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(2 grace days used)

Scenario 1: Constant Q

To better work with the differential equation given to us I simplified it down in order to easier use the quadratic formula. The full simplification is given in the appendix. Simplifying differential equation (2) gives:

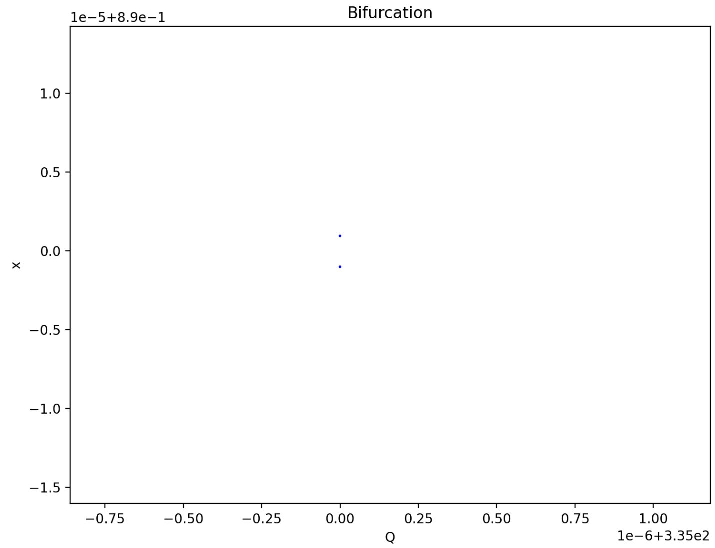
$$\frac{dx}{dt} = 0 = (1-x)\left(\frac{10{,}000}{27}x^2 - \frac{17{,}800}{27}x - \frac{1{,}124}{27} + Q\right)$$

To find a bifurcation diagram, start by plotting phase lines. Plotting a few in the range of Q=325, 335, and 350, will give you a pretty good idea of what the bifurcation should look like. The calculations for Q=325 are shown in the appendix but the general format uses the following two equations. First calculate the fixed points, then find the direction of arrows on the phase line using the second equation, which determines dx/dt at the specific point x=a.

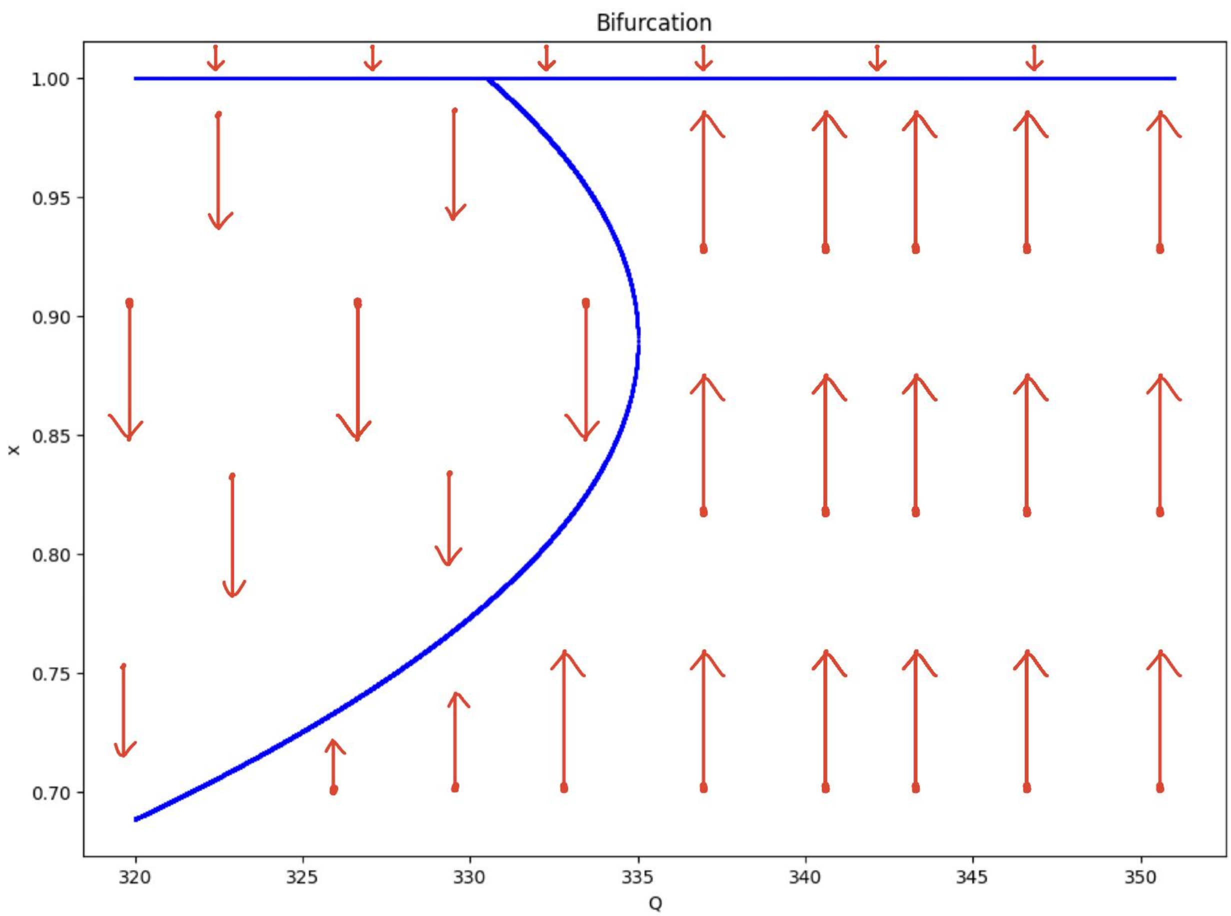
$$x = 1 \text{ or } \frac{\frac{17{,}800}{27} \pm \sqrt{\left(\frac{17{,}800}{27}\right)^2 - 4\left(\frac{10{,}000}{27}\right)\left(-\frac{1{,}124}{27} + Q\right)}}{2\left(\frac{10{,}000}{27}\right)}$$

