Calculus Notes

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A way to describe functions. To analyse relationships.

Derivative

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' is the set of points in the domain of f for which the limit exists.

What does it mean for a function to be differentiable?

If f' exists at a particular x, we say that f **is differentiable** (has a derivative) at x. If f' exists at every point in the domain of f, we call f differentiable.

Continuity

A function $f(x)$ is **continuous at a point** $x = c$ iff it meets the following three conditions [1].

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim x \to cf(x)$ exists (f has a limit as x tends towards c).
3. $\lim x \to cf(x) = f(c)$ (the limit equals the function value).

[1] We define a continuous function as one that is continuous at every point in its domain.

Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x then the composite function [2] $(f \circ g)(x) = f(g(x))$ is differentiable at x and

[2] Composition occurs when the output of one function is used as the input to another function.

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

An implicit curve arises when two values are related to eachother in a non-standard geometrical form, not a singular independent and dependent plot.

Derivative of Inverse functions

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If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point.

Related rates problem example

The problem of finding a rate of change of a variable from other known rates of change is called a related rates problem.

- We know the volume of the cone as $V = \frac{1}{3}\pi r^2 h$.

  The problem is to find $\frac{dh}{dt}$.

- We **assume that each function r, h and V are differentiable at a given t**. [3]

The goal is to **not keep track of multiple variables** so we need a relationship between h and r to stop tracking about r. This relationship turns out to be $r = \frac{h}{2}$.

[3] You might think that we could just do the product rule on r^2h but this implies that r and h are known functions. You actually get

$$r^2\frac{dh}{dt} + h\frac{dr}{dt}$$

when you do this, which doesn't help matters.

Linearization

If f is differentiable at $x = a$, then $L(x)$ is an approximating function called the **linearization** of f at a defined as:

$$L(x) = f(a) + f'(a)(x - a)$$

A quadratic approximation adds more accuracy by determining the slope of the slope of the initially linear approximation.

$$f(x) = f(a) + f(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

Differentials

Let $y = f(x)$ be a differentiable function, dx be an independent variable, the differential dy is

y is always the dependent variable. In this case it depends on dx and x.

Geometrically, this is a linear approximation of f around a.

$$dy = f'(x)dx$$

Determining the shape of a function

Let f be a function with domain D. f has an **absolute max value** on the domain D at point c if

$$f(x) \leq f(c)$$

and an **absolute min value** on D at point c if

$$f(x) \geq f(c)$$

Local extreme values can be found by setting the derivative to zero.

A **Criticial point** An interior point of the domain of a function f where f' is zero or undefined.

The absolute extrema for f are the largest and smallest of these critical points for the function f.

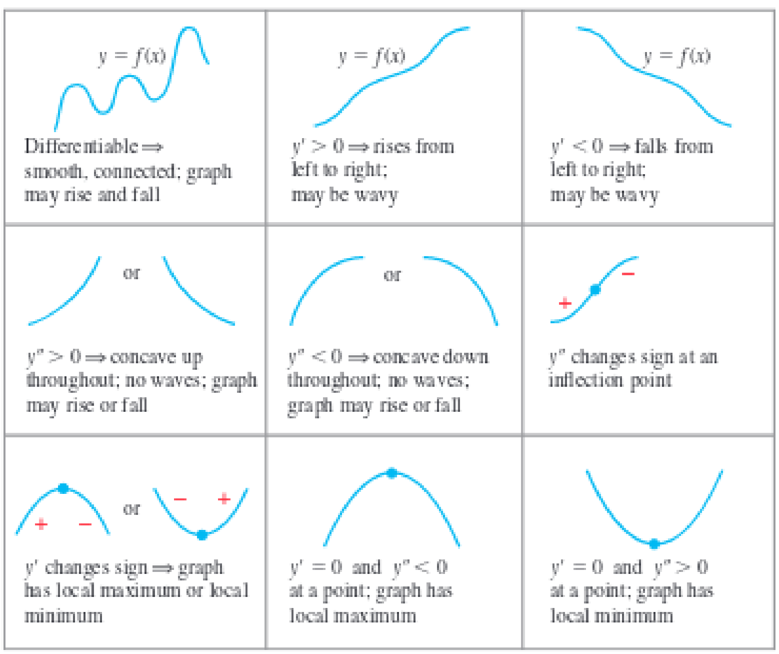


Figure 1: We test if a function is concave up/down by getting its second derivative. If this is greater than zero over an interval then its concave up.

Inverse functions

Useful when flipping axis

Even and Odd functions

A function is **even** if $f(x) = f(-x)$ for every $x \in X$ its domain e.g. x^2.

A function is **odd** if $f(-x) = -f(x)$ for every $x \in X$ its domain e.g. x^3.

A polynomial with an even exponent is an even function. One with an odd exponent is odd.

The product of an even and odd function is odd.

Mean Value Theorem (weak point)

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

This is just the slope of the secant line joining a to b.

