MATH 228: Project 1

A. Borum

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1 Introduction

This project focuses on a model of polar ice caps and the effect of solar radiation on the ice cap's size. Using the methods we've developed for analyzing first-order differential equations, your job is to predict the behavior of an ice cap under different assumptions on the amount of solar radiation.

2 Background

In this section, we describe a model of how solar radiation affects the growth or melting of an ice cap. The amount of solar radiation received at a particular latitude influences the temperature at that latitude. Furthermore, if the temperature at a particular latitude is low enough for an ice cap to form, the ice cap will affect how much solar radiation is absorbed or reflected back into the atmosphere. This coupling between temperature and latitude is described by the following differential equation[1].

$$\frac{\partial T}{\partial t} = D\left(2x\frac{\partial T}{\partial x} + (1-x^2)\frac{\partial^2 T}{\partial x^2}\right) - A - BT + QS(x)a(T). \tag{1}$$

In this differential equation, t denotes time measured in centuries, and x denotes the sine of latitude as shown in Figure 1 (i.e., $x = 1 = \sin(90°)$ corresponds to the north pole and $x = 0 = \sin(0°)$ corresponds to the equator). The remaining variables are described below.

Q	Average solar radiation received by one square meter at the Earth's equator
$S(x)$	Average annual sunlight at latitude x
$a(T)$	Reflectivity of water/ice at temperature T
A, B, D	Empirical constants determined by experimental observations

Equation (1) is a partial differential equation since it involves derivative with respect to both t and x. One way to simplify this equation is to only consider the latitude corresponding to the ice cap's boundary. The behavior of this boundary is qualitatively captured by the first-order ordinary differential equation

$$\frac{dx}{dt} = (1-x)\left(Q - 335 + 3\left(\frac{(x-0.89)}{0.09}\right)^2\right), \tag{2}$$

where x denotes the sine of the latitude of the ice cap's boundary (i.e., $x = 0$ corresponds to an ice cap covering the entire northern hemisphere, and $x = 1$ corresponds to having no ice cap.) The variable Q is still a measure of solar radiation and t is still time in centuries.

[1] G.R. North, The Small Ice Cap Instability in Diffusive Climate Models, *Journal of Atmospheric Science*, 41: 3390-3395, 1984.

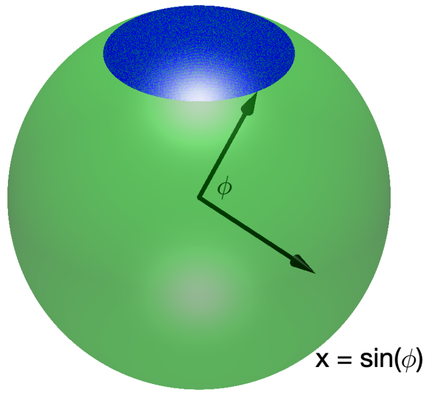

Figure 1: Schematic of an ice cap whose boundary is at latitude ϕ.

3 Problem

Equation (2) is challenging to solve analytically, either by hand or using Sage. However, we can analyze the qualitative behavior of the solutions, and we can use Sage to approximate the solutions. We will focus on two scenarios described below.

Scenario 1: Constant Q

If we assume that Q is constant, we can analyze the equilibrium points of equation (2). We can also plot the bifurcation diagram showing how these equilibrium points change for different values of Q. In your project report, you should address the following topics/questions:

a. Compute the fixed points of equation (2) and determine their stability. Describe the process you used to determine stability and include supporting evidence (e.g., plots, phase lines, etc.).

b. Based on your work in part (a), sketch the bifurcation diagram for equation (2) with values of Q in the range $[320, 350]$. If any bifurcations or other interesting phenomena occur, you should label these in your diagram and compute the values of x and Q at which they occur.

c. Using the Sage code on the last page of this document, plot the slope field and the solutions of equation (2) starting at different initial conditions. Do this for a few different values of Q and include these plots in your report. Finally, discuss if the solutions from Sage agree with your predictions from (a) and (b).

