Simple Linear Regression

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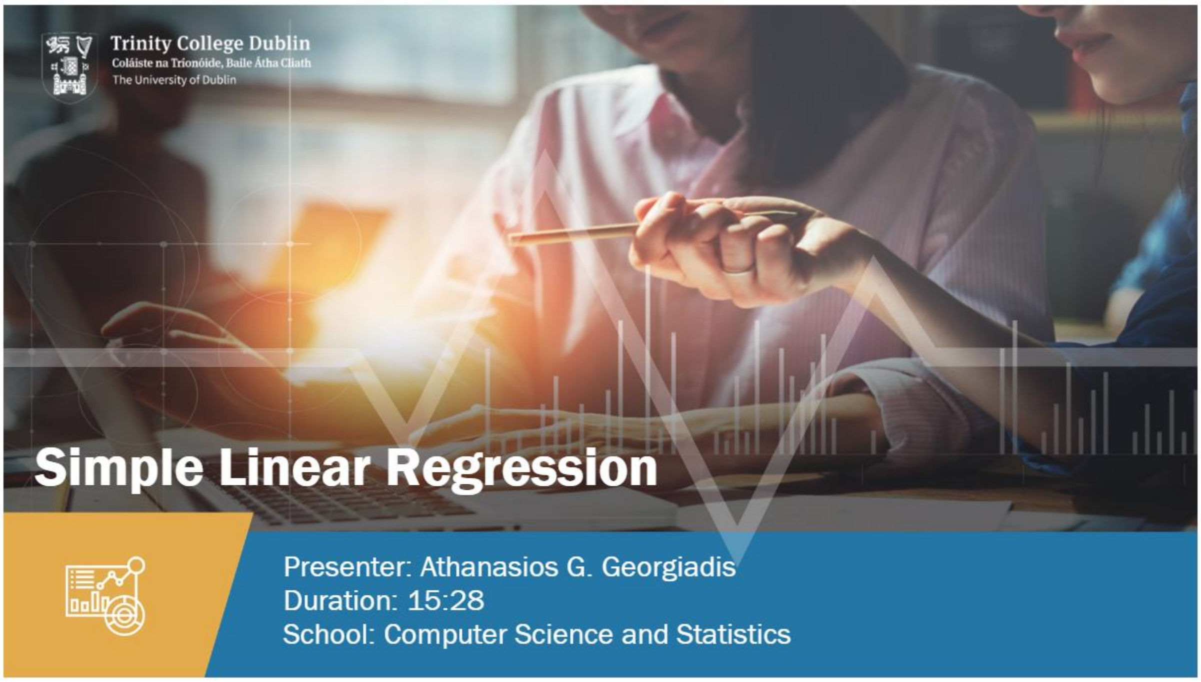
Slide 1: Introduction

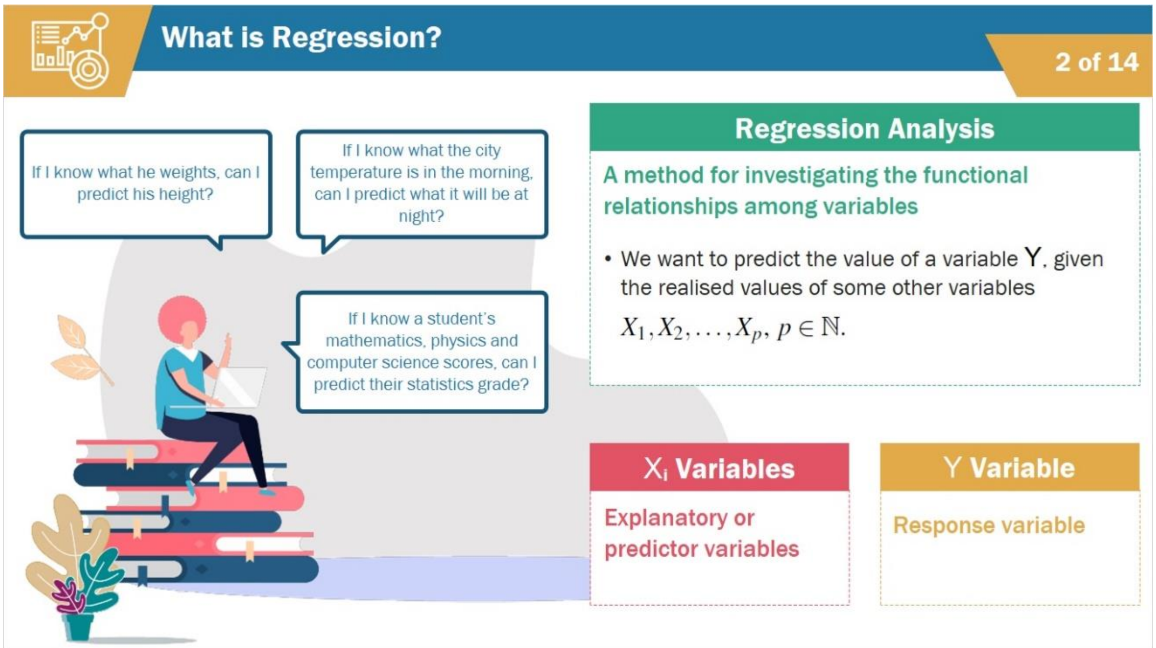


Hello, and welcome. My name is Athanasios Georgiadis and I'm the instructor for this presentation on simple linear regression.

During this presentation we will explain what is meant by regression, focus on the simple linear regression, and elaborate on the principle of least squares.

We will present a representative example of linear regression and we will conclude with diagnostics.

Slide 2: What is Regression?



Let's start with some motivational questions:

If I knew the weight of a person, could I predict his height?

If I knew the temperature in a city at a certain time in the morning could I predict the temperature at night-time?

If I knew the grade of a Student in Mathematics, Physics and Computer Science, could I predict hers or his grade in Statistics?

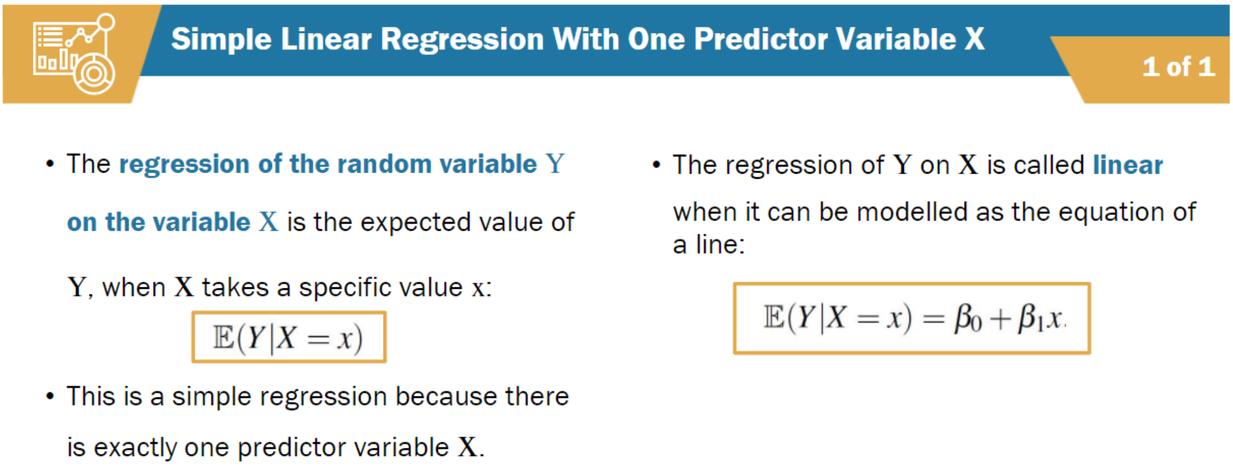
Regression analysis is a method for investigating the functional relationship among variables.

In Regression analysis, we try to predict the value of a variable Y, given the realized values of some other variables X1, X2, . . . , Xp.

The Xi-variables are called the explanatory or predictor variables.

The Y -variable is called the response variable.

Slide 3: Simple Linear Regression with One Predictor Variable X



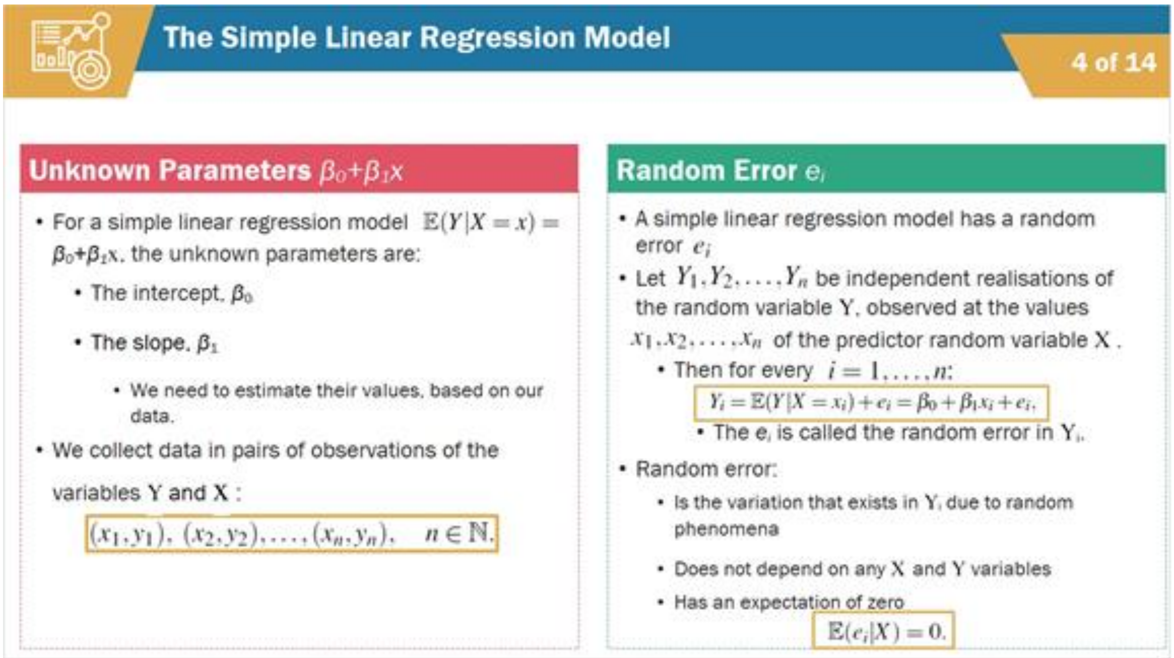
In this presentation we focus on the case when we have one predictor variable *X*.

