y_\tau - \Phi_\tau Y_\xi, \end{aligned} \tag{5.6.3}$$

and for (5.6.2):

$$\begin{aligned} &U\Phi_\xi - \frac{1}{2} X_\xi (U^2 + V^2) + gYX_\xi + \Phi_\tau X_\xi - \Phi_\xi X_\tau \\ &= U(UX_\xi + VY_\xi) - X_\xi U^2 + \frac{1}{2} X_\xi (U^2 - V^2) + gYX_\xi + \Phi_\tau X_\xi - \Phi_\xi X_\tau \\ &= UVY_\xi + \frac{1}{2} X_\xi (U^2 - V^2) + gYX_\xi + \Phi_\tau X_\xi - \Phi_\xi X_\tau. \end{aligned} \tag{5.6.4}$$

As $\delta\widetilde{S} = 0$ these equal zero, hence we may write:

$$\begin{aligned} \frac{1}{2} Y_\xi (U^2 - V^2) - UVX_\xi - gYY_\xi &= \Phi_\tau Y_\xi - \Phi_\xi Y_\tau \\ \frac{1}{2}(U^2 - V^2) X_\xi + UVY_\xi + gYX_\xi &= \Phi_\xi X_\tau - \Phi_\tau X_\xi, \end{aligned} \tag{5.6.5}$$

which is the dynamic free-surface boundary condition on $\eta = 0$ [2].

Consider the dynamic free-surface boundary condition (4.7.5):

$$\Phi_\tau + J^{-1}\left((X_\eta Y_\tau - Y_\eta X_\tau)\Phi_\xi + (Y_\xi X_\tau - X_\xi Y_\tau)\Phi_\eta\right) + \frac{1}{2}(U^2+V^2) + gY = 0. \tag{5.6.6}$$

Using (4.3.1), (4.3.2), and defining $R = \Phi_\tau + \frac{1}{2}(U^2+V^2) + gY$ [2], we see that (5.6.6) can be expressed in form:

$$\begin{aligned} R + J^{-1}F_1\Phi_\xi + J^{-1}F_2\Phi_\eta = 0 \qquad \text{on} \quad \eta = 0 \\ = R + J^{-1}(F_1\Phi_\xi + F_2\Phi_\eta) = R + J^{-1}[(UX_\xi + VY_\xi)(X_\eta Y_\tau - Y_\eta X_\tau) \\ +(UX_\eta + VY_\eta)(Y_\xi X_\tau - X_\xi Y_\tau)] = R - J^{-1}[JUX_\tau + JVY_\tau] = R - UX_\tau - VY_\tau = 0. \end{aligned} \tag{5.6.7}$$

The surface is non-degenerate, meaning for unit normal $\boldsymbol{n} = (n_1, n_2)$ at free-surface $\eta = 0$ this equation is only satisfied if [2]:

$$\boldsymbol{n}\cdot(R - UX_\tau - VY_\tau) = 0. \tag{5.6.8}$$

Using the kinematic condition and $\ell\boldsymbol{n} = (-Y_\xi, X_\xi)$ from (4.4.1):

$$\begin{aligned} \ell n_1(UX_\tau - VY_\tau) = -Y_\xi(UX_\tau - VY_\tau) &= -UY_\xi X_\tau - Y_\tau(UX_\xi + VY_\xi) + UX_\xi Y_\tau \\ &= -\Phi_\xi Y_\tau - U^2X_\xi + UVX_\xi, \\ \ell n_2(UX_\tau - VY_\tau) = X_\xi(UX_\tau - VY_\tau) &= X_\tau(UX_\xi + VY_\xi) - VX_\tau Y_\xi + VX_\xi Y_\tau \\ &= \Phi_\xi X_\tau + V^2X_\xi - UVY_\xi, \end{aligned} \tag{5.6.9}$$

using identity $\Phi_\xi = UX_\xi + VY_\xi$. As $\ell n_1 R = -Y_\xi R$, $\ell n_2 R = X_\xi R$, we substitute these expressions into (5.6.7):

$$\begin{aligned} -Y_\xi R &= -\Phi_\xi Y_\tau - U^2X_\xi + UVX_\xi \\ \implies -Y_\xi\Phi_\tau - Y_\xi\frac{1}{2}(U^2+V^2) - gYY_\xi &= -\Phi_\xi Y_\tau - U^2X_\xi + UVX_\xi \\ \implies \frac{1}{2}Y_\xi(U^2-V^2) - UVX_\xi - gYY_\xi &= \Phi_\tau Y_\xi - \Phi_\xi Y_\tau, \end{aligned} \tag{5.6.10}$$

$$\begin{aligned} X_\xi R &= \Phi_\xi X_\tau + V^2X_\xi - UVY_\xi \\ \implies X_\xi\Phi_\tau + X_\xi\frac{1}{2}(U^2+V^2) + gYX_\xi &= \Phi_\xi X_\tau + V^2X_\xi - UVY_\xi \\ \implies \frac{1}{2}(U^2-V^2)X_\xi + UVY_\xi + gYX_\xi &= \Phi_\xi X_\tau - \Phi_\tau X_\xi. \end{aligned} \tag{5.6.11}$$

(5.6.10) and (5.6.11) are the expressions recovered in (5.6.5).

5.7 The hidden conservation law of terms δx and δy

From (5.3.5), (5.3.6), and (5.3.7) we take the δx terms:

