Lecture notes for Introduction to Modern Algebra I: Lecture 16 + 17

1 Characters

1.1 Group characters for finite abelian groups

Characters will be play a central role in the classification of finite abelian groups. Consider the set $C = \{z \in \mathbb{C}^* \mid |z| = 1\}$ in the complex plane.

Proposition 1. *C is a subgroup of $\mathbb{C}^* = \{z \in \mathbb{C} \mid z \neq 0\}$ and $\xi^{-1} = \bar{\xi}$ (complex conjugate).*

Proof. The identity element 1 of $\mathbb{C}^*$ is in C. Also for $z_1, z_2 \in \mathbb{C}$ we have $|z_1 z_2| = |z_1||z_2|$, and the set C is closed under multiplications. For the inverse we have:

$$z \in C \Rightarrow z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \bar{z} \in C.$$

And we have checked all conditions for C being a subgroup. □

Proposition 2. *Every finite subgroup of C is cyclic.*

Proof. Let D be a subgroup of C. If $D = 1$ then $D = \langle 1 \rangle$ and we are done. Otherwise, suppose that $D \neq 1$ and choose the element $\delta \in D$ such that the positive angle θ to the x-axis s minimal. If the element $z \in D$ is strictly $\delta^k < z < \delta^{k+1}$ for some value k, then the element $z\delta^{-k} \in D$ would have a smaller angle. The conclusion is that, for any $z \in D$ there exists a power k such that $z = \delta^k$. □

Example 3. The group C_n of complex n-roots of unity is a finite subgroup of C.

Remark 4. Infinite subgroups of C not need to be cyclic. Take for example an irrational number θ and the group K generated by -1 and $e^{2\pi i\theta}$. If this group were to be cyclic generated by an element $\delta = e^{2\pi i\phi}$, we will have integers k, l such that

$$e^{2\pi i\theta} = \delta^k = e^{2k\pi i\phi} \text{ and } -1 = \delta^l = e^{2l\pi i\phi}.$$

This will give $\theta = k\phi + m$ and $\frac{1}{2} = l\phi + n$ for integers m, n. Since, θ is irrational and $1/2$ is a rational number we arrive to a contradiction. In the group just described we cannot define an element with minimum positive angle.

Definition 5. Let G be a finite abelian group. A character of G is a group homomorphism $\chi \colon G \longrightarrow C$. The set of all characters of G is denoted by $\widehat{G}$.

Proposition 6. *The set of characters has a natural structure of abelian group.*

Proof. Define $\chi * \chi'$ by the map multiplication $(\chi * \chi')(x) = \chi(x)\chi'(x)$. It is also a group homomorphism $G \longrightarrow C$. Also put $\chi^{-1} = \bar{\chi}$ and take the identity element as the constant function 1. $\square$

Remark 7. The group of characters of a finite abelian group is finite. Let $x \in G$ and n be the order of the group G. We have $1 = \chi(1) = \chi(x^n) = (\chi(x))^n$. Hence $\chi(x)$ is an n-th root of unity in C, there are at most n choices of $\chi(x)$ for each $x \in G$ and the number of characters is finite.

Proposition 8. *If G is cyclic, $\widehat{G} \cong G$.*

Proof. Let χ be a character on G and $G = \langle g \rangle$ of order n. Since $\chi(g)^n = 1$, we know that $\chi(g)$ is a complex n-root of unity, that is $\chi(g) \in C_n$. Let us define a map $\rho\colon C_n \longrightarrow \hat{G}$, $\rho(\xi) = \chi_\xi$, where $\chi_\xi(g^k) = \xi^k$.
First of all, the map is well defined since $g^k = g^{k'} \Rightarrow n|k - k' \Rightarrow \xi^k = \xi^{k'}$. We need to check that: χ_ξ is a character: $\chi_\xi(g^k g^l) = \chi_\xi(g^{k+l}) = \xi^{k+l} = \xi^k \xi^l = \chi_\xi(g^k)\chi_\xi(g^l)$.
The map is a group homomorphism: $\chi_{\xi\nu}(g^k) = \xi^k \nu^k = \chi_\xi(g^k)\chi_\nu(g^k)$.
The map is surjective: since each character $\chi = \chi_\xi$ for some $\xi \in C_n$.
The map is injective: since $\chi_\xi = 1 \Rightarrow \xi^k = 1$ for all $k \Rightarrow \xi = 1$. $\square$

Proposition 9. *Let G be a finite abelian group and $H \subset G$ a subgroup. Every character χ_0 on H can be extended to a character on G.*

