[Higher Math] Midterm Review

Relations

- Equivalence Relation (?)
- Types of relations
- Functions as relations

Functions

General Notes

Definition: Take two sets, X and Y. A function $f : X \to Y$ (a function from X to Y) is a collection of ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

- The condition for these ordered pairs to be considered a function is that they follow the Vertical Line Test (VLT). Anoterh way to put this is that each input only ever has one output!

More formally, we can say that $\forall x \in X$, the function (x, y) only ever has one y. In these scenarios, we tend to use the "negative" definition, and say that if $(x_1, y_1) \in f$ and $(x_1, y_2) \in f$, then $y_1 = y_2$.

Some important definitions are as follows:

- **Image (Range):** The image of $f : X \to Y$ is basically all the values in Y that values $x \in X$ map to. In order words, it's the "range" of the function.
    - However, we use the term image because this refers to the "range" of a specific set. You can imagine that you have a function $f(x) = x^2$ defined over all the real numbers. The range is $(0, \infty)$, but let's say that we only cared about the inputs $\{1, 2, 3\}$. Then, the image of this set of the function would be $\{1, 4, 9\}$.
- **PreImage (Domain):** This is a fancy word for domain, but it's basically like all the elements $x \in X$ that map to values $y \in Y$. In the above example, take something like $\{1, 4, 9\}$ as output values for the function $f(x) = x^2$. Then, we can say the preimage of this set is $\{1, 2, 3\}$ because those are the values that map to those outputs.

There are three types of "mappings". Consider the function $f : A \to B$ where $a \in A$ and $b \in B$.

- **Injective (one-to-one):** If $f(a) = f(b)$, then $a = b$. Another way to say this is that every input value only maps to one output value. You will never have a case where your function has something like $f(1) = f(3)$.
- **Surjective (onto):** For all $b \in B$, there exists some $a \in A$ such that $f(a) = b$. This is basically saying that every possible output (in set B) guarentees to have some input from set A that can map to it.
- **Bijective:** Both surjective and injective!
  - NOTE: If they ever ask about the inverse of a function, this is only possible with a bijective function! If you think about it, it makes sense why you can only have an inverse (that is a function) if the properties of injectivity and surjectivity are satisfied.

Example Problems

Problem 1 (HW 5): Give the following definitions:

- a function $f : X \to Y$

A function $f : X \to Y$ (a function from X to Y) is a collection of ordered pairs (x, y) such that $x \in X$ and $y \in Y$, and $\forall x \in X$, the function (x, y) only ever has one y.

- for $A \subset X$, the set $f(A)$

$$f(A) = \{f(a) \in Y \ : \ \forall a \in A\} \tag{1}$$

- for $C \subset Y$, the set $f^{-1}(C)$

$$f^{-1}(C) = \{a \in A \ : \ \forall c \in C, f(a) = c\} \tag{2}$$

Injective Proof: By definitino of injectivity, we have that if $f(a) = f(b)$, then $a = b$.

Take any element of $x \in A \cap B$. It should be obvious that $f(x) \in f(A)$ and $f(x) \in f(B)$. Now, take any element $y \in f(A) \cap f(B)$. Assume that $x_1 \in A \implies f(x_1) = y$ and $x_2 \in B \implies f(x_2) = y$ Then, $f(x_1) = f(x_2) = y$ and we know that $x_1 = x_2$ and so that $x_1 \in A \cap B$. This means that double inclusion so we good.

Limits

$$(|a| - |b|)^2 = |a|^2 + |b|^2 - 2|a||b| \implies |a| + |b| \tag{3}$$

Example 4 HW 7

$0 < |x| < \delta \implies 0 < |x - M| < \epsilon$. Assume that $\delta = \frac{1}{x}$. this means that $x\sin(x)$ and around $x = 0$,

Fix some M. Now, define $x = \frac{1}{\pi(M+1)} \to f(x_0) = \pi(M+1) > M$. Thus, your bound fails.

Limits:

$$3x^2 - 3 < \epsilon \iff \delta = \sqrt{} \tag{4}$$