Lecture notes for Modern Algebra I: Lecture 18

1 Abelian groups

1.1 Classification of abelian groups

Definition 1. Let H be the subgroup of a group G that is generated by a subset of elements $\{g_i \in G \mid i \in I\}$. Then $h \in H$ exactly when it is a product of the form $h = g_{i_1}^{\alpha_1} \dots g_{i_n}^{\alpha_n}$, where the g_{i_k}'s are not necessarily different and $\alpha_i \in \mathbb{Z}$.

Definition 2. Let p be a prime, we define a group G to be a **p-group** if every element in G has as its order a power of p.

Remark 3. A finite group is a p-group if and only if its order is a power of p

Theorem 4. *(Fundamental Theorem of Finite Abelian Groups) Every finite abelian group G is isomorphic to a direct product of cyclic groups of the form*

$$\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \dots \times \mathbb{Z}_{p_n^{\alpha_n}},$$

where the p_i are primes not necessarily distinct.

Lemma 5. *If H is a cyclic group, then $\hat{H} \cong H$ is also cyclic. That is $\hat{H} = \langle \chi_0 \rangle$ for some character χ_0. The kernel of χ_0 is trivial, i.e.,* $\ker(\chi_0) = 1$.

Proof. If $h \neq 1$ is in $\ker(\chi_0)$. We can find a character χ such that $\chi(h) \neq 1$, but then we will have a contradiction since $1 \neq \chi(h) = \chi_0^k(h) = (\chi_0(h))^k = 1$. $\square$

Theorem 6. *Let G be a finite abelian group and H be a cyclic subgroup of maximal order. Then $G = H \times K$ for some subgroup K. This means G splits as direct product of H and K.*

Proof. Let $H = \langle h \rangle$, for an element $h \in G$ of maximal order m. Consider the group of characters $\hat{H} = \langle \chi_0 \rangle$ where χ_0 has the same order m and $\ker(\chi_0) = 1$. The character χ_0 can be extended to a character $\chi \colon G \longrightarrow C$. Suppose that χ has order n. Then $\chi(g)^n = 1$ for all $g \in G$, hence $\chi_0(h)^n = 1$ for all $h \in H$ and $\chi_0^n = 1 \Rightarrow m|n \Rightarrow m \leq n$. On the other hand $\chi(G)$ is a cyclic subgroup of C, hence we can choose $g \in G$ such that $\chi(G) = \langle \chi(g) \rangle$. Since the order of χ is n, we have that $\chi^k \neq 1$ for $0 < k < n$. We have then $\chi(g^k) = \chi^k(g) \neq 1 \Rightarrow g^k \neq 1$ in the range $0 < k < n$. The order of g is therefore $m \geq n$, and the two must be equal $m = n$.
We obtained then $|H| = |\chi(G)|$. Let us denote

$$K = \ker(\chi) \Rightarrow H \cap K = \ker(\chi_0) = 1.$$

and we have:

$$H = H/\{1\} = H/(H \cap K) \cong HK/K \subset G/K \cong \chi(G).$$

By the statement on the order of the groups H and $\chi(G)$, more than subset $\subset$ we must have equality in the previous line and this forces $HK \cong G$. By the conditions on the internal direct product it must be $G \cong H \times K$. $\square$

Theorem 7. *(Fundamental Theorem of Finite Abelian Groups) Every finite abelian group G is isomorphic to a direct product of cyclic groups of the form*

$$\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_n^{\alpha_n}}.$$

Proof. We proceed by induction on the order of the finite abelian group G. If the order of G is 1, G is no product of no cyclic group. If $|G| > 1$, there exist a maximal cyclic group $H_1 \subset G$ of order > 1 and by theorem 6, it must be $G = H_1 \times K$ with $|K| < |H_1|$. By induction hypothesis, the finite abelian group K can be expressed as direct product of cyclic groups $K \cong H_2 \times \cdots \times H_r \Rightarrow G \cong H_1 \times H_2 \times \cdots \times H_r$.
Now suppose that $H_i = H = C_n$ is a cyclic group of order $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ then we can decompose:

$$H \cong C_{p_1^{\alpha_1}} \times \cdots \times H_{p_k^{\alpha_k}}$$

obtaining our decomposition with cyclic groups and prime powers as orders. $\square$

Corollary 8. *Any finite abelian group G is isomorphic to the direct product of cyclic p-groups.*

Example 9. List abelian groups of order 32. In general, the number of non-isomorphic abelian groups of order n is given by the function $N_a(n)$. In the special case of $n = p^c$ for a prime number p, we can get the list using the number of finite sequence of positive numbers $b_1 \geq b_2 \geq \cdots \geq b_r$ such that $b_1 + b_2 + \cdots + b_r = c$. For example $N_a(32) = N_a(2^5) = 7$.

Decomposition	Group
5	$\mathbb{Z}_{32}$
4+1	$\mathbb{Z}_{16} \times \mathbb{Z}_2$
3+2	$\mathbb{Z}_8 \times \mathbb{Z}_4$
3+1+1	$\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
2+2+1	$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
2+1 +1 +1	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$
1+1 + 1 + 1 +1	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Practice Questions:

1. Write down the list of Abelian groups of order 2022 and 2048.