Lecture notes for Modern Algebra I: Lecture 24

1 Burside Counting Theorem

Suppose that G is a finite group and X is a finite G-set. We would like to count the orbits under the action of G. Let us recall that we can define:

Definition 1. The stabilizer of an element $x \in X$ is the subgroup G_x of G defined as

$$G_x = \{g \in G \mid gx = x\}.$$

Definition 2. The fixed point set of an element $g \in G$ is the subset X_g of X defined as

$$X_g = \{x \in X \mid gx = x\}.$$

Also we will put the equivalence relation determined by elements being on the same orbit:

Definition 3. For $x, y \in X$, we consider the G-equivalence relation $x \sim y$ if there exist $g \in G$ such that $gx = y$.

Lemma 4. *If $x \sim y$ then the stabilizer sets satisfy $G_x \cong G_y$ and in particular $|G_x| = |G_y|$.*

Proof. Let $y = gx$ and define a map $\phi_g \colon G_x \longrightarrow G_y$ defined by $\phi_g(a) = gag^{-1}$. First of all we can observe that

$$\phi_g(a)y = gag^{-1}y = gax = gx = y$$

Also $\phi_g(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \phi_g(a)\phi_g(b)$. Our map is therefore a group homomorphism with inverse $\phi_a^{-1} \colon G_y \longrightarrow G_x$ defined by $\phi_a^{-1}(b) = g^{-1}bg$ for every element $b \in G_y$. Therefore we have an isomorphism and the theorem follows. $\square$

As mentioned earlier, we aim to count orbits under the action of a finite group G on a finite set X.

Theorem 5. *(Burnside counting theorem) Let G be a finite group acting on a set X and let k denote the number of orbits of X. Then*

$$k = \frac{1}{|G|} \sum_{g \in G} |X_g|.$$

Proof. We look at all the fixed points x of elements $g \in G$; that is, we are counting pairs (g, x) such that $gx = x$. There are two different ways to view it, as fixed points:

$$\sum_{g \in G} |X_g|$$

and as stabilizer subgroups is

$$\sum_{x \in X} |G_x|.$$

We have then the identity

$$\sum_{g \in G} |X_g| = \sum_{x \in X} |G_x|.$$

On the other hand for $y \in O_x$, the sum $\sum_{y \in O_x} |G_y| = |O_x||G_x| = |G : G_x||G_x| = |G|$. Adding up over all orbits O_x, we get

$$\sum_{g \in G} |X_g| = \sum_{x \in X} |G_x| = k|G|.$$

and dividing by $|G|$ we obtain the result. $\square$

1.1 Applications

Let G be a finite group acting on a finite set $X = \{x_1, \ldots, x_n\}$ We can apply Burnside theorem to count the number of ways in which the elements of X can be colored using a fix amount of colors. For example consider the action of $G = \mathbb{D}_4$ on the vertices $X = \{1, 2, 3, 4\}$ of a square. Let us count the number of ways those vertices can be colored using two colors: black and white. Notice that we can sometimes obtain equivalent colorings by simply applying a rigid motion to the square. For instance, if we color one of the vertices black and the remaining three white, it does not matter which vertex was colored black since a rotation will give an equivalent coloring. The elements of $\mathbb{D}_4$ are:

$$\begin{array}{cccc} (1) & (13) & (24) & (1432) \\ (1234) & (12)(34) & (14)(23) & (13)(24) \end{array}$$

A coloring is a map $f : X \longrightarrow \{B, W\}$. We are going to consider the set $\tilde{X}$ of maps $f : X \longrightarrow \{B, W\}$. The action of G on X defines an action of a permutation group $\tilde{G}$ on $\tilde{X}$. This action is defined by $\tilde{\sigma}(f) = f \circ \sigma$. **The number of colorings up to $\tilde{G}$-action is the number of orbits on $\tilde{X}$** or, what is the same, the number of $\tilde{G}$-equivalent classes on $\tilde{X}$:

1. $\tilde{X}_{(1)} = \tilde{X}$ since the identity fixes every possible coloring. $|\tilde{X}_{(1)}| = 2^4 = 16$.

2. $\tilde{X}_{(1234)}$ consist of $f \in \tilde{X}$ such that $f(1) = f(2) = f(3) = f(4)$ and $|\tilde{X}_{(1234)}| = 2$.

3. $|\tilde{X}_{(1234)}| = 2$.

4. $\tilde{X}_{(12)(34)}$ consist of f such that $f(1) = f(2)$ and $f(3) = f(4)$ and $|\tilde{X}_{(12)(34)}| = 4$.

5. $|\tilde{X}_{(13)(24)}| = 4$.

6. $|\tilde{X}_{(14)(23)}| = 4$.

7. $\tilde{X}_{(13)}$ consist of maps f such that $f(1) = f(3)$ and we have $|\tilde{X}_{(13)}| = 2^3 = 8$.

8. $|\tilde{X}_{(24)}| = 8$.

Now, we can use Burside formula to count the number of orbits on $\tilde{X}$ under the action by $\tilde{G}$:

$$k = \frac{1}{8}(16 + 2 + 2 + 4 + 4 + 4 + 8 + 8) = 48/8 = 6.$$

Proposition 6. *Let G be a permutation group of X, Y any set and $\tilde{X}$ the set of maps from X to Y. There exist a permutation group $\tilde{G}$ acting on $\tilde{X}$, where $\tilde{\sigma} \in \tilde{G}$ is defined by $\tilde{\sigma}(f) = f \circ \sigma$ for $\sigma \in G$ and $f \in \tilde{X}$. Furthermore, if n is the number of cycles in the cycle decomposition of σ, then $|X_\sigma| = |Y|^n$.*

Proof. For each permutation σ of X he map $\tilde{\sigma} \colon \tilde{X} \longrightarrow \tilde{X}$. Also, is If we were to have $\tilde{\sigma}(f) = \tilde{\sigma}(g)$, for elements $f, g \in \tilde{X}$, then

$$f \circ \sigma = g \circ \sigma \Rightarrow f \circ \sigma(x) = g \circ \sigma(x) \, \forall \, x \in X$$

and since σ is a permutation $\sigma \colon X \longrightarrow X$, it must be bijective and $f = g$. On the other hand, the map $G \to \tilde{G}$ that maps $\sigma \mapsto \tilde{\sigma}$ is a bijection. Since the equality

$$\tilde{\sigma} = \tilde{\sigma'} \Rightarrow f \circ \sigma = f \circ \sigma',$$

for all functions $f \colon X \longrightarrow Y$ would give $\sigma = \sigma'$.
Suppose that σ is written as product of disjoint cycles as $\sigma = \sigma_1 \circ \cdots \circ \sigma_n$. Any $f \in \tilde{X}_\sigma$ will have the same value on each cycle of σ and therefore $|\tilde{X}_\sigma| = |Y|^n$. $\square$