$$
\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial \xi}[J^{-2}(x_\eta^2+y_\eta^2)y_\eta\phi_\xi^2]-\frac{1}{2}\frac{\partial}{\partial \eta}[J^{-2}(x_\eta^2+y_\eta^2)y_\xi\phi_\xi^2]-\frac{\partial}{\partial \eta}[J^{-1}x_\eta\phi_\xi^2]\\
&-\frac{\partial}{\partial \xi}[J^{-2}(x_\eta x_\xi+y_\eta y_\xi)y_\eta\phi_\xi\phi_\eta]+\frac{\partial}{\partial \eta}[J^{-2}(x_\eta x_\xi+y_\eta y_\xi)y_\xi\phi_\xi\phi_\eta]\\
&+\frac{\partial}{\partial \eta}[J^{-1}x_\xi\phi_\xi\phi_\eta]+\frac{\partial}{\partial \xi}[J^{-1}x_\eta\phi_\xi\phi_\eta]\\
&+\frac{1}{2}\frac{\partial}{\partial \xi}[J^{-2}(x_\xi^2+y_\xi^2)y_\eta\phi_\eta^2]-\frac{1}{2}\frac{\partial}{\partial \eta}[J^{-2}(x_\xi^2+y_\xi^2)y_\xi\phi_\eta^2]-\frac{\partial}{\partial \xi}[J^{-1}x_\xi\phi_\eta^2]\\
&=\frac{\partial}{\partial \xi}\Big[\frac{1}{2}y_\eta\left(J^{-2}(x_\eta^2+y_\eta^2)\phi_\xi^2-2J^{-2}(x_\eta x_\xi+y_\eta y_\xi)\phi_\xi\phi_\eta+J^{-2}(x_\xi^2+y_\xi^2)\phi_\eta^2\right)\\
&\quad+\phi_\eta(J^{-1}x_\eta\phi_\xi-J^{-1}x_\xi\phi_\eta)\Big]+\frac{\partial}{\partial \eta}\Big[\phi_\xi(J^{-1}x_\xi\phi_\eta-J^{-1}x_\eta\phi_\xi)\\
&-\frac{1}{2}y_\xi\left(J^{-2}(x_\xi^2+y_\xi^2)y_\xi\phi_\eta^2-2J^{-2}(x_\eta x_\xi+y_\eta y_\xi)\phi_\xi\phi_\eta+J^{-2}(x_\eta^2+y_\eta^2)\phi_\xi^2\right)\Big]\\
&=\frac{\partial}{\partial \xi}\left[\frac{1}{2}(u^2+v^2)y_\eta-v\phi_\eta\right]-\frac{\partial}{\partial \eta}\left[\frac{1}{2}(u^2+v^2)y_\xi-v\phi_\xi\right]\\
&=\frac{\partial}{\partial \xi}\left[\frac{1}{2}(u^2-v^2)y_\eta-uvx_\eta\right]-\frac{\partial}{\partial \eta}\left[\frac{1}{2}(u^2-v^2)y_\xi-uvx_\xi\right]\\
&=y_\eta\frac{\partial}{\partial \xi}\left[\frac{1}{2}(u^2-v^2)\right]-y_\xi\frac{\partial}{\partial \eta}\left[\frac{1}{2}(u^2-v^2)\right]-x_\eta\frac{\partial}{\partial \xi}[uv]+x_\xi\frac{\partial}{\partial \eta}[uv]\\
&=\frac{\partial}{\partial x}\left[\frac{1}{2}(\phi_x^2-\phi_y^2)\right]+\frac{\partial}{\partial y}[\phi_x\phi_y]=\phi_{xx}\phi_x-\phi_{xy}\phi_y+\phi_{xy}\phi_y+\phi_x\phi_{yy}=\phi_x\Delta\phi,
\end{aligned}
\tag{5.7.1}
$$

as $\frac{1}{2}(J^{-2}g_{22}\phi_\xi^2-2J^{-2}g_{12}\phi_\xi\phi_\eta+J^{-2}g_{11}\phi_\eta^2)=\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi=\phi_x^2+\phi_y^2=u^2+v^2$. For the δy terms in (5.3.5), (5.3.6), and (5.3.7):

$$
\begin{aligned}
&-\frac{1}{2}\frac{\partial}{\partial \xi}[J^{-2}(x_\eta^2+y_\eta^2)x_\eta\phi_\xi^2]+\frac{1}{2}\frac{\partial}{\partial \eta}[J^{-2}(x_\eta^2+y_\eta^2)x_\xi\phi_\xi^2]-\frac{\partial}{\partial \eta}[J^{-1}y_\eta\phi_\xi^2]\\
&+\frac{\partial}{\partial \xi}[J^{-2}(x_\eta x_\xi+y_\eta y_\xi)x_\eta\phi_\xi\phi_\eta]-\frac{\partial}{\partial \eta}[J^{-2}(x_\eta x_\xi+y_\eta y_\xi)x_\xi\phi_\xi\phi_\eta]\\
&+\frac{\partial}{\partial \eta}[J^{-1}y_\xi\phi_\xi\phi_\eta]+\frac{\partial}{\partial \xi}[J^{-1}y_\eta\phi_\xi\phi_\eta]\\
&-\frac{1}{2}\frac{\partial}{\partial \xi}[J^{-2}(x_\xi^2+y_\xi^2)x_\eta\phi_\eta^2]+\frac{1}{2}\frac{\partial}{\partial \eta}[J^{-2}(x_\xi^2+y_\xi^2)x_\xi\phi_\eta^2]-\frac{\partial}{\partial \xi}[J^{-1}y_\xi\phi_\eta^2]
\end{aligned}
$$

.

$$
\begin{aligned}
&= \frac{\partial}{\partial \xi}\Bigg[-\frac{1}{2}x_\eta\left(J^{-2}(x_\eta^2+y_\eta^2)\phi_\xi^2 - 2J^{-2}(x_\eta x_\xi + y_\eta y_\xi)\phi_\xi\phi_\eta + J^{-2}(x_\xi^2+y_\xi^2)\phi_\eta^2\right) \\
&\qquad +\phi_\eta(J^{-1}y_\eta\phi_\xi - J^{-1}y_\xi\phi_\eta)\Bigg] + \frac{\partial}{\partial \eta}\Bigg[\phi_\xi(J^{-1}y_\xi\phi_\eta - J^{-1}y_\eta\phi_\xi) \\
&\quad +\frac{1}{2}x_\xi\left(J^{-2}(x_\xi^2+y_\xi^2)y_\xi\phi_\eta^2 - 2J^{-2}(x_\eta x_\xi + y_\eta y_\xi)\phi_\xi\phi_\eta + J^{-2}(x_\eta^2+y_\eta^2)\phi_\xi^2\right)\Bigg] \\
&\qquad = -\frac{\partial}{\partial \xi}\left[\frac{1}{2}(u^2+v^2)x_\eta - u\phi_\eta\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{2}(u^2+v^2)x_\xi - u\phi_\xi\right] \qquad (5.7.2) \\
&\qquad = -\frac{\partial}{\partial \xi}\left[\frac{1}{2}(v^2-u^2)x_\eta - uvy_\eta\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{2}(v^2-u^2)x_\xi - uvy_\xi\right] \\
&\quad = -x_\eta\frac{\partial}{\partial \xi}\left[\frac{1}{2}(v^2-u^2)\right] + x_\xi\frac{\partial}{\partial \eta}\left[\frac{1}{2}(v^2-u^2)\right] + y_\eta\frac{\partial}{\partial \xi}[uv] - y_\xi\frac{\partial}{\partial \eta}[uv] \\
&= \frac{\partial}{\partial y}\left[\frac{1}{2}(\phi_y^2-\phi_x^2)\right] + \frac{\partial}{\partial x}[\phi_x\phi_y] = \phi_{yy}\phi_y - \phi_{xy}\phi_x + \phi_{xy}\phi_x + \phi_y\phi_{xx} = \phi_y\Delta\phi,
\end{aligned}
$$