The *regression of the random variable Y on the variable X* is the expected value of *Y* , when *X* takes a specific value *x*, and it is referred as a *simple regression*, because we have exactly one predictor variable *X*.

The regression of *Y* on *X* is called *linear* when it can be modelled as the equation of a line

$E(Y | X = x) = \beta_0 + \beta_1 x.$

Slide 4: The Simple Linear Regression Model



For a simple linear regression model, the *unknown parameters* β_0 and β_1 are referred as the *intercept* and the *slope* of the model respectively.

These are unknown population parameters. We need to *estimate* their values based on our data. For this purpose, we collect data in pairs of observations of the variables X and Y:

$(x_1, y_1), \ (x_2, y_2), \ldots, (x_n, y_n),$

for some data size n.

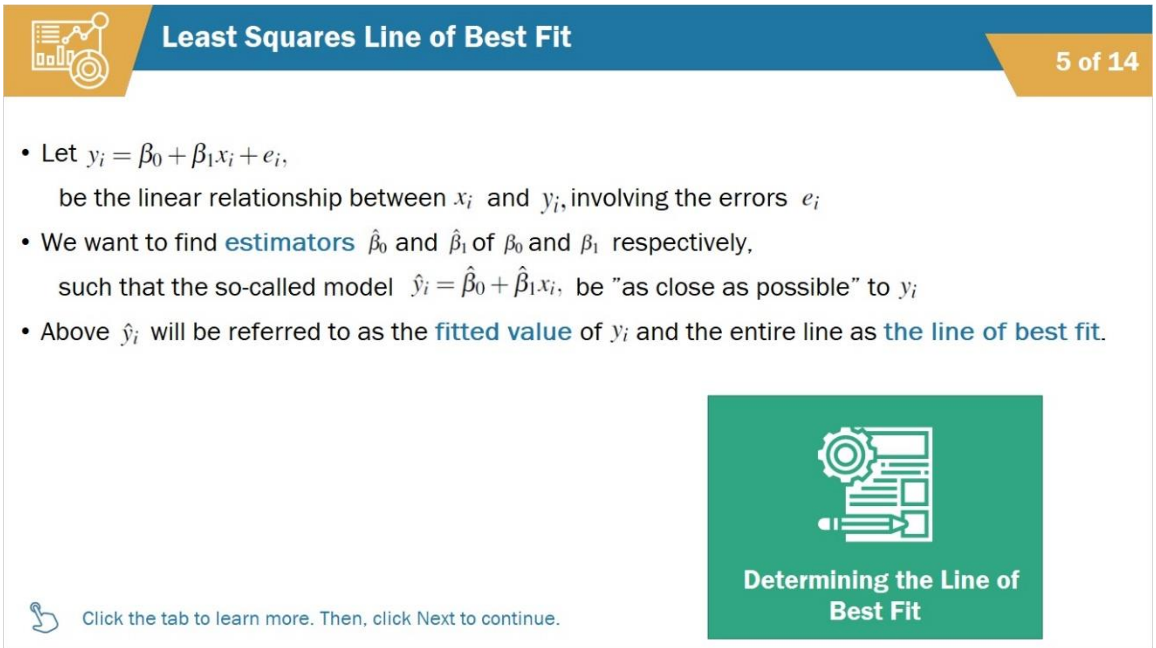
Let $Y_1, Y_2, \ldots, Y_n$ be independent realizations of the random variable Y, observed at the values $x_1, x_2, \ldots, x_n$ of the predictor random variable$X$. Then for every $i = 1, \ldots, n$:

$Y_i = \mathrm{E}(Y \mid X = x_i) + e_i,$

this e_i is called the *random error* in Y_i.

The random error represents the variation that exists in Y_i due to random phenomena that cannot be predicted or explained. The random error does not depend on any of X and Y variables and its expectation is zero.

Slide 5: Least Squares Line of Best Fit



Let:

$y_i = \beta_0 + \beta_1 x_i + e_i$,

be the linear relationship between x_i and y_i, involving the errors e_i.

We aim to find *estimators* $\hat{\beta}_0$ and $\hat{\beta}_1$ of β_0 and β_1 respectively, such that the so-called fitted model

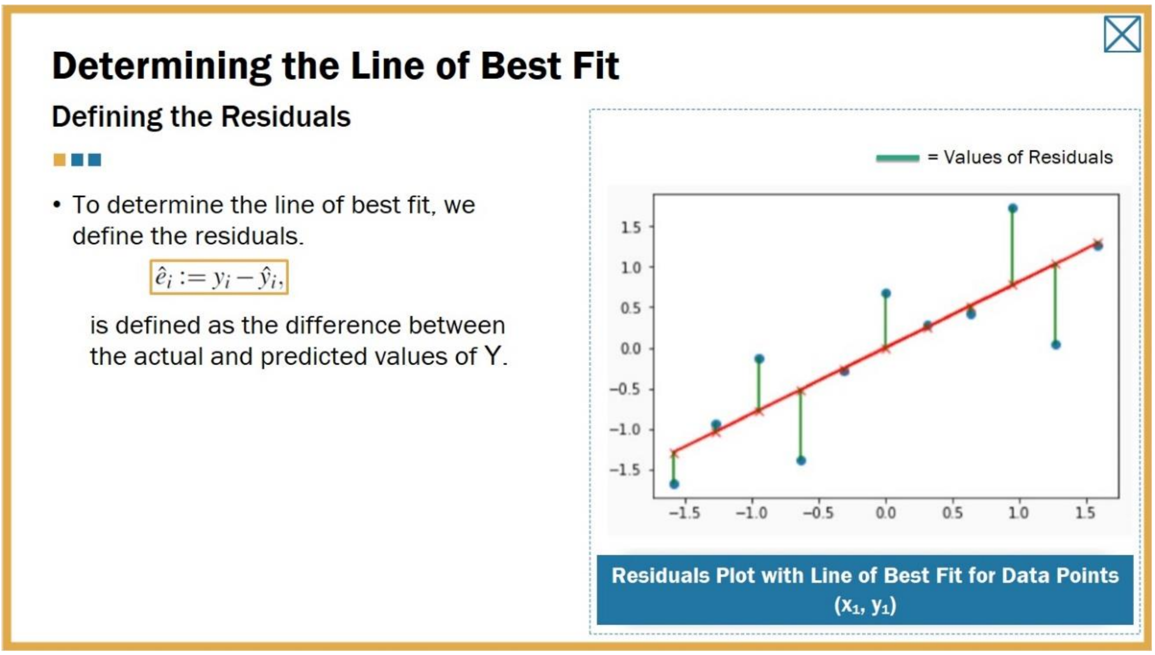
$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$,

is "as close as possible" to y_i. Above $\hat{y}_i$ will be referred as the *fitted*

value of y_i and the entire line as the *line of best fit*.

Click the tab to find out more about the process of determining the line of best fit. When you are ready, click next to continue.

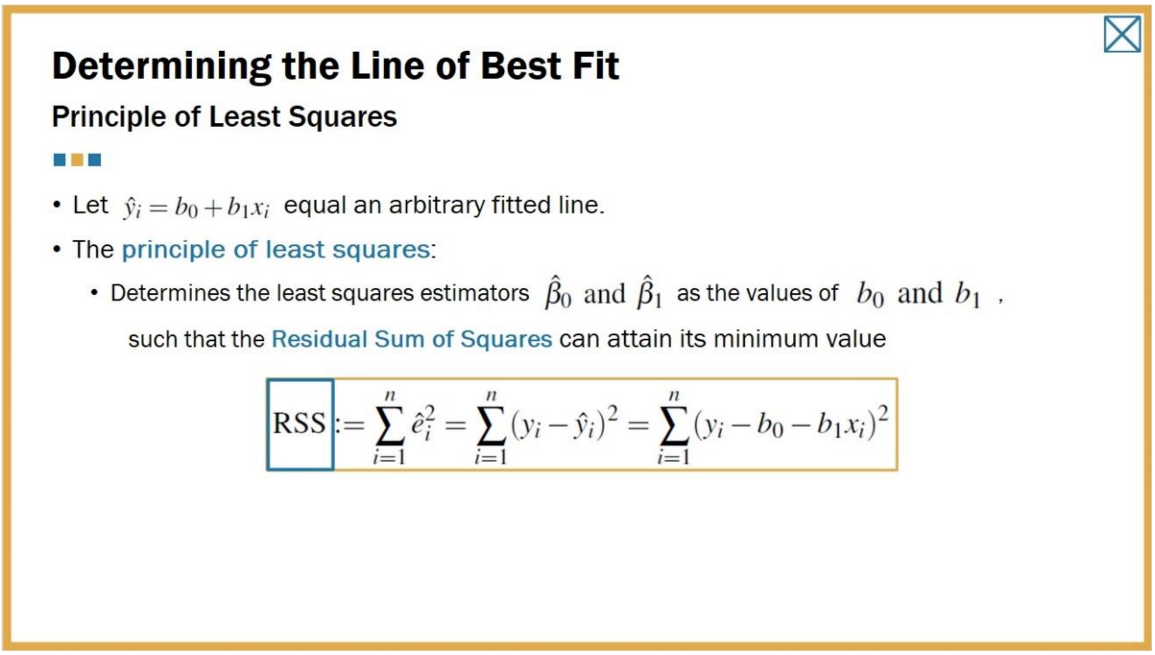
Tab 1: Determining the Line of Best Fit



For determining the line of best fit, we need to define the *residuals*.The residual $\hat{e}_i$ is defined as the difference between the actual and the predicted values of Y.

