Exam 1 Topics

Solution Methods ⇒
- Seperation of values
- Integrating factor
- Exact equation

Qualitative Analysis ⇒
- Direction fields, solution curves
- Phase lines
- Equilibrium points, bifurcation

Theoretical Questions ⇒
- Existence and uniqueness
- Interval of existence

Applications ⇒
- Population models
- Cooling
- Mixing

take home

Integrals must know

$$\int (a \pm x)^n \, dx \qquad \int \sin x \, dx$$

$$\int e^{ax+b} \, dx \qquad \int \cos x \, dx$$

Problem 1.a HW 4

$y' = ty^3 \quad y(0) = 1$

$f(t,y) = ty^3$

$\frac{\partial f}{\partial y} = 3ty^2$

continuous for any t, y

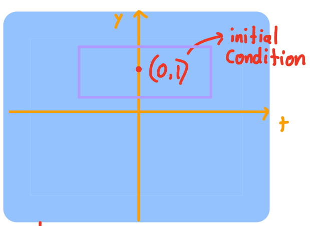
$\frac{dy}{dt} = ty^3$

$\int \frac{1}{y^3} dy = \int t \, dt$

$y = \frac{1}{\sqrt{1-t^2}}$

exists for t in $(-1,1) \subset (-\infty, \infty)$

contained inside



continuous everywhere

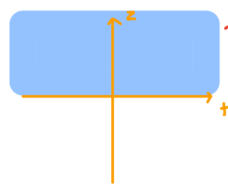
Problem 1.b HW4

$z' = \sqrt{z} \quad z(0) = 3$

$f(t,z) = \sqrt{z}$

$\frac{\partial f}{\partial z} = \frac{1}{2}(z)^{-1/2}$

continuous for all t and $z > 0$



continuous here

$\frac{dz}{dt} = \sqrt{z}$

$z = \frac{(t + \sqrt{12})^2}{4}$

interval of existence is $t \in (-\infty, \infty)$

Problem 1.c HW 4

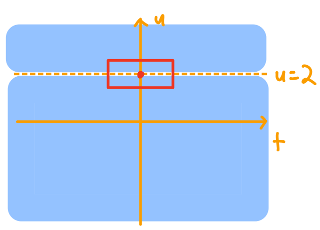
$u' = \frac{t}{u-2} \qquad u(0) = 2$

$f(t,u) = \frac{t}{u-2}$

$\frac{\partial f}{\partial u} = \frac{-t}{(u-2)^2}$

not continuous @ $u = 2$

can't apply E+U Theorem



$\frac{du}{dt} = \frac{t}{u-2}$

$u^2 - 4u - t^2 - 2C = 0$

$u = \frac{4 \pm \sqrt{4^2 - 4(-t^2 - 2C)}}{2} \qquad C = -2$

solved out

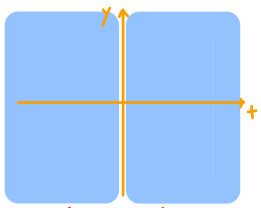
$u = 2 \pm 2t$

2 solutions ∴ not unique

Example

$y' = -\frac{1}{t}y + 1 \qquad y(0) = 1$

$\left.\begin{array}{l} f(t,y) = -\frac{1}{t}y + 1 \\ \frac{\partial f}{\partial y} = -\frac{1}{t} \end{array}\right\}$ discontinuous at $t=0$



