Lecture notes for Modern Algebra I: Lecture 23

1 Solvable groups

1.1 Subnormal series and solvable groups

Definition 1. A subnormal series of a group G is a sequence of subgroups:

$$G = H_n \supset H_{n-1} \supset \cdots \supset H_0 = \{e\},$$

where H_{i-1} is a normal subgroup of H_i. A group G is solvable when it has a subnormal series such that successive quotients H_i/H_{i-1} are abelian.

Remark 2. In general the length of a subnormal series is not unique. We could have:

$$\mathbb{Z}_6 \triangleright 1 \quad \text{or} \quad \mathbb{Z}_6 \triangleright \mathbb{Z}_3 \triangleright 1,$$

of lengths one and two. However, a theorem of Jordan-Hölder states that all maximal subnormal series, i.e. one whose factors are simple groups, must have the same lengths.

Example 3. An abelian group G is trivially solvable with subnormal series $G \supset \{e\}$.

Example 4. We could prove that the group S_4 is solvable. On the other hand for $n > 4$, the only normal subgroup of S_n is A_n. The group A_n is a simple normal subgroup. therefore the only possibility for a subnormal series is:

$$S_n \supset A_n \supset \{1\}$$

with non-abelian factor A_n.

Theorem 5. *The dihedral group D_n is solvable, as the subgroup of rotations R_n is a normal subgroup isomorphic to $\mathbb{Z}_n$ and we have $D_n \triangleright R_n \triangleright 1$ with factor group $D_n/R_n \cong \mathbb{Z}_2$.*

Lemma 6. *Let A, B, K subgroups of G with $A \triangleleft B$ and $K \triangleleft G$. Put $\overline{A} = AK/K$ and $\overline{B} = BK/K$. Then $\overline{A} \triangleleft \overline{B}$ and the quotient groups satisfy $\dfrac{\overline{B}}{\overline{A}}$ is isomorphic to a factor group of $\dfrac{B}{A}$.*

Proof. One can check that $AK \triangleleft BK$. By the third isomorphism theorem, we have then $\overline{A} \triangleleft \overline{B}$ and

$$\frac{\overline{B}}{\overline{A}} = \frac{BK}{K} \Big/ \frac{AK}{K} \cong \frac{BK}{AK} = \frac{B(AK)}{AK} \cong \frac{B}{B \cap AK} \cong \frac{B}{A} \Big/ \frac{B \cap AK}{A}.$$

The later being a factor group of B/A. $\square$

Proposition 7. *The factor group of a solvable group is solvable. In general if we have a series:*

$$G = H_n \supset H_{n-1} \supset \cdots \supset H_0,$$

with $K \triangleleft G$ and $H_0 \subset K$, we have that G/K is solvable.

Proof. For $k = 0, 1, \ldots, n$ put $F_k = H_k K/K$. We get:

$$G = GK = H_n K \supset H_{n-1}K \supset \cdots \supset H_0 K \quad \Rightarrow \quad G/K \supset H_{n-1}K/K \supset \cdots \supset H_0 K/K.$$

The series for the quotient is $G/K = F_n \supset F_{n-1} \supset \cdots \supset F_0 = \{1\}$.
By lemma 6 we have that the group F_{k-1} is normal in F_k and F_k/F_{k-1} is a factor group of the abelian group G_k/G_{k-1}. $\square$

Lemma 8. *Let A, B, H subgroups of G with $A \triangleleft B$. Put $\underline{A} = A \cap H$ and equally $\underline{B} = B \cap H$. Then $\underline{A} \triangleleft \underline{B}$ and the quotient groups satisfy $\dfrac{\underline{B}}{\underline{A}}$ is a subgroup of $\dfrac{B}{A}$.*

Proof. It is not hard to check $\underline{A} \triangleleft \underline{B}$. Also we have

$$\underline{B}/\underline{A} = (B \cap H)/(A \cup H) = (B \cap H)/A \cup (B \cup H) \cong A(B \cup H)/A$$

and this last one $A(B \cup H)/A$ is a subgroup of B/A. $\square$

Proposition 9. *A subgroup H of a solvable group G is also solvable.*

Proof. Let G be a solvable group and H a subgroup. Suppose that

$$G = G_n \supset G_{n-1} \supset \cdots \supset G_0 = \{e\}$$

with G_n/G_{n-1} abelian. For $k = 0, \ldots, n$, put $H_k = G_k \cap H$, we will have then

$$H = H_n \supset H_{n-1} \supset \cdots \supset H_0 = \{e\}.$$

Also H_n/H_{n-1} is a subgroup of the abelian group G_n/G_{n-1} by lemma 8. $\square$

