Lecture notes for Introduction to Modern Algebra I: Lecture 13 + 14

1 Isomorphism theorems

1.1 Isomorphism theorems

Let $\varphi\colon G \longrightarrow G'$ be a homomorphism of groups and $H = \ker(\varphi)$. From the previous lecture, we know that there exist a surjective group homomorphism $q\colon G \longrightarrow G/H$. We have a converse in the form:

Theorem 1. *(First isomorphism theorem) Let $\varphi\colon G \longrightarrow G'$ be a homomorphism of groups. Suppose that φ is surjective and let H be the kernel of φ. Then G' is isomorphic to the quotient group G/H.*

Proof. Let $x \in G$ and $\bar{x} = xH$ the equivalence class in the quotient G/H. Let us define the map $\tilde{\varphi}\colon G/H \longrightarrow G'$ by the formula $\tilde{\varphi}(\bar{x}) = \varphi(x)$. To check that the map $\tilde{\varphi}$ is well defined we observe that:

$$\bar{x} = \bar{x'} \Rightarrow xx'^{-1} \in H = \ker(\varphi) \Rightarrow \varphi(x) = \varphi(x'),$$

and $\tilde{\varphi}(\bar{x}) = \tilde{\varphi}(\bar{x'})$. □

Example 2. Let G be a cyclic group with generator g. Define a map by $\varphi\colon \mathbb{Z} \longrightarrow G$ by $\varphi(n) = g^n$. This map is a surjective homomorphism since .

$$\varphi(n+m) = g^{n+m} = g^n g^m = \varphi(n)\varphi(m).$$

Clearly φ is onto. If $|g| = m$, then $\ker(\varphi) = m\mathbb{Z}$ and $\mathbb{Z}/\ker(\varphi) \cong \mathbb{Z}_m \cong G$. On the other hand, if the order of the generator g is infinite, then $\ker(\varphi) = 0$ and $G \cong \mathbb{Z}$ are isomorphic. Hence, two cyclic groups are isomorphic exactly when they have the same order. Up to isomorphism, the only cyclic groups are $\mathbb{Z}$ and $\mathbb{Z}_m$.

Remark 3. (Universal property of the quotient G/H) The first isomorphism theorem is reflection of a deeper property of the quotient. Let $\varphi\colon G \longrightarrow G'$ be a homomorphism of groups and H any normal subgroup of G. The following two are equivalent:

(1) The map φ annihilates H, that is, $\varphi(H) = e'$.

(2) The map φ factors through $q\colon G \longrightarrow G/H$ in the sense that there exist a group homomorphism $\theta\colon G/H \longrightarrow G'$ such that $\varphi = \theta \circ q$.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "q"'] & G' \\ G/H \arrow[ur, dashed, "\theta"] & \end{tikzcd}$$

Theorem 4. *(Second Isomorphism Theorem). Let G be a group, H a subgroup and N a normal subgroup.*

$$H/(H \cap N) \cong HN/N.$$

Proof. Consider the natural map $G \longrightarrow G/N$ restricted to the subgroup H. The image is the union of classes:

$$HN = N \cup h_1 N \cup h_2 N \cup \dots$$

Since N is normal, the set $HN \subset G$ is a subgroup with N as a normal subgroup. Hence

$$HN/N = \{N, h_1 N, \dots\}$$

and the image of the quotient map restricted to H is HN/N. On the other hand the kernel of this map is $H \cap N$. By the first theorem: $H/(H \cap N) \cong HN/N$. □

Theorem 5. *(Third Isomorphism Theorem) Let $K \subset H$ be two normal subgroups of a group G. Then*

$$G/H \cong (G/K)\,/\,(H/K)\,.$$

Proof. Consider the map $\varphi\colon G/K \longrightarrow G/H$ sending $xK \mapsto xH$. We need to show that:

(1) The map φ is well defined: if $x'K = xK$ then $x' = xk$ for some $k \in K$ and so because $K \subset H$ we have $x'H = xH$.

(2) The map φ is a group homomorphism with image G/H, since we have

$$xKx'K = xx'K \mapsto xx'H = xHx'H.$$

(3) The kernel of the map is $\ker(\varphi) = H/K$.

And the result will follow from the first isomorphism theorem. □

Example 6. By the Third Isomorphism Theorem, we know that

$$\mathbb{Z}/m\mathbb{Z} \cong (\mathbb{Z}/mn\mathbb{Z})/(m\mathbb{Z}/mn\mathbb{Z}).$$

As a consequence for example the order $|m\mathbb{Z}/mn\mathbb{Z}| = mn/m = n$.

Practice Questions:

1. Find the quotient G/H given that $G = \mathbb{Z}_4$ and $H = \{0, 2\}$. Write down the multiplication table for G/H