In a hypothetical example, we draw the line of the best fit (in red) for some data-points (x_i, y_i) and mark the values of the residuals(in green).

Tab 1.1: Principle of Least Squares

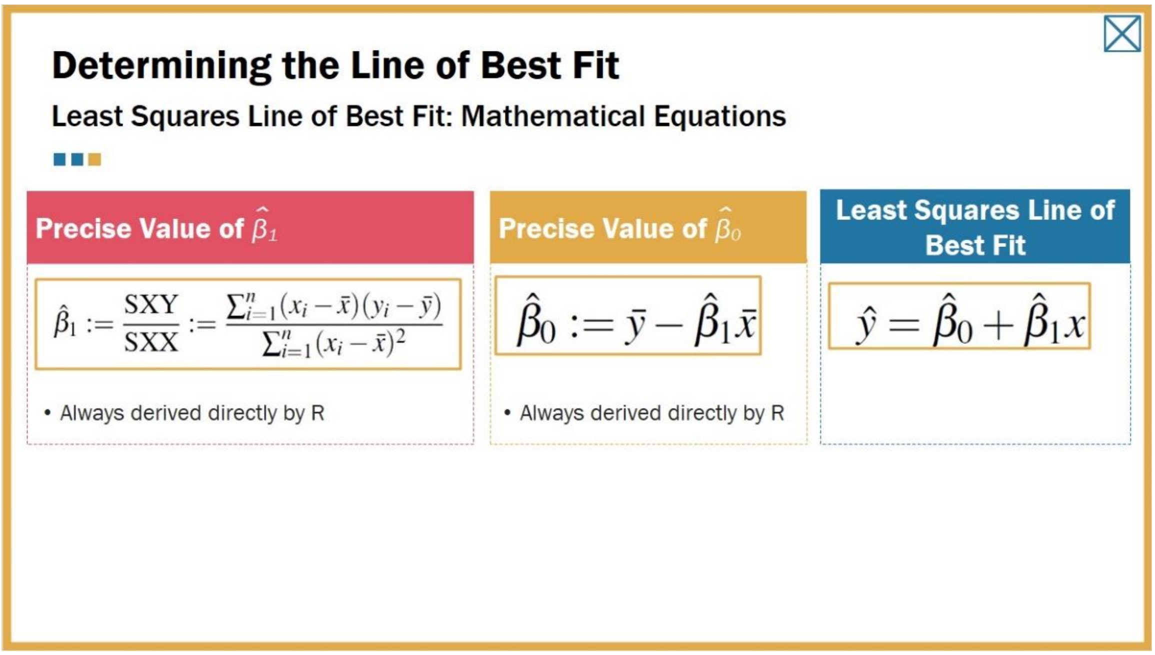


Let $\hat{y}_i = b_0 + b_1 x_i$ an arbitrary fitted line.

The *Principle of Least Squares* determines the *least squares estimators*

$\hat{\beta}_0$ and $\hat{\beta}_1$ as the values of the arbitrary b_0 and b_1 such that the *Residual Sum of Squares* RSS, presented explicitly this equation can attain its minimum value.

Tab 1.2: **Least Squares Line of Best Fit: Mathematical Equations**

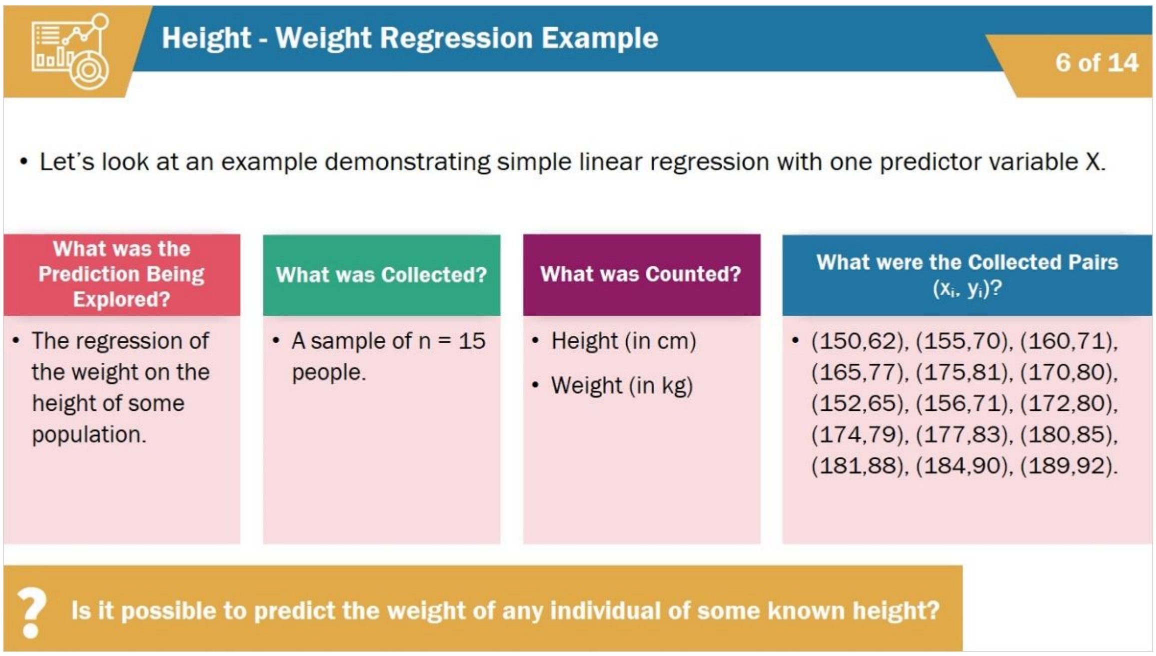


The precise values $\hat{\beta}_1$ and $\hat{\beta}_0$, that minimise RSS, can be found using mathematical techniques and are presented in these equations . We will always derive them directly by R language.

The *least squares line of best fit* takes the form

$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x.$

Slide 6: **Height to Weight Regression Example**



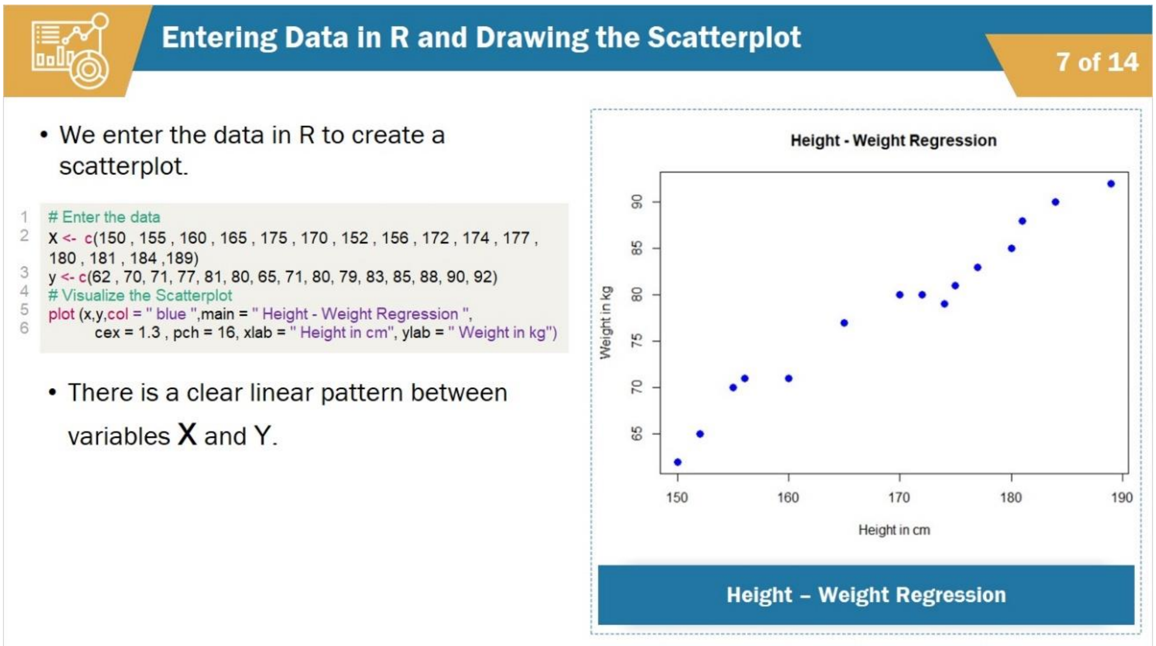
We will now look at an example that demonstrates simple linear regression with one predictor variable X.

Scientists are exploring the regression of the Weight on the Heightof some population. In other words, they want to predict the Weightof any individual of some known Height.

They collected a sample of n = 15 people and they counted their Height (in cm) and Weight (in kilos). The collected pairs (x_i, y_i) are presented here:

If we want to establish whether we can make this weight – height prediction, we need to undertake the regression analysis!