hence from (5.7.1) and (5.7.2) we conclude [2]:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{1}{2}(\phi_x^2-\phi_y^2)\right] + \frac{\partial}{\partial y}[\phi_x\phi_y] &= \phi_x\Delta\phi, \\
\frac{\partial}{\partial x}[\phi_x\phi_y] + \frac{\partial}{\partial y}\left[\frac{1}{2}(\phi_y^2-\phi_x^2)\right] &= \phi_y\Delta\phi.
\end{aligned} \qquad (5.7.3)
$$

As from (3.4.2) we have the field equation for fluid motion $\Delta\phi = 0$, the right hand sides vanish, giving the conservation laws [2]:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{1}{2}(\phi_x^2-\phi_y^2)\right] + \frac{\partial}{\partial y}[\phi_x\phi_y] &= 0, \\
\frac{\partial}{\partial x}[\phi_x\phi_y] + \frac{\partial}{\partial y}\left[\frac{1}{2}(\phi_y^2-\phi_x^2)\right] &= 0.
\end{aligned} \qquad (5.7.4)
$$

6 Two dimensional floating rigid body

6.1 Deriving the extra terms

The aim in this section is to extend Luke's variational principle to the two-dimensional problem of interactions between water-waves and a floating rigid body. We study this problem in physical variables and leave the transformation of the corresponding variational principle to curvilinear coordinates for further studies of future work.

Now, if we add a floating container into the water, we have the floating rigid body problem. While we still use Bernoulli's equation, we now have the addition of multiple new terms and thus, new boundary conditions. Recall from (3.5.2) that:

$$S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{0}^{h(x,t)} p(x,y,t)dydxdt = \int_{t_1}^{t_2} \int_{\Omega} p(x,y,t)d\Omega dt, \tag{6.1.1}$$

where for simplicity we define the two-dimensional region Ω as the region in the x and y plane bounded by $[x_1, x_2]$ and $[0, h(x,t)]$, respectively. With the floating rigid body we now have new term additive to S that describes the body. This is defined as the time integral between $[t_1, t_2]$ of the rigid body kinetic energy (K_{body}) subtracted by the potential energy (P_{body}). Defining the new Lagrangian action integral as $\widehat{S}$, we now have:

$$\widehat{S} = \int_{t_1}^{t_2} \int_{\Omega} p(x,y,t)d\Omega dt + \int_{t_1}^{t_2} (K_{body} - P_{body})dt. \tag{6.1.2}$$

Define $\boldsymbol{X} = (X, Y)$ to be a fixed coordinate system, with $\boldsymbol{x} = (x, y)$ being a coordinate system that moves and rotates with the floating rigid body. We can relate the fixed and moving coordinates using a proper rotation matrix. For a proper rotation matrix $\boldsymbol{Q}(t)$, by definition we have that [4]:

$$\begin{aligned} \boldsymbol{Q}^T\boldsymbol{Q} &= \mathrm{I}, \\ det[\boldsymbol{Q}] &= 1. \end{aligned} \tag{6.1.3}$$

For $\boldsymbol{q} = (q_1, q_2)$ being a translation vector describing the position of the origin of the moving rigid body coordinate system relative to the fixed coordinate system. With this we can now define the relationship between coordinates of a point in the fixed and moving frame, and differentiate it with respect to t to find the rigid body velocity:

$$\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q} \implies \dot{\boldsymbol{X}} = \dot{\boldsymbol{Q}}\boldsymbol{x} + \dot{\boldsymbol{q}}. \tag{6.1.4}$$

Hence we define the kinetic energy of the rigid body [4]:

$$K_{body} = \int_{\Omega} \frac{1}{2}\rho||\dot{\boldsymbol{X}}||^2 d\Omega. \tag{6.1.5}$$

Because of the determinant identity in (6.1.3), we have that:

$$\begin{aligned}||\dot{\boldsymbol{X}}||^2 = \dot{\boldsymbol{X}} \cdot \dot{\boldsymbol{X}} = \boldsymbol{Q}\dot{\boldsymbol{X}} \cdot \boldsymbol{Q}\dot{\boldsymbol{X}} = \boldsymbol{Q}^{-1}\dot{\boldsymbol{X}} \cdot \boldsymbol{Q}^{-1}\dot{\boldsymbol{X}} = \boldsymbol{Q}^T\dot{\boldsymbol{X}} \cdot \boldsymbol{Q}^T\dot{\boldsymbol{X}} = ||\boldsymbol{Q}^T\dot{\boldsymbol{X}}||^2 \\ \implies K_{body} = \int_\Omega \frac{1}{2}\rho||\boldsymbol{Q}^T\dot{\boldsymbol{X}}||^2 d\Omega.\end{aligned} \tag{6.1.6}$$

