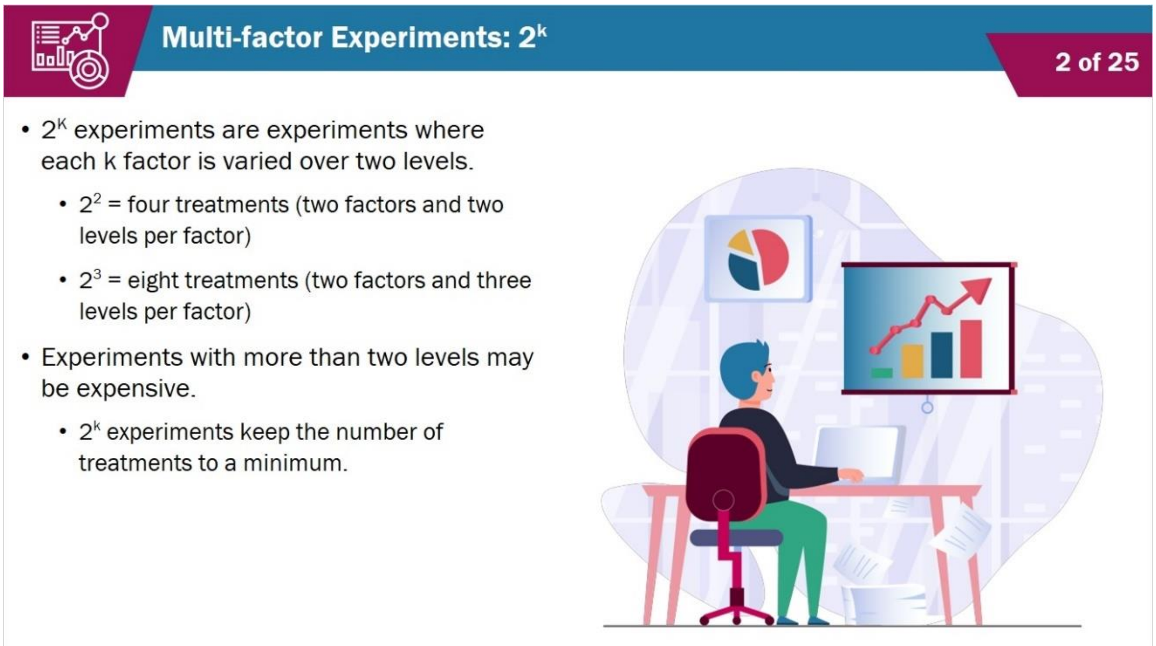
Two-level Factors

Slide 1: ## Introduction

Hello and welcome. My name is James Ng, and I will lead you through this presentation on two-level factors, focusing on the basic principles of the design and analysis of 2^k experiments, the dual role of the design matrix and how to apply various methods to perform an analysis of 2^k experiments with no replicates.

Slide 2: ## Multi-factor Experiments: 2^k



In this session, the focus is on multi-factor experiments where each factor is varied over just two levels, referred to as 2^k experiments, denoting experiments with k factors each with 2 levels.

Note that the number of distinct treatments, that is, factor level combinations, in such experiments is 2^k. Thus, with 2 factors and 2 levels per factor, there are $2^2 = 4$ treatments, with 3 such factors there are $2^3 = 8$ treatments, etc. This provides one rationale for using such experiments, namely, keeping the number of treatments to a minimum. For example, with just 3 factors,