$$\left.\frac{dx}{dt}\right|_{x=a} = (1-a)\left(\frac{10{,}000}{27}(a)^2 - \frac{17{,}800}{27}(a) - \frac{1{,}124}{27} + Q\right) = \text{number to determine direction on phase line}$$

Using a python program I solved for the x in each equation with a Q range of [320,350]. I had a step of 0.001, meaning I calculated each x value for 320, 320.001, 320.002 … etc. The code is attached in the appendix. Using a program allows me to manipulate the graph in order to view extremely small points, such as at Q=335 and x = .89, which is a crucial point on this bifurcation diagram. This is that point super zoomed in:

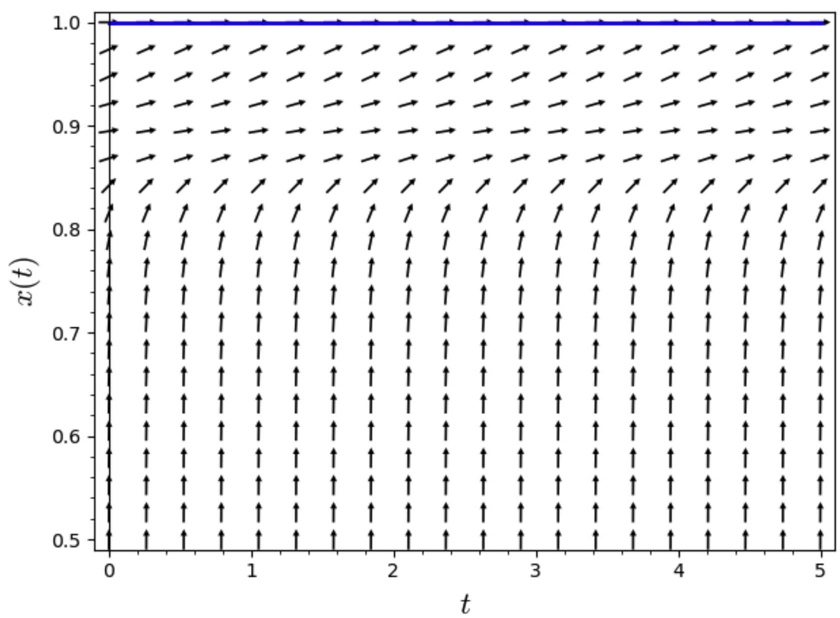


The bifurcation diagram given by the program:

