Lecture Notes for Introduction to Modern Algebra I: Lecture 10

1 Dihedral groups

Suppose that we have number the vertices of a regular n-gon by $\{1, 2, \ldots, n\}$. Notice that there are exactly n choices to replace the first vertex. If we replace the first vertex by k, then the second vertex must be replaced either by vertex $k+1$ or by vertex $k-1$ (to be a rigid motion); hence, there are at most $2n$ possible rigid motions of the regular n-gon. The group of symmetries of the regular n-polygon is denoted by $\mathbb{D}_n$.

Let us denote, for $k = 0, 1, \ldots, n-1$, by r_k the counter-clock rotation with angle

$$\theta_k = \frac{360° k}{n}.$$

Also, for $k = 1, 2, \ldots, n$, we denote by s_k, the reflexion around the axis of symmetry through vertex k. If k is even there are only $n/2$ such different reflexions. On the other hand if k is odd, there will be n such reflexions. The reflexions s_k satisfy $s_k^2 = 1$ and the rotation $r_k^n = 1$. We denote $s_1 = s$ and $r_1 = r$.

Lemma 1. *The elements r, s satisfy the relation $sr^j = r^{-j}s$.*

Proof. Let s be the reflection around the axis of symmetry through vertex 1. Take any vertex k and consider the action of both maps on k mod n:

$$sr^j(k) \equiv s(k+j) \equiv 2-k-j \equiv r^{-j}(2-k) \equiv r^{-j}s(k).$$

And this would take care of the relation we wanted to prove. □

Theorem 2. *The group $\mathbb{D}_n$, with $n \geq 3$, consists of all products of the two elements r and s, satisfying the relations*

$$r^n = 1, \qquad s^2 = 1 \quad \textit{and} \quad srs = r^{-1}.$$

Proof. Any rigid motion t of the n-gon replacing the first vertex by the vertex k, must replace the second vertex by an adjacent vertex to k. If the second vertex goes to $k+1$, then $t = r^k$. If the second vertex is replaced by $k-1$, then $t = r^k s$. Hence, r and s generate $\mathbb{D}_n$ and

$$\mathbb{D}_n = \{1, r, r^2, \ldots, r^{n-1}, s, rs, \ldots, r^{n-1}s\},$$

by lemma 1. □

Remark 3. The rotation r_k admits a matrix representation:

$$r_k = \begin{pmatrix} \cos\left(\dfrac{360° k}{n}\right) & -\sin\left(\dfrac{360° k}{n}\right) \\ \sin\left(\dfrac{360° k}{n}\right) & \cos\left(\dfrac{360° k}{n}\right) \end{pmatrix}$$

We can check that the matrices that we obtain in this case are elements of $\mathrm{Sl}_2(\mathbb{R})$. In general we say that we have a representation of a group G when we have a good map (group homomorphism)

$$\rho\colon G \longrightarrow \mathrm{Gl}(V),$$

for a vector space V over $\mathbb{C}$. When the dimension of V is 1, this particular case of representation is called a character.

Practice Questions:

1. Find a 2-dimensional matrix representation for the reflexions of $\mathbb{D}_n$.