Gravitational Lensing: Introductory Notes

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May 8, 2017

When light from a far away object (the source) passes by a nearer massive object (the lens), the path of the light undergoes a deflection due to the curvature of space around the lens. This is a direct result of general relativity but the mathematics describing the situation are remarkably simple.

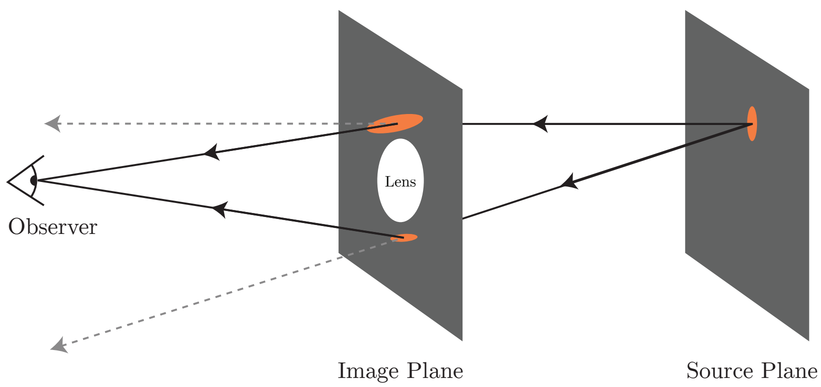

Figure 1: A lensing event. The source is in the background. Two images of the source are produced in the foreground on a plane passing through the lens called the image plane.

Thin Lens Approximation

The distance from observer (us) to the lens, and from the lens to the source are both many orders of magnitude larger than the extension of the lens in space. As such, we can project the entire lens onto the plane of the sky and treat it as a 2D object. All of our work will take place in the plane of the lens, projected onto the sky. It is on this 2D surface that images of the background source form. As such it is referred to as the *image plane*.

The entire geometry of a lensing event is outlined in the figure on the next page. The most important quantity on the diagram is α. This is simply called the **deflection angle** and it contains all of the physics of the problem.