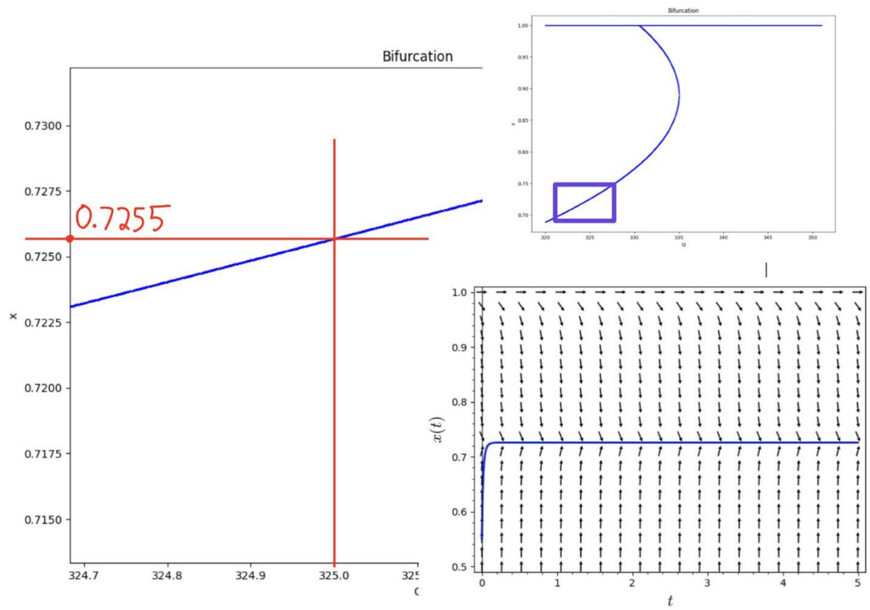


As you can see, we have a constant fixed point at x = 1. That is because of the (1 - x) in the front of the differential equation. Something interesting to note, at Q = 330, the field lines flip from going down to up towards x(t) = 1. After Q = 330, x = 1 becomes a stable point.

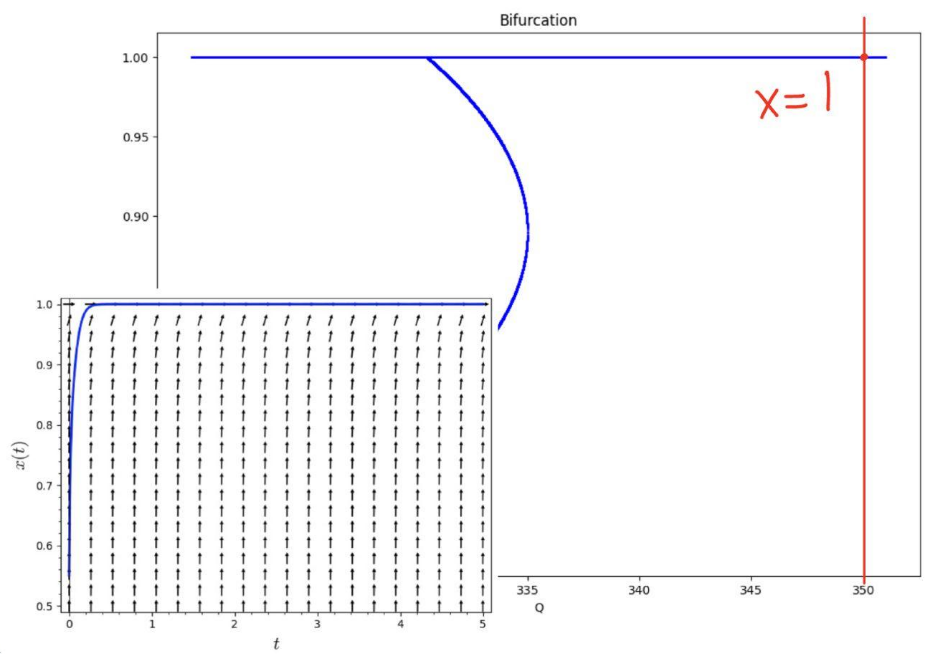
Some interesting phenomena occur when our Q value is 335. This is the last Q value that will have more than one solution. It is not as obvious with Q = 335, so I chose Q = 335.3, which best shows the change in slope lines at around x = 0.89.



