# ST3009: Week 8 Assignment

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## Question 1

(a) Only a fraction of students will respond so there may be a bias in the results. Responses may not be entirely honest as many people might not want to admit that they are "studying to pass".
(b) The random experiment that can be repeated many times in this case is the is individual response to the poll $X_i \in \{0, 1\}$. This relates to the issues outlined in part (a) in that, through only a fraction of people responding, we do not garner accurate results.

A better way to design the experiment would be to make poll responses mandatory so that every student is sampled thus improving the overall accuracy of $Y$. Anonymous responses could be allowed in order to facilitate more honest responses.

## Question 2

(a) Two Bernoulli random variables, $X$ and $Y$, are identically distributed if and only if the cumulative distribution functions of $X$ and $Y$ are equal for every value of $x \in \mathbb{R}$:

$$P(X \leq x) = P(Y \leq x) \qquad \forall x \in \mathbb{R}$$

Using the standard definition of independence for two events $E$ and $F$, $P(E \cap F) = P(E) \cdot P(F)$, we know that two Bernoulli random variables, $X$ and $Y$, are independent if and only if:

$$P(X \leq x \cap Y \leq y) = P(X \leq x) \cdot P(Y \leq y) \qquad \forall x, y \in \mathbb{R}$$

(b) Since each $X_i$ has $\mu = 0.1$ we know that $P(X_i = 1) = 0.1$ for all $i$. Thus

$$Y = \frac{1}{N}\sum_{i=1}^{N} X_i = \frac{1}{N}(X_i \cdot N) = X_i$$

$Y$ is the empirical mean and, given that each $X_i$ is a random variable and since $Y = X_i$, we know that $Y$ is also a random variable. The mean of $Y$ is the same as the mean of any $X_i$ which is 0.1.

We can find the variance of $Y$ like so:

$$Var(Y) = Var(\frac{1}{N}\sum_{i=1}^{N} X_i) = \frac{1}{N^2}\sum_{i=1}^{N} Var(X_i) = \frac{1}{N}Var(X_i)$$

First we must calculate $Var(X_i)$:

$$\begin{aligned} Var(X_i) &= E[{X_i}^2] - E[X_i]^2 \\ &= \big(P(X_i = 1) \cdot 1^2 + P(X_i = 0) \cdot 0^2\big) - (0.1)^2 \\ &= (0.1 \cdot 1^2 + 0.9 \cdot 0^2) - (0.01) \\ &= 0.09 \end{aligned}$$

Therefore, given $N = 100$:

$$Var(Y) = \frac{0.09}{100}. = 0.0009$$

(c) Given $N = 100$, $\mu = 0.1$ and $\sigma = \sqrt{0.0009} = 0.03$ we can calculate the 95% confidence interval using Chebyshev's inequality like so:

$$\begin{gathered} 0.1 - \frac{0.03}{\sqrt{0.05 \cdot 100}} \leq Y \leq 0.1 + \frac{0.03}{\sqrt{0.05 \cdot 100}} \\ 0.1 - 0.0134 \leq Y \leq 0.1 + 0.0134 \\ 0.0866 \leq Y \leq 0.1134 \end{gathered}$$

(d) Again, given $N = 100$, $\mu = 0.1$ and $\sigma = 0.03$ we can calculate the 95% confidence interval $(2\sigma)$ using the CLT like so:

$$0.1 - 2 \cdot \frac{0.03}{\sqrt{100}} \leq Y \leq 0.1 + 2 \cdot \frac{0.03}{\sqrt{100}}$$
$$0.1 - 0.006 \leq Y \leq 0.1 + 0.006$$
$$0.094 \leq Y \leq 0.106$$