From the diagram we can see two things; firstly that

$$D_s\theta = D_s\beta + D_{ds}\alpha, \tag{1}$$

and,

$$\alpha = \frac{D_{ds}}{D_s}\hat{\alpha} \tag{2}$$

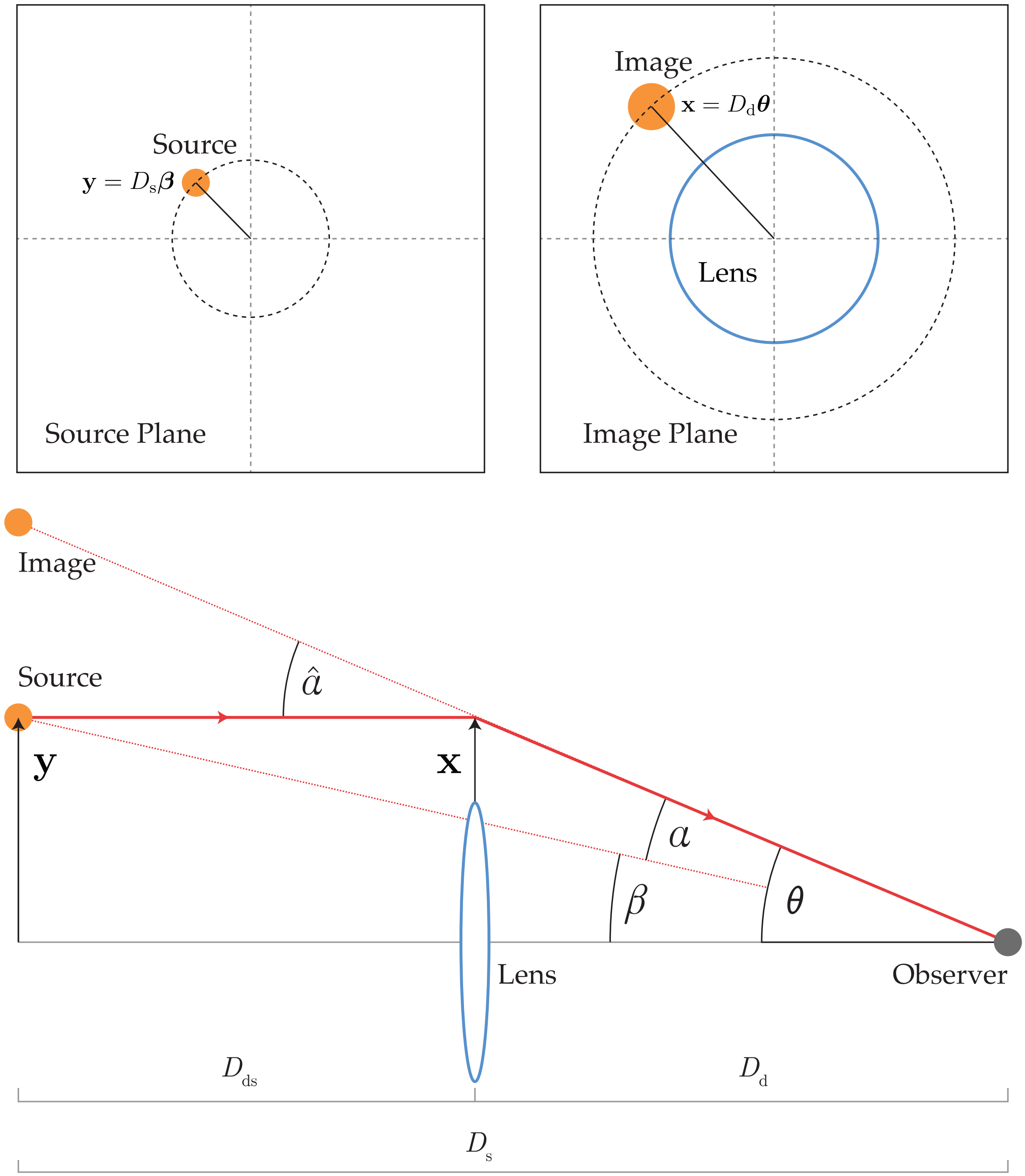

Figure 2: The geometry of a lensing event.

We put one into the other and get something called the lens equation;

$$\beta = \theta - \alpha(\theta) \tag{3}$$

This is just a coordinate transformation: it relates a position on the image plane θ to a position on the source plane β. If we know the brightness in the source plane at a given position, then we just transform onto the imgae plane via $\alpha(\theta)$ to find the brightness in the image plane (i.e. the actual image we would observe).

Note now how α is a function of θ. This is because different mass distributions produce different deflection angles that can vary with distance from the mass, and have angular properties (we'll get into elliptical lenses later).

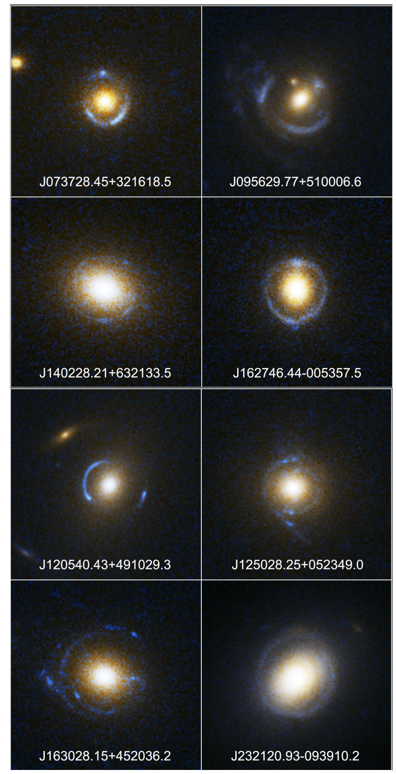

Figure 3: Some real Einstein rings found in the SLACS project!