Slide 7: Entering the Data in R and Drawing the Scatterplot



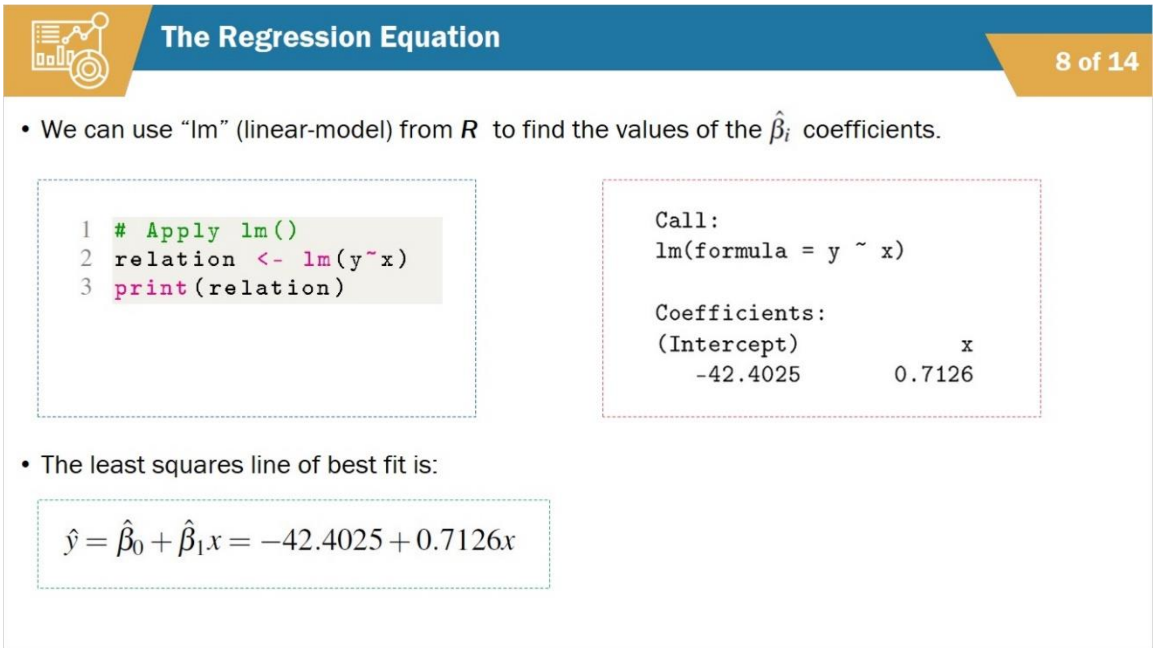
Creating the scatterplot is “Step 1” in regression analysis.

To obtain the scatterplot, we enter the data in R using the following command.

This visual representation of the data may indicate the existence of a linear (or other) relation between the variables.

Here we observe a clear linear pattern between the variables X which is the Height and Y which is the Weight.

Slide 8: The Regression Equation

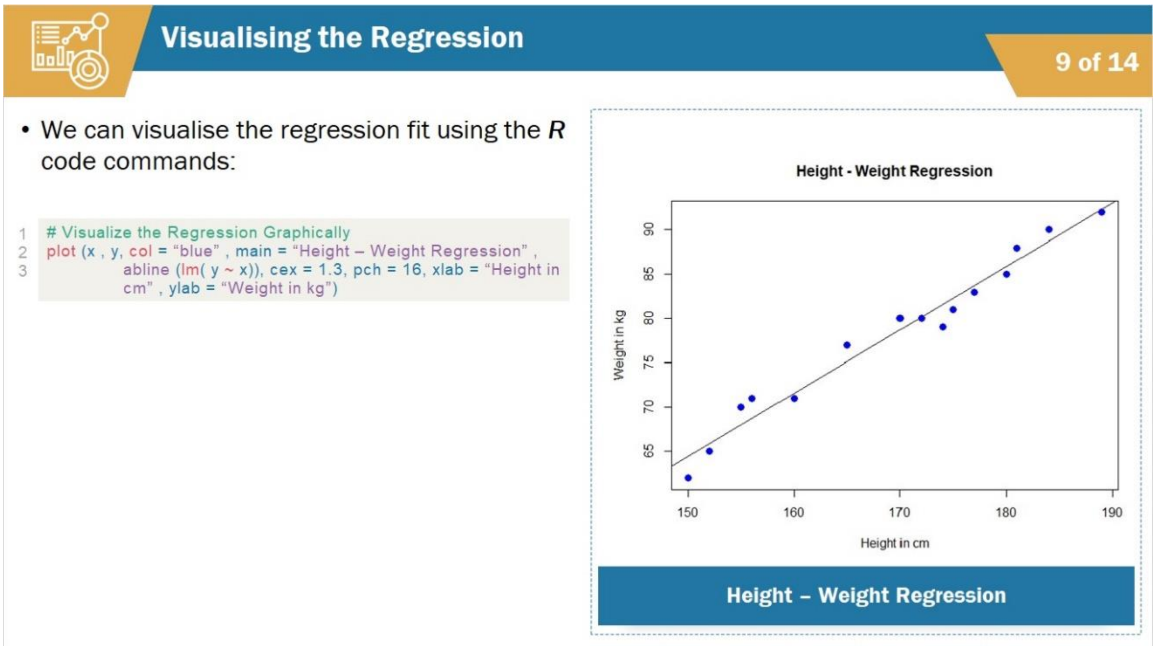


We can use "lm" (linear-model) command from R, as in the frame below, to find the values of the $\hat{\beta}_i$ coefficients.

Here the equation of the least squares line of best fit is:

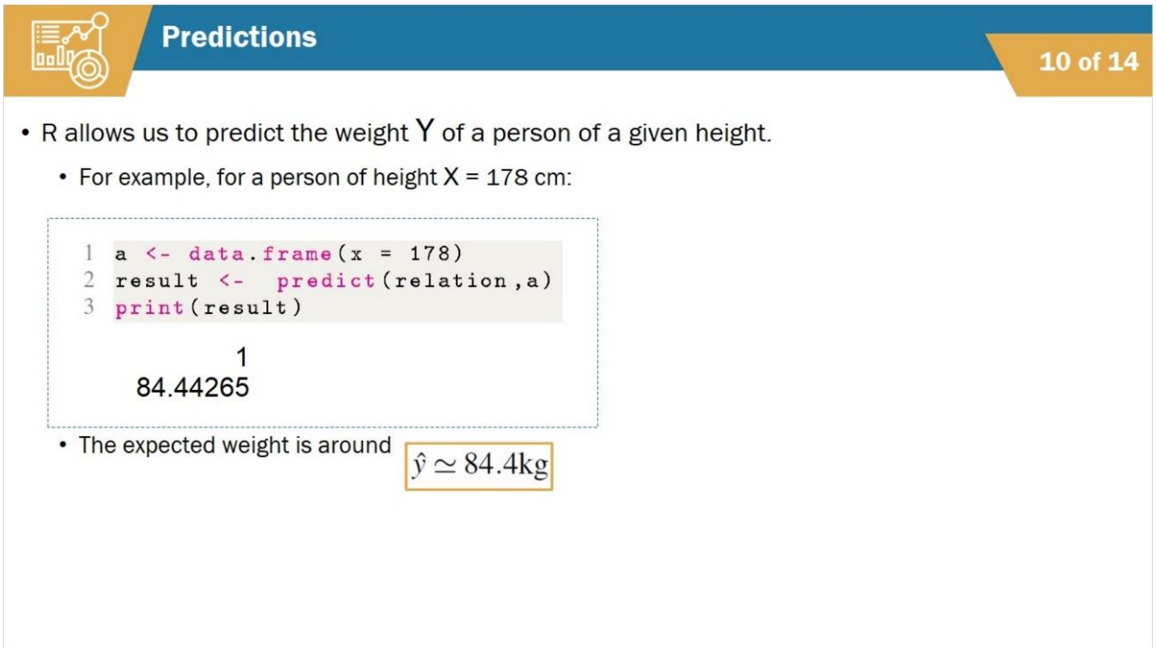
$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x = -42.4025 + 0.7126x.$

Slide 9: Visualising the Regression



We can visualise the regression fit using the commands given below.

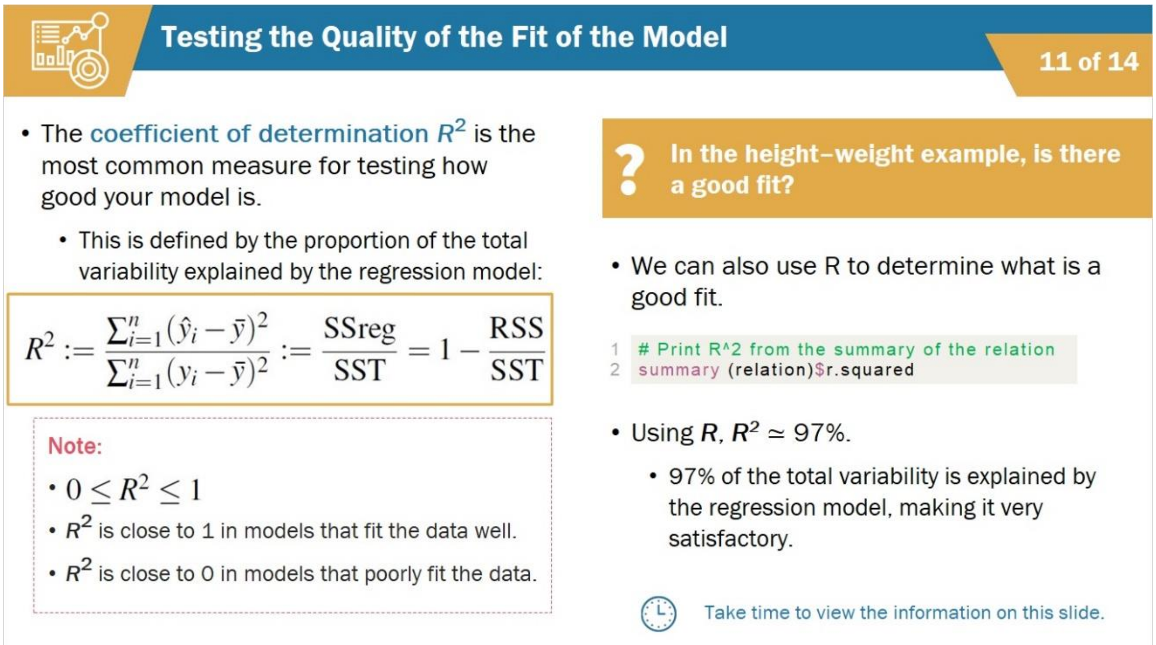
Slide 10: Predictions



R language allows us to further to predict the Weight *Y* of a person of a given Height.

