$mx''+cx'+kx=0$

$A_1 = x(0)$ $\quad A = \sqrt{A_1^2 + A_2^2}$ ⟹ Amplitude

$\zeta = \frac{c}{2\sqrt{km}}$ $\quad A_2\omega_n = x'(0)$ $\quad \omega_n = \sqrt{\frac{k}{m}}$ ⟹ Natural Frequency

$\lambda = -\omega_n\zeta \pm \omega_n\sqrt{\zeta^2-1}$ $\quad T = \frac{2\pi}{\omega_n}$ ⟹ period

Equations

$0<\zeta<1$: underdamped

$$x(t) = e^{-\zeta\omega_n t}\left(A_1\cos(\omega_n\sqrt{1-\zeta^2}\,t) + A_2\sin(\omega_n\sqrt{1-\zeta^2}\,t)\right)$$

$\zeta>1$: overdamped ⟹ no oscillations

$$x(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}$$

$$x(t) = A_1e^{(-\zeta\omega_n+\sqrt{\zeta^2-1})t} + A_2e^{(-\zeta\omega_n-\sqrt{\zeta^2-1})t}$$

$\zeta=1$: critically damped

$\lambda_{1,2} = -\omega_n$

$$x(t) = A_1e^{-\omega_n t} + A_2te^{-\omega_n t}$$

$\zeta=0$: no damping

imaginary λ

$$x(t) = A_1\sin(\omega_n t) + B_1\cos(\omega_n t)$$

Resonance

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = \sin(\omega t)$$

$\omega = \omega_n$

Beats

$\omega \approx \omega_n$ but not $\omega = \omega_n$

$$\frac{1}{2\omega\delta}\sin(\delta t)$$

$\bar{\omega} = \frac{\omega_n+\omega}{n}$ $\quad \delta = \frac{\omega_n-\omega}{2}$

Express as

$$x(t) = A\sin(\omega t + \phi)$$

$A = \sqrt{A^2+B^2}$ $\quad \phi = \tan^{-1}\left(\frac{B}{A}\right)$

After Exam 2 - Question 1

$\overset{m}{1}x'' + \overset{c}{4}x' + \overset{k}{4}x = 3e^{-2t}$ $\quad = 3te^{-2t}$

Find the general solution

$\zeta = \frac{c}{2\sqrt{km}} = \frac{4}{2\sqrt{4\cdot 1}} = 1$ ← critically damped

eigenvalues → $\lambda = -2$

$x_h(t) = A_1e^{-2t} + A_2te^{-2t}$

$x_p(t) = At^2e^{-2t}$ $\quad \underbrace{t^2(At+B)e^{-2t}}_{At^2e^{-2t}+Bte^{-2t}}$

Why not $(At^2+Bt+C)e^{-2t}$?

$x_p' = 2Ate^{-2t} - 2At^2e^{-2t}$

$x_p'' = 2Ae^{-2t} - 4Ate^{-2t} - 4Ate^{-2t} + 4At^2e^{-2t}$

$$\begin{aligned} x_p'' + 4x_p' + 4x_p &= (2Ae^{-2t} - 8Ate^{-2t} + 4At^2e^{-2t}) \\ &+ 4(2Ate^{-2t} - 2At^2e^{-2t}) \\ &+ 4(At^2e^{-2t}) \\ &= \underbrace{(4A-8A+4A)}_{=0}t^2e^{-2t} + \underbrace{(-8A+8A)}_{=0}te^{-2t} + \underbrace{(2A)}_{=3}e^{-2t} \end{aligned}$$

$\Rightarrow A = \frac{3}{2}$

$$x(t) = A_1e^{-2t} + A_2te^{-2t} + \frac{3}{2}t^2e^{-2t}$$

(b) For this problem, $f(t,z) = \sqrt{z}$, and $\frac{\partial f}{\partial z} = \frac{-1}{2\sqrt{z}}$. Although this function is discontinuous at $z=0$, out initial condition is $z(0)=3$. Both of these functions are continuous near $t=0$ and $z=3$, so we will have a unique solution. Using separation of variables, we can find the solution as follows.

$$\begin{aligned} \int \frac{dz}{\sqrt{z}} &= \int dt + C \\ 2\sqrt{z} &= t + C \\ z &= (t+C)^2/4 \\ y &= (t+\sqrt{12})^2/4 \end{aligned}$$

We have found the unique solution, and its interval of existence is $-\infty < t < \infty$.

