CS4004: Assignment 1

Conor McCauley - 17323203

November 3, 2020

Question 1

(a) As no contradictions occurred we know that the following semantic entailment is falsifiable and therefore it does not hold:

	$\neg p$	$\vee$	$(q$	$\rightarrow$	$p)$	$\models$	$\neg p$	$\wedge$	q
1		T						F	
1.1	T	T	F	T	F		T	F	F
1.2	F	T	T	T	T		F	F	T
1.3	F	T	F	T	T		F	F	F

(b) As a contradiction occurred we know that the following semantic entailment is not falsifiable and therefore it does hold:

	$\models$	$((A$	$\rightarrow$	$B)$	$\rightarrow$	$A)$	$\rightarrow$	A
1							F	
1.1		T	F	F	T	F	F	F

Question 2

(a)

1. $\neg(A_1 \vee A_2)$, *premise*
2. A_1, *assumption*
3. $A_1 \vee A_2$, $\vee i_1(2)$
4. $\perp$, $\neg e(1, 3)$
5. A_2, *assumption*
6. $A_1 \vee A_2$, $\vee i_2(5)$
7. $\perp$, $\neg e(1, 6)$
8. $\neg A_1$, $\neg i(2 - 4)$
9. $\neg A_2$, $\neg i(5 - 7)$
10. $\neg A_1 \wedge \neg A_2$, $\wedge i(8, 9)$

(b) We will refer to the rule we derived in (a) as follows:

$$\frac{\neg(A_1 \vee A_2)}{\neg A_1 \wedge \neg A_2}\mathcal{R}_1$$

Before proving the given sequent we will first attempt to prove the following (it will make our final proof less convoluted):

$$\neg(A_1 \rightarrow A_2) \vdash A_1 \wedge \neg A_2$$

1. $\neg(A_1 \to A_2)$, *premise*
2. $\neg A_1$, *assumption*
3. A_1, *assumption*
4. $\bot$, $\neg e(2,3)$
5. A_2, $\bot e(4)$
6. $A_1 \to A_2$, $\to i(3-5)$
7. $\bot$, $\neg e(1,6)$
8. A_1, $PBC(2-7)$
9. A_2, *assumption*
10. $A_1 \to A_2$, $\to i(8,9)$
11. $\bot$, $\neg e(1,10)$
12. $\neg A_2$, $\neg i(9-11)$
13. $A_1 \wedge \neg A_2$, $\wedge i(8,12)$

We will refer to this derived rule as follows:

$$\frac{\neg(A_1 \to A_2)}{A_1 \wedge \neg A_2}\mathcal{R}_2$$

1. $\neg((p \to q) \vee (r \to p))$, *assumption*
2. $\neg(p \to q) \wedge \neg(r \to p)$, $\mathcal{R}_1(1)$
3. $\neg(p \to q)$, $\wedge e_1(2)$
4. $\neg(r \to p)$, $\wedge e_2(2)$
5. $p \wedge \neg q$, $\mathcal{R}_2(3)$
6. $r \wedge \neg p$, $\mathcal{R}_2(4)$
7. p, $\wedge e_1(5)$
8. $\neg p$, $\wedge e_2(6)$
9. $\bot$, $\neg e(7,8)$
10. $(p \to q) \vee (r \to p)$, $PBC(1-9)$

(c) We can use the rules derived from (a) and (b), $\mathcal{R}_1$ and $\mathcal{R}_2$, to prove the given sequent:

1. $(A \to B) \to A$, *assumption*
2. $\neg(A \vee \neg(A \to B))$, *assumption*
3. $\neg A$, $\mathcal{R}_1(2)$
4. $\neg\neg(A \to B)$, $\mathcal{R}_1(2)$
5. $A \to B$, $\neg\neg e(4)$
6. A, $\to e(5,1)$
7. $\bot$, $\neg e(3,6)$
8. $A \vee \neg(A \to B)$, $PBC(2-7)$
9. A, *assumption*
10. A, $COPY(9)$
11. $\neg(A \to B)$, *assumption*
12. $A \wedge \neg B$, $\mathcal{R}_2(11)$
13. A, $\wedge e_1(12)$
14. A, $\vee e(8, 9-10, 11-13)$
15. $((A \to B) \to A) \to A$, $\to i(1-14)$