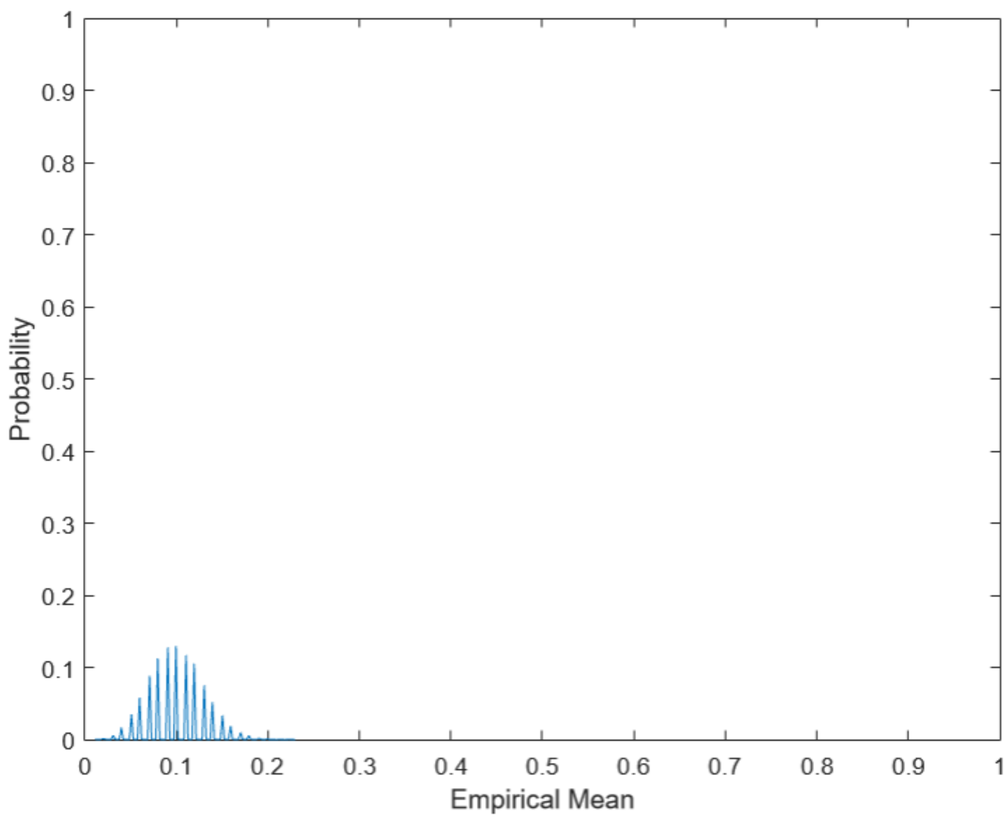
We can see that the CLT gave us a tighter approximate bound than Chebyshev's inequality. However, one of the drawbacks of CLT is that it only provides an approximation related to the finite value of $N$. Chebyshev's inequality provided us with an actual bound which works for any value of $N$ but it is looser than CLT.

(e) Using the following code we can run the numerical simulation 10,000 times and estimate the PMF and 95% confidence interval:

```
S = 10000;          % number of simulations to run
N = 100;            % number of BRVs to generate
R = zeros(S, 1);    % simulation results
M = 0.1;            % mean of Y

% run simulations
for s = 1:S
    R(s) = sum((rand(1, N) < M)) / N; % store empirical mean
end

% plot data
[n, es] = histcounts(R, N, 'Normalization','probability');
es = es(2:end) - (es(2) - es(1)) / 2;
plot(es, n);

% plot settings
axis([0 1 0 1]);
xticks(0:0.1:1);
yticks(0:0.1:1);
xlabel('Empirical Mean');
ylabel('Probability');

% estimate 95% CI
m = mean(R);
```

```
sd = std(R);
chebLower = m - (sd / sqrt(N * 0.05));
chebUpper = m + (sd / sqrt(N * 0.05));
cltLower = m - (2 * (sd / sqrt(N)));
cltUpper = m + (2 * (sd / sqrt(N)));

fprintf("Chebyshev: %.5f <= Y <= %.5f \n", chebLower, chebUpper);
fprintf("CLT:       %.5f <= Y <= %.5f \n", cltLower, cltUpper);
```

This produces the following probability mass function:




It also produces the following estimate for the 95% confidence interval:

$$\text{Chebyshev: } 0.08656 \leq Y \leq 0.11323$$
$$\text{CLT: } 0.09393 \leq Y \leq 0.10585$$

The answers given in (c) and (d) are almost identical to their respective simulated results which tells us that the simulation is extremely accurate when run 10,000 times.