The picture below shows the phase diagram for Q = 325. As you can see, the diagram shows that there are two equilibrium points; x = 1 and roughly x = 0.72. We can compare each graph by taking a cross-section of our bifurcation diagram at our specific Q value, and then look for where our Q has x solutions. In this case, the bifurcation shows that there are two equilibrium points, at x = 1 and x = 0.7255. With an initial condition of x(0) = .55, we can see that the line approaches our x value of 0.7255.



Another Q value that we should look at is any Q value that is greater than 335. Let's look at 350. Sage gives back a phase diagram that shoots up to x = 1, which makes sense. As we have more solar radiation hit the earth, the ice caps are going to melt at an increasingly alarming rate. The bifurcation diagram doesn't show the magnitude of the change in temperature, but it does show that the differential equation has its only equilibrium point at x = 0.



Scenario 2: Variable Q

Let’s view our differential equation with a variable Q. A good starting point is Q = 330, and we’ll calculate the equilibrium point the same as we did with the other points. As we can see, we have 1 and 0.0778…, unstable and stable respectively. (Full calculation in appendix)

Calculating Equilibrium Points

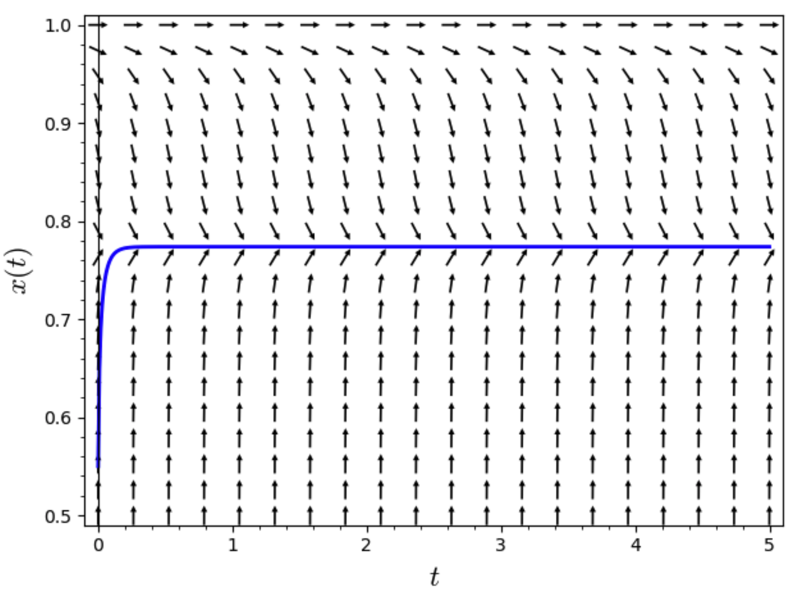
$$\left.\frac{dx}{dt}\right|_{x=1.5} = (1-1.5)\left(\frac{10{,}000}{27}(1.5)^2 - \frac{17{,}800}{27}(1.5) - \frac{1124}{27} + 330\right) = -66.407$$

$$\left.\frac{dx}{dt}\right|_{x=.8} = (1-.8)\left(\frac{10{,}000}{27}(.8)^2 - \frac{17{,}800}{27}(.8) - \frac{1124}{27} + 330\right) = -0.4$$

$$\left.\frac{dx}{dt}\right|_{x=.5} = (1-.5)\left(\frac{10{,}000}{27}(.5)^2 - \frac{17{,}800}{27}(.5) - \frac{1124}{27} + 330\right) = 25.66$$

$\therefore$ W/ a Q value of 330 the stable equilibrium point is $x \approx 0.77381\ldots$ and the unstable point is $x = 1$