For example, for a person of Height *X* = 178cm, the expected Weight $\hat{y}$ is around 84.4kilos, as we can extract using this command in R.

Slide 11: Testing the Quality of the Fit of the Model



Our next aim is to test the quality of the fit of the model, as this will have an impact on how much we can trust our model.

The most common measure for test how good is your model is the *coefficient of determination* R^2. As you can see from this equation, this is defined by the proportion of the total variability explained by the regression model. Note that

R^2 is a number between 0 and 1.

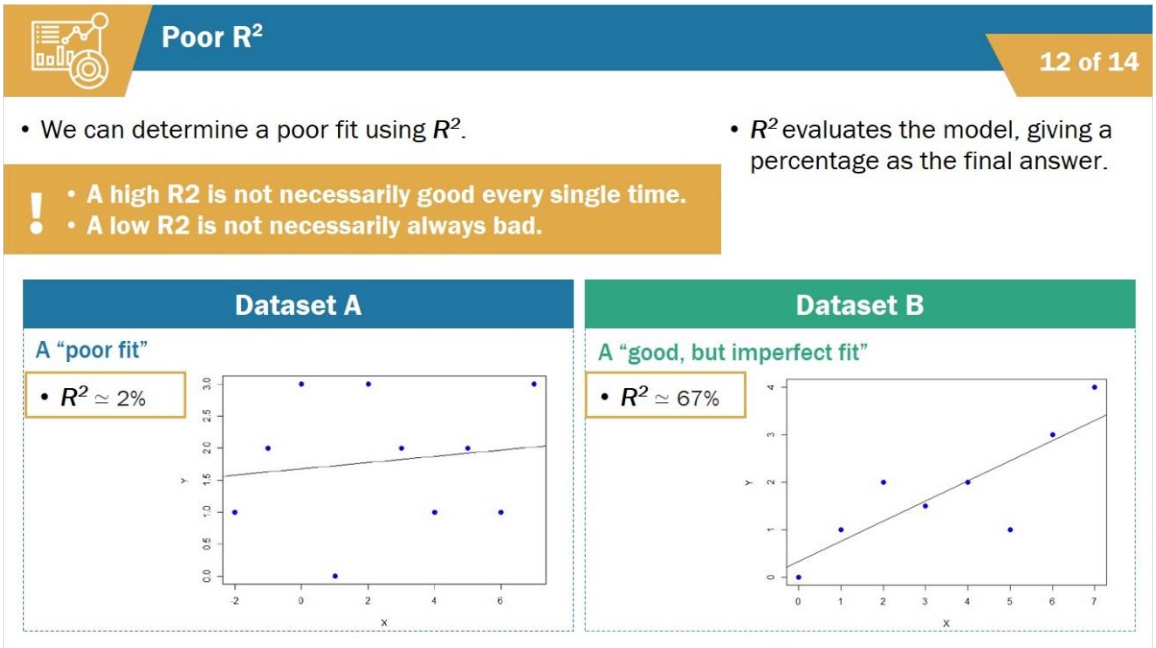
For models that fit the data well, R^2 is close to 1.

For models that poorly fit the data, R^2 is close to 0.

On the Height-Weight example, using R, we find that R^2 is around 97%. This means that around 97% of the total variability is explained by the regression model. Which is very satisfactory!

Take time to view the information on this slide. When you are ready, click next to continue.

Slide 12: **Poor R^2**



Let's take a look at the following dataset A in the graph, which presents $R^2 \simeq 2\%$. This agrees completely with the feeling of a "poorfit" that we all have when we see the scatterplot and the line of bestfit.

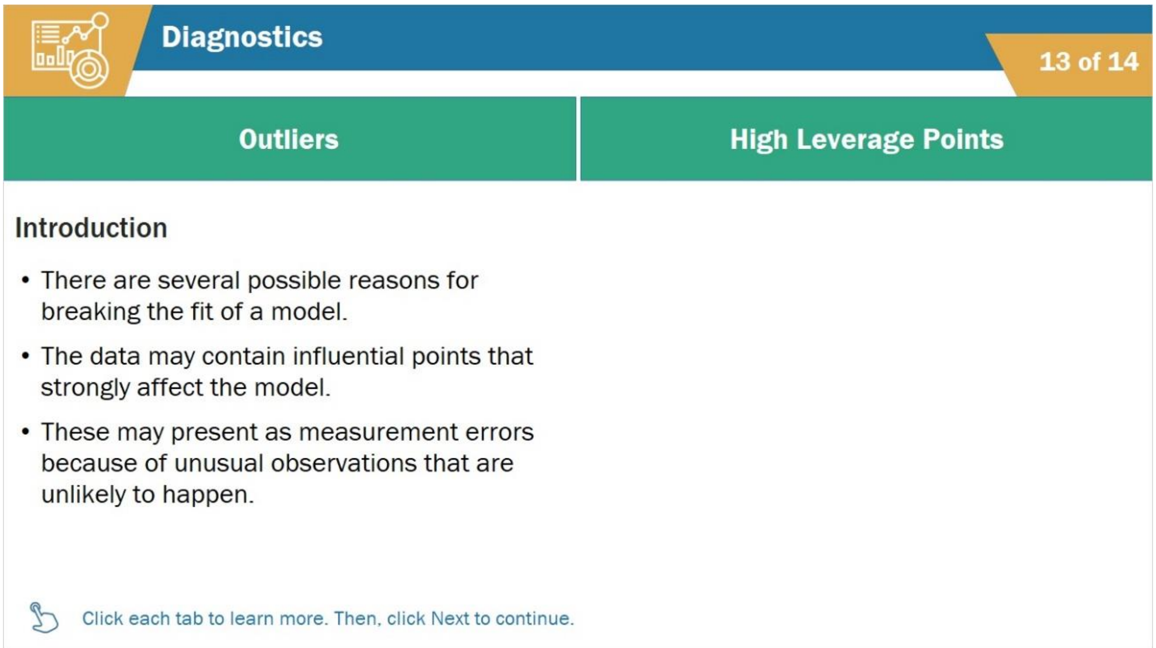
Finally, dataset B with $R^2 \simeq 67\%$ is presented in the graph on the right.

We observe a fit that still looks good, but not perfect. It is clearly better than the example with R^2~2% and worse than the Height-Weight example, where R^2~97%.

R^2 evaluates the model, giving a percentage as final answer.

As an important final remark here, we mention that sometimes a high R^2 is not necessarily good every single time and a low R^2 is not necessarily always bad.

Slide 13: Diagnostics



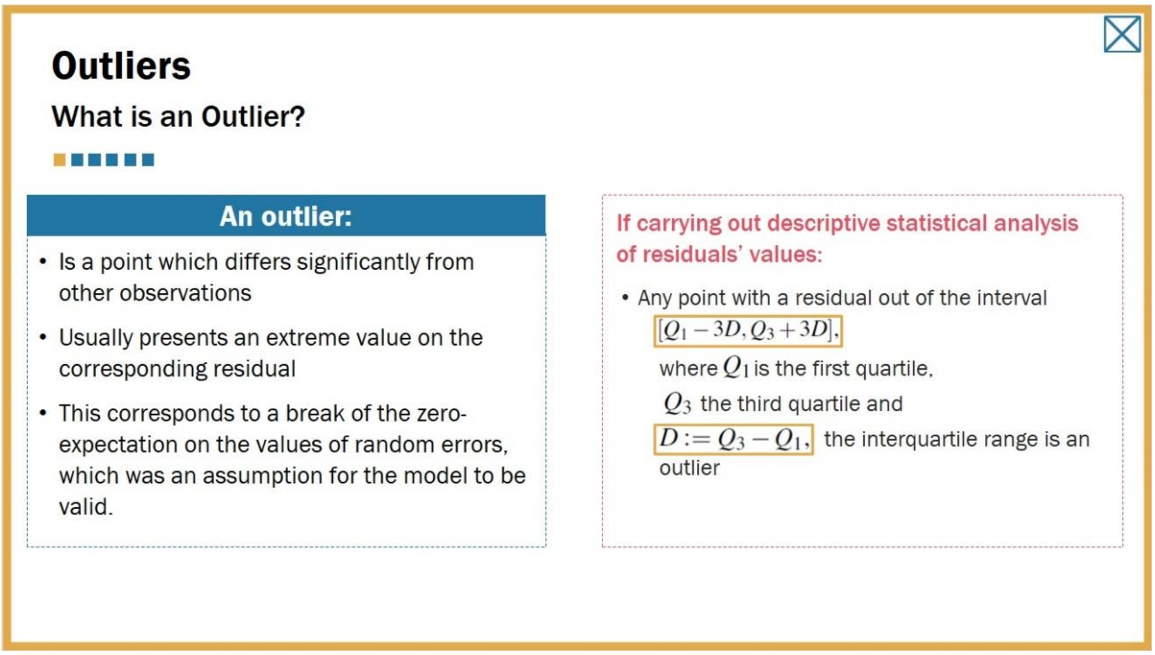
There are several possible reasons for breaking the fit of a model.We proceed to explore them now.