Einstein Radius and Einstein Rings

For a point mass, it can be shown that the deflection angle at an angular distance θ from the axis is given by

$$\alpha = \frac{4GM}{c^2}\frac{D_{ds}}{D_s}\frac{1}{\theta}. \tag{4}$$

If we define a quantity called the *Einstein Radius*, θ_E to be

$$\theta_E^2 = \frac{4GM}{c^2}\frac{D_{ds}}{D_d D_s} \tag{5}$$

then we can rewrite the lens equation (eq. 3) as;

$$\beta = \theta - \frac{\theta_E^2}{\theta}. \tag{6}$$

The lens equation has the solutions in terms of θ;

$$\theta_{\pm} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 \pm 4\theta_E^2}\right) \tag{7}$$

This is quite informative we can see that for a given source there are always **two** image positions. When the source is actually on the optical axis ($\beta = 0$) we can see the image appears at $\theta = \pm\theta_E$. So the image becomes a ring formed at the Einstein radius, i.e. an *Einstein ring*. The Einstein radius is the essential quantity in lensing and describes the scale of any gravitational lens. The figure on the next page shows different image configurations from different source positions for a point mass.

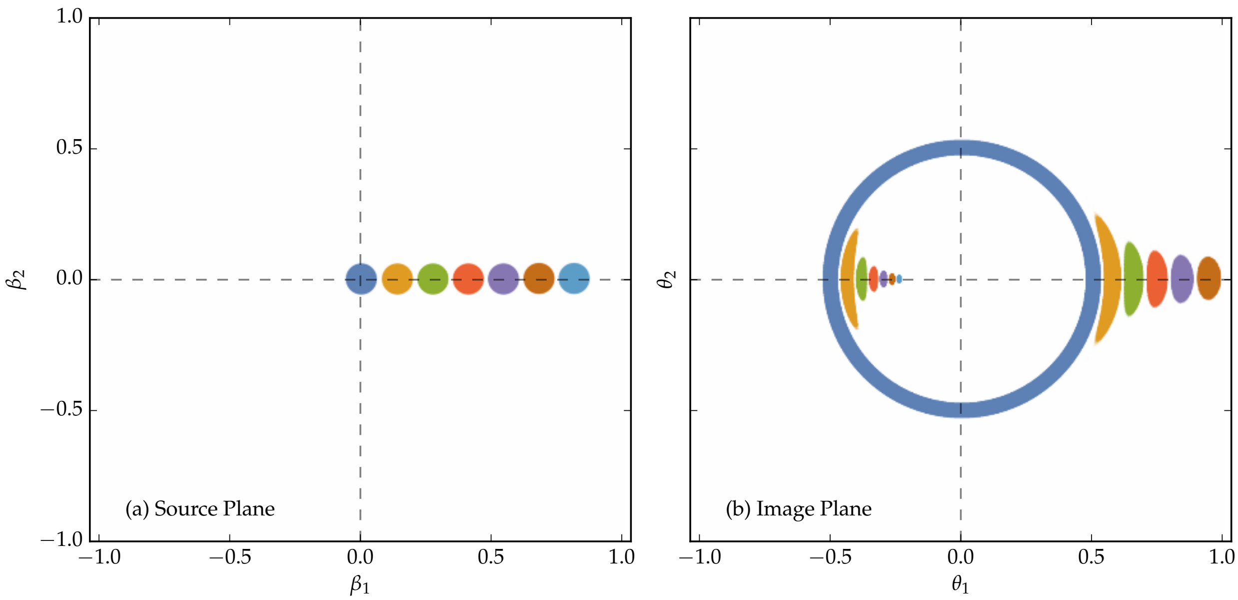

Figure 4: Different images produced by different source positions. When the source is on the axis an Einstein Ring is formed (blue). The lens equation has two solutions and so there are always two images; one inside and one outside the Einstein radius.

