ST3009: Week 3 Assignment

Conor McCauley - 17323203

February 17, 2020

Question 1

(a) Each roll must come up with the exact given number so there's a $\frac{1}{6}$ probability that each of the 6 rolls is right.

$$\left(\frac{1}{6}\right)^6 = \frac{1}{46656} \approx 0.0000214$$

(b) There are $\binom{6}{4}$ ways to order the '3' rolls. The probability for each of the '3' rolls is $\frac{1}{6}$ and the probability for the other rolls is $\frac{5}{6}$. This is just an application of the binomial probability law.

$$\binom{6}{4} \cdot \left(\frac{1}{6}\right)^4 \cdot \left(\frac{5}{6}\right)^2 = 0.00803$$

(c) There are $\binom{6}{1}$ ways to order the rolls. The probability for the '1' roll is $\frac{1}{6}$ and the probability for the other rolls is $\frac{5}{6}$.

$$\binom{6}{1} \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^5 = 0.40187$$

(d) The probability that we get at least a single 1 is the inverse of the probability that we get exactly zero '1's.

$$1 - \left(\frac{5}{6}\right)^6 = 0.6651$$

Question 2

The probability of the event A occurring, $P(A)$, is $\frac{1}{6}$. The only way for two dice rolls to sum to 2 is if both rolls are a 1, therefore

$$P(B) = \frac{1}{6} \cdot \frac{1}{20} = \frac{1}{120}$$

The probability that both A and B occur is the probability that the six-sided die rolls a 1 *and* the 20-sided die rolls a 1, thus

$$P(A \cap B) = \frac{1}{6} \cdot \frac{1}{20} = \frac{1}{120}$$

The formal definition of independence states that two events E and F are independent if

$$P(E \cap F) = P(E) \cdot P(F)$$

It is clear that the events A and B are **not** independent since

$$P(A \cap B) = \frac{1}{120}$$

$$P(A) \cdot P(B) = \frac{1}{6} \cdot \frac{1}{120} = \frac{1}{720}$$

$$P(A \cap B) \neq P(A) \cdot P(B)$$

Question 3

(a) On the hacker's first try there is a $\frac{n-1}{n}$ chance that she is unsuccessful and on her second try the chance that she is unsuccessful is $\frac{n-2}{n-1}$ (since she removed the incorrect password from the previous try), etc. On her k-th try the probability that she **is** successful is $\frac{1}{n-k+1}$. Therefore, the probability that she succeeds on her k-th try is

$$\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdots \frac{1}{n-k+1}$$

It is clear that the numerator of the first factor is cancelled out by the denominator of the second factor, etc. This leaves us with

$$\frac{1}{n}$$

(b) Substituting the values into the formula from the previous question gives us the answer

$$\frac{1}{6} = 0.1\overline{6}$$

(c) For the first $k-1$ attempts the probability that the hacker is unsuccessful is $\frac{n-1}{n}$. Then, on the k-th attempt, the probability that she **is** successful is $\frac{1}{n}$. Therefore, the generalised probability that the first success occurs on the k-th try is

$$\left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$$

(d) Substituting the values into the formula from the previous question gives us the answer

$$\left(\frac{5}{6}\right)^{2} \cdot \frac{1}{6} = 0.115\overline{740}$$

Question 4

(a) The probability that a robot fails at least one of the tests is the inverse of the probability that it passes all of the tests.

$$1 - 0.3^3 = 0.973$$

(b) As in the previous question, the probability that a human fails at least one of the tests is the inverse of the probability that the person passes all of the tests.

$$1 - 0.95^3 = 0.142625$$

(c) I will refer to the event that a visitor is a robot as R and the event that a visitor gets flagged as F.

– Likelihood - the probability that a visitor is flagged given that they are a robot:

$$P(F \mid R) = 0.973$$

– Prior - the probability that a visitor is a robot prior to receiving any new evidence:

$$P(R) = 0.1$$

– Evidence - calculated using marginalisation:

$$\begin{aligned} P(F) &= P(F \mid R) \cdot P(R) + P(F \mid \overline{R}) \cdot P(\overline{R}) \\ &= 0.973 \cdot 0.1 + 0.1426 \cdot (1 - 0.1) = 0.22564 \end{aligned}$$

Therefore, the probability that the visitor is a robot given that they've been flagged is

$$P(R \mid F) = \frac{P(F \mid R) \cdot P(R)}{P(F)} = \frac{0.973 \cdot 0.1}{0.22564} = 0.43121$$