with 3 levels each there are $3^3 = 27$ possible treatments,

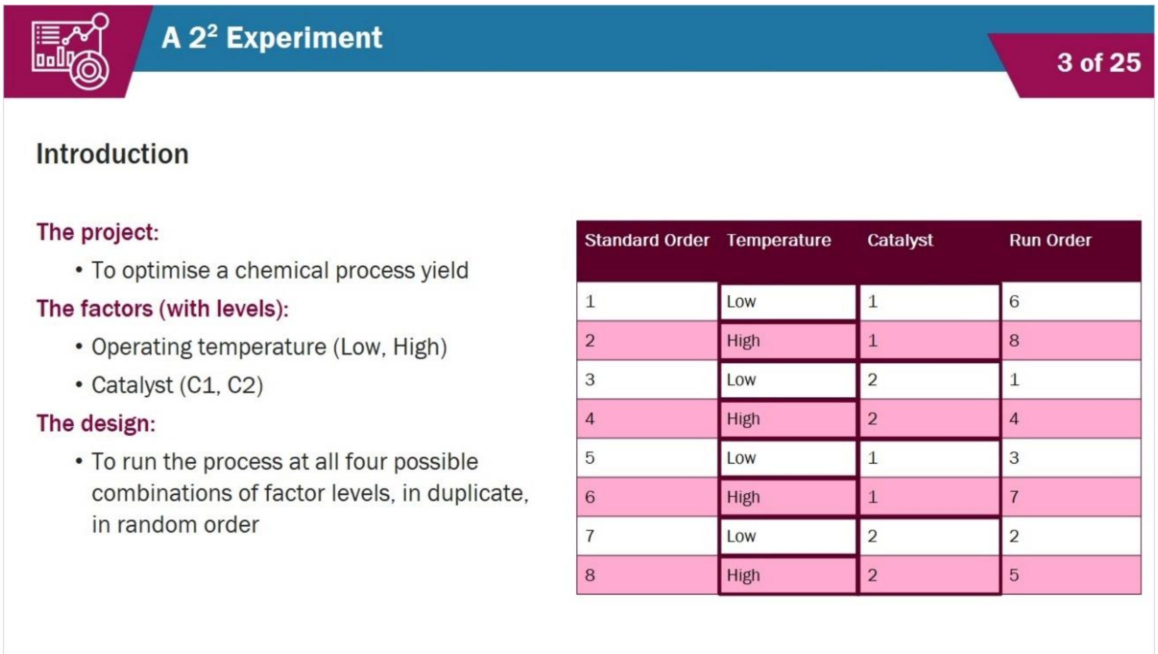
with 4 levels each there are $4^3 = 64$ treatments,

with 5 levels each there are $5^3 = 125$ treatments,

with 3, 4 and 5 levels, respectively, there are 3x4x5 = 60 treatments.

Where experiments are expensive and / or resources available for experiments are scarce, such numbers may be prohibitive.

Slide 3: A 2^2 Experiment



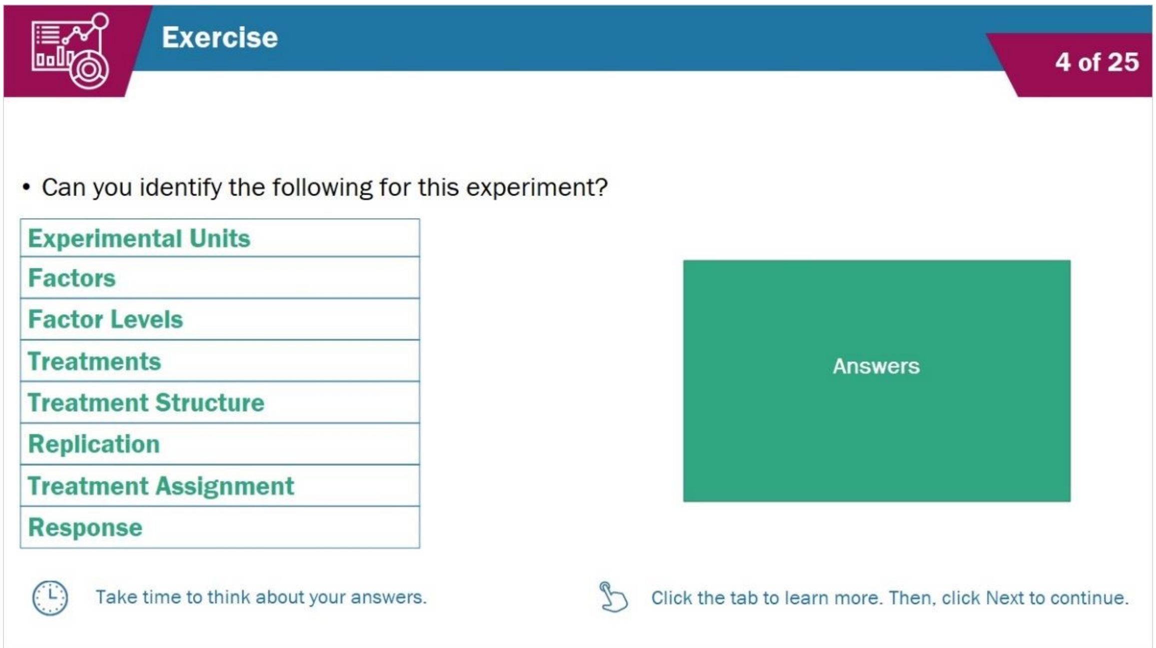
Standard Order	Temperature	Catalyst	Run Order
1	Low	1	6
2	High	1	8
3	Low	2	1
4	High	2	4
5	Low	1	3
6	High	1	7
7	Low	2	2
8	High	2	5