Simulating Lensing

To simulate a lensing event we need to do a few distinct things. First, we need a grid of pixels that will become our image plane. We will describe the coordinates of these pixels with a vector $\mathbf{x}$. To produce an image all we have to do is find the intensity of each of these image pixels, $I(\mathbf{x})$.

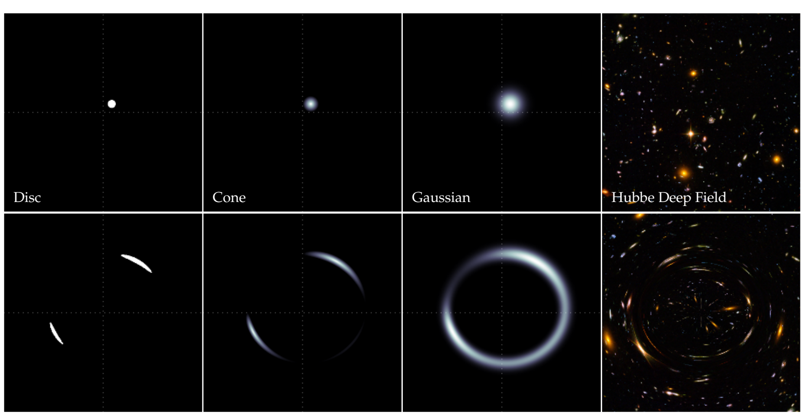

Figure 5: Lensing of different source types; a disc, a cone, a Gaussian, and a photo.

We also need a set of coordinates in the source plane which we'll similarly describe by the vector $\mathbf{y}$. The source plane has its own intensity; this might for example be a Gaussian, a flat disc or even a photo. The source can be anything as long as every position on the source $\mathbf{y}$ has some intensity $S(\mathbf{y})$.

Once we've defined our source then we know what $S(\mathbf{y})$ is. However, to produce an image we need $I(\mathbf{x})$. This is just a coordinate transformation, which is provided by the lens equation (eq. 3) a relationship between source and image coordinates, according to a deflection angle function (this contains all the physics).

This gives us a new form for the image plane;

$$I(\mathbf{x}) = S(\mathbf{x} - \alpha(\mathbf{x})) \tag{8}$$

i.e. the brightness of all the pixels in the image is found by transforming those pixels back to the source plane via the deflection and taking that source brightness.

The Singular Isothermal Ellipsoid

We have seen that for a point mass $\alpha(\mathbf{x})$ has a very simple form. For more complicated masses (i.e. those that look like actual galaxies!) α becomes a bit more complex. Most lenses are early-type galaxies. These are young galaxies still in the midst of forming and settling down. As such, they can be highly elliptical but their mass is *virialised* meaning the matter in the galaxy acts like a hot **isothermal** gas.

This type of galaxy has a well defined mass distribution and Kormann (1994) provides an analytic solution to the deflection angle for this type of galaxy. The deflection angle has components in both directions on the image plane and so becomes a **vector**. It's given by;

$$\alpha_1 = -\frac{\sqrt{f}}{f'} \operatorname{arcsinh}\left(\frac{f'}{f} \cos\phi\right) \tag{9}$$

$$\alpha_2 = -\frac{\sqrt{f}}{f'} \arcsin\left(f' \sin\phi\right) \tag{10}$$

where f is the ratio between the minor and major axes of the elliptical mass distribution, $f' = \sqrt{1 - f^2}$ and ϕ is the angle between the vector $\mathbf{x}$ and the x-axis.

All of the above can be calculated in a completely vectorised way in NumPy making the calculations very quick. With an elliptical lens the images become a bit more complicated, certain configurations will now produce four images rather than two. See the diagram on the next page for some examples.

