Lecture 20 3/28/2024

Modular Forms:

Defn: A weakly meromorphic modular form (WMMMF) of weight k is a func $f: \mathfrak{H} \to \mathbb{C}$ which is mero on $\mathfrak{H}$ and satis

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \qquad (*)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ and $k \in \mathbb{Z}$, $k \geq 0$.

Claim: For such a func to exist, $k \equiv 0 \pmod 2$

Pf: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{Z})$, so for with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ have $\left(\frac{az+b}{cz+d}\right) = z$, so need $(-1)^k = (-1)^k = 1 \implies k \equiv 0 \pmod 2$

Since $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $SL(2,\mathbb{Z})$, we can prove $(*)$ if

$$\begin{cases} f(z+1) = f(z) & \text{[periodic]} \\ f\left(-\frac{1}{z}\right) = z^k f(z) \end{cases}$$

Since f is periodic, it has Fourier expansion

$$f(z) = \sum_n a_n e^{2\pi i n z}$$

Pf (Expansion works): Let f holo and periodic on $\mathfrak{H}$. Fix y

$$\implies f(x+iy) = \sum_n a_n(y)\, e^{2\pi i n x}$$

$$a_n(y) = \int_0^1 f(x+iy)\, e^{-2\pi i n x}\, dx$$

Basically just analytically continue this real expansion when $y > 0$ to rest of $\mathbb{C}$.

We say f is holo on $\mathfrak{H}^* = \mathfrak{H} \cup \{\infty\}$ if f holo on $\mathfrak{H}$ and $\lim_{y \to \infty} f(iy) \neq \infty$.

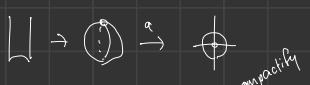
Say f holo on $\mathfrak{H}$ and has pole at ∞ if $\lim_{y \to \infty} f(iy) = \infty$

Assume f holo or mero. on $\mathfrak{H}$ and periodic. Then it has expan.

$$f(z) = \sum a_n\, e^{2\pi i n z}$$

Let $q = e^{2\pi i z}$, $z \in \mathfrak{H} \implies |q| < 1$

Defn $\tilde{f}(q) = \sum_{-M \leq n \leq N} a_n q^n$. Then $\tilde{f}$ is mero.



Eisenstein series (of wt. $k \equiv 0 \pmod 2$) Let $z \in \mathfrak{H}$, $k \geq 4$, $k \in \mathbb{Z}$. Then defn

$$E_k(z) = \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d) \neq (0,0)}} \frac{1}{(cz+d)^k}$$

Claim: $E_k(z)$ holo on $\mathfrak{H} \cup \{\infty\}$. $E_k\left(\frac{az+b}{cz+d}\right) = (cz+d)^k E_k(z)$

Assuming convergence $\lim_{y \to \infty} E_k(x+iy) = 2$, take $c = 0$, so $d \neq 1$. If $c \neq 0$, the term $\to 0$.

Now prove convergence. For each $N=1,2,\dots$, consider $(c,d)=1$ with $\max(|c|,|d|)=N$. There are at most $4N$ such pairs. Then

$$\sum_{c,d} \frac{1}{|cz+d|^k} \le \sum_{N=1}^{\infty} \frac{4N}{(Ky\cdot N)^k}$$

$$\ll_k \sum_N \frac{1}{N^{k-1}} \qquad [k\ge 4]$$

Let $\gamma=\begin{pmatrix} * & * \\ c & d \end{pmatrix}$, $g\in\beta$. Let $j(\gamma,g)=cg+d$. For $\gamma_1,\gamma_2\in SL(2,\mathbb{Z})$

$$j(\gamma_1\gamma_2, g) = j(\gamma_1, \gamma_2 g)\, j(\gamma_2, g) \qquad [\text{cocycle cond.}]$$

Can write

$$E_k(g) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \frac{1}{j(\gamma,g)^k}$$

Then for $\alpha\in SL(2,\mathbb{Z})$, have

$$E_k(\alpha g) = j(\alpha, g)^k\, E_k(g)$$

from

$$\frac{1}{j(\gamma,\alpha g)^k} = \frac{j(\alpha,g)^k}{j(\gamma\alpha, g)^k} \qquad [\text{cocycle}]$$

$$\Rightarrow \sum_\gamma \frac{1}{j(\gamma,\alpha g)^k} = j(\alpha,g)^k \sum_\gamma \frac{1}{j(\gamma\alpha,g)^k} = j(\alpha,g)^k \sum_\gamma \frac{1}{j(\gamma,g)^k}$$

(multi. by α is iso.)

Two types of hols. mod forms. (show 2nd later)

① $E_k(g)$, $k=4,6,8,\dots$ $\qquad E_k(i\infty)=2$

② cusp forms $f\left(\frac{ag+b}{cg+d}\right)=(cg+d)^k f(g)$, $\lim_{g\to\infty} f(g)=0$

Can expand as

① Eis. series! $\qquad E_k(g) = 2 + \sum_{n\ge 1}^{\infty} a_n(n)\, e^{2\pi i n g}$

② Cusp form! $\qquad f(g) = \sum_{n\ge 1}^{\infty} a_n e^{2\pi i n g}$