Lecture Notes for Introduction to Modern Algebra I: Lecture 5 + 6

1 Subgroups

1.1 Subgroups of a group

Definition 1. Let $(G, *)$ be a group. A subset $H \subset G$ is a subgroup if the restriction of the operation $*$ to H forms a group. This means:

(1) The neutral element $e \in H$.

(2) For $x \in H$, the inverse element $x^{-1} \in H$.

(3) For $x, y \in H \Rightarrow x * y \in H$.

We use the notation $H \leqslant G$, to say that H is a subgroup of G.

Example 2. Every group has a trivial subgroup given $H = \{e\} \leqslant G$.

Example 3. A subspace W of a vector space V is a subgroup for the addition operation. For example the set of solutions in $\mathbb{R}^n$ to a system of homogeneous linear equations is naturally a subgroup of the vector space $(\mathbb{R}^n, +)$. On the other hand, the set of solutions to a non-homogeneous system of equations is not a subgroup of $(\mathbb{R}^n, +)$, since it does not contain the neutral element $\vec{0}$.

Proposition 4. *$H \leqslant G$ is a subgroup iff $H \neq \emptyset$ and for all $x, y \in H \Rightarrow x * y^{-1} \in H$.*

Proof. If H is a subgroup, then $H \neq \emptyset$ and for $x, y \in H \Rightarrow x * y^{-1} \in H$. On the other hand, if $x \in H \Rightarrow x * x^{-1} = e \in H$. Also, $x \in H \Rightarrow e * x^{-1} = x^{-1} \in H$ and $x, y \in H \Rightarrow x * (y^{-1})^{-1} = x * y \in H$. □

Example 5. The even integers is a subgroup of the integers: $2\mathbb{Z} \leqslant \mathbb{Z}$.

Example 6. The special linear group of matrices with determinant 1 is a subgroup of the group of invertible matrices

$$\mathrm{Sl}_n(\mathbb{R}) \leqslant \mathrm{Gl}_n(\mathbb{R}).$$

Example 7. For positive integers m, n such that $m|n$, the m-roots of unity are a subgroup of the n-roots of unity, that is $\Phi_m \leqslant \Phi_n$, for $m|n$. At the same time, the group Φ_n for any n is a subgroup of $C = \{z \in \mathbb{Z} \,|\, |z| = 1\}$ (which is a subgrop $\mathbb{C}^*$).

Example 8. The dihedral group $\mathbb{D}_n$ is a subgroup of the symmetric group S_n for $n \geq 3$. For $n > 3$, not all permutations define a rigid motion of the n-dimension polygon.

Example 9. For $n = 3$, the dihedral group $\mathbb{D}_3$ is the whole group S_3 of permutations. If we denote rotations by ρ_1, ρ_2 and $\rho_3 = id$ and reflections by μ_1, μ_2 and μ_3, the correspondence between elements is:

$$\rho_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\,2\,3) \quad \rho_2 = \rho_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\,3\,2) \quad \rho_3 = id$$

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2\,3) \quad \mu_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\,3) \quad \mu_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\,2)$$

The subgroups of $\mathbb{D}_3$ are presented in the Fig 1.

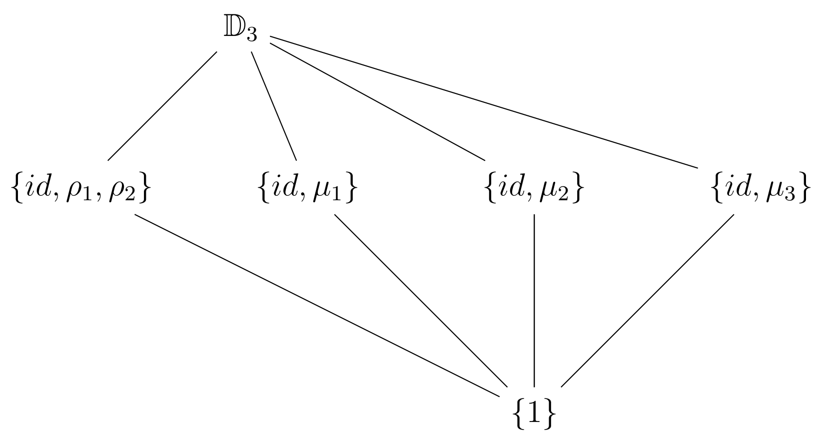

Figure 1: Subgroup Diagram of $\mathbb{D}_3$

Example 10. Given subgroups $\{H_i\}_{i \in I}$, the intersection set $\cap_i H_i$ is also a subgroup.

Example 11. (Centralizer subgroup) Let G be a group and $S \subset G$, as subset of G. The centralizer of the set S in G, defined as

$$C_S(G) = \{y \in G \,|\, yx = xy \quad \forall x \in S\},$$

is the subgroup of the elements of G that commute with all elements of S. The center $Z(G) = C_G(G)$, of the the group G, is the subgroup of elements commuting with all members of G.

Example 12. (Conjugate elements and conjugate subgroups) Let $x \in G$ be an element of G. Define an inner automorphism $\varphi_x \colon G \longrightarrow G$ by the operation $\varphi_x(y) = x * y * x^{-1}$, for $y \in G$. An inner automorphism defines the conjugation action and we say that $\varphi_x(y)$ is conjugate to y. Conjugation defines an equivalence relation in G and a partition of G in classes called **conjugacy classes**. At the same time, given a subgroup $H \leqslant G$, the set of conjugates

$$\varphi_x(H) = xHx^{-1} = \{xhx^{-1} \,|\, h \in H\}$$

is a subgroup of G (not necessarily different from H).

Practice Questions:

1. Let G be a group. Show that the centralizer of a subset $S \subset G$ is a subgroup of G.

2. Find all subgroups for the groups $\mathbb{V}_4$ and $\mathbb{Z}_4$ of 4 elements.

3. Show that the intersection of subgroups gives you again a subgroup.