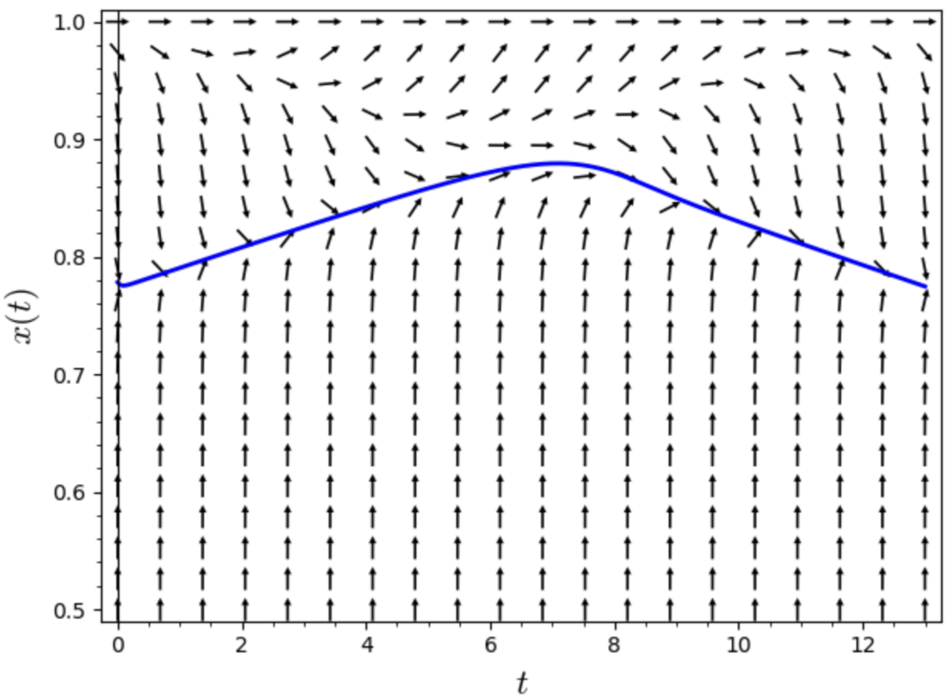
If we look at the phase diagram for Q=330 in Sage, we get the same result. We set an initial condition to x(0)=.55, and our line approaches 0.77831… .



Our variable Q equation will look like:

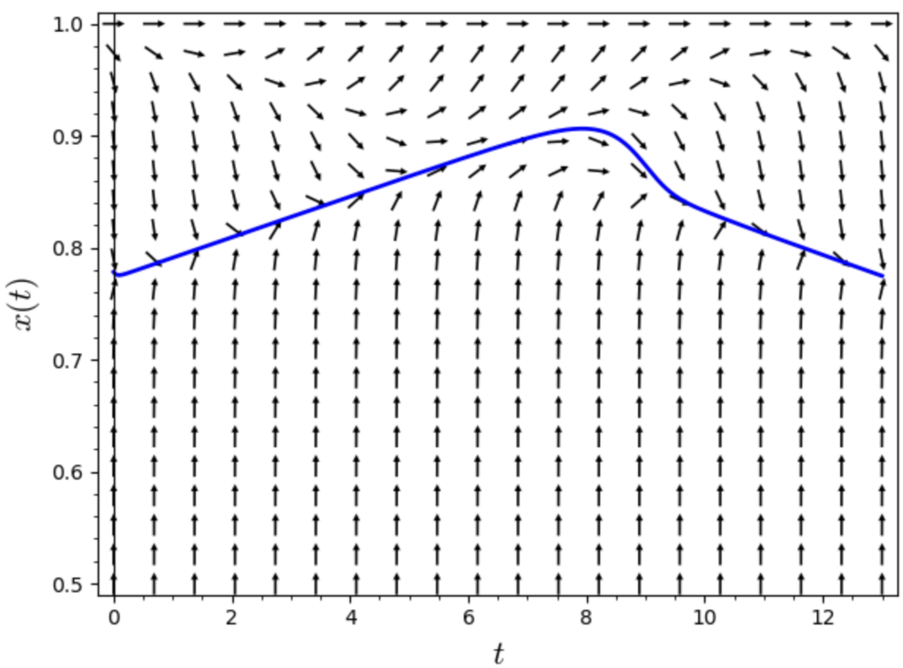
$$Q(t) = 330 + (Q_{\max} - 330)\frac{4t(N-t)}{N^2}.$$

Where N is the number of centuries and Qmax is the maximum number of solar radiation above 330. I chose N to be 13 centuries because the phase diagram flows well and nicely shows the field lines, and Qmax to be 335. Starting at x(0)=0.77831, x(t) finds its way back to 0.77831 after 13 centuries. With any number of centuries you choose, Q(t) will eventually go back to 330, and x(t) back to 0.77831.