(c) For this problem, $f(t,u) = t/(u-2)$, and $\frac{\partial f}{\partial u} = -t/(u-2)^2$. Both of these functions are discontinuous at $u=2$, which is the location of our initial conditions. We therefore cannot conclude that a unique solution exists. If we attempt to solve using separation of variables, we find the following.

$$\begin{aligned} \int (u-2)\,du &= \int t\,dt + C \\ \frac{1}{2}u^2 - 2u &= \frac{1}{2}t^2 + C \\ u^2 - 4u - t^2 - 2C &= 0 \\ u &= \frac{4 \pm \sqrt{4^2 - 4(-t^2-2C)}}{2} \end{aligned}$$

Applying the initial condition gives

$$u(0) = \frac{4 \pm \sqrt{4^2+8C}}{2} = 2 \Longrightarrow C = -2,$$

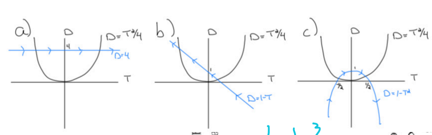
and the possible solutions are

$$y(t) = 2 \pm \sqrt{t^2}.$$

(c) We have $T = \operatorname{tr}(A) = 2a$ and $D = \det(A) = 1-4a^2$, so as a changes, we have a parabola along $D = 1-T^2$ in the trace-determinant plane. As a increases, we will first have a saddle, then a nodal sink, then a spiral sink, then a center, then a spiral source, then a nodal source, and finally another saddle. The center occurs when $T=0$, which gives $a=0$. The transition from saddle to nodal sink and from nodal source to saddle occurs when $D=0$, which gives $a=\pm 1/2$. The transition from nodal sink to spiral sink and spiral source to nodal source occurs when $D = T^2/4$, which gives $1-4a^2 = (2a)^2/4$, or $a = \pm 1/\sqrt{5}$. We therefore have

$$\begin{aligned} a < -1/2 &\Longrightarrow \text{saddle} \\ -1/2 < a < -1/\sqrt{5} &\Longrightarrow \text{nodal sink} \\ -1/\sqrt{5} < a < 0 &\Longrightarrow \text{spiral sink} \\ a = 0 &\Longrightarrow \text{center} \\ 0 < a < 1/\sqrt{5} &\Longrightarrow \text{spiral source} \\ 1/\sqrt{5} < a < 1/2 &\Longrightarrow \text{nodal source} \\ 1/2 < a &\Longrightarrow \text{saddle} \end{aligned}$$

The trace-determinant plane is shown below.



$y' = ty^3$

$y(0) = 1$

problem, $f(t,y) = ty^3$, and $\frac{\partial f}{\partial y} = 3ty^2$. Both of these functions are continuous for ∞) and $y \in (-\infty,\infty)$, so we can conclude that a unique solution exists on some erval. Using separation of variables, we can find the solution as follows.

$$\begin{aligned} \int \frac{dy}{y^3} &= \int t\,dt + C \\ -\frac{1}{2y^2} &= \frac{1}{2}t^2 + C \\ y &= \frac{\pm 1}{\sqrt{-2C-t^2}} \\ y &= \frac{1}{\sqrt{1-t^2}} \end{aligned}$$

at we must use the "+" solution to achieve the given initial condition. We have found ue solution, and its interval of existence is $-1 < t < 1$.

(a) From problem 1, we have the homogeneous solution

$$x_h(t) = e^{-t}\left(A_1\sin(\sqrt{3}t) + A_2\cos(\sqrt{3}t)\right).$$

Note that we cannot use the values of A_1 and A_2 found in problem 1 since we will be adding additional terms to this solution. Since t^2 does not appear in the homogeneous solution, the method of undetermined coefficients suggests that we try a particular solution of the form

$$x_p(t) = At^2 + Bt + C.$$

Plugging this solution into the differential equation gives

$$\begin{aligned} 2(2A) + 4(2At+B) + 8(At^2+Bt+C) &= t^2 \\ (8A)t^2 + (8A+8B)t + (4A+4B+8C) &= t^2 \end{aligned}$$

which is satisfied when $A = 1/8$, $B = -1/8$, and $C = 0$. The general solution is

$$x(t) = e^{-t}\left(A_1\sin(\sqrt{3}t) + A_2\cos(\sqrt{3}t)\right) + \frac{1}{8}t^2 - \frac{1}{8}t.$$

Using the given initial conditions leads to

$$\begin{aligned} x(0) = A_2 = 0 &\quad\Longrightarrow\quad A_2 = 0 \\ x'(0) = \sqrt{3}A_1 - \frac{1}{8} = 1 &\quad\Longrightarrow\quad A_1 = \frac{9}{8\sqrt{3}}, \end{aligned}$$

and our final solution becomes

$$x(t) = \frac{9}{8\sqrt{3}}e^{-t}\sin(\sqrt{3}t) + \frac{1}{8}t^2 - \frac{1}{8}t.$$