Say we have the angular velocity vector $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$, then we may define $\widehat{\boldsymbol{\Omega}} = \boldsymbol{Q}^T\dot{\boldsymbol{Q}}$. But for our two-dimensional problem, rotation is about the $\widehat{\boldsymbol{z}}$ axis. Hence the rotation matrix becomes:

$$\begin{aligned}\dot{\boldsymbol{Q}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \boldsymbol{Q}^T\dot{\boldsymbol{Q}} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot{\theta}\sin\theta & -\dot{\theta}\cos\theta & 0 \\ \dot{\theta}\cos\theta & -\dot{\theta}\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix},\end{aligned} \tag{6.1.7}$$

and so $\boldsymbol{\Omega} = (0, 0, \dot{\theta})$, i.e., $\Omega_3 = \dot{\theta}$, $\Omega_2 = \Omega_1 = 0$. As a skew-symmetric matrix, i.e., $\widehat{\boldsymbol{\Omega}}^T = -\widehat{\boldsymbol{\Omega}}$. By making the relation that for some vector $\boldsymbol{a} \in \mathbb{R}^3$, $\widehat{\boldsymbol{\Omega}}\boldsymbol{a} = \boldsymbol{\Omega} \times \boldsymbol{a}$, negating the integral we can transform the kinetic energy equation [4]:

$$\begin{aligned}\frac{1}{2}\rho||\boldsymbol{Q}^T\dot{\boldsymbol{X}}||^2 = \frac{1}{2}\rho||\boldsymbol{Q}^T\dot{\boldsymbol{Q}}\boldsymbol{x} + \boldsymbol{Q}^T\dot{\boldsymbol{q}}||^2 \\ = \frac{1}{2}\rho(\widehat{\boldsymbol{\Omega}}\boldsymbol{x} + \boldsymbol{Q}^T\dot{\boldsymbol{q}}) \cdot (\widehat{\boldsymbol{\Omega}}\boldsymbol{x} + \boldsymbol{Q}^T\dot{\boldsymbol{q}}) = \frac{1}{2}\rho(\boldsymbol{\Omega} \times \boldsymbol{x} + \boldsymbol{Q}^T\dot{\boldsymbol{q}}) \cdot (\boldsymbol{\Omega} \times \boldsymbol{x} + \boldsymbol{Q}^T\dot{\boldsymbol{q}}) \\ = \frac{1}{2}\rho(||\boldsymbol{\Omega} \times \boldsymbol{x}||^2 + 2(\boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \boldsymbol{Q}^T\dot{\boldsymbol{q}} + ||\boldsymbol{Q}^T\dot{\boldsymbol{q}}||^2).\end{aligned} \tag{6.1.8}$$

Taking just the term $||\boldsymbol{\Omega} \times \boldsymbol{x}||^2$ we may simplify this by using Lagrange's identity for vectors $\boldsymbol{A}, \boldsymbol{B}$: $||\boldsymbol{A} \times \boldsymbol{B}||^2 = (\boldsymbol{A} \cdot \boldsymbol{A})(\boldsymbol{B} \cdot \boldsymbol{B}) - (\boldsymbol{A} \cdot \boldsymbol{B})^2$:

$$\begin{aligned}\int_\Omega \frac{1}{2}\rho||\boldsymbol{\Omega} \times \boldsymbol{x}||^2 d\Omega = \int_\Omega \frac{1}{2}\rho((\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})(\boldsymbol{x} \cdot \boldsymbol{x}) - (\boldsymbol{\Omega} \cdot \boldsymbol{x})^2) d\Omega \\ = \frac{1}{2}\boldsymbol{\Omega} \cdot \int_\Omega \rho(||\boldsymbol{x}||^2\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \boldsymbol{x})\boldsymbol{x}) d\Omega = \frac{1}{2}\boldsymbol{\Omega} \cdot \left[\int_\Omega \rho(||\boldsymbol{x}||^2\mathrm{I} - \boldsymbol{x}\boldsymbol{x}^T) d\Omega\right] \boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathrm{I}_b\boldsymbol{\Omega},\end{aligned} \tag{6.1.9}$$

where I is the diagonal identity matrix and $\mathrm{I}_b = \int_\Omega \rho(||\boldsymbol{x}||^2\mathrm{I} - \boldsymbol{x}\boldsymbol{x}^T) d\Omega$ is the mass moment of inertia symmetric matrix [4]. By (6.1.6), $||\boldsymbol{Q}^T\dot{\boldsymbol{q}}||^2 = ||\dot{\boldsymbol{q}}||^2$. Substituting this and (6.1.9) into (6.1.8) gives:

$$K_{body} = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathrm{I}_b\boldsymbol{\Omega} + \int_\Omega \left(\rho(\boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \boldsymbol{Q}^T\dot{\boldsymbol{q}} + \frac{1}{2}\rho||\dot{\boldsymbol{q}}||^2\right) d\Omega. \tag{6.1.10}$$

This can be further simplified by defining the centre of mass of the rigid body relative to the moving coordinate system attached to the body [4]:

$$\overline{\boldsymbol{x}} = \frac{\int_\Omega \rho \boldsymbol{x} d\Omega}{\int_\Omega \rho d\Omega} \implies \overline{\boldsymbol{x}} \int_\Omega \rho d\Omega = m\overline{\boldsymbol{x}} = \int_\Omega \rho \boldsymbol{x} d\Omega, \tag{6.1.11}$$

where $m = \int_\Omega \rho d\Omega$ is the mass of the rigid body. The first term $\frac{1}{2}\boldsymbol{\Omega} \cdot \mathrm{I}_b \boldsymbol{\Omega}$ becomes:

$$\mathrm{I}_b = \int_\Omega \rho(||\boldsymbol{x}||^2 \mathrm{I} - \boldsymbol{x}\boldsymbol{x}^T) d\Omega = \int_\Omega \rho \left(\begin{bmatrix} x^2+y^2 & 0 & 0 \\ 0 & x^2+y^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} - \begin{bmatrix} x^2 & xy & 0 \\ xy & y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) d\Omega$$

$$= \int_\Omega \rho \begin{bmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} d\Omega \implies \frac{1}{2}\boldsymbol{\Omega} \cdot \mathrm{I}_b \boldsymbol{\Omega} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \cdot \int_\Omega \rho \begin{bmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} d\Omega$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \cdot \int_\Omega \rho \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}(x^2+y^2) \end{bmatrix} d\Omega = \frac{1}{2} \int_\Omega \rho \dot{\theta}^2 (x^2+y^2) d\Omega = \frac{1}{2} m(\overline{x}^2 + \overline{y}^2)\dot{\theta}^2 = \frac{1}{2}\mathbb{I}\dot{\theta}^2, \tag{6.1.12}$$

where $\mathbb{I} = m(\overline{x}^2 + \overline{y}^2)$ [4]. Therefore for the two-dimensional wave problem, (6.1.10) becomes:

$$\begin{aligned} K_{body} &= \frac{1}{2}\mathbb{I}\dot{\theta}^2 + \int_\Omega \rho \left((\boldsymbol{\Omega} \times \boldsymbol{x}) \cdot \boldsymbol{Q}^T \dot{\boldsymbol{q}} + \frac{1}{2}||\dot{\boldsymbol{q}}||^2 \right) d\Omega \\ &= \frac{1}{2}\mathbb{I}\dot{\theta}^2 + m(\boldsymbol{\Omega} \times \overline{\boldsymbol{x}}) \cdot \boldsymbol{Q}^T \dot{\boldsymbol{q}} + \frac{1}{2} m ||\dot{\boldsymbol{q}}||^2 \end{aligned}$$

$$= \frac{1}{2}\mathbb{I}\dot{\theta}^2 + m \left(\begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix} \times \begin{bmatrix} \overline{x} \\ \overline{y} \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ 0 \end{bmatrix} + \frac{1}{2} m (\dot{q_1}^2 + \dot{q_2}^2)$$

$$= \frac{1}{2}\mathbb{I}\dot{\theta}^2 + m \begin{bmatrix} -\dot{\theta}\overline{y} \\ \dot{\theta}\overline{x} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{q}_1\cos\theta + \dot{q}_2\sin\theta \\ -\dot{q}_1\sin\theta + \dot{q}_2\cos\theta \\ 0 \end{bmatrix} + \frac{1}{2} m (\dot{q_1}^2 + \dot{q_2}^2),$$

$$K_{body} = \frac{1}{2}\mathbb{I}\dot{\theta}^2 - m\overline{y}\dot{\theta}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta) + m\overline{x}\dot{\theta}(-\dot{q}_1\sin\theta + \dot{q}_2\cos\theta) + \frac{1}{2} m (\dot{q_1}^2 + \dot{q_2}^2). \tag{6.1.13}$$

Next, we may also define potential energy [4]:

$$P_{body} = \int_\Omega \rho g[\boldsymbol{X}\cdot(0,1,0)]d\Omega = \int_\Omega \rho g[(\boldsymbol{Qx}+\boldsymbol{q})\cdot(0,1,0)]d\Omega = \int_\Omega \rho g[(\boldsymbol{Qx})\cdot(0,1,0)+q_2]d\Omega$$
$$= \int_\Omega \rho g\left[\begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ 0 \end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix} + q_2\right]d\Omega = \int_\Omega \rho g(x\sin\theta + y\cos\theta + q_2)d\Omega,$$
$$P_{body} = mg(\overline{x}\sin\theta + \overline{y}\cos\theta + q_2)$$
$$, \tag{6.1.14}$$

where height is defined trivially defined as $\boldsymbol{X}\cdot(0,1,0)$. Hence finally, the Lagrangian for the rigid body becomes:

$$\widehat{S} = \int_{t_1}^{t_2}\int_\Omega p(x,y,t)d\Omega dt + \int_{t_1}^{t_2}(K_{body}-P_{body})dt = \int_{t_1}^{t_2}\int_\Omega p(x,y,t)d\Omega dt + \int_{t_1}^{t_2}\Bigg(\frac{1}{2}\mathbb{I}\dot{\theta}^2$$
$$-m\overline{y}\dot{\theta}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta) + m\overline{x}\dot{\theta}(-\dot{q}_1\sin\theta + \dot{q}_2\cos\theta) + \frac{1}{2}m({\dot{q}_1}^2 + {\dot{q}_2}^2) - mg(\overline{x}\sin\theta + \overline{y}\cos\theta + q_2)\Bigg)dt. \tag{6.1.15}$$

6.2 Variation of the kinetic energy

First, consider the $\frac{1}{2}\mathbb{I}\dot{\theta}^2$ term:

$$\delta\left(\frac{1}{2}\mathbb{I}\dot{\theta}^2\right) = \frac{1}{2}\mathbb{I}\delta\dot{\theta}^2 = \frac{1}{2}\mathbb{I}(\dot{\theta}\delta\dot{\theta} + \dot{\theta}\delta\dot{\theta}) = \mathbb{I}\dot{\theta}\delta\left(\frac{d\theta}{dt}\right) = \mathbb{I}\dot{\theta}\frac{d}{dt}(\delta\theta) = \mathbb{I}\frac{d}{dt}(\dot{\theta}\delta\theta) - \mathbb{I}\ddot{\theta}\delta\theta. \tag{6.2.1}$$

Using this term in the time integral allows us to simplify the term further using the endpoint condition $\delta\theta(t_2) - \delta\theta(t_1) = 0$:

$$\mathbb{I}\int_{t_1}^{t_2}\left[\frac{d}{dt}(\dot{\theta}\delta\theta) - \ddot{\theta}\delta\theta\right]dt = \mathbb{I}[\dot{\theta}\delta\theta]_{t_1}^{t_2} + \mathbb{I}\int_{t_1}^{t_2} -\ddot{\theta}\delta\theta dt = \int_{t_1}^{t_2} -\mathbb{I}\ddot{\theta}\delta\theta dt. \tag{6.2.2}$$

Next, consider the $-m\overline{y}\dot{\theta}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)$ term. As $\overline{y}$ is fixed it has no variation.

$$\delta[-m\overline{y}\dot{\theta}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)] = -m\overline{y}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)\delta\dot{\theta} - m\overline{y}\dot{\theta}\delta(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)$$
$$= -m\overline{y}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)\delta\dot{\theta} - m\overline{y}\dot{\theta}(\dot{q}_1\delta\cos\theta + \cos\theta\delta\dot{q}_1 + \dot{q}_2\delta\sin\theta + \sin\theta\delta\dot{q}_2)$$
$$= -m\overline{y}\frac{d}{dt}[(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)\delta\theta] + m\overline{y}\frac{d}{dt}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)\delta\theta$$
$$-m\overline{y}\dot{\theta}\left(-\dot{q}_1\sin\theta\delta\theta + \frac{d}{dt}(\cos\theta\delta q_1) - \frac{d}{dt}(\cos\theta)\delta q_1 + \dot{q}_2\cos\theta\delta\theta + \frac{d}{dt}(\sin\theta\delta q_2) - \frac{d}{dt}(\sin\theta)\delta q_2\right)$$
$$= m\overline{y}\frac{d}{dt}(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)\delta\theta - m\overline{y}\dot{\theta}\left(-\dot{q}_1\sin\theta\delta\theta + \dot{\theta}\sin\theta\delta q_1 + \dot{q}_2\cos\theta\delta\theta - \dot{\theta}\cos\theta\delta q_2\right)$$
$$-m\overline{y}\left(\frac{d}{dt}[(\dot{q}_1\cos\theta + \dot{q}_2\sin\theta)\delta\theta] - \dot{\theta}\frac{d}{dt}(\cos\theta\delta q_1) + \dot{\theta}\frac{d}{dt}(\sin\theta\delta q_2)\right). \tag{6.2.3}$$