Scenario 2: Variable Q

Next, we will consider the behavior of equation (2) when Q is changing with time. Specifically, we will analyze the solution of (2) when Q begins at 330, rises to a value of $Q_{\max}$, and then returns to 330 over the course of N centuries. Your report should address the following topics/questions:

d. First compute the stable equilibrium point for $Q = 330$. Then select the number of centuries N over which you would like to study the model, and select a maximum solar radiation $Q_{\max}$ that is above 330. To describe a rise from $Q = 330$ to $Q_{\max}$ and a return to 330 over N centuries, we can use the function

$$Q(t) = 330 + (Q_{\max} - 330)\frac{4t(N-t)}{N^2}. \tag{3}$$

Note that this function satisfies $Q(0) = 330$, $Q(N) = 330$, and it reaches a maximum of $Q(N/2) = Q_{\max}$.

e. Using your Sage code from part (c), replace the constant value for Q in your code with the function in equation (3) (using your chosen values of N and $Q_{\max}$). Then simulate the solution with this time-varying $Q(t)$ starting at the equilibrium point you found in part (d). Include a plot of $x(t)$ in your report, and discuss the behavior you observe after N centuries.

f. For this last portion, you should address at least one of the following questions:

   – Using the Sage code from part (e), find the largest value of $Q_{\max}$ (to 2 decimal places) so that x returns (approximately) to its initial value after N centuries. What behavior do you observe if $Q_{\max}$ is larger than this value?

   – Suppose that over the course of 30 centuries, Q rises slowly at a constant rate of 0.2/century, i.e., $Q(t) = 330 + 0.2t$. Does x change at a constant rate as well, or does x undergo an abrupt change at some point? Can you explain the behavior you observe in terms of your bifurcation diagram?

   – Select a different time-varying function for $Q(t)$ (other than equation (3) or $Q(t) = 330 + 0.2t$). Give a justification for why this function is physically meaningful, simulate equation (2) with your chosen $Q(t)$, and describe the behavior you observe.

4 Project Report

The report is due by 1:30PM on October 12, and it must address items (a)-(f) above. The report should be typed (but handwritten equations are ok), and it should highlight your approach to the questions descried above, your results, and your conclusions. Longer computations, if needed, can be placed in an appendix of the report. The report should not be written in the style of a homework assignment (e.g., it should not have section headings of "part a," "part b," etc.), but instead in the style of a short lab or research report. You will submit the report as a pdf on Moodle.

5 Evaluation

Your report will be evaluated based on the following rubric:

- (10%) The report has the correct format, as described above
- (15% each) Items (a)-(f)
  - (10%) Content: Complete and thoughtful responses are given to all questions, and claims are supported with evidence such as computations, plots, or simulations.
  - (5%) Presentation: The submission is organized into a cohesive report, and the writing is both clear and concise.

```
t, x = var('t, x') #declare t and x as variables
Q = 330 #specify Q
N = 5 #number of centuries

f(t,x) = (1-x)*(Q-335+3*((x-0.89)/.1)^2) #define the differential equation

S = plot_slope_field(f, #plot the slope field for f and call the plot S
                     (t, 0, N), #show t between 0 and N
                     (x, 0.5, 1), #show x between 0.5 and 1
                     headaxislength=3, #set the arrow's head size
                     headlength=3, #set the arrow's head length
                     axes_labels=['$t$','$x(t)$']) #label the axes

#NOTE: additional solutions for other initial conditions can be added by copying
#the code below and changing the initial condition

S += desolve_rk4(f, x, #solve the differential equation x'=f(t,x) and add it to our plot S
                 ics=[0,1], #with the initial condition x(0)=1
                 ivar=t, #the independet variable is t
                 step=.01, #use a step size of 0.01
                 output='plot', #visualizet the solution on a plot
                 end_points=[0, N], #show the solution for t between 0 and 10
                 thickness=2) #set the line thickness

S.show(xmin=0, xmax=N, #show the solution with t between 0 and N
       ymin=0.5, ymax=1) #and x between 0.5 and 1
```