The project is concerned with the optimisation of a chemical process yield. Two factors are involved in this study, temperature with levels Low and High, and catalyst with levels C1 and C2. Following a review, it is decided to use a 2-factor design in which the process is run at all four possible combinations of temperature and catalyst. To allow for the estimation of random error / chance variation, the full set of four combinations was run in duplicate, giving eight runs in total.

The run settings may be listed as in the table. In the list, "Standard Order" means that the levels in the leftmost factor (Temperature) vary fastest, changing from row to row, while those in the rightmost factor (Catalyst) vary slowest, changing every second row. More specifically, the Temperature column in the table has the pattern “low, high, low, high” changing every row whereas the Catalyst column has the pattern “1, 1, 2, 2” change every 2 rows.

To minimise the chances that an unknown factor varying from run to run might bias the results, the order in which the various combinations were run was randomised.

Slide 4: Exercise



Can you identify the following for this experiment?

Experimental units

Factors

Factor Levels

Treatments

Treatment Structure

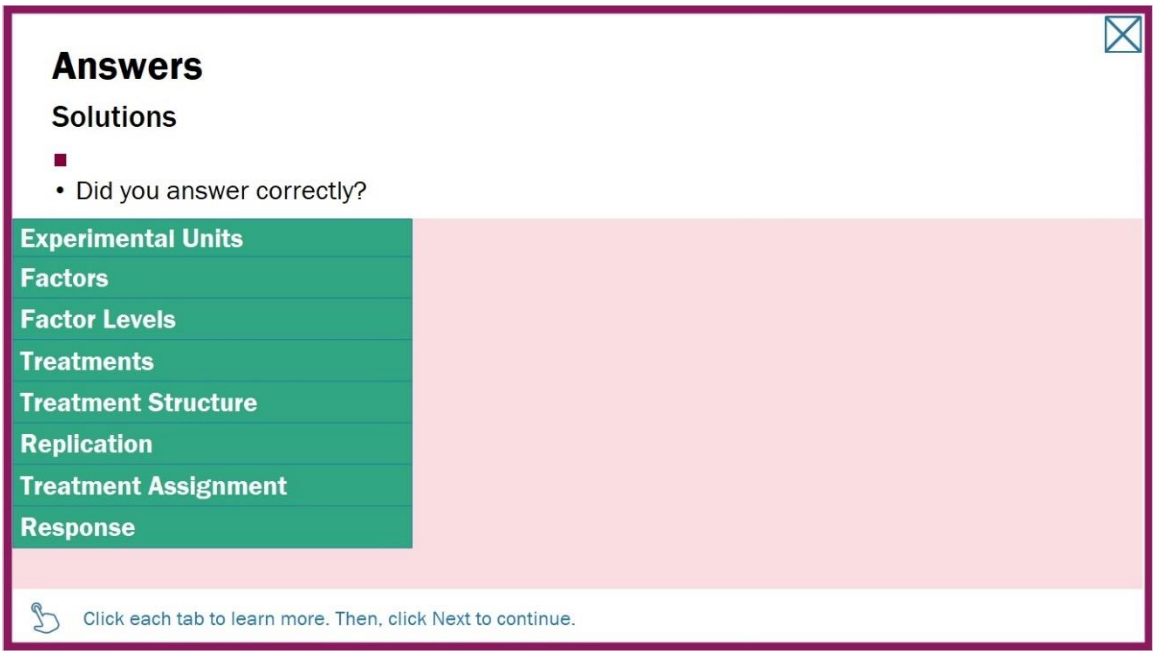
Replication

Treatment Assignment

Response

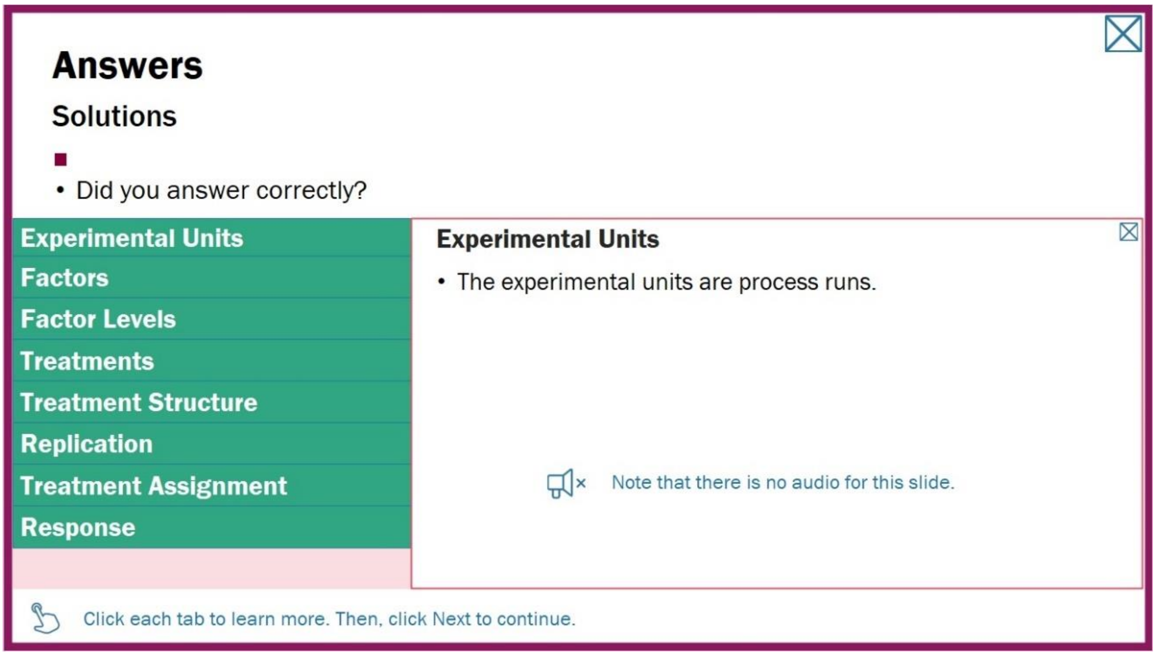
Take time to think about your responses, then click the tab to check your answers.