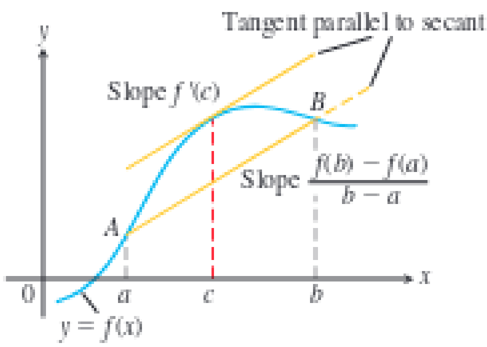


FIGURE 4.13 Geometrically, the Mean Value Theorem says that somewhere between a and b the curve has at least one tangent parallel to the secant joining A and B.

If f is continuous on $[a, b]$, then at some point c in $[a, b]$:

$$f(c) = \frac{1}{b - a} \int_a^b f(x)dx$$

Especially because its continuous, this makes sense. At some point your function must take the average value.

There are 3 corrolaries to the mean value theorem:

1. **Only constant functions have zero derivates**. If $f'(c)$ is zero for all x in the interval $[a, b]$ then $f(b)$ is equal to $f(a)$.
2. **Matching curvatures** if $f'(x) = g'(x)$ for all $x \in [a, b]$ then there exists a constant C for all $x \in [a, b]$. $f - g$ is a constant function on the interval.

A function that is increasing/decreasing on an interval is said to be **monotonic** on that interval [4].

[4] If $f'(x) > 0$ at each point in the interval $[a, b]$ where x is in this interval then f is increasing over that interval.

Newtons method (weak point)

Use tangent lines of the graph $f(x)$ to approximate a solution for $f(x) = 0$.

You're trying to guess the root of a function. Each new 'guess' should bring you closer to the root.

If we look at the linearization of a line at point x_n,

$$y = f(x_n) + f'(x_n)(x - x_n)$$

.

Provided $f'(x_n)$ is not equal to zero, rearranging gets us

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is finding the intercept of the tangent line to a point $(x_n, f(x_n))$.

Antiderivative

A function F is an antiderivative of f on an interval if $F'(x) = f(x)$ where $x \in I$.

The process of recovering the function F from its derivative is called *antidifferentiation.*

Finding an antiderivative for a function $f(x)$ is the same problem as finding a solution to the equation

$$\frac{dy}{dx} = f(x)$$

This is a differential equation because we have an **unknown y that is being differentiated.** In this case, y is the antiderivative of f.

Indefinite integrals

The **collection of all antiderivatives of a function** f is called the indefinite integral with respect to x denoted

the function f is called the integrand and dx the variable of integration.

$$\int f(x)dx$$

If $P = (x_0, x_1, ..., x_n)$ is a partition[5] of some interval $[a, b]$ A

[5] $a = x_0 \leq x_1 \leq x_2 \leq ... \leq x_n = b$.

Riemann sum S of f over $[a, b]$ with partition P is defined as

$$S = \sum_{i=1}^{n} f(c)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and $c \in [x_i - x_{i-1}]$.

Definite Integral

If J is the limit of the Riemann sum for f. The definite integral is

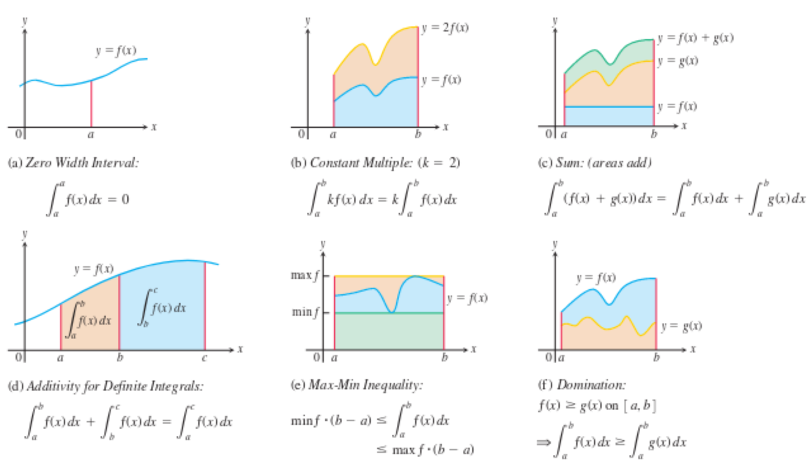
$$J = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k$$

where $||P||$ is the largest of all the sub interval widths. We rewrite J as

$$\int_a^b f(x)dx$$

What makes a function integrable?

If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral exists. f is integrable over $[a, b]$.



Mean of integrable function

The average value of the integral of a function f on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Fundamental Theorem of Calculus (weak enough)

On one hand, we have a way to determine the slope of a tangent line at a point in the derivative. On the other hand, we've a way to determine the area under a line (or between it and the x-axis) with the definite integral. The fundamental theorem relates these two concepts.

THINK ABOUT a function $g(x)$ that we define as the definite integral from a to x for some x:

$$g(x) = \int_a^x f(t)dt$$

t is a dummy variable to calculate the integral (so maybe the partition in the Riemann sum).