If this term is substituted back into the time integral, endpoint conditions of $\delta\theta(t_2) - \delta\theta(t_1) = 0$, $\delta q_1(t_2) - \delta q_1(t_1) = 0$, and $\delta q_2(t_2) - \delta q_2(t_1) = 0$ will eliminate the entire first term on the bottom row of (6.2.3), and through differentiation by parts will eliminate $-\frac{d}{dt}(\dot{\theta}\cos\theta\delta q_1)$ and $\frac{d}{dt}(\dot{\theta}\sin\theta\delta q_2)$ on the other two. Therefore, this leaves:

$$= m\overline{y}(\ddot{q}_1\cos\theta - \dot{q}_1\dot{\theta}\sin\theta + \ddot{q}_2\sin\theta + \dot{q}_2\dot{\theta}\cos\theta)\delta\theta$$
$$-m\overline{y}\dot{\theta}\left(-\dot{q}_1\sin\theta\delta\theta + \dot{\theta}\sin\theta\delta q_1 + \dot{q}_2\cos\theta\delta\theta - \dot{\theta}\cos\theta\delta q_2 + \ddot{\theta}\cos\theta\delta q_1 - \ddot{\theta}\sin\theta\delta q_2\right) \tag{6.2.4}$$

$$= m\overline{y}\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right)\delta\theta - m\overline{y}\dot{\theta}\left(\dot{\theta}\sin\theta + \ddot{\theta}\cos\theta\right)\delta q_1 + m\overline{y}\dot{\theta}\left(\dot{\theta}\cos\theta + \ddot{\theta}\sin\theta\right)\delta q_2.$$

Now considering the term $m\overline{x}\dot{\theta}(-\dot{q_1}\cos\theta + \dot{q_2}\sin\theta)$. Notably, this is a very similar term as $m\overline{y}\dot{\theta}(\dot{q_1}\cos\theta + \dot{q_2}\sin\theta)$, so our method is similar if we use endpoint conditions:

$$\begin{aligned}
\delta[m\overline{x}\dot{\theta}(-\dot{q_1}\sin\theta + \dot{q_2}\cos\theta)] &= m\overline{x}(-\dot{q_1}\sin\theta + \dot{q_2}\cos\theta)\delta\dot{\theta} + m\overline{x}\dot{\theta}\delta(-\dot{q_1}\sin\theta + \dot{q_2}\cos\theta) \\
&= m\overline{x}\frac{d}{dt}\left[(-\dot{q_1}\sin\theta + \dot{q_2}\cos\theta)\delta\theta\right] - m\overline{x}\frac{d}{dt}(-\dot{q_1}\sin\theta + \dot{q_2}\cos\theta)\delta\theta \\
&\quad + m\overline{x}\dot{\theta}(-\dot{q_1}\delta\sin\theta - \sin\theta\delta\dot{q_1} + \dot{q_2}\delta\cos\theta + \cos\theta\delta\dot{q_2}) \\
&= -m\overline{x}(-\ddot{q_1}\sin\theta - \dot{q_1}\dot{\theta}\cos\theta + \ddot{q_2}\cos\theta - \dot{q_2}\dot{\theta}\sin\theta)\delta\theta \\
+m\overline{x}\dot{\theta}&\left(-\dot{q_1}\cos\theta\delta\theta - \frac{d}{dt}(\sin\theta\delta q_1) + \frac{d}{dt}(\sin\theta)\delta q_1 - \dot{q_2}\sin\theta\delta\theta + \frac{d}{dt}(\cos\theta\delta q_2) - \frac{d}{dt}(\cos\theta)\delta q_2\right) \\
&= m\overline{x}(\ddot{q_1}\sin\theta - \ddot{q_2}\cos\theta)\delta\theta \\
+m\overline{x}&\left(-\frac{d}{dt}(\dot{\theta}\sin\theta\delta q_1) + \ddot{\theta}\sin\theta\delta q_1 + \dot{\theta}^2\cos\theta\delta q_1 + \frac{d}{dt}(\dot{\theta}\cos\theta\delta q_2) - \ddot{\theta}\cos\theta\delta q_2 + \dot{\theta}^2\sin\theta\delta q_2\right) \\
&= m\overline{x}(\ddot{q_1}\sin\theta - \ddot{q_2}\cos\theta)\delta\theta + m\overline{x}\dot{\theta}(\dot{\theta}\cos\theta + \ddot{\theta}\sin\theta)\delta q_1 + m\overline{x}\dot{\theta}(\dot{\theta}\sin\theta - \ddot{\theta}\cos\theta)\delta q_2.
\end{aligned} \tag{6.2.5}$$

Finally, the $\frac{1}{2}m(\dot{q_1}^2 + \dot{q_2}^2)$ term:

$$\begin{aligned}
\delta\left[\frac{1}{2}m(\dot{q_1}^2 + \dot{q_2}^2)\right] &= \frac{1}{2}m(\delta\dot{q_1}^2 + \delta\dot{q_2}^2) = m(\dot{q_1}\delta\dot{q_1} + \dot{q_2}\delta\dot{q_2}) \\
= m\left[\frac{d}{dt}(\dot{q_1}\delta q_1 + \dot{q_2}\delta q_2) - \ddot{q_1}\delta q_1 - \ddot{q_2}\delta q_2\right] &= -m\ddot{q_1}\delta q_1 - m\ddot{q_2}\delta q_2.
\end{aligned} \tag{6.2.6}$$

