Lecture notes for Modern Algebra I: Lecture 19

1 Group actions

1.1 Group actions

Definition 1. Let X be a set and G be a group. A (left) action of G on X is a map $G \times X \longrightarrow X$ given by $(g, x) \mapsto gx$, where

1. $ex = x$ for all $x \in X$;
2. $(g_1 * g_2)x = g_1(g_2x)$ for all $g_1, g_2 \in G$.

When G acts on X, we say that X is a G-set.

Remark 2. In a G-set X every element $g \in G$, determines a bijection $g \colon X \longrightarrow X$ defined by $x \mapsto gx$. The inverse is given by $x \mapsto g^{-1}x$.

Example 3. The group $\mathbb{D}_4$ acts on the vertex of the square.

Example 4. Any group G acts on itself $X = G$ by left multiplication $g \mapsto f_g(x) = gx$ or right multiplication $g \mapsto f_g(x) = xg$.

Example 5. Let G be a group and suppose that $X = G$. If H is a subgroup of G, then G is an H-set under conjugation; that is, we can define an action of H on G,

$$H \times G \longrightarrow G \qquad (h, g) = \varphi_h(g) = hgh^{-1}.$$

Proposition/Definition 6. *Let G be a group and X a G-set. The stabilizer of an element $x \in X$ is the subgroup G_x of G defined as*

$$G_x = \{g \in G \mid gx = x\}.$$

Definition 7. Let G be a group and X a G-set. The fixed point set of an element $g \in G$ is the set

$$X_g = \{x \in X \mid gx = x\}.$$

Definition 8. Let G be a group and X a G-set. The orbit of an element $x \in X$ is the set

$$O_x = \{gx \mid g \in G\}.$$

Proposition 9. *Let G be a finite group and X a finite G-set. If $x \in X$, then the order of the orbit $|O_x| = [G : G_x]$.*

Proof. We know that $|G|/|G_x|$ is the number of left cosets of G_x in G by Lagrange's Theorem. We will define a bijective map φ between the orbit O_x of x and the set of left cosets L_{G_x} of G_x in G. Consider an element $y \in O_x$. Then there exists an element g in G such that $gx = y$. Define the map φ by $\varphi(y) = gG_x$. To show that φ is one-to-one, assume that $\varphi(y) = \varphi(y')$. Then

$$\varphi(y) = g_1G_x = g_2G_x = \varphi(y'),$$

where $g_1x = y$ and $g_2x = y'$. Since they are equal $g_1G_x = g_2G_x$, there exists an element $g \in G_x$ such that $g_2 = g_1g$ and using $g \in G_x$ we get

$$y' = g_2x = g_1gx = g_1x = y.$$

Consequently, the map φ is one-to-one. Finally, we must show that the map φ is onto. Let gGx be a left coset. If $gx = y$,then $\varphi(y) = gG_x$. $\square$

1.2 Actions as permutations

A permutation of a set X is a bijective function from $X \to X$. The group of permutations with function compositions is denoted by S_X.

Theorem 10. *Let G be a group and X a set:*

(1) For an action of G on X we get a map $\pi_g\colon X \longrightarrow X$, for element $g \in G$ defined by $\pi_g(x) = gx$. The map $g \mapsto \pi_g$ determines a group homomorphism $\pi\colon G \longrightarrow S_X$ into the group of permutations $X \to X$.

(2) Conversely, for each group homomorphism $\pi\colon G \longrightarrow S_X$, we get a group action of G on X given by $gx = \pi_g(x)$.

Proof. (1) From the properties of group actions, we have the two equations $\pi_g\pi_h = \pi_{gh}$ and $\pi_e = e_X$. When we use $h = g^{-1}$ (and then g^{-1} in place of g), we obtain

$$\pi_g\pi_{g^{-1}} = \pi_{gg^{-1}} = \pi_e = e_X \quad \text{and} \quad \pi_{g^{-1}}\pi_g = \pi_{g^{-1}g} = \pi_e = e_X.$$

This is saying that the maps $\pi_g\colon X \longrightarrow X$ are actually bijections and $g \mapsto \pi_g$ determines a group homomorphism.
(2) Suppose that $g \mapsto \pi_g$ is a group homomorphism from $G \longrightarrow S_X$. From the properties of group homomorphisms $\pi_g\pi_h = \pi_{gh}$ and $\pi_e = e_X$. The map $G \times X \longrightarrow X$ defined by $gx = \pi_g(x)$ is then an action of G on X. $\square$

Corollary 11. *Every group G is isomorphic to a subgroup of S_G.*

Proof. Consider the group G acting on itself by left multiplication. Consider the associated $\pi\colon G \longrightarrow S_G$. By the first isomorphism theorem $\mathrm{Im}(\pi)$ is a subgroup of G and

$$G/\ker(\pi) \cong \mathrm{Im}(\pi).$$

In this situation however, the kernel of the map π is trivial since $g \in \ker(\pi) \Rightarrow gx = x \Rightarrow g = 1$ and $G \cong \mathrm{Im}(\pi) \leqslant S_G$. $\square$

1.3 Class equation

Let X be a finite G-set and X_G be the set of fixed points in X; that is,

$$X_G = \{x \in X \mid gx = x \, \forall g \in G\}$$

Since the orbits of the action partition X,

$$|X| = |X_G| + \sum_{i=1}^{k} |O_{x_i}|,$$

for representatives x_i of the non-trivial orbits.
Consider the special case of a group G acting on itself by conjugation $y \mapsto xyx^{-1}$, we get the class equation:

$$|G| = |Z(G)| + \sum_{i=1}^{k} |G : C(x_i)|,$$

where $Z(G)$ is the center of the group and $C(x_i)$ is the centralizer of the element x_i in G.

Example 12. The conjugacy classes for S_3 are:

$$\{(1)\} \qquad \{(1\,2\,3), (1\,3\,2)\} \qquad \{(1\,2), (1\,3), (2\,3)\}$$

and the class equation $6 = 1 + 2 + 3$.

Practice Questions:

1. Show that is a prime number p divides the order of G and does not divide any of the order of the centralizers $C(x_i)$ it must divide the order of the center.

2. Show that a group of order p^m, where p is prime, must have a non-trivial center.