What is $g'(x)$?

$$g'(x) = \lim_{h \to 0} \frac{g(x+h)f(x)}{h}$$
$$g'(x) = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

Analysing the numerator geometrically, it can be approximated as $f(x)h$ as h gets smaller and smaller. Therefore

$$g'(x) = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} = f(x)$$

If f is continuous on $[a, b]$, then we define a function $g(x) = \int_a^x f(t)dt$ that is also continuous on $[a, b]$ and differentiable on $[a, b]$. Its derivative is $f(x)$ [6]

[6] Can think of the function $f(x)$ as implicit something we're 'circling' around. Jumping between derivatives and integrals to find it.

$$g'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

$g'(x) = f(x)$ therefore $g(x)$ is an antiderivative of $f(x)$.

Part 2

The following theorem allows us to solve definite integrals more easily than computing Riemann sums.

If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Net Change Theorem

$$F(b) - F(a) = \int_a^b F'(x)$$

Substitution method (subtler than you think)

Evaluating

$$\int f(g(x))g'(x)dx$$

That is, we assume that f is some product of using the chain rule.

1. Substitute $u = g(x)$ and $du = \frac{du}{dx}dx = g'(x)dx$ to obtain $\int f(u)du$.
2. Integrate with respect to u.
3. Replace u by g(x).

Chapter 6 (Applications of Integration)

> The solid generated by rotating (or revolving) a plane region about an axis in its plane is called a solid of revolution.

THE GOAL is to define $A(x)$ as some **integrable cross sectional area of the function**.

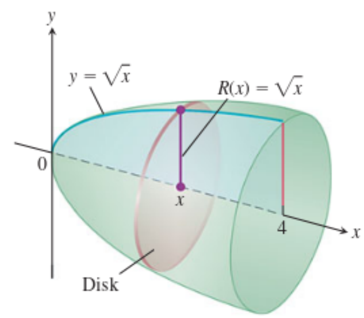


Figure 2: If we revolve the curve $y = \sqrt{x}, 0 \leq x \leq 4$ around the x-axis its volume would be the sum of disks define as

$$V = \int_a^b \pi(\sqrt{x})^2$$

The Disk Method

If we're rotating around the x-axis, we can use the volume by disks approach. The Area function is defined as the area of a disk. This is dependent on the radius at a given x.

$$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)]^2$$

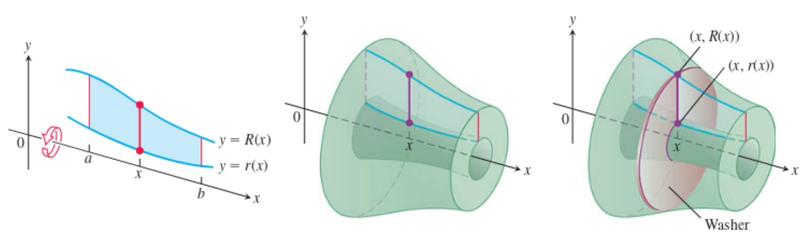


Figure 3: **Washer Method** If the region to be revolved around an axis does not border that axis then there is a volume of space between it and the axis of revolution to be accounted for.

$$V = \int_a^b A(x)dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)$$

The Shell method

The volume of the solid generated by revolving the region between the x-axis and the graph of $y = f(x) \geq 0, L \leq a \leq x \leq b$ about x = L is[7]

$$V = \int_a^b 2\pi(x - L)f(x)dx$$

[7] Where $x - L$ is the shell radius and $f(x)$ the shell height.

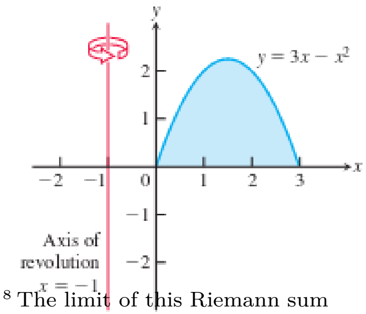


Figure 4: The Washer method fails in the case of an offset parabola. Calculating the inner and outer radius is non-trivial: $x^2 = 3x - y$.

In figure 4 we take a partition P of $[a, b]$ and attempt to revolve this partition about the line L. We define a cylindrical volume for the kth partition

$$\Delta V_k = 2\pi(1 + x_k)(2x_k - x_k^2)\Delta x_k$$

We approximate the volume of the whole region rotated by summing these smaller volumes.[8]

[8] The limit of this Riemann sum becomes the definite integral.

$$V \approx \sum_{k=1}^{n} \Delta V_k$$

Integration by parts

Simplifying integrals of the from

$$\int f(x)g(x)dx$$

Integration by parts says that

$$\int udv = uv - \int vdu$$

Where $u = f(x)$ and v $= g(x)$.

For definite integrals the formula is

$$\int_a^b f(x)g'(x) = f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx$$

Evaluating Integrals

Look out for symmetric intervals. Then determining whether the resulting integrand is even or odd can simplify evaluation.

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad \text{(any number } k)$
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \qquad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \qquad (x > a > 0)$