1.2 Simple groups

Simple groups are groups with no nontrivial normal subgroups. It is not easy to find non-trivial examples of simple groups. We have the following theorem:

Theorem 10. *The alternating group, A_n, is simple for $n \geq 5$.*

To be able to get the proof, we need to do first three lemmas. In the first lemma we prove that the group A_n is generated by 3-cycles for $n \geq 3$. In the second lemma, we show that we can generate A_n for $n \geq 3$ using only 3-cycles of the form $(i\,j\,k)$ for i, j fix and k moving in $\{1, 2, \ldots, n\}$.

Lemma 11. *The alternating group A_n is generated by 3-cycles for $n \geq 3$.*

Proof. To show that the 3-cycles generate A_n, we need only show that any pair of transpositions can be written as the product of 3-cycles. Since $(a\,b) = (b\,a)$, every pair of transpositions must be one of the following:

$$(a\,b)(a\,b) = \mathrm{id} \qquad (a\,b)(c\,d) = (a\,c\,b)(a\,c\,d) \qquad (a\,b)(a\,c) = (a\,c\,b).$$

□

Lemma 12. *Let N be a normal subgroup of A_n, where $n \geq 3$. If N contains a 3-cycle, then $N = A_n$.*

Proof. We can improve our result in Lemma 11 and prove that A_n for $n \geq 3$ is generated by 3-cycles of the form $(i\,j\,k)$ for fixed numbers i, j and k moving in $\{1, 2, \ldots, n\}$. To do this, we use the equations:

$$(i\,a\,j) = (i\,j\,a)^2 \qquad (i\,a\,b) = (i\,j\,b)(i\,j\,a)^2 \qquad (j\,a\,b) = (i\,j\,b)^2(i\,j\,a)$$

$$\text{and finally } \ (a\,b\,c) = (i\,j\,a)^2(i\,j\,c)(i\,j\,b)^2(i\,j\,a).$$

If the non-trivial normal subgroup N of A_n, contains a 3-cycle of the form $(i\,j\,a)$, then, as a subgroup it must also contain the square $(i\,j\,a)^2$. Also as a normal subgroup it must contain all the conjugates of the square:

$$(i\,j\,k) = [(i\,j)(a\,k)](i\,j\,a)^2[(i\,j)(a\,k)]^{-1}.$$

As a consequence $(i\,j\,k) \in N$ for all $k = 1, 2, \ldots, n$ and $N = A_n$. □

Lemma 13. *For $n \geq 5$, every nontrivial normal subgroup N of A_n contains a 3-cycle.*

Proof. There several cases that we must consider for an arbitrary element $\sigma \in A_n$, as long as $n \geq 5$.

(1) σ is a 3-cycle.

(2) σ is the product of disjoint cycles and one of them has length $r > 3$, this is $\sigma = \tau(a_1\,a_2 \ldots a_r)$.

(3) For some $\tau \in A_n$, σ is the disjoint product $\sigma = \tau(a_1\,a_2\,a_3)(a_4\,a_5\,a_6)$.

(4) The element $\sigma = \tau(a_1\,a_2\,a_3)$, where τ is a product of an even amount of disjoint two cycles.

(5) The element σ is just product of disjoint two cycles $\sigma = \tau(a_1\,a_2)(a_3\,a_4)$, where the permutation τ is disjoint product of 2-cycles.

If we are in case (2), meaning that we contain a cycle of length $r > 3$ in the cycle decomposition of σ, we can write:

$$\begin{aligned}\sigma^{-1}(a_1\,a_2\,a_3)\sigma(a_1\,a_2\,a_3)^{-1} &= \sigma^{-1}(a_1\,a_2\,a_3)\sigma(a_1\,a_3\,a_2)\\ &= (a_1\,a_2\ldots a_r)^{-1}\tau^{-1}(a_1\,a_2\,a_3)\tau(a_1\,a_2\ldots a_r)(a_1\,a_3\,a_2)\\ &= (a_1\,a_r\ldots a_2)(a_1\,a_2\,a_3)(a_1\,a_2\ldots a_r)(a_1\,a_3\,a_2)\\ &= (a_1\,a_3\,a_r)\end{aligned}$$

