Answers

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1.

$$H = \frac{\Omega_x}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{\Omega_y}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\Omega_x}{2} - \frac{\Omega_y}{2}i \\ \frac{\Omega_x}{2} + \frac{\Omega_y}{2}i & 0 \end{bmatrix} = \frac{\Omega_x + \Omega_y}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

As the $\frac{\Omega_x+\Omega_y}{2}$ term is a scalar, we can find the the eigenvalues and eigenvectors for $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and multiply by the scalar:

$$A - \lambda I = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{bmatrix}$$

$$det\begin{bmatrix} 0-\lambda & -i \\ i & 0-\lambda \end{bmatrix} = \lambda^2 - 1$$

$$(\lambda - 1)(\lambda + 1) = 0$$

Therefore:

$$\lambda = -1$$

$$\lambda = +1$$

These are the eigenvalues for the matrix $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. We can now work out the eigenvectors:

$$\lambda = +1$$

$$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 - x_2 i = 0$$

$$x_1 i - x_2 = 0$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$
$$\lambda = -1$$
$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_1 - x_2 i = 0$$
$$x_1 i + x_2 = 0$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

The eigenvectors remain the same, however, if we multiply the lambda values by the scalar $\frac{\Omega_x + \Omega_y}{2}$, we obtain the eigenvalues for H.

Eigenvalues:

$$\frac{\Omega_x + \Omega_y}{2}, \frac{\Omega_x - \Omega_y}{2}$$

Eigenvectors:

$$\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Check:

$$\begin{bmatrix} 0 & \frac{\Omega_x}{2} - \frac{\Omega_y}{2} i \\ \frac{\Omega_x}{2} + \frac{\Omega_y}{2} i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{\Omega_x + \Omega_y}{2} \begin{bmatrix} 1 \\ i \end{bmatrix}$$
$$\begin{bmatrix} 0 & \frac{\Omega_x}{2} - \frac{\Omega_y}{2} i \\ \frac{\Omega_x}{2} + \frac{\Omega_y}{2} i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{\Omega_x - \Omega_y}{2} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

2.

$$U(t)\Psi = \begin{bmatrix} 0 & exp\left[\frac{\Omega_x}{2} + \frac{\Omega_y}{2} i(t)\right] \\ exp\left[\frac{\Omega_x}{2} - \frac{\Omega_y}{2} i(t)\right] & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} exp\left[\frac{\Omega_x}{2} + \frac{\Omega_y}{2} i(t)\right] b \\ exp\left[\frac{\Omega_x}{2} - \frac{\Omega_y}{2} i(t)\right] a \end{bmatrix}$$