ST3009: Week 5 Assignment

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Question 1

(a) If one marble is taken out of the box then the probability that the next marble taken out is the same colour as the first one is $\frac{4}{9}$ as there are only 4 marbles of the same colour remaining and only 9 marbles left in total. The probability that the marbles are different colours is $\frac{5}{9}$.

Therefore, the expected value of the winnings, $E[W]$, can be calculated like so:

$$E[W] = \left(\frac{4}{9} \cdot 1.10\right) + \left(\frac{5}{9} \cdot -1.00\right) = -0.0\overline{6}$$

(b) We can calculate the variance of the winnings, $Var(W)$, using the following formula, where μ is the mean value of the winnings:

$$Var(W) = \sum_{i=1}^{n} (w_i - \mu)^2 \cdot p(w_i)$$

Since $\mu = E[W]$ we can use our answer from the previous question to find the variance:

$$Var(W) = \left((1.10 - -0.0\overline{6})^2 \cdot \frac{4}{9}\right) + \left((-1.00 - -0.0\overline{6})^2 \cdot \frac{5}{9}\right) = 1.0888$$

Question 2

(a) We can calculate $E[X_i]$ using our method from 1(a):

$$E[X_i] = (0.6 \cdot 1) + (0.4 \cdot 0) = 0.6$$

We can calculate $Var(X_i)$ using the method from 1(b) given that $\mu = 0.6$:

$$Var(X_i) = \big((1-0.6)^2 \cdot 0.6\big) + \big((0-0.6)^2 \cdot 0.4\big) = 0.24$$

(b) $E[Y]$ is the sum of the expected values for each individual voter i - in other words, it is the expected number of voters given a sample of n people.

Since we know that:

$$Y = X_1 + \cdots + X_n$$

We can use the linearity of expectation to find that:

$$\begin{aligned} E[Y] &= E[X_1] + \cdots + E[X_n] \\ &= E[X_i] \cdot n \\ &= 0.6n \end{aligned}$$

It is clear that $E[X]$ and $E[Y]$ are different since $0.6n \neq 0.6$ where $n > 1$.

(c) Using the linearity of expectation we know that $E[\frac{1}{n}Y] = \frac{1}{n} \cdot E[Y]$. Given that $E[Y]$ is the expected number of voters given a sample of n people we can deduce that $\frac{1}{n} \cdot E[Y]$ is the proportion of people who voted:

$$E[\frac{1}{n}Y] = \frac{1}{n} \cdot 0.6n = 0.6$$

(d) From (b) we know that $Y = X \cdot n$ which means that $\frac{1}{n}Y = X$:

$$Var(\frac{1}{n}Y) = Var(X)$$

Question 3

(a) The probability that the first ball is white, $P(X_1 = 1)$, is $\frac{5}{13}$ and the probability that it's red, $P(X_1 = 0)$, is $\frac{8}{13}$.

The probability that both balls are red is:

$$P(X_1 = 0, X_2 = 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{14}{39}$$

The probability that the first ball is red and the second is white is:

$$P(X_1 = 0, X_2 = 1) = \frac{8}{13} \cdot \frac{5}{12} = \frac{10}{39}$$

The probability that the first ball is white and the second is red is:

$$P(X_1 = 1, X_2 = 0) = \frac{5}{13} \cdot \frac{8}{12} = \frac{10}{39}$$

The probability that the both balls are white is:

$$P(X_1 = 1, X_2 = 0) = \frac{5}{13} \cdot \frac{4}{12} = \frac{5}{39}$$

We can use these probabilities to calculate the joint probability mass function:

	$x_1 = 0$	$x_1 = 1$	$P(X_2 = x_2)$
$x_2 = 0$	$\frac{14}{39}$	$\frac{10}{39}$	$\frac{8}{13}$
$x_2 = 1$	$\frac{10}{39}$	$\frac{5}{39}$	$\frac{5}{13}$
$P(X_1 = x_1)$	$\frac{8}{13}$	$\frac{5}{13}$	1

(b) The formal definition of independence states that two events E and F are independent if

$$P(E \cap F) = P(E) \cdot P(F)$$

We know from the previous question that $P(X_1) = \frac{5}{13}$ and that $P(X_2) = \frac{5}{13}$:

$$P(X_1) \cdot P(X_2) = \frac{5}{13} \cdot \frac{5}{13} = \frac{25}{169}$$

We also know that $P(X_1 \cap X_2) = \frac{5}{39}$. We can therefore say that X_1 and X_2 are **not** independent as

$$\frac{25}{169} \neq \frac{5}{39}$$

(c) From part (a) we know that $P(X_2 = 0) = \frac{8}{13}$ and that $P(X_2 = 1) = \frac{5}{13}$:

$$E[X_2] = \left(\frac{8}{13} \cdot 0\right) + \left(\frac{5}{13} \cdot 1\right) = \frac{5}{13} = 0.38461$$

(d) Similarly, from part (a) we know that $P(X_2 = 0|X_1 = 1) = \frac{10}{39}$ and that $P(X_2 = 1|X_1 = 1) = \frac{5}{39}$. We must also divide the result of the following equation by $P(X_1 = 1)$ since we're not interested in the proportion where $P(X_1 = 0)$:

$$E[X_2|X_1 = 1] = \frac{\left(\frac{10}{39} \cdot 0\right) + \left(\frac{5}{39} \cdot 1\right)}{\frac{5}{13}} = \frac{13}{39} = 0.\overline{3}$$