6.3 Variation of the potential energy

Considering the potential energy term:

$$\begin{aligned}
\delta[-mg(\overline{x}\sin\theta + \overline{y}\cos\theta + q_2)] &= -mg(\overline{x}\delta\sin\theta + \overline{y}\delta\cos\theta + \delta q_2)] \\
&= mg(\overline{y}\sin\theta - \overline{x}\cos\theta)\delta\theta - mg\delta q_2.
\end{aligned} \tag{6.3.1}$$

6.4 Boundary conditions of the floating body model

We may add each variation term back together, discounting the terms of S:

$$\begin{aligned}\int_{t_1}^{t_2}\Big[&-\ddot{\theta}\mathbb{I}\delta\theta + m\overline{y}\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right)\delta\theta - m\overline{y}\dot{\theta}\left(\dot{\theta}\sin\theta + \ddot{\theta}\cos\theta\right)\delta q_1\\ &+m\overline{y}\dot{\theta}\left(\dot{\theta}\cos\theta + \ddot{\theta}\sin\theta\right)\delta q_2 + m\overline{x}(\ddot{q}_1\sin\theta - \ddot{q}_2\cos\theta)\delta\theta\\ &+m\overline{x}\dot{\theta}(\dot{\theta}\cos\theta + \ddot{\theta}\sin\theta)\delta q_1 + m\overline{x}\dot{\theta}(\dot{\theta}\sin\theta - \ddot{\theta}\cos\theta)\delta q_2\\ &-m\ddot{q}_1\delta q_1 - m\ddot{q}_2\delta q_2 + mg(\overline{y}\sin\theta - \overline{x}\cos\theta)\delta\theta - mg\delta q_2\Big]dt.\end{aligned} \tag{6.4.1}$$

Separating this into $\delta\theta$, δq_1, and δq_2 terms gives three equations:

$$\begin{aligned}\delta\theta: &\quad -\mathbb{I}\ddot{\theta} + m\overline{y}\left(\ddot{q}_1\cos\theta + \ddot{q}_2\sin\theta\right) + m\overline{x}(\ddot{q}_1\sin\theta - \ddot{q}_2\cos\theta) + mg(\overline{y}\sin\theta - \overline{x}\cos\theta),\\ \delta q_1: &\quad -m\overline{y}\dot{\theta}\left(\dot{\theta}\sin\theta + \ddot{\theta}\cos\theta\right) + m\overline{x}\dot{\theta}(\dot{\theta}\cos\theta + \ddot{\theta}\sin\theta) - m\ddot{q}_1,\\ \delta q_2: &\quad m\overline{y}\dot{\theta}\left(\dot{\theta}\cos\theta + \ddot{\theta}\sin\theta\right) + m\overline{x}\dot{\theta}(\dot{\theta}\sin\theta - \ddot{\theta}\cos\theta) - m\ddot{q}_2 - mg.\end{aligned} \tag{6.4.2}$$

These are equations of motion for the rigid body but without the contribution from exterior waves, as we have have not yet considered the integral $S = \int_{t_1}^{t_2}\int_\Omega p(x,y,t)d\Omega dt$. Recall the Lagrangian for the wave-body problem in physical space (6.1.5). The contribution from exterior waves on the equations of motion for the rigid body appears when we take the variation of the Lagrangian for the fluid with a time-dependant boundary curve. As we studied before without a rigid body, the time-dependent boundary curve is the free-surface of the fluid. But with a floating rigid body the time-dependent boundary curve is the free-surface of the fluid and the curve for the rigid body [4]. Hence for RBV defined as the rigid body variations, s as the wetted rigid body surface, and $\boldsymbol{n} = (n_1, n_2)$ as the rigid body normal vector, taking the variations causes one extra term to appear as we now consider not only the pressure on the free-surface, but also against the rigid body:

$$\begin{aligned}\delta\widehat{S} &= \delta\int_{t_1}^{t_2}\int_\Omega -(\phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) + gy)d\Omega dt + \text{RBV}\\ &= -\int_{t_1}^{t_2}\int_{x_1}^{x_2}[\phi_t + \frac{1}{2}(\phi_x^2+\phi_y^2) + gy]\Big|^{y=h}\delta h dx dt - \int_{t_1}^{t_2}\int_\Omega(\delta\phi_t + (\phi_x\delta\phi_x + \phi_y\delta\phi_y))d\Omega dt\\ &\quad + \int_{t_1}^{t_2}\int_s P(\delta\boldsymbol{X}_s\cdot\boldsymbol{n})ds dt + \text{RBV}.\end{aligned} \tag{6.4.3}$$

Here we have that:

$$\begin{aligned}\delta\boldsymbol{X}_s\cdot\boldsymbol{n} &= \delta(\boldsymbol{Q}\boldsymbol{x}_s + \boldsymbol{q})\cdot\boldsymbol{n}\\ &= (\dot{\boldsymbol{Q}}\boldsymbol{x}_s\delta\theta + \delta\boldsymbol{q})\cdot\boldsymbol{n} = \dot{\boldsymbol{Q}}\boldsymbol{x}_s\cdot\boldsymbol{n}\delta\theta + \delta\boldsymbol{q}\cdot\boldsymbol{n},\end{aligned} \tag{6.4.4}$$

for wetted surface coordinates relative to the fixed frame $\boldsymbol{X}_s$ and wetted surface coordinates relative to the body frame $\boldsymbol{x}_s$.