Question 3

(a) $\forall x.(T(x) \to \exists y.(G(y) \wedge P(y,x)))$
(b) $\forall x.(G(x) \to \exists y.(T(y) \wedge P(x,y)))$
(c) $\neg\exists x.(G(x) \wedge \exists y.(T(y) \wedge P(x,y)) \wedge \exists z.(T(z) \wedge P(x,z)) \wedge \neg(y = z))$

(d) $\exists x.(M(x, mu, rm) \land \neg W(mu, rm, x))$
(e) $\neg\exists x.(\neg(x = ik) \land P(x, mu) \land \forall y.((TG(mu, y) \land PG(ik, y)) \to PG(x, y)))$
(f) $\forall x.((\neg(x = ik) \land P(x, mu)) \to \exists y.(TG(mu, y) \land PG(ik, y) \land \neg PG(x, y)))$

Question 4

(a)

1. $\exists x.(P(x) \land Q(x))$, *premise*

   x_0
   2. $(P(x_0) \land Q(x_0)) = (P(x) \land Q(x))[x_0/x]$, *assumption*
   3. $P(x')[x_0/x'] = P(x_0)$, $\land e_1(2)$
   4. $Q(x')[x_0/x'] = Q(x_0)$, $\land e_2(2)$
   5. $\exists x'.P(x')$, $\exists i(3)$
   6. $\exists x'.Q(x')$, $\exists i(4)$
   7. $\exists x'.P(x') \land \exists x'.Q(x')$, $\land i(5, 6)$

8. $\exists x'.P(x') \land \exists x'.Q(x')$, $\exists e(1, 2-7)$

(b) First we will prove the following sequent:

$$\neg\exists x.A \vdash \neg A$$

1. $\neg\exists x.A$, *premise*
   2. $A[t/x] = A$, *assumption*
   3. $\exists x.A$, $\exists i(2)$
   4. $\bot$, $\neg e(3, 1)$
5. $\neg A$, $\neg i(2-4)$

We will refer to this derived rule as follows:

$$\frac{\neg\exists x.A}{\neg A}\mathcal{R}_3$$

We will also make use of one of the derived rules from question 2, $\mathcal{R}_2$.

1. $\forall x.P(x) \to S$, *premise*
   2. $\neg\exists x.(P(x) \to S)$, *assumption*
   3. $\neg(P(x) \to S)$, $\mathcal{R}_3(2)$
   4. $P(x) \land \neg S$, $\mathcal{R}_2(3)$
   5. $P(x)$, $\land e_1(4)$
   6. $\neg S$, $\land e_2(4)$

      x_0
      7. $P(x)[x_0/x] = P(x_0)$, $COPY(5)$
   8. $\forall x.P(x)$, $\forall i(7)$
   9. S, $\to e(1, 8)$
   10. $\bot$, $\neg e(9, 6)$
11. $\exists x.(P(x) \to S)$, $PBC(2-10)$

(c)

1. $\neg\forall x.\neg P(x)$, *premise*
2. $\neg\exists x.P(x)$, *assumption*
x_0
3. $P(x)[x_0/x] = P(x_0)$, *assumption*
4. $\exists x.P(x)$, $\exists i(3)$
5. $\perp$, $\neg e(4,2)$
6. $(\neg P(x))[x_0/x] = \neg P(x_0)$, $\neg i(3-5)$
7. $\forall x.\neg P(x)$, $\forall i(3-6)$
8. $\perp$, $\neg e(7,1)$
9. $\exists x.P(x)$, $PBC(2-8)$

Question 5

8 hours.