(b) From problem 1, we have the homogeneous solution

$$x_h(t) = A_1e^{-t} + A_2e^{-9t}.$$

Note that we cannot use the values of A_1 and A_2 found in problem 1 since we will be adding additional terms to this solution. Since e^{-t} does appear in the homogeneous solution, the method of undetermined coefficients suggests that we try a particular solution of the form

$$x_p(t) = Ate^{-t}.$$

Plugging this solution into the differential equation gives

$$\begin{aligned} (-2Ae^{-t} + Ate^{-t}) + 10(Ae^{-t} - Ate^{-t}) + 9(Ate^{-t}) &= 3e^{-t} \\ (-2A+10A)e^{-t} + (A-10A+9A)te^{-t} &= 3e^{-t} \end{aligned}$$

which is satisfied when $A = 3/8$. The general solution is

$$x(t) = A_1e^{-t} + A_2e^{-9t} + \frac{3}{8}te^{-t}.$$

Using the given initial conditions leads to

$$x(0) = A_1 + A_2 = 1 \qquad x'(0) = -A_1 - 9A_2 + \frac{3}{8} = 0,$$

which is satisfied when $A_1 = 69/64$ and $A_2 = -5/64$. Our final solution becomes

$$x(t) = \frac{69}{64}e^{-t} - \frac{5}{64}e^{-9t} + \frac{3}{8}te^{-t}.$$

(c) From problem 1, we have the homogeneous solution

$$x_h(t) = A_1e^{-t} + A_2te^{-t}.$$

Note that we cannot use the values of A_1 and A_2 found in problem 1 since we will be adding additional terms to this solution. Since $t\sin(2t)$ does appear in the homogeneous solution, the method of undetermined coefficients suggests that we try a particular solution of the form

$$x_p(t) = (At+B)\sin(2t) + (Ct+D)\cos(2t).$$

Plugging this solution into the differential equation gives

$$\begin{aligned} &3(4A\cos(2t) - 4(At+B)\sin(2t) - 4C\sin(2t) - 4(Ct+D)\cos(2t)) + \\ &6(A\sin(2t) + 2(At+B)\cos(2t) + C\cos(2t) - 2(Ct+D)\sin(2t)) + \\ &3((At+B)\sin(2t) + (Ct+D)\cos(2t)) \\ &= t\sin(2t), \end{aligned}$$

which can be rearranged to give

$$\begin{aligned} &(12A - 12D + 6C + 12B + 3D)\cos(2t) + (-12C - 12B + 6A - 12D + 3B)\sin(2t) + \\ &(-12C + 12A + 3C)t\cos(2t) + (-12A - 12C + 3A)t\sin(2t) = t\sin(2t), \end{aligned}$$

which is satisfied when

$$12A + 12B + 6C - 9D = 0 \quad 6A - 9B - 12C - 12D = 0 \quad 12A - 9C = 0 \quad -9A - 12C = 1.$$

The solution of this linear system is $A = -1/25$, $B = 22/375$, $C = -4/75$, and $D = -4/375$. The general solution is then

$$x(t) = A_1e^{-t} + A_2te^{-t} + \left(\frac{-1}{25}t + \frac{22}{375}\right)\sin(2t) + \left(\frac{-4}{75}t + \frac{-4}{375}\right)\cos(2t).$$

Using the given initial conditions leads to

$$\begin{aligned} x(0) = A_1 - \frac{4}{375} = 1 &\quad\Longrightarrow\quad A_1 = \frac{379}{375} \\ x'(0) = -A_1 + A_2 + 2\frac{22}{375} - \frac{4}{75} = 1 &\quad\Longrightarrow\quad A_2 = \frac{146}{75}. \end{aligned}$$

Our final solution becomes

$$x(t) = \frac{379}{375}e^{-t} + \frac{146}{75}te^{-t} + \left(\frac{-1}{25}t + \frac{22}{375}\right)\sin(2t) + \left(\frac{-4}{75}t + \frac{-4}{375}\right)\cos(2t).$$

The complex solution corresponding to $\lambda_2 = -3i$ is

$$\vec{x}_2(t) = e^{\lambda_2 t}\vec{v}_2 = e^{-3it}\begin{bmatrix}1\\-i\end{bmatrix}.$$

Breaking this solution into its real and imaginary parts gives

$$\vec{x}_2(t) = \begin{bmatrix}e^{-3it}\\-ie^{-3it}\end{bmatrix} = \begin{bmatrix}\cos(-3t)+i\sin(-3t)\\-i(\cos(-3t)+i\sin(-3t))\end{bmatrix} = \begin{bmatrix}\cos(3t)-i\sin(3t)\\-i\cos(3t)-\sin(3t)\end{bmatrix} = \begin{bmatrix}\cos(3t)\\-\sin(3t)\end{bmatrix} + i\begin{bmatrix}-\sin(3t)\\-\cos(3t)\end{bmatrix}.$$

We can now form the general solution by considering linear combinations of the real and imaginary parts of $x_2(t)$.

$$x(t) = c_1\begin{bmatrix}\cos(3t)\\-\sin(3t)\end{bmatrix} + c_2\begin{bmatrix}-\sin(3t)\\-\cos(3t)\end{bmatrix}.$$

By absorbing the negative sign in the second vector into the constant c_2, we can rewrite this as

$$x(t) = c_1\begin{bmatrix}\cos(3t)\\-\sin(3t)\end{bmatrix} + c_2\begin{bmatrix}\sin(3t)\\\cos(3t)\end{bmatrix},$$

which is equivalent to the solution found in class.