Chapter 2. Probability

0.1 Notes

Def 2.5 *A function $\mathbb{P}$ that assigns a real number $\mathbb{P}(A)$ to each event A, called the* ***probability*** *of A. We also call P the* ***probability distribution*** *or* ***probability measure****. To qualify as a probability, P has to satisfy three axioms:*

Axiom 1. $\mathbb{P}(A) \geq 0$ for every A
Axiom 2. $\mathbb{P}(\Omega) = 1$
Axiom 3. If $A_1, A_2, \ldots$ are disjoint, then $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

0.2 Problems

1. Fill in the details of the proof of Theorem 2.8. Also, prove the monotone decreasing case.

Thm. 2.8 (Continuity of Probabilities). *If $A_n \to A$ then $\mathbb{P}(A_n) \to \mathbb{P}(A)$ as $n \to \infty$.*

Proof. Suppose that A_n is monotone increasing so that $A_1 \subset A_2 \cdots$. Let $A = \lim_{n\to\infty} A_n$. Since A_n is monotone increasing, $A = \bigcup_{i=1}^{\infty} A_n$. Define a sequence B_n as follows.

$$B_i = \begin{cases} A_1, \ i = 1 \\ A_i \setminus \bigcup_{j=1}^{i-1} A_i, \ i > 1 \end{cases}$$

Let $1 \leq i < j$. By definition of B_j, $B_j \cap A_i = \emptyset$ and $B_i \subset A_i$, hence $B_i \cap B_j = \emptyset$. So, the terms of B_n are disjoint. Since A_n is monotone increasing, $A_k = \bigcup_{i=1}^{k} A_i$. Now, let $x \in A_k$ for some $k \geq 1$. If $x \notin A_j$ for all $j < k$, then $x \in B_k$ and hence $x \in \bigcup_{i=1}^{k} B_i$. Suppose $x \in A_j$ for some $j < k$, and take j to be the smallest such value. Then $x \in B_j$ and so $x \in \bigcup_{i=1}^{k} B_i$. Hence $A_k \subset \bigcup_{i=1}^{k} B_i$. Now, suppose $x \in \bigcup_{i=1}^{k} B_i$. Then $x \in B_j$ for some $1 \leq j \leq k$. Since $B_j \subset A_j$, we get $x \in A_j$, and since $A_j \subset A_k$ we have $x \in A_k$. Hence, $\bigcup_{i=1}^{k} B_i \subset A_k$, and therefore $A_k = \bigcup_{i=1}^{k} B_i$. Then by Axiom (3)*,

$$\mathbb{P}(A_n) = \mathbb{P}\left(\bigcup_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \mathbb{P}(B_i)$$

and again by Axiom (3),

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{P}(B_i) = \sum_{i=1}^{\infty} \mathbb{P}(B_i) = \mathbb{P}\left(\bigcup_{i=1}^{\infty}\right) = \mathbb{P}(A)$$

$\square$

2. Prove the statements in equation (2.1).

(a) $\mathbb{P}(\emptyset) = 0$.

Proof. Let $A \subset \Omega$. Note that $A = A \cup \emptyset$, so by Axiom (3),

$$\mathbb{P}(A) = \mathbb{P}(A \cup \emptyset) = \mathbb{P}(A) + \mathbb{P}(\emptyset) \Rightarrow \mathbb{P}(\emptyset) = \mathbb{P}(A) - \mathbb{P}(A) = 0$$

□

(b) $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

Proof. Assume $A \subset B$. We can write $B = A \cup (B \setminus A)$. By Axiom (3),

$$\mathbb{P}(B) = \mathbb{P}(A \cup (B \setminus A)) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

By Axiom (1), $\mathbb{P}(A \cup (B \setminus A)) \geq 0$, hence

$$\mathbb{P}(A) = \mathbb{P}(B) - \mathbb{P}(A \cup (B \setminus A)) \leq \mathbb{P}(B)$$

□

(c) $0 \leq \mathbb{P}(A) \leq 1$

Proof. This follows from (a) and (b), since for all A, $\emptyset \subset A \subset \Omega$. □

(d) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

Proof. By definition $A^c = \Omega \setminus A$ and $\Omega = A \cup (\Omega \setminus A) = A \cup A^c$. By Axiom (3)

$$\mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c) = 1 \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

□

(e) $A \cap B = \emptyset \Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Proof. This follows immediately from Axiom (3). □

3. Let Ω be a sample space and let $A_1, A_2, \ldots$ be events in Ω. Define $B_n = \bigcup_{i=n}^{\infty} A_i$ and $C_n = \bigcap_{i=n}^{\infty} A_i$.

(a) Show that $B_1 \supset B_2 \supset \cdots$ and $C_1 \subset B_2 \subset \cdots$.

Proof. Let $1 \leq m < n$. Then

$$B_m = \bigcup_{i=m}^{\infty} A_i = \bigcup_{i=m}^{n-1} A_i \cup \bigcup_{i=n}^{\infty} A_i = \bigcup_{i=m}^{n-1} A_i \cup B_n \Rightarrow B_m \supset B_n$$

Similarly,

$$C_m = \bigcap_{i=m}^{\infty} A_i = \bigcap_{i=m}^{n-1} A_i \cap \bigcap_{i=n}^{\infty} A_i = \bigcap_{i=m}^{n-1} A_i \cap C_n \Rightarrow C_m \subset C_n$$

□

(b) Show that $\omega \in \bigcap_{n=1}^{\infty} B_n$ if and only if ω belongs to an infinite number of events $A_{i_1}, A_{i_2}, \ldots$.

Proof. ($\Rightarrow$) Suppose $\omega \in \bigcap_{n=1}^{\infty} B_n$. Then $\omega i n B_1$. □

4.

5. Suppose we toss a fair coin until we get exactly two heads. Describe the sample space S. What is the probability that exactly k tosses are required?

Let t^n denote a n consecutive tails. Then tossing a fair coin until we get exactly two heads can be denoted as $t^{n_1} h t^{n_2} h$, where $n_1, n_2 \geq 0$. Fix some $k \geq 2$. There are

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