continuous everywhere but $t=0$

$$\frac{dy}{dt} = -\frac{1}{t}y + 1$$

$$\frac{dy}{dt} + \frac{1}{t}y = 1$$

$$ye^{\int \frac{1}{t}dt} = \int 1 e^{\int \frac{1}{t}dt}\,dt + C$$

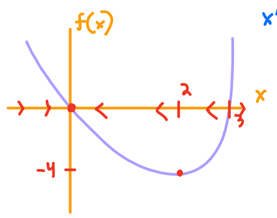
$$yt = \frac{1}{2}t^2 + C$$

$$y = \frac{t}{2} + \frac{C}{t} \qquad y(0) = 1$$

$$1 = \frac{0}{2} + \frac{C}{0} \rightarrow \text{no solution}$$

:(

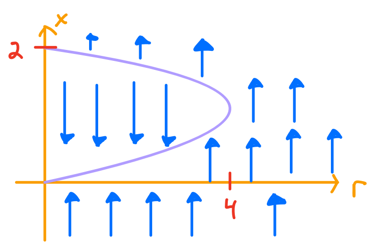
Bifurcation



$x' = f(x)$

Sketch bifurcation for the equation $x' = f(x) + r$

As r increase the equilibrium points move closer as $f(x)$ moves up

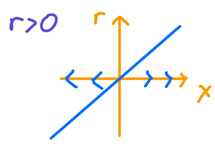


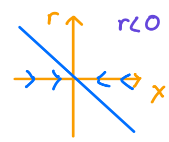
$x' = \frac{4-x}{t-3} \qquad x(0) = 2$

Solve using: sep vars
int factor
exact

$$\frac{dx}{dt} = \frac{4-x}{t-3}$$

$x' = rx$





$x' = \frac{4-x}{t-3}$ $x(0)=2$

sep vars

$\frac{dx}{dt} = \frac{4-x}{t-3}$

$\int \frac{1}{4-x}\,dx = \int \frac{1}{t-3}\,dt$

$-\ln|4-x| = \ln|t-3| + C$

$\frac{1}{4-x} = (t-3)C$

$\frac{C}{t-3} = 4-x$

$x = 4 - \frac{C}{t-3}$ $x(0)=2$

$C=-6$ $\boxed{x = 4 + \frac{6}{t-3}}$

exact

$M(x,t) + N(x,t)\frac{dx}{dt} = 0$

$(-4+x) + (t-3)\frac{dx}{dt} = 0$

$a(t,x) = -4+x$ $b(t,x) = t-3$

$a_x(t,x) = 1 = b_t(t,x)$

$H(x,t) = -4t - 3x + xt = C$

$-6 = C$

$-4t - 3x + xt = -6$

$\boxed{x = \frac{4t-6}{t-3}}$

int factor $x' + p(t)x = g(t)$

$\frac{dx}{dt} - \frac{4-x}{t-3} = 0$

has to be positive idk why $\rightarrow$ $p(t) = \frac{1}{t-3}$ $x = -4 + x$

$g(t) = 0$

$(x-4)e^{\int \frac{1}{t-3}dt} = \int 0\, e^{\int \frac{1}{t-3}dt}\,dt + C$

$(x-4)e^{\ln|t-3|} = C$

$(x-4)(t-3) = C$ $x(0)=2$ $6 = C$

$(x-4)(t-3) = 6$

$\boxed{x = \frac{6}{t-3} + 4}$

$$u_0(t) = 1$$

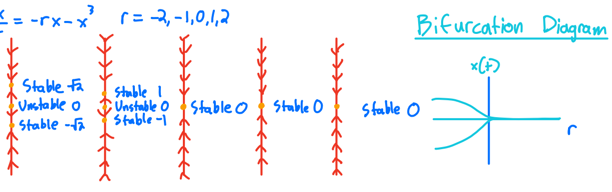
$$\begin{aligned} u_1(t) &= 1 + \int_0^t 2tu_0(t)\,dt \\ &= 1 + \int_0^t 2t\,dt \\ &= 1 + t^2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= 1 + \int_0^t 2tu_1(t)\,dt \\ &= 1 + \int_0^t 2t(1+t^2)\,dt \\ &= 1 + t^2 + \frac{t^4}{2} \end{aligned}$$

$$\begin{aligned} u_3(t) &= 1 + \int_0^t 2tu_2(t)\,dt \\ &= 1 + \int_0^t 2t(1+t^2+\frac{t^4}{2})\,dt \\ &= 1 + t^2 + \frac{t^4}{2} + \frac{t^6}{6} \end{aligned}$$

$\frac{dx}{dt} = -rx - x^3$ $r = -2, -1, 0, 1, 2$

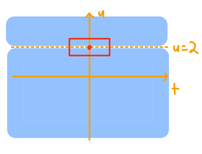
Bifurcation Diagram



Problem 1.c   HW 4

$u' = \frac{t}{u-2}$ $u(0)=2$

$f(t,u) = \frac{t}{u-2}$
$\frac{\partial f}{\partial u} = \frac{-t}{(u-2)^2}$ not continuous @ $u=2$



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$\frac{du}{dt} = \frac{t}{u-2}$

$u^2 - 4u - t^2 - 2C = 0$

$u = \frac{4 \pm \sqrt{4^2 - 4(-t^2-2C)}}{2}$ $C=-2$ solved out

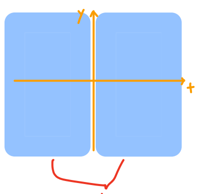
$u = 2 \pm 2t$

2 solutions ∴ not unique

Example

$y' = -\frac{1}{t}y + 1$ $y(0)=1$

$f(t,y) = -\frac{1}{t}y + 1$
$\frac{\partial f}{\partial y} = -\frac{1}{t}$ discontinuous at $t=0$



continuous everywhere but $t=0$

$\frac{dy}{dt} = -\frac{1}{t}y + 1$

$\frac{dy}{dt} + \frac{1}{t}y = 1$

$ye^{\int \frac{1}{t}dt} = \int 1e^{\int \frac{1}{t}dt}\,dt + C$

$yt = \frac{1}{2}t^2 + C$

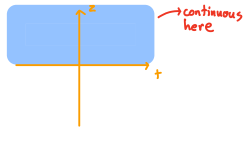
$y = \frac{t}{2} + \frac{C}{t}$ $y(0)=1$

$1 = \frac{0}{2} + \frac{C}{0}$ → no solution :(

Problem 1.b   HW4

$z' = \sqrt{z}$ $z(0)=3$

$f(t,z) = \sqrt{z}$
$\frac{\partial f}{\partial z} = \frac{1}{2}(z)^{-1/2}$ continuous for all t and $z>0$

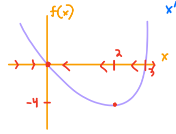


continuous here

$\frac{dz}{dt} = \sqrt{z}$

$z = \frac{(t+\sqrt{12})^2}{4}$ interval of existence is $t \in (-\infty, \infty)$

$x' = f(x)$ Sketch bifurcation for the equation $x' = f(x) + r$



As r increase the equilibrium points move closer as f(x) moves up