Lecture notes for Modern Algebra I: Lecture 25

1 Rotation groups on solids

1.1 Rotation group of the 3-dimensional cube.

We can investigate the groups of rigid motions of geometric objects other than a regular n-sided polygon to obtain interesting examples of permutation groups.

Proposition 1. *The group of rotations of the* $3D$*-cube contains* 24 *elements.*

Proof. Suppose that we center our cube at $(0, 0, 0)$ and denote the vertex of the cube by $\{1, 2, 3, 4, 5, 6, 7, 8\}$. We have the following possibilities for rotations in $\mathbb{R}^3$ that leave the cube invariant:

	Angle(s)	Number of rotations
rotate about opposite faces	$90°, 180°, 270°$	9 (3 for each pair)
rotate about opposite vertices	$\pm 120°$	8 (2 for each pair)
rotate about opposite lines	$180°$	6 (1 for each pair)
identity	$0°$	1

We get a total of $9 + 8 + 6 + 1 = 24$ elements in the group of rotations. □

Proposition 2. *The group of rotations of a* $3D$*-cube is* S_4*.*

Proof. From the above proposition, we already know that the group of rotations of the three dimensional cube has 24 elements, the same number of elements as there are in S_4. There are exactly four diagonals in the cube. If we label these diagonals a, b, c, and d, we must show that the motion group of the cube will give us any permutation of the diagonals. If we can obtain all of these permutations, then S_4 and the group of rotations in the cube must be the same. To obtain a transposition we can rotate the cube $180°$ about the axis joining the midpoints of opposite edges. There are six such axes, giving all transpositions in S_4. Since every element in S_4 is the product of a finite number of transpositions, the motion group of a cube must be S_4. □

1.2 Platonic solids

There are exactly five convex regular solid figures in $\mathbb{R}^3$. Each is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. The Euler formula for polyhedrons says that

$$V - E + F = 2$$

If we denote by m the number of edges that appear in one face and by n, the number of edges that meet on one vertex. We will have

$$Fm = 2E \qquad Vn = 2E \quad \Rightarrow \quad \frac{2E}{n} - E + \frac{2E}{m} = 2$$

If we divide by E, we will get the inequality $\frac{2}{n} + \frac{2}{m} > 1$ giving only a finite list of possible pairs of $(m, n) = \{(3,3), (4,3), (3,4), (3,5), (5,3)\}$

m, n	Name	Vertex	Eges	Faces	Type of Faces
$3, 3$	Tetrahedron	4	6	4	Triangles
$4, 3$	Cube	8	12	6	Squares
$3, 4$	Octahedron	6	12	8	Triangles
$5, 3$	Dodecahedron	20	30	12	Penthagons
$3, 5$	Icosahedron	12	30	20	Triangles

For each of the platonic solids, we are going to study the action of the rotational symmetry group on the set of vertices, the set of edges and the set of faces of our solid. For example for the cube we have already seen that:

Action	Orbit #	Stab #	$\lvert G\rvert$
Faces	6	4	24
Edges	12	2	24
Vertices	8	3	24

1.3 The tetrahedron

Consider the tetrahedron. For this solid, we can enumerate the vertices as $\{1, 2, 3, 4\}$ and list the different elements in the group of rotations:

Permutation	Type
$(1,2,3,4) \mapsto (1,2,3,4)$	identity
$(1,2,3,4) \mapsto (1,3,4,2)$	rotation (fix vertex 1)
$(1,2,3,4) \mapsto (1,4,2,3)$	rotation (fix vertex 1)
$(1,2,3,4) \mapsto (4,2,1,3)$	rotation (fix vertex 2)
$(1,2,3,4) \mapsto (3,2,4,1)$	rotation (fix vertex 2)
$(1,2,3,4) \mapsto (2,4,3,1)$	rotation (fix vertex 3)
$(1,2,3,4) \mapsto (4,1,3,2)$	rotation (fix vertex 3)
$(1,2,3,4) \mapsto (3,1,2,4)$	rotation (fix vertex 4)
$(1,2,3,4) \mapsto (2,3,1,4)$	rotation (fix vertex 4)
$(1,2,3,4) \mapsto (2,1,4,3)$	rotation fixes edges 12 and 34
$(1,2,3,4) \mapsto (3,4,1,2)$	rotation fixes edges 13 and 24
$(1,2,3,4) \mapsto (4,3,2,1)$	rotation fixes edges 14 and 23