Tab 1: Answers



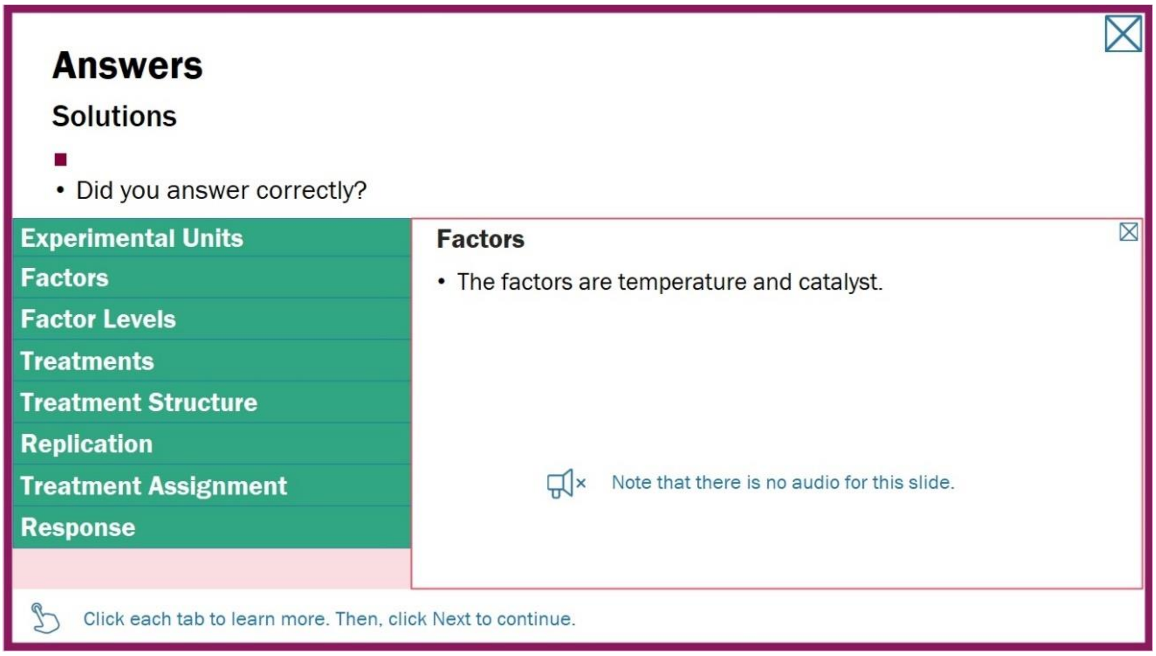
Click each tab to view the correct answer. Then click next to continue.

Tab 1.1: Experimental Units



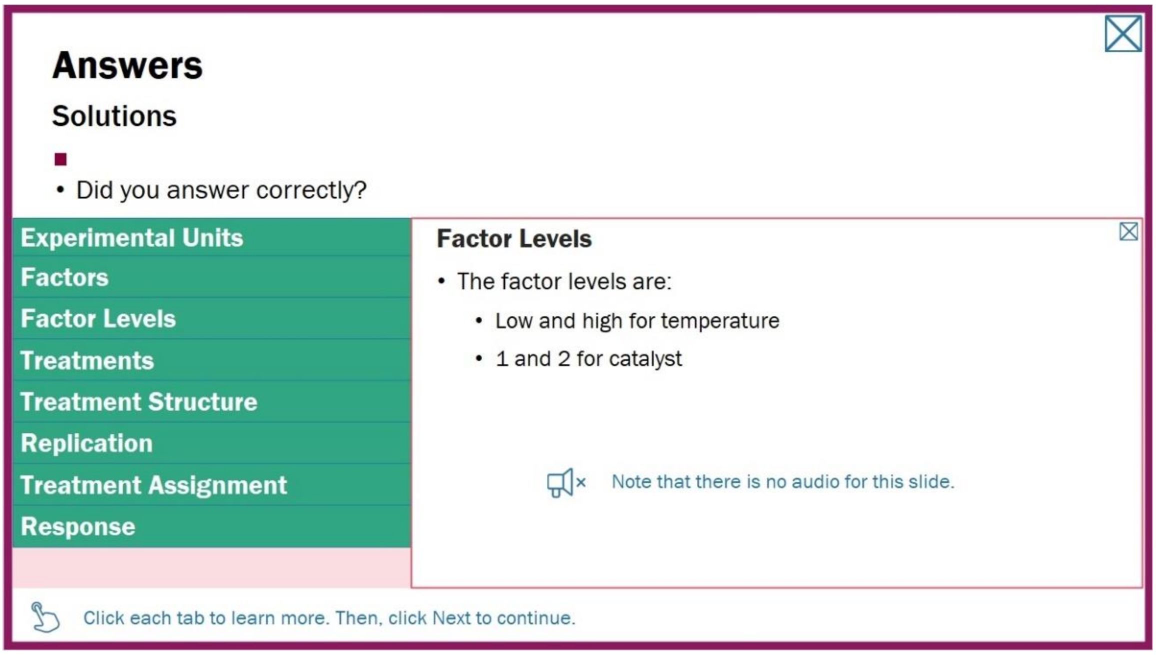
The experimental units are process runs.

Tab 1.2: Factors



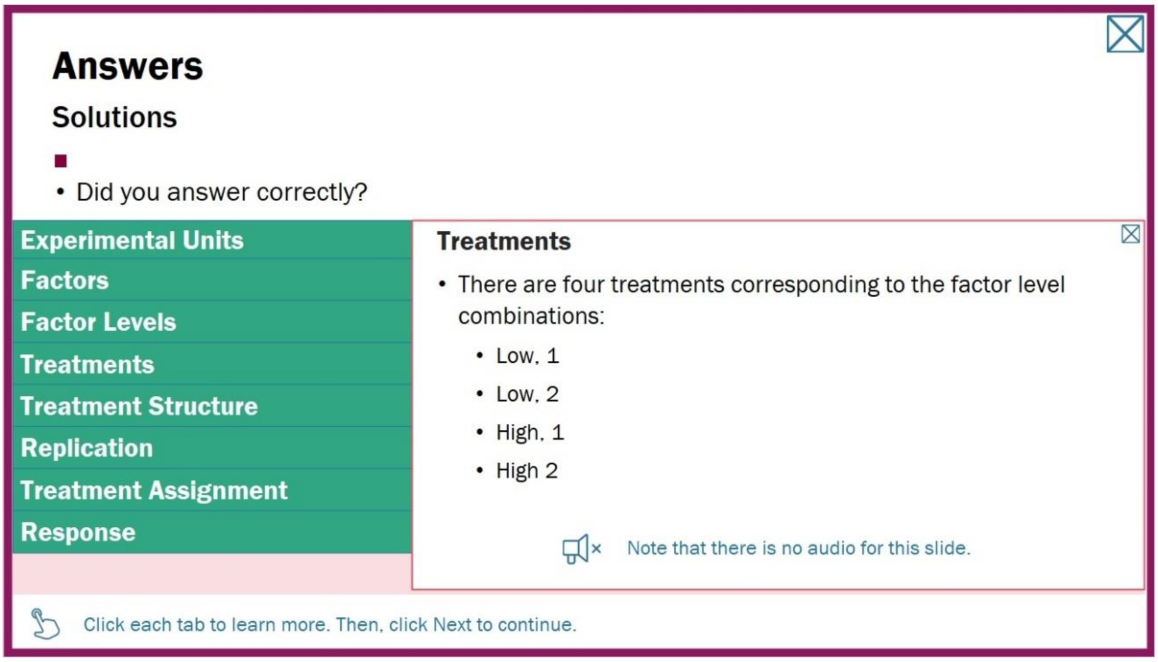
The factors are temperature and catalyst.

Tab 1.3: Factor Levels



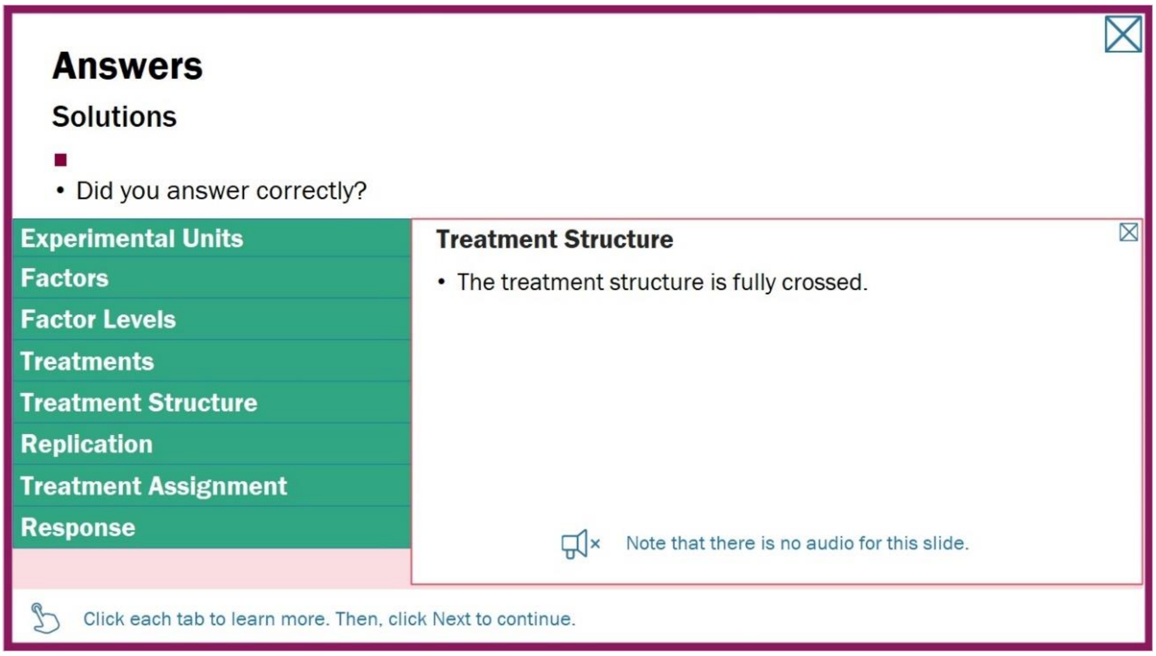
The factor levels are low and high for temperature, and 1 and 2 for catalyst.

Tab 1.4: Treatments



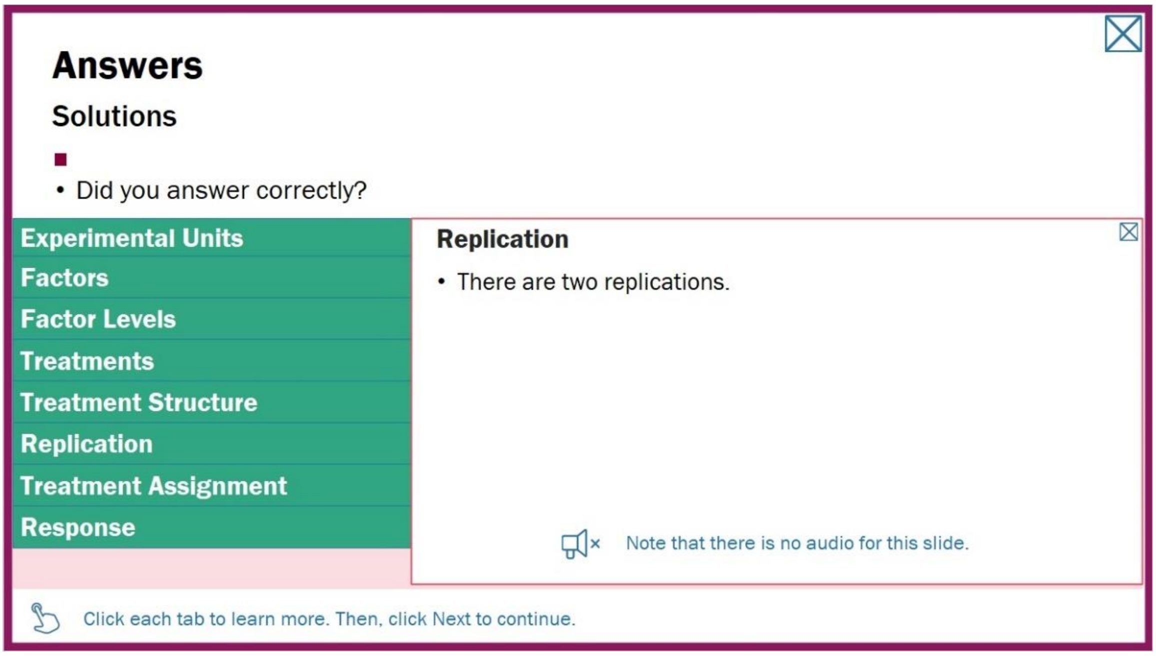
There are 4 treatments corresponding to the factor level combinations: low, 1; low, 2; high, 1; high, 2.

Tab 1.5: Treatment Structure



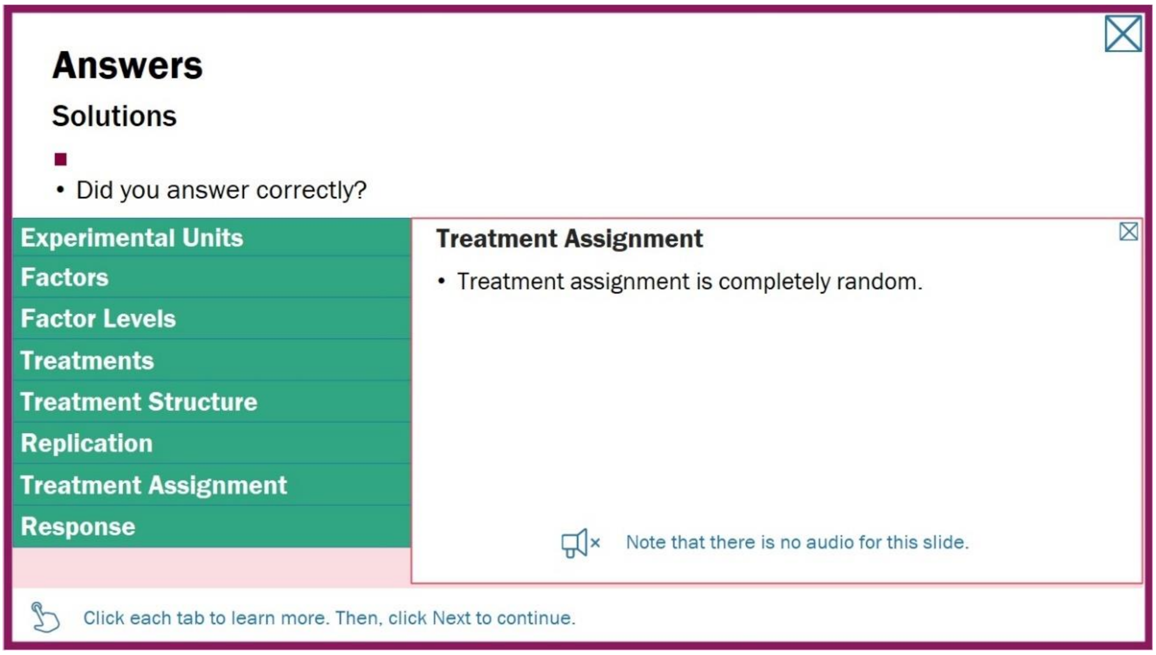
The treatment structure is fully crossed.

Tab 1.6: Replication