Our data may contain *influential points* that strongly affect the model. Such points may pop up as measurement errorsbecause of very unusual observations that are unlikely to happen.

Let us focus in two of them: the *outliers* and the points of *high leverage*.

Click each tab to learn more. When you are ready, click next to continue.

Tab 1: Outliers



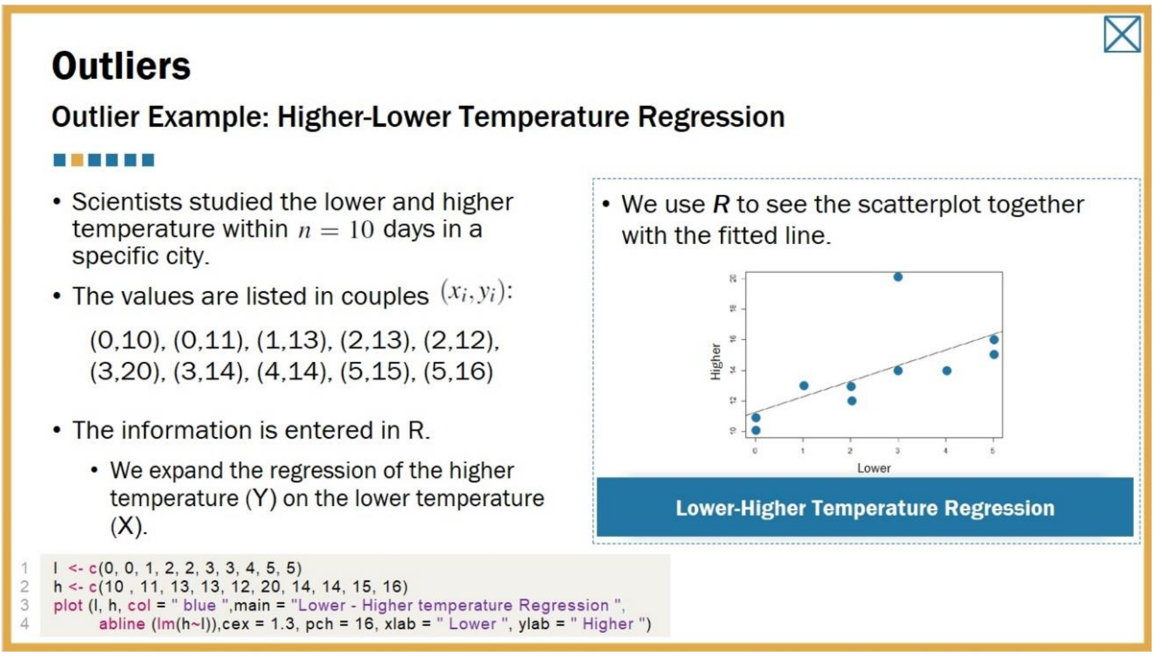
We start with the Outliers.

A point is referred as an *outlier* when it differs significantly from other observations. Such a point usually presents an extreme value on the corresponding residual.

This corresponds to a break of the zero-expectation on the values of random errors, which was an assumption for the model to be valid.

An empirical rule asserts that when we do the descriptive statistical analysis of the residuals' values, then any point with a residual out of the interval $[Q_1 - 3D, Q_3 + 3D]$, where Q_1 is the first quartile, Q_3 is the third quartile and $D := Q_3 - Q_1$, the interquartile range, is an outlier. In different books, slightly different intervals may appear.

Tab 1.1: Outlier Example: Higher-Lower Temperature Regression



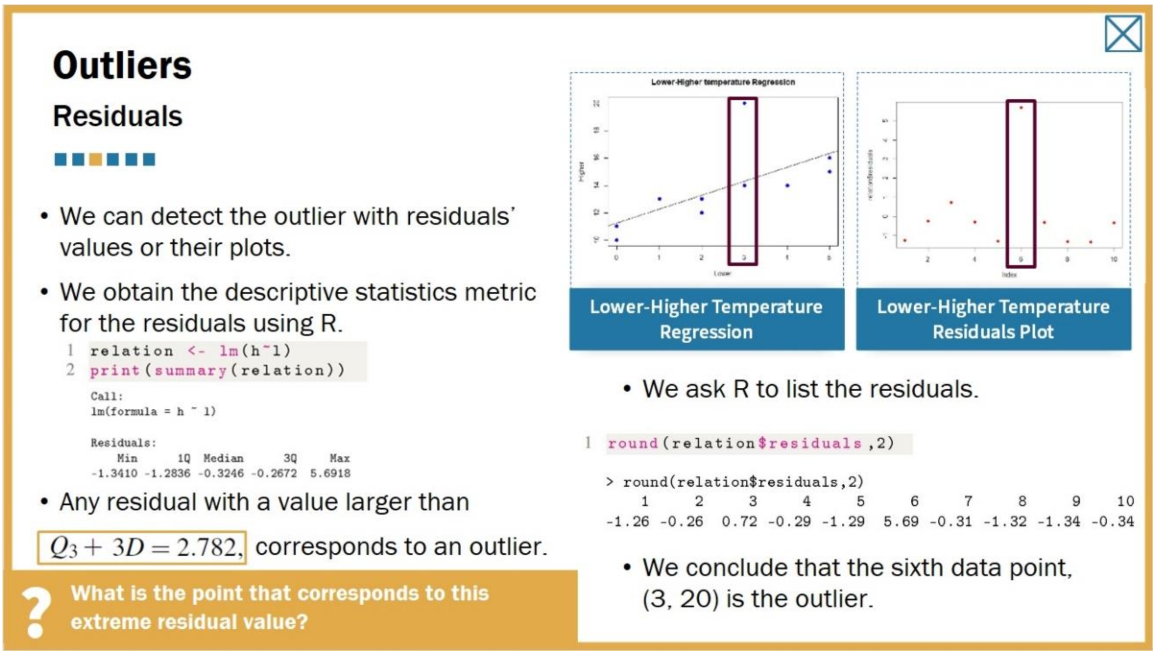
Let us see an example.

Scientists studied the Lower and Higher temperature within $n = 10$ days in a specific City. The values they found are listed as the couples below.

We assist them by entering everything in R and expanding the regression of the Higher temperature (Y) on the Lower temperature (X).

We use R to see the scatterplot together with the fitted line.

Tab 1.2: Residuals



The outlier appears clearly in the graph (for x = 3), but it couldalso be detected with the values of the residuals or their plots.

We obtain the descriptive statistics metric for the residuals using R.

R gives the first and third quartile. Under the empirical rule, any residual with a value larger than $Q_3 + 3D$, which here equals to 2.782, corresponds to an outlier.

Here there exists such a value, with a residual around 5! What is the point that corresponds to this extreme residual value? We ask R to list the residuals and we conclude that the 6th data point; (3, 20) is the outlier.

The outlier corresponding point appears clearly in the plot of the residuals too.

Tab 1.3: Is There Any Treatment for Outliers?

Outliers

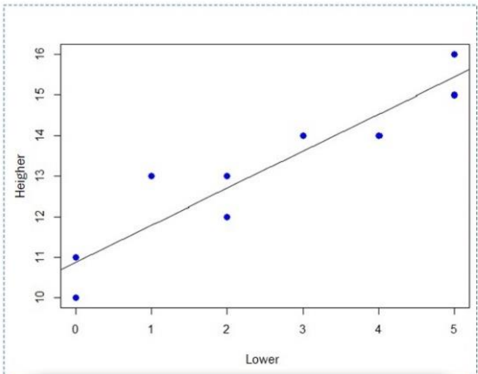
Is There Any Treatment for Outliers?

? How do we deal with outliers?

- We can carefully exclude it from the dataset and re-analyse the restricted data.

```
l <- c(0, 0, 1, 2, 2, 3, 4, 5, 5)
h <- c(10 , 11, 13, 13, 12, 14, 14, 15, 16)
plot (l, h, col = " blue ",main = "Lower - Higher temperature Regression ",
      abline (lm(h~l)), cex = 1.3 , pch = 16, xlab = " Lower ", ylab = " Heigher ")
```

- We remove the point and re-apply the usual commands in *R*.
  - We now have a nice fit.



Lower-Higher Temperature Regression

Is there any recommended treatment for dealing with outliers?

When an outlier is identified, we may (carefully) exclude it from thedataset and re-analyse, as seen here. We remove the point and reapply the usual commands in R.

Now we have a nice fit.