The group of rotations obtained is isomorphic to A_4 and we can get the stabilizer and orbits of elements like:

Action	Orbit #	Stab #	$\|G\|$
Faces	4	3	12
Edges	6	2	12
Vertices	4	3	12

1.4 The dodecahedron

The rotations of $\mathbb{R}^3$ that fix the dodecahedron are given by rotation around one of the following:

1. One of the 6 lines joining two opposite pentagonal faces (responsible for 4 elements of order 5).

2. One of the 15 lines connecting midpoints of opposite edges (giving elements of order one as rotation with angle $180°$).

3. On of the 10 lines connecting opposite vertices (giving 2 elements of order 3 as rotations with angle $\pm 120°$)).

So we have a total of elements equal to $1 + 6(4) + 15 + 10(2) = 60$. On the other hand, the stabilizer and orbits for elements of dodecahedron look like:

Action	Orbit #	Stab #	$\|G\|$
Faces	12	5	60
Edges	30	2	60
Vertices	20	3	60

We could even proof the following theorem:

Theorem 3. *The group of rotations R_D of the dodecahedron is the the alternating group A_5.*

Proof. Similar to the way rotations permute opposite pairs of vertices of the cube, rotations of the dodecahedron permute five inscribed cubes amongst each other. Observe that the edges of each cube are diagonals of every pentagonal face of the dodecahedron. Each of the five possible diagonals on every pentagonal face corresponds to one of the five inscribed permutable cubes. We can number each cube by numbering these diagonals on the topmost face of the dodecahedron, starting at the nearest diagonal, and labeling them from 1 to 5 in a clockwise fashion. Now, note that the cube has axes of symmetry that intersect opposite vertices in pairs; there are 10 such axes for the 20 vertices. Moreover, as each vertex connects three edges, and they must map to each other in a rotational symmetry about that vertex, these axes have only 2 rotational symmetry elements of 120 and 240 degrees. Therefore there are 20 total rotational symmetries among these axes, and we can show they correspond to the 20 3-cycles in the alternating group A_5. Choosing one such axes of symmetry, we can see that its rotations fix the two inscribed cubes whose N axes intersect the same two vertices. Note that the N axis has rotational symmetries of 120 and 240 degrees, equivalent to the rotations exhibited by the dodecahedron we investigate. Now, there are three remaining inscribed cubes not-fixed by rotations on each axes of symmetry and thus must be sent to each other. These cubes can be represented by their numbered face diagonal per the labeling scheme above. Thus each rotation among these axes directly corresponds to a permutation of three cubes, or a 3-cycle in S_5. In fact, as there exist 20 unique rotational symmetries along the 10 diagonals, 20 unique 3-cycles can be expressed. There are a total of 20 unique 3-cycles possible in S_5, so these rotational elements must correspond to all 3-cycles in S_5. As 3-cycles generate A_n for $n \geq 3$, we get our result. □

Definition 4. The dual of a regular polyhedron is obtained by joining the center of adjacent faces and taking these new segments as edges of a new polyhedron.

m, n	Name	Vertex	Eges	Faces	Type of Faces	Dual
$3, 3$	Tetrahedron	4	6	4	Triangles	Tetrahedron
$4, 3$	Cube	8	12	6	Squares	Octahedron
$3, 4$	Octahedron	6	12	8	Triangles	Cube
$5, 3$	Dodecahedron	20	30	12	Pentagons	Icosahedron
$3, 5$	Icosahedron	12	30	20	Triangles	Dodecahedron

Remark 5. The tetrahedron is self-dual.

Remark 6. The faces of a polyhedron correspond to the vertices of the dual polyhedron and vice versa. The number of edges remain constant when calculating the dual. The **group of symmetries of a dual solid is the same group as the original solid since they share the same axis and planes of symmetry**.

Corollary 7. *The group R_O of rotations of the octahedron is S_4 and the group R_I of rotations of the Icosahedron is A_5.*