Since $\sigma \in N$ and N is normal, the product $\sigma^{-1}(a_1\,a_2\,a_3)\sigma(a_1\,a_2\,a_3)^{-1}$ is also in N and the 3-cycle $(a_1\,a_3\,a_r)$ must be in N.
If we are in case (4), the element $\sigma = \tau(a_1\,a_2\,a_3) \in N$ and the square of the element $(a_1\,a_3\,a_2) = \tau(a_1\,a_2\,a_3)\tau(a_1\,a_2\,a_3)\sigma^2$ must also be in N. Hence we have a 3-cycle in N.
If we are in case (3) $\sigma = \tau(a_1\,a_2\,a_3)(a_4\,a_5\,a_6)$ and, since N is a normal subgroup, the product of σ^{-1} and the conjugate $(a_1\,a_2\,a_4)\sigma(a_1\,a_2\,a_4)^{-1}$ must be an element of N.

$$\begin{aligned}\sigma^{-1}(a_1\,a_2\,a_4)\sigma(a_1\,a_2\,a_4)^{-1} &= [\tau(a_1\,a_2\,a_3)(a_4\,a_5 a_6)]^{-1}(a_1\,a_2\,a_4)\tau(a_1\,a_2\,a_3)(a_4\,a_5\,a_6)(a_1\,a_2\,a_4)^{-1}\\ &= (a_4\,a_6\,a_5)(a_1\,a_3\,a_2)\tau^{-1}(a_1\,a_2\,a_4)\tau(a_1\,a_2,\,a_3)(a_4\,a_5\,a_6)(a_1\,a_4\,a_2)\\ &= (a_4\,a_6,a_5)(a_1\,a_3\,a_2)(a_1\,a_2\,a_4)(a_1\,a_2\,a_3)(a_4\,a_5\,a_6)(a_1\,a_4\,a_2)\\ &= (a_1\,a_4\,a_2\,a_6\,a_3).\end{aligned}$$

In this way we have found an element like in case (2) that belongs to N.
The only remaining possible case (that is not (1)) is case (5) where σ is a disjoint product of the form

$$\sigma = \tau(a_1\,a_2)(a_3\,a_4),$$

and τ is the product of an even number of disjoint 2-cycles. But $\sigma^{-1}(a_1\,a_2\,a_3)\sigma(a_1\,a_2\,a_3)^{-1}$ is in N since the element $(a_1\,a_2\,a_3)\sigma(a_1\,a_2\,a_3)^{-1}$ is in N; and so

$$\begin{aligned}\sigma^{-1}(a_1\,a_2\,a_3)\sigma(a_1\,a_2\,a_3)^{-1} &= \tau^{-1}(a_1\,a_2)(a_3\,a_4)(a_1\,a_2\,a_3)\tau(a_1\,a_2)(a_3\,a_4)(a_1\,a_2\,a_3)^{-1}\\ &= (a_1\,a_3)(a_2\,a_4),\end{aligned}$$

is also an element of N. Since $n \geq 5$, we can find $b \in \{1, 2, \ldots, n\}$ such that the element $b \neq a_1, a_2, a_3, a_4$. Let $\mu = (a_1\,a_3\,b)$ Then

$$\mu^{-1}(a_1\,a_3)(a_2\,a_4)\mu(a_1\,a_3)(a_2\,a_4) \in N$$

$\mu^{-1}(a_1\,a_3)(a_2\,a_4)\mu(a_1\,a_3)(a_2\,a_4) = (a_1\,b\,a_3)(a_1\,a_3)(a_2\,a_4)(a_1\,a_3\,b)(a_1\,a_3)(a_2, a_4) = (a_1\,a_3\,b).$

And we are able to find a 3-cycle in N. □

Theorem 14. *The alternating group, A_n, is simple for $n \geq 5$.*

Proof. Let N be a normal subgroup of A_n. By Lemma 13, N contains a 3-cycle. By Lemma 12, $N = A_n$; therefore, A_n contains no proper nontrivial normal subgroups for $n \geq 5$. □

Practice Questions:

1. Prove that a group that is simple non abelian cannot be solvable.

2. Prove that the direct product of solvable groups G and G' is solvable. Hint: The proof can be done directly for the direct product, constructing a subnormal abelian series for the product from the series of G and G'. A more general approach would be to prove a converse for our propositions in class, namely: If H is a normal subgroup of G and both H and G/H are solvable, then G itself is solvable. This would take care of the direct product (and much more) as a particular case.