Tab 1.4: The New Fit: Regression Equation and R^2

Outliers

The New Fit: Regression Equation and R^2

```
relation <- lm(h~l)
print ( relation )
```

```
Coefficients:
(Intercept)            1
    10.8824       0.9118
```

- The new data, excluding the outlier, is used to obtain the regression equation:

Higher = 10.88824 + 0.9118 Lower

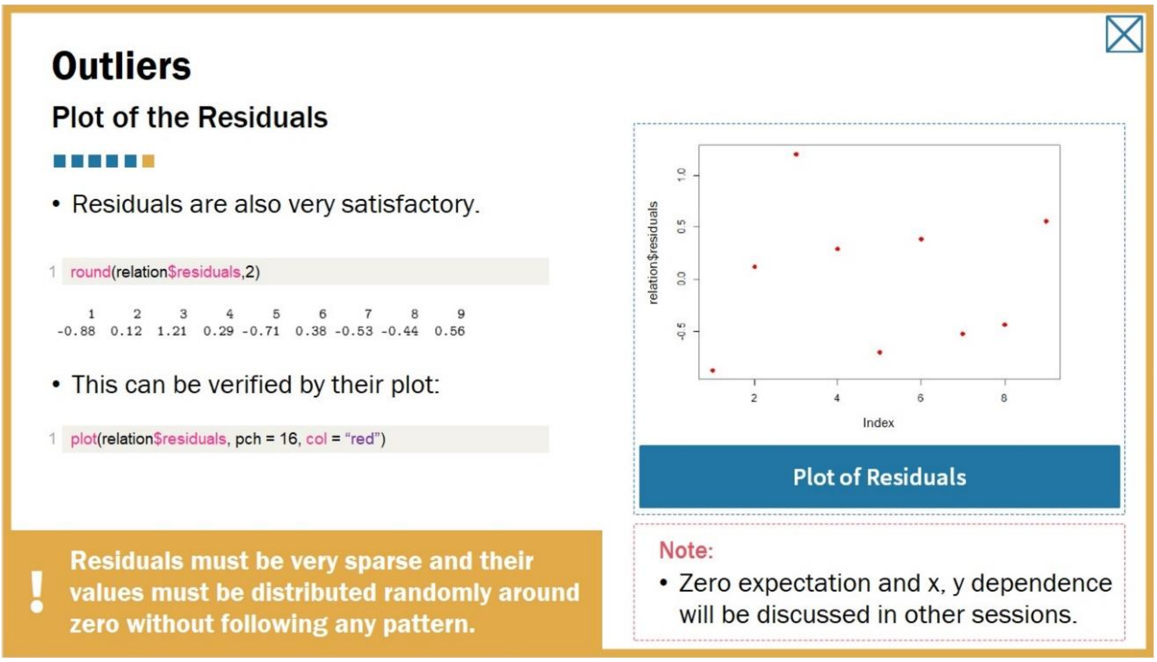
- R^2 is around 87% which is very satisfactory.

```
summary ( relation )$r. squared
```

[1] 0.8696833

We refit and the regression equation is obtained by the new data that excludes the outlier as in this first equation. We can see that R^2 is around 87% which is very satisfactory.

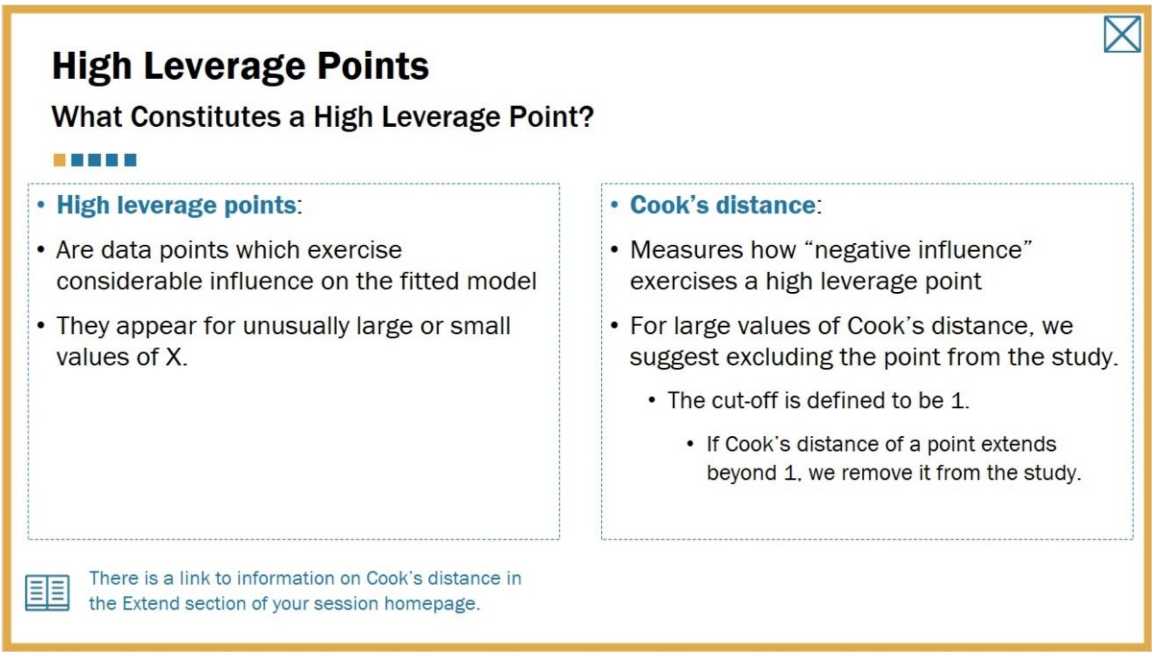
Tab 1.5: Plot of the Residuals



The same holds for the residuals, as it can be observed by their values or their plot.

Note that the residuals must be very sparse, and their values must be distributed randomly around zero without following any pattern (as it happens above). We will discuss the zero expectation and x, y dependence in more detail in the next sessions.

Tab 2: High Leverage Points



Another reason for breaking the fit of a model may be because of high leverage points.

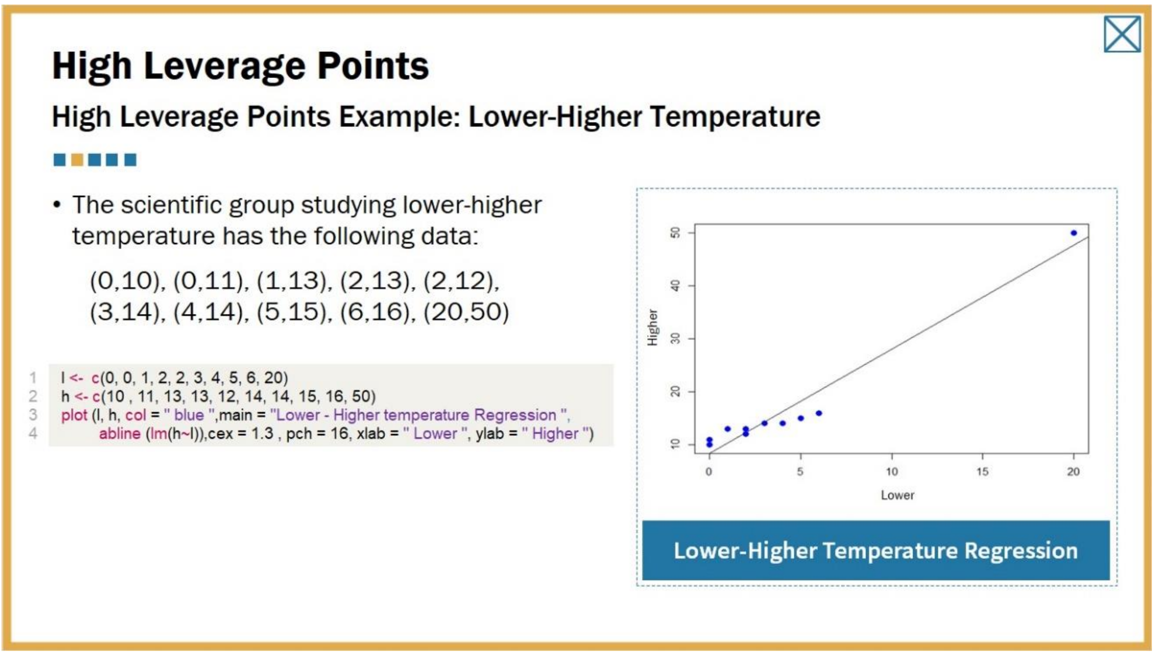
Data points which exercise considerable influence on the fitted model are called *high leverage points*.

Such points appear for unusually large or small values of x. Cook's distance measures how "negative influence" exercises a high leverage point.

There is a link to more information on Cook's distance in the Extend section of your session homepage.

For large values of Cook's distance, it is suggested to exclude the point from the study. The cut-off is defined to be 1. If Cook's distance of a point extends 1, we remove it from the study.

