1 External product and internal direct product

1.1 Isomorphisms

Definition 1. A map $\varphi\colon (G, *) \longrightarrow (G', .)$ is an isomorphism if it satisfies the following two conditions:

(1) The map φ is bijective.

(2) For all elements $x, x \in G$, $\varphi(x * y) = \varphi(x).\varphi(x')$.

Two groups $(G, *)$ and $(G', \star)$ are isomorphic to each other if a group isomorphism exists between them. We denoted by $G \cong G'$.

Example 2. The groups $(\mathbb{Z}, +)$ and $(2\mathbb{Z}, +)$ are isomorphic: $\mathbb{Z} \cong 2\mathbb{Z}$. We can use the invertible additive map $n \mapsto 2n$ between them.

Example 3. The dihedral group $\mathbb{D}_3$ and the symmetric group S_3 are isomorphic. As they are groups of finite order and $\mathbb{D}_3 \subset S_3$, the identity map is the bijective map between them.

Example 4. Let G be a group and $x \in G$. The pair of conjugate subgroups H and $x^{-1}Hx$ are isomorphic, via the inner automorphism $\varphi_x\colon G \longrightarrow G$.

Remark 5. Suppose $\psi\colon G \cong G'$. Then, corresponding elements x and $\psi(x)$ have the same order.

1.2 External direct product

Definition 6. Given groups $(G, *)$ and $(H, .)$, we can construct a group that is the direct product of G and H. As a set, the direct product is just the Cartesian product $G \times H$ together with the operation $(g, h)(g', h') = (g * g', h.h')$. The group $G \times H$ is called the external direct product of G and H.

Example 7. $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\} \cong V_4$. Compare the tables:

$$\mathbb{V}_4 = \begin{array}{c|cccc} & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array} \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 = \begin{array}{c|cccc} & (0,0) & (1,0) & (0,1) & (1,1) \\ \hline (0,0) & (0,0) & (1,0) & (0,1) & (1,1) \\ (1,0) & (1,0) & (0,0) & (0,1) & (0,1) \\ (0,1) & (0,1) & (1,1) & (0,0) & (1,0) \\ (1,1) & (1,1) & (0,1) & (1,0) & (0,0) \end{array}$$

The isomorphism is $\varphi(e) = (0,0), \varphi(a) = (1,0), \varphi(b) = (0,1)$ and $\varphi(c) = (1,1)$.

Remark 8. The direct product of abelian groups is always abelian.

Proposition 9. *The order of an element (g, g') in the product $G \times G'$ is the least common multiple $lcm(ord(g_1), ord(g_1))$.*

Proof. Let $n = \mathrm{ord}(g)$, $m = \mathrm{ord}(g')$ and $k = \mathrm{lcm}(n, m)$. We have $k(g, g') = (e, e')$, hence $\mathrm{ord}(g, g')|k$. On the other hand $\mathrm{ord}(g, g')(g, g') = (e, e')$, which means $\mathrm{ord}(g, g')(g) = e$ and $\mathrm{ord}(g, g')(g') = e'$. The order $\mathrm{ord}(g, g')$ is hence a multiple of both n and m and we get $k|\mathrm{ord}(g, g')$. $\square$

Corollary 10. $\mathbb{Z}_n \times \mathbb{Z}_m \cong \mathbb{Z}_{mn}$ *if and only if* $gcd(m, n) = 1$.

Corollary 11. *Let* $n = p_1^{n_1} \dots p_k^{n_k}$. *Then* $\mathbb{Z}_n \cong \mathbb{Z}_{p_1^{n_1}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$.

Corollary 12. *For a square free number* $n = p_1 p_2 \dots p_k$, *we have only one group* $\mathbb{Z}_n \cong \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_k}$ *of order* n. *On the other hand, for* n *divisible by a square, we can create different abelian groups with the same order.*

1.3 Internal direct product

Definition 13. Let G be a group with subgroups H and K satisfying the following conditions:

(a) $G = HK = \{hk \,|h \in H \,\text{and}\, k \in K\}$

(b) $H \cap K = \{e\}$.

(c) $hk = kh$ for all $h \in H$ and for all $k \in K$.

Then G is the internal direct product of H and K.

Remark 14. We will see in coming lectures that we can replace (c) by using a special type of subgroups of G called **normal subgroups**. We could take (c') H and K are normal subgroups of G. Then, conditions (c')+(b) implies (c) since:

$$h^{-1}k^{-1}hk \in H \cap K = \{e\}$$

Example 15. The group S_3 has a subgroups $H = \langle(123)\rangle$ of order three and several subgroups of order two, for example $K = \langle(12)\rangle$. However, elements of K and H do not commute and $S_3 \neq H \times K$. Among other things K is not normal in S_3.

Example 16. The dihedral group $\mathbb{D}_6$ is an internal direct product of its two subgroups:

$$H = \{\mathrm{id}, r^3\} \qquad K = \{\mathrm{id}, r^2, r^4, s, r^2s, r^4s\}.$$

Condition (a) can be checked directly $HK = \{\mathrm{id}, r, r^2, r^3, r^4, r^5, s, rs, r^2s, r^3s, r^4s, r^5s\}$.
Also, the property $sr^3 = r^3s$ of the dihedral group gives (c).
Now when we pay further attention to the group K we see that the elements r^2 and r^4 are of order three. Also, the elements s, r^2s, r^4s are all of order two. The subgroup K is therefore $K \cong S_3$ and $\mathbb{D}_6 \cong \mathbb{Z}_2 \times S_3$.

Theorem 17. *Let G be the internal direct product of subgroups H and K. Then G is isomorphic to $H \times K$.*

Proof. It would be sufficient to check that the map $\varphi\colon H \times K \longrightarrow G$ given by $\varphi(h, k) = hk$ satisfies:

1. It is a group homomorphism since:
$$\varphi(h, k) \cdot \varphi(h', k') = hkh'k' = hh'kk' = \varphi(hh', kk').$$
2. It is surjective: Property (a).
3. It is injective: $\varphi(h, k) = \varphi(h, k) \Rightarrow hk = h'k' \Rightarrow h'^{-1}h = k'k^{-1} \in H \cap K \Rightarrow h'^{-1}h = e = k'k^{-1} \Rightarrow (h, k) = (h', k')$.

$\square$

Practice Questions:

1. Show that if G and G' are groups, we have copies $G \cong G_0 \subset G \times G'$ and $G' \cong G'_0 \subset G \times G'$ such that G_0, G'_0 are normal subgroups of $G \times G'$.