Proof. We proceed by induction on the order of the quotient group $|G/H|$. If $|G/H| = 1$, then $G = H$, the character χ_0 is already a character of G. If the order of the quotient $|G/H| > 1$, choose a class gH of G/H such that $gH \neq H$ and denote by $M/H = \langle gH \rangle$ the cyclic group of G/H generated by gH. If $M \neq G$, both $|M/H|$ and G/M are both of order less than G/H and hence we can extend χ_0, first to a character of M and then to a character of the whole G. We may assume then that $M = G$ and $G/H = M/H = \langle gH \rangle$ is cyclic. Let $n = |G/H|$. We have then that $g^n H = H$ and hence $g^n \in H$. Choose a complex number ξ such that $\chi_0(g^n) = \xi^n$ and define the map:

$$\chi(g^r h) = \xi^r \chi_0(h),$$

for all $n > r \geq 0$ and $h \in H$. The map is well defined because the cosets

$$H_r = \{g^r h \,|\, 0 \leq r < n, h \in H\}$$

form a partition of G. We can prove that the map defined this way is a character on the whole G and it extendes χ_0. $\square$

Corollary 10. *Let G be a finite abelian group and $h \in G$ such that $h \neq 1$. Then, there exist a character $\chi \in \widehat{G}$ such that $\chi(h) \neq 1$.*

Theorem 11. *(Orthogonality relations) Let G be a finite abelian group:*

1. *For χ and χ' characters we have* $\sum_{g\in G}\overline{\chi(g)}\chi'(g) = \begin{cases} |G| & \text{if } \chi=\chi' \\ 0 & \text{if } \chi\neq\chi' \end{cases}$

2. *For g and g' in G have* $\sum_{\chi\in\hat{G}}\overline{\chi(g)}\chi(g') = \begin{cases} |\hat{G}| & \text{if } g=g' \\ 0 & \text{if } g\neq g' \end{cases}$

Proof. Consider the character $\xi = \overline{\chi}\chi'$. If $\xi \equiv 1$, then $\xi(g) = 1$ for all g and $\sum_{g\in G}\xi(g) = |G|$. On the other hand if ξ is not identically 1, there exist $g_1 \in G$ such that $\xi(g_1) \neq 1$ and we will have:

$$\xi(g_1)\sum_{g\in G}\xi(g) = \sum_{g\in G}\xi(g_1 g) = \sum_{g\in G}\xi(g).$$

Since $\xi(g_1)\neq 1$, it must be the case that $\sum_{g\in G}\xi(g) = 0$.
The other direction is similar: put now $h = g^{-1}g'$ and compute $\sum_{\chi\in\hat{G}}\chi(h)$. If $h=1$, we get $\sum_{\chi\in\hat{G}}\chi(h) = |\hat{G}|$. On the other hand if $h\neq 1$, there is a character $\chi_1(h)\neq 1$ and

$$\chi_1(h)\sum_{\chi\in\hat{G}}\chi(h) = \sum_{\chi\in\hat{G}}\chi_1(h)\chi(h) = \sum_{\chi\in\hat{G}}(\chi_1\chi)(h) = \sum_{\chi\in\hat{G}}\chi(h)$$

and since $\chi_1(h)\neq 1$ it must be the case that $\sum_{\chi\in\hat{G}}\chi(h)$=0. □

Theorem 12. *For each finite abelian group the orders $|\widehat{G}| = |G|$.*

Proof. We do a summation in two different ways:

$$|\widehat{G}| + 0\cdots+0 = \sum_{g\in G}\sum_{\chi\in\widehat{G}}\chi(g) = \sum_{\chi\in\widehat{G}}\sum_{g\in G}\chi(g) = |G|. + 0 + \cdots + 0$$

to get equality of the orders. □

1.2 Character tables

The characters of a finite group form a character table which encodes much useful information about the group G. Each row is labelled by a character and the entries in the row are the values of that character on elements of the group. The columns are labelled by elements of G (in a more general setting this would be representative of conjugacy classes). It is customary to label the first row by the trivial character χ_0, which is the trivial action of G given by $\rho(g) = 1$ for all $g\in G$.

	$C_0 = 1$	$C_1 = g_1$	$C_2 = g_2$	...
$\chi_0 = 1$	1	1	1	...
χ_1	1	$\chi_1(g_1)$	$\chi_1(g_2)$	...
χ_2	1	$\chi_2(g_1)$	$\chi_2(g_2)$	...
...	...	...	...	...

For example, suppose that ω is a primitive third root of unity. Then the character table of of the cyclic group C_3 can be represented by:

	1	g	g^2
$\chi_0 = 1$	1	1	1
χ_1	1	ω	ω^2
χ_2	1	ω^2	ω

Practice Questions:

1. Describe the table character for the cyclic group C_5.