CS3081: Assignment 3

Conor McCauley - 17323203

April 13, 2020

Answers

Question 4.26

(i) (a) = 4, (b) = 7.

Question 6.13

(iii) $530W$

Question 8.8

(ii) $O(h^2)$.

Question 8.9

(ii) (a) 4940 (males) and 10681 (females) (b) 673601 (males) and 277987 (females).

Solutions

Question 4.26

The following very simple Matlab code calculates the infinity norm of a given matrix. It returns 4 and 7 for the matrices in (a) and (b), respectively.

```
A = [
    4 -1 0 1 0;
    -1 4 -1 0 1;
    0 -1 4 -1 0;
    1 0 -1 4 -1;
    0 1 0 -1 4
];
N = InfinityNorm(A);
disp(N);

function N = InfinityNorm(A)
    % return the largest sum of a row's absolute values
    N = max(sum(abs(A), 2));
end
```

Question 6.13

We can use the following generalised formula to calculate the n^{th}-order polynomial in the Lagrange form:

$$f(x) = \sum_{i=1}^{n} y_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

This results in the following formula for the tabular values given in the question and $x = 26$:

$$\begin{aligned} f(26) = 320 \cdot & \frac{(26-22)(26-30)(26-38)(26-46)}{(14-22)(14-30)(14-38)(14-46)} \\ +490 \cdot & \frac{(26-14)(26-30)(26-38)(26-46)}{(22-14)(22-30)(22-38)(22-46)} \\ +540 \cdot & \frac{(26-14)(26-22)(26-38)(26-46)}{(30-14)(30-22)(30-38)(30-46)} \\ +500 \cdot & \frac{(26-14)(26-22)(26-30)(26-46)}{(38-14)(38-22)(38-30)(38-46)} \\ +480 \cdot & \frac{(26-14)(26-22)(26-30)(26-38)}{(46-14)(46-22)(46-30)(46-38)} \end{aligned}$$

$$= 320 \cdot \frac{-5}{128} + 490 \cdot \frac{15}{32} + 540 \cdot \frac{45}{64} + 500 \cdot \frac{-5}{32} + 480 \cdot \frac{3}{128} = 530W$$

Question 8.8

We can use Taylor series expansions to rewrite the terms in the formula. The Taylor series in the case of this finite difference formula can be defined as follows (we use hk rather than $x - x_0$ as they are equivalent in this case since the difference is equal to k steps of size h):

$$f'(x_{i+k}) = f(x_i) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_i)}{n!}(hk)^n$$

This gives us the following expansions:

$$f'(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2}f''(x_i) + \frac{h^3}{6}f'''(x_i) + \cdots$$
$$f'(x_{i+3}) = f(x_i) + 3hf'(x_i) + \frac{9h^2}{2}f''(x_i) + \frac{27h^3}{6}f'''(x_i) + \cdots$$

As there are three points in the initial formula: x_i, x_{i+1} and x_{i+3}, we can treat the fourth term in the expansion as the remainder like so:

$$f'(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2}f''(x_i) + \frac{h^3}{6}f'''(\xi_1),\ x_i \leq \xi_1 \leq x_{i+1}$$

$$f'(x_{i+3}) = f(x_i) + 3hf'(x_i) + \frac{9h^2}{2}f''(x_i) + \frac{27h^3}{6}f'''(\xi_2),\ x_i \leq \xi_2 \leq x_{i+3}$$

Then we can solve both of the equations for $f'(x_i)$:

$$f'(x_i) = \frac{f'(x_{i+1}) - f(x_i)}{h} - \frac{h}{2}f''(x_i) - \frac{h^2}{6}f'''(\xi_1)$$
$$f'(x_i) = \frac{f'(x_{i+3}) - f(x_i)}{3h} - \frac{3h}{2}f''(x_i) - \frac{3h^2}{2}f'''(\xi_2)$$

From these equations we can see that the largest exponent of h in both cases is h^2 which tells us that the order of the truncation error in this finite difference formula is $O(h^2)$.

Question 8.9

(a) The formula for three-point backward differences with unequally spaced points is

$$f'(x_{i+2}) = y_i \frac{x_{i+2} - x_{i+1}}{(x_i - x_{i+1})(x_i - x_{i+2})} + y_{i+1} \frac{x_{i+2} - x_i}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})}$$
$$+ y_{i+2} \frac{2x_{i+2} - x_i - x_{i+1}}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})}$$

With $x_{i+2} = 2006$ we can substitute the tabular values for male doctors to calculate the rate of changes:

$$f'(2006) = 638182 \cdot \frac{2006 - 2003}{(2002 - 2003)(2002 - 2006)} + 646493 \cdot \frac{2006 - 2002}{(2003 - 2002)(2003 - 2006)}$$
$$+ 665647 \cdot \frac{2 \cdot 2006 - 2002 - 2003}{(2006 - 2002)(2006 - 2003)}$$
$$= 4939.92$$

Similarly, for female doctors:

$$f'(2006) = 215005 \cdot \frac{2006 - 2003}{(2002 - 2003)(2002 - 2006)} + 225042 \cdot \frac{2006 - 2002}{(2003 - 2002)(2003 - 2006)}$$
$$+ 256257 \cdot \frac{2 \cdot 2006 - 2002 - 2003}{(2006 - 2002)(2006 - 2003)}$$
$$= 10681.0$$

(b) The formula for three-point central differences with unequally spaced points is

$$f'(x_{i+1}) = y_i \frac{x_{i+1} - x_{i+2}}{(x_i - x_{i+1})(x_i - x_{i+2})} + y_{i+1} \frac{2x_{i+1} - x_i - x_{i+2}}{(x_{i+1} - x_i)(x_{i+1} - x_{i+2})}$$
$$+ y_{i+2} \frac{x_{i+1} - x_i}{(x_{i+2} - x_i)(x_{i+2} - x_{i+1})}$$

We can substitute our answers from (a) into $f'(x_{i+1})$ and solve for y_{i+2}. For male doctors:

$$4940 = 646493 \cdot \frac{2006 - 2008}{(2003 - 2006)(2003 - 2008)} + 665647 \cdot \frac{2 \cdot 2006 - 2003 - 2008}{(2006 - 2003)(2006 - 2008)}$$
$$+ y_{i+2} \frac{2006 - 2003}{(2008 - 2003)(2008 - 2006)}$$
$$y_{i+2} = \frac{4940 + 86199.07 + 110941.17}{0.3} = 673600.8$$

Similarly, for female doctors:

$$10681 = 225042 \cdot \frac{2006 - 2008}{(2003 - 2006)(2003 - 2008)} + 256257 \cdot \frac{2 \cdot 2006 - 2003 - 2008}{(2006 - 2003)(2006 - 2008)}$$
$$+ y_{i+2} \frac{2006 - 2003}{(2008 - 2003)(2008 - 2006)}$$
$$y_{i+2} = \frac{10681 + 30005.6 + 42709.5}{0.3} = 277987.0$$

We can also calculate the percentage errors between the actual and predicted values:

$$E_{males} = \frac{|673601 - 677807|}{\frac{673601+677807}{2}} = 0.0062 = 0.62\%$$
$$E_{females} = \frac{|277987 - 276417|}{\frac{277987+276417}{2}} = 0.0056 = 0.56\%$$