The variations then become:

$$
\begin{aligned}
&-\int_{t_1}^{t_2}\int_{x_1}^{x_2}[\phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2)+gy]|^{y=h}\delta h dxdt-\int_{t_1}^{t_2}\int_{x_1}^{x_2}(\delta\phi_t+(\phi_x\delta\phi_x+\phi_y\delta\phi_y))d\Omega dt\\
&\quad+\int_{t_1}^{t_2}\int_s P(\delta\boldsymbol{X}_s\cdot\boldsymbol{n})dsdt+\text{RBV}=-\int_{t_1}^{t_2}\int_{x_1}^{x_2}[\phi_t+\frac{1}{2}(\phi_x^2+\phi_y^2)+gy]|^{y=h}\delta h dxdt\\
&\quad+\int_{t_1}^{t_2}\int_{x_1}^{x_2}(h_t+h_x\phi_x-\phi_y)\delta\phi|^{y=h}+\int_{t_1}^{t_2}\int_{x_1}^{x_2}\phi_y|_{y=0}\delta\phi dxdt+\int_{t_1}^{t_2}\int_\Omega\Delta\phi\delta\phi d\Omega dt\\
&+\int_{t_1}^{t_2}\int_s P(\dot{\boldsymbol{Q}}\boldsymbol{x}_s\cdot\boldsymbol{n})\delta\theta dsdt+\int_{t_1}^{t_2}\int_s P\boldsymbol{n}\cdot\delta\boldsymbol{q}dsdt+\int_{t_1}^{t_2}\int_s\left(\dot{\boldsymbol{X}}\cdot\boldsymbol{n}-\frac{\partial\phi}{\partial\boldsymbol{n}}\right)\delta\phi dsdt+RBV,
\end{aligned}
\tag{6.4.5}
$$

where the bottom, dynamic, kinematic, and field equations are all recovered [2] [5], as well as the contact boundary condition [2]:

$$
\dot{\boldsymbol{X}}\cdot\boldsymbol{n}-\frac{\partial\phi}{\partial\boldsymbol{n}}=0\implies\frac{\partial\phi}{\partial\boldsymbol{n}}=\boldsymbol{\nabla}\phi\cdot\boldsymbol{n},\tag{6.4.6}
$$

and the boundary conditions for the rigid body:

$$
-\mathbb{I}\ddot{\theta}+m\overline{y}\left(\ddot{q_1}\cos\theta+\ddot{q_2}\sin\theta\right)+m\overline{x}(\ddot{q_1}\sin\theta-\ddot{q_2}\cos\theta)+mg(\overline{y}\sin\theta-\overline{x}\cos\theta)+\int_s P(\dot{\boldsymbol{Q}}\boldsymbol{x}_s\cdot\boldsymbol{n})ds=0,\tag{6.4.7}
$$

$$
-m\overline{y}\dot{\theta}\left(\dot{\theta}\sin\theta+\ddot{\theta}\cos\theta\right)+m\overline{x}\dot{\theta}(\dot{\theta}\cos\theta+\ddot{\theta}\sin\theta)-m\ddot{q_1}+\int_s Pn_1\delta q_1 ds=0,\tag{6.4.8}
$$

$$
m\overline{y}\dot{\theta}\left(\dot{\theta}\cos\theta+\ddot{\theta}\sin\theta\right)+m\overline{x}\dot{\theta}(\dot{\theta}\sin\theta-\ddot{\theta}\cos\theta)-m\ddot{q_2}-mg+\int_s Pn_2\delta q_2 ds=0.\tag{6.4.9}
$$

7 Concluding remarks

Under cartesian coordinates we have derived governing equations (3.6.1), (3.6.2), (3.6.3), and (3.6.4) in two-dimensions. The cartesian model would be simple and the most useful approach for a near-constant free-surface with little to no distortion, as it can be assumed $h(x,t)$ and ϕ have small derivatives, thus we may linearise the boundary-value problem about a level free-surface:

$$
\begin{array}{cc}
\text{Dynamic free-surface boundary condition} & \phi_t+gy=0\\
\text{Kinematic free-surface boundary condition} & h_t-\phi_y=0\\
\text{Bottom boundary condition} & \phi_y=0\\
\text{Field equation for fluid motion} & \Delta\phi=0
\end{array}
\tag{7.0.1}
$$

The same may be done for the floating rigid body equations (6.4.7), (6.4.8), and (6.4.9), as an undistorted free-surface will not rotate or move the rigid body to a large degree. As the free-surface becomes more distorted, we cannot assume a near constant $h(x,t)$ or ϕ and the curvilinear model becomes more suitable, though requires the solving of more differential equations in governing equations (5.2.7), (5.2.10), (5.2.15), and (5.6.5).

In the future I plan to move the floating rigid body model to a curvilinear moving mesh two dimensional model as done with the classical wave problem, or a cartesian three dimensional model. In preliminary thought for the former, the boundary conditions would not be modified to a large degree, as it is assumed that time $t = \tau$, and θ, q_1, and q_2 all depend solely on time. For the latter, this would become more complex due to the new three dimensional definitions for rotation, rigid body position, and Luke's Lagrangian.

Further, I also plan to map a moving mesh to the three dimensional classical wave problem, defining the new map [2]:

$$(\xi, \eta, \kappa, \tau) \mapsto \begin{bmatrix} x(\xi, \eta, \kappa, \tau) \\ y(\xi, \eta, \kappa, \tau) \\ z(\xi, \eta, \kappa, \tau) \\ t(\tau) \end{bmatrix}, \tag{7.0.2}$$

with free surface defined [2]:

$$\begin{bmatrix} X(\xi, \eta, \tau) \\ Y(\xi, \eta, \tau) \\ Z(\xi, \eta, \tau) \end{bmatrix} = \lim_{\kappa \to 0} \begin{bmatrix} x(\xi, \eta, \kappa, \tau) \\ y(\xi, \eta, \kappa, \tau) \\ z(\xi, \eta, \kappa, \tau) \end{bmatrix}, \tag{7.0.3}$$

as $\kappa \to 0$. Considering the three dimensional Bernoulli's pressure equation, computational space volume integral, and time integral, this model becomes more complex in both the cartesian and curvilinear coordinate system requiring its own study in the future.

References

1. Frank Boyer, Pierre Fabrie, Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models (2013), p. 38

2. Thomas J. Bridges, Neil M. Donaldson, Variational Principles for Water Waves from the Viewpoint of a Time-Dependent Moving Mesh, Mathematika 57 (2011), pp. 147–173

3. Etienne Guyon, Jean-Pierre Hulin, Luc Petit, Catalin D. Mitescu, Physical Hydrodynamics, Oxford University Press (2001), pp. 210-220

4. Darryl D. Holm, Tanya Schumah, Cristina Stoica, David C. P. Ellis, Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions (2009), pp. 12-16, pp. 42-49, pp. 103-145

5. J. C. Luke, A variational principle for a fluid with a free-surface, J. Fluid Mech. (1967), vol. 27, part 2, pp. 395-397

6. Philip M. Morse, Herman Feshbach, Methods of Theoretical Physics (1953), pp. 37