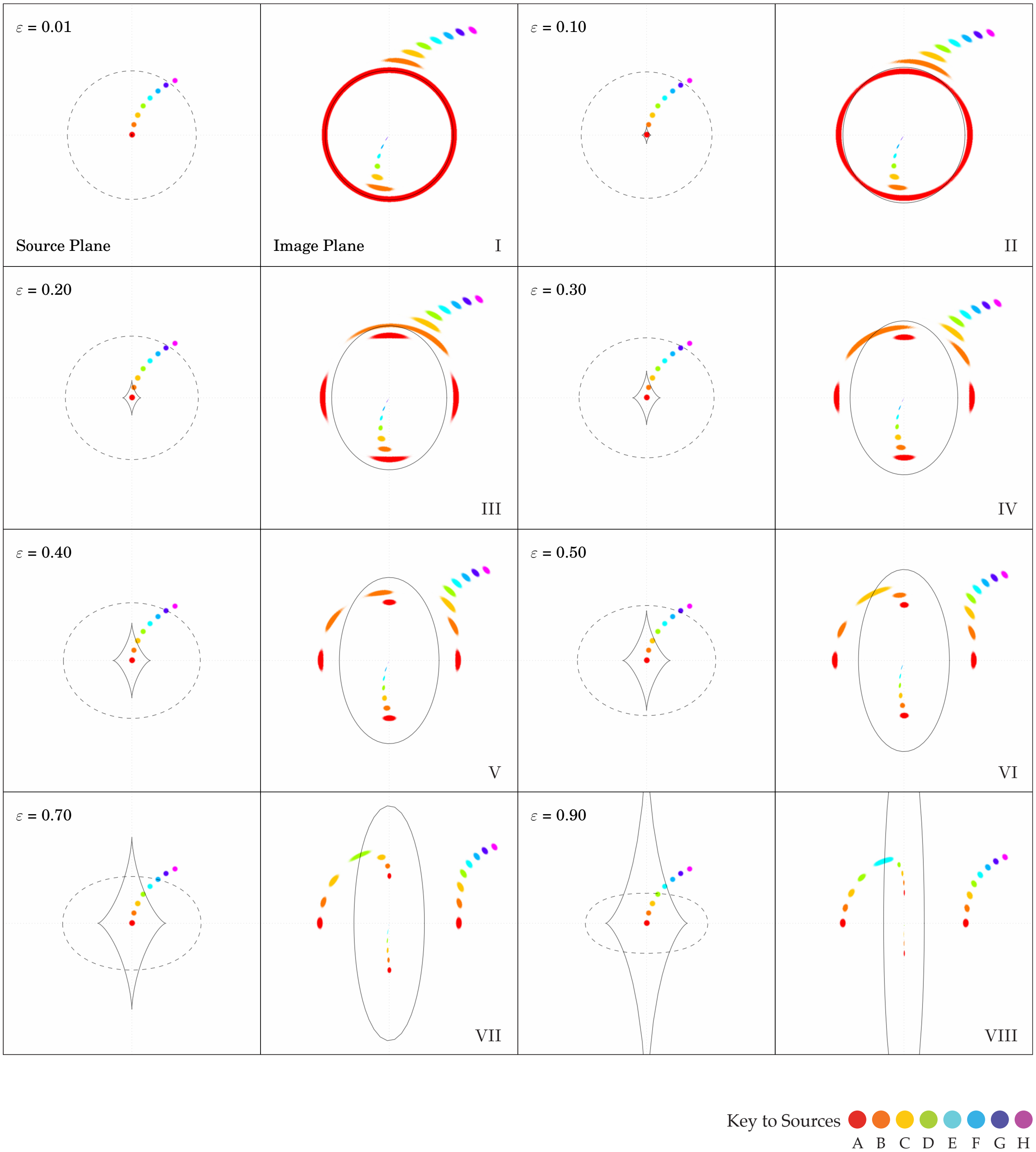

Figure 6: $\varepsilon = 1 - f = 1 - b/a$ is called the *ellipticity* so $\varepsilon = 0$ represents a circular lens. We see the more elliptical lenses can produce complicated four image configurations.