Tab 2.1: Example: Lower-Higher Temperature

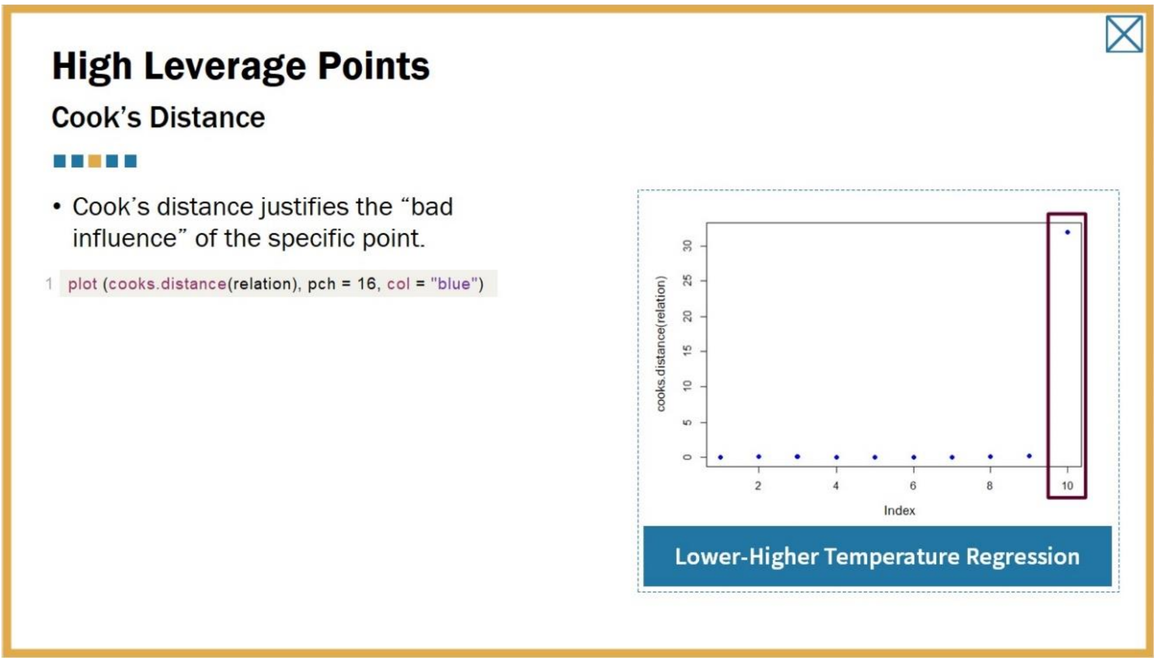


Let's take a look at an example.

Assume that the scientific group which studies the Lower-Higher temperature problem, has the following data.

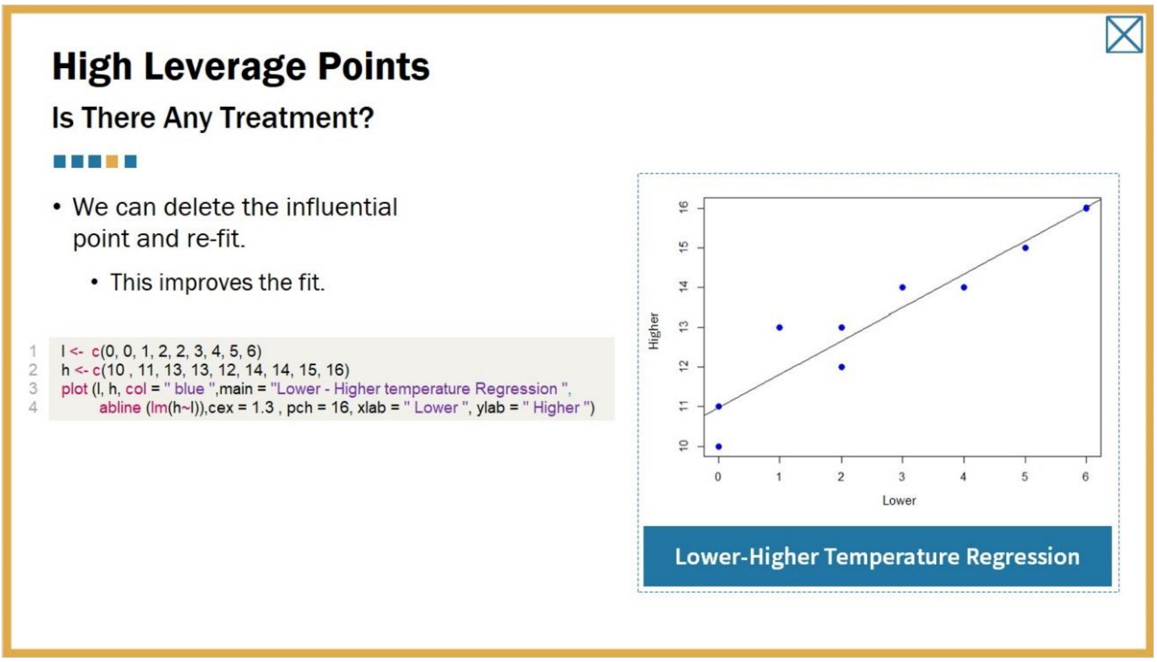
We draw the least squares line (with the usual commands in R) andwe observe that it is "dictated" by the point with $x = 20$.

Tab 2.2: Cook's Distance



Cook's distance justifies the "bad influence" of the specific point, as it can be observed by the graph obtained by R, since the value of Cook's distance is huge in this point.

Tab 2.3: Is There Any Treatment?

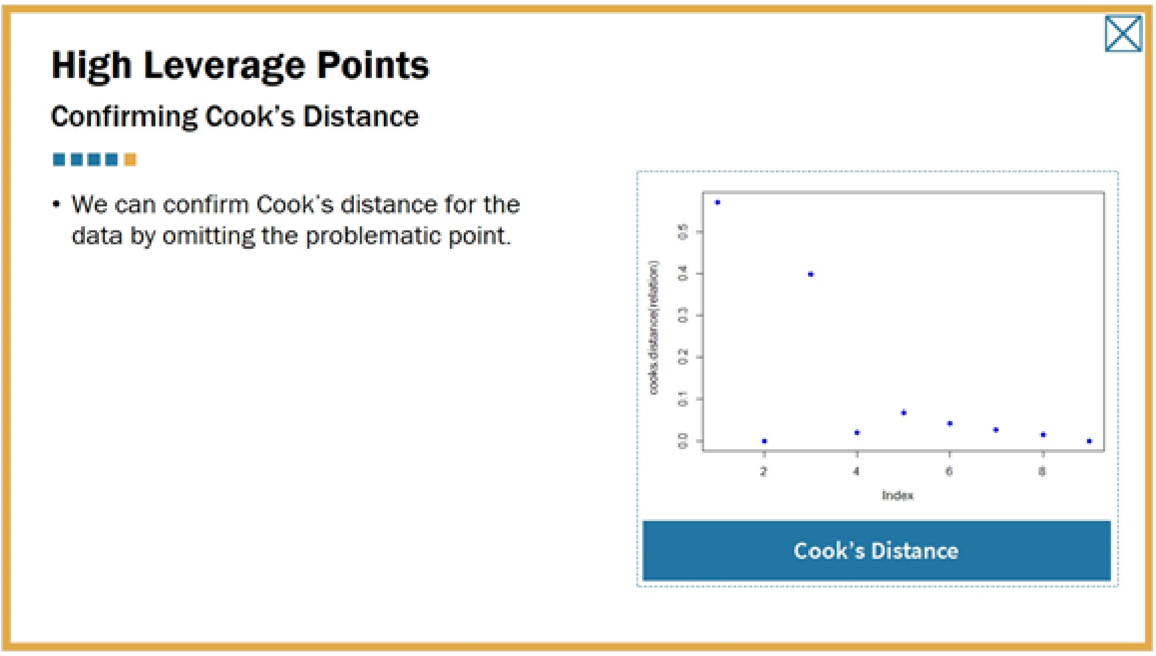


Is there any treatment for this high leverage point?

We delete the influential point from the dataset and re-fit:

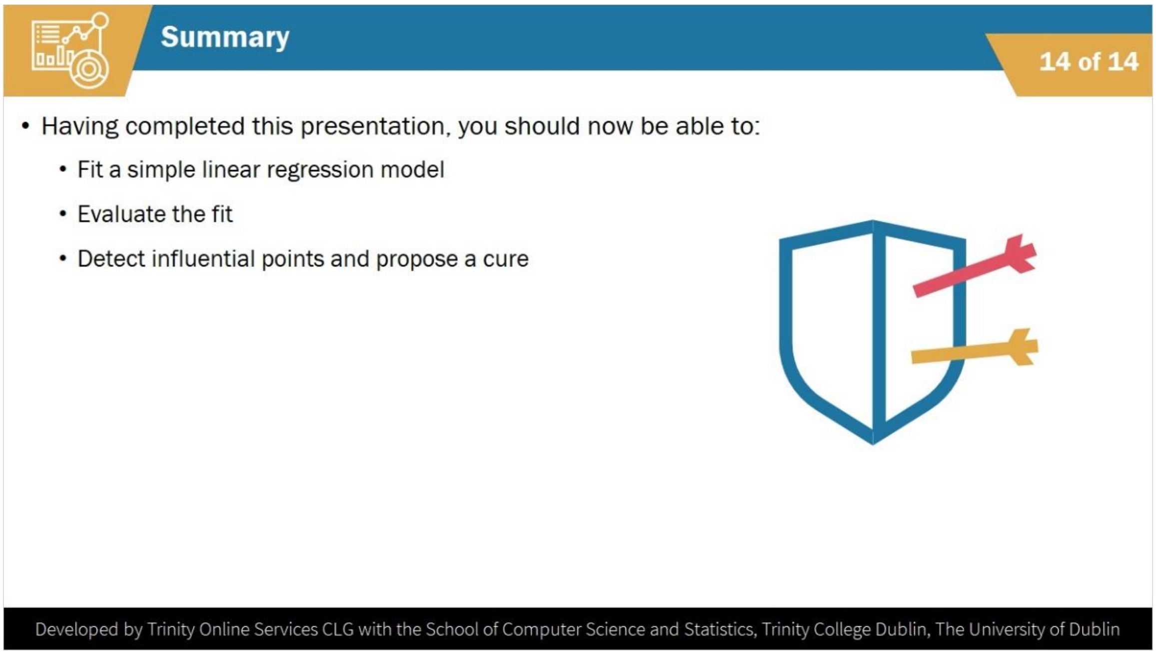
It is far better now!

Tab 2.4: **We Confirm Cook's Distance**



In addition, we can also confirm Cook's distance for the data by omitting the problematic point.

Slide 14: **Summary**



Having completed this presentation, you should now be able to:

- Fit a simple linear regression model,
- Evaluate the fit, and
- Detect influential points and propose a cure.