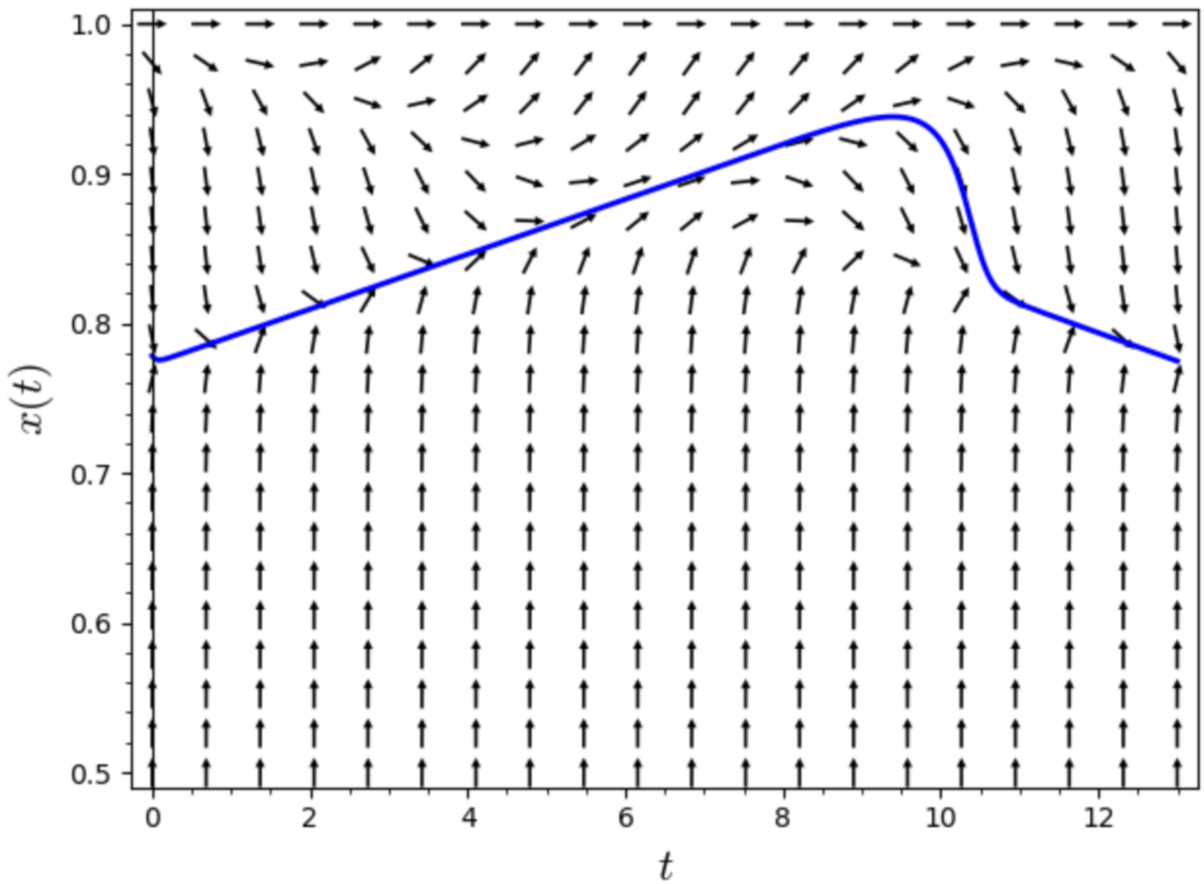


Using the Sage code from part (e), find the largest value of Qmax (to 2 decimal places) so that x returns (approximately) to its initial value after N centuries. What behavior do you observe if Qmax is larger than this value?

Finding the largest Qmax value that will return x to its initial value starts with a guess. My first guess had to be close to 335, because of the special behavior that this Q value has. My first guess was 335.15, which gave a phase diagram that follows:



As the diagram shows, x(t) is right on the verge of crossing over as the hump at the top of the curve drops off sharper than it did at Qmax = 335. After a large amount of guessing and checking, I found that the final Qmax value for which x(t) will return back to its initial value was Qmax = 335.17:


1.0
0.9
0.8
0.7
0.6
0.5
$x(t)$
0
2
4
6
8
10
12
t

Appendix:

Solving the phase line for Q=325:

$$Q = 325$$

$$x = 1 \text{ or } \frac{\frac{17{,}800}{27} \pm \sqrt{\left(\frac{17{,}800}{27}\right)^2 - 4\left(\frac{10{,}000}{27}\right)\left(-\frac{1{,}124}{27} + Q\right)}}{2\left(\frac{10{,}000}{27}\right)}$$

$$x = 1, 1.05, 0.725$$

x cannot be greater than 1 ∴ solutions are

$$x = 1, 0.725$$

$$\left.\frac{dx}{dt}\right|_{x=1.5} = (1-1.5)\left(\frac{10{,}000}{27}(1.5)^2 - \frac{17{,}800}{27}(1.5) - \frac{1124}{27} + 325\right) = -63.9$$

$$\left.\frac{dx}{dt}\right|_{x=.8} = (1-.8)\left(\frac{10{,}000}{27}(.8)^2 - \frac{17{,}800}{27}(.8) - \frac{1124}{27} + 325\right) = -1.4$$

$$\left.\frac{dx}{dt}\right|_{x=.7} = (1-.7)\left(\frac{10{,}000}{27}(.7)^2 - \frac{17{,}800}{27}(.7) - \frac{1124}{27} + 325\right) = 1.01$$

1 = unstable
0.725 = stable

Differential Equation (2):

$$\frac{dx}{dt} = (1-x)\left(Q - 335 + 3\left(\frac{x-0.89}{0.09}\right)^2\right)$$

Simplification of differential equation (2):

$$\frac{dx}{dt} = (1-x)\left(Q - 335 + 3\left(\frac{(x-0.89)}{.09}\right)^2\right)$$

$$= (1-x)\left(Q - 335 + 3\left(\frac{(x-0.89)^2}{0.09^2}\right)\right)$$

$$= (1-x)\left(Q - 335 + 3\left(\frac{x^2 - 1.78x + 0.7921}{0.0081}\right)\right)$$

$$= (1-x)\left(Q - 335 + \frac{10{,}000}{27}x^2 - \frac{17{,}800}{27}x + \frac{7921}{27}\right)$$

$$\frac{dx}{dt} = 0 = (1-x)\left(\frac{10{,}000}{27}x^2 - \frac{17{,}800}{27}x - \frac{1124}{27} + Q\right)$$

Calculating equilibrium points at Q=330:

$$\boxed{\text{Finding stable equilibrium point for } Q = 330}$$

$$\frac{dx}{dt} = (1-x)\left(\frac{10{,}000}{27}x^2 - \frac{17{,}800}{27}x - \frac{1124}{27} + 330\right)$$

$$\Downarrow$$

$$x = 1 \text{ or } \frac{\frac{17{,}800}{27} \pm \sqrt{\left(\frac{17{,}800}{27}\right)^2 - 4\left(\frac{10{,}000}{27}\right)\left(-\frac{1{,}124}{27} + 330\right)}}{2\left(\frac{10{,}000}{27}\right)}$$

$$\therefore$$

$$x = 1,\ 0.77381,\ 1.00618$$

↳ x cannot be greater than 1

∴ x solutions are

$$x = 1, 0.77381$$

Code for calculating bifurcation diagram:

```
import matplotlib.pyplot as plt
import numpy as np
import math
```

```python
ones = []
Q = np.arange(320.0, 351.0, 0.001)

for x in range(len(Q)):
   ones.append(1)

xPlus = []
xNegative = []

def findXPlus(q):
   first = (17800/27) * (17800/27)
   second = 4 * (10000/27) * (-1124/27 + q)
   bottom = 2 * (10000/27)
   if first-second > 0:
       xPoint = ( (17800/27) + math.sqrt(first - second) ) / bottom
       if xPoint > 1:
           xPlus.append( 1 )
       else:
           xPlus.append( xPoint )
   else:
       xPlus.append(1)

def findXNegative(q):
   first = (17800/27) * (17800/27)
   second = 4 * (10000/27) * (-1124/27 + q)
   bottom = 2 * (10000/27)
   if first-second > 0:
       xNegative.append( ( (17800/27) - math.sqrt(first - second) ) / bottom )
   else:
       xNegative.append(1)


for i in Q:
   findXPlus(i)

for i in Q:
   findXNegative(i)

plt.scatter(Q, xPlus, c="blue", s=1)
plt.scatter(Q, xNegative, c="blue", s=1)
plt.scatter(Q, ones, c="blue", s=1)
```

```
plt.title("Bifurcation")
plt.xlabel("Q")
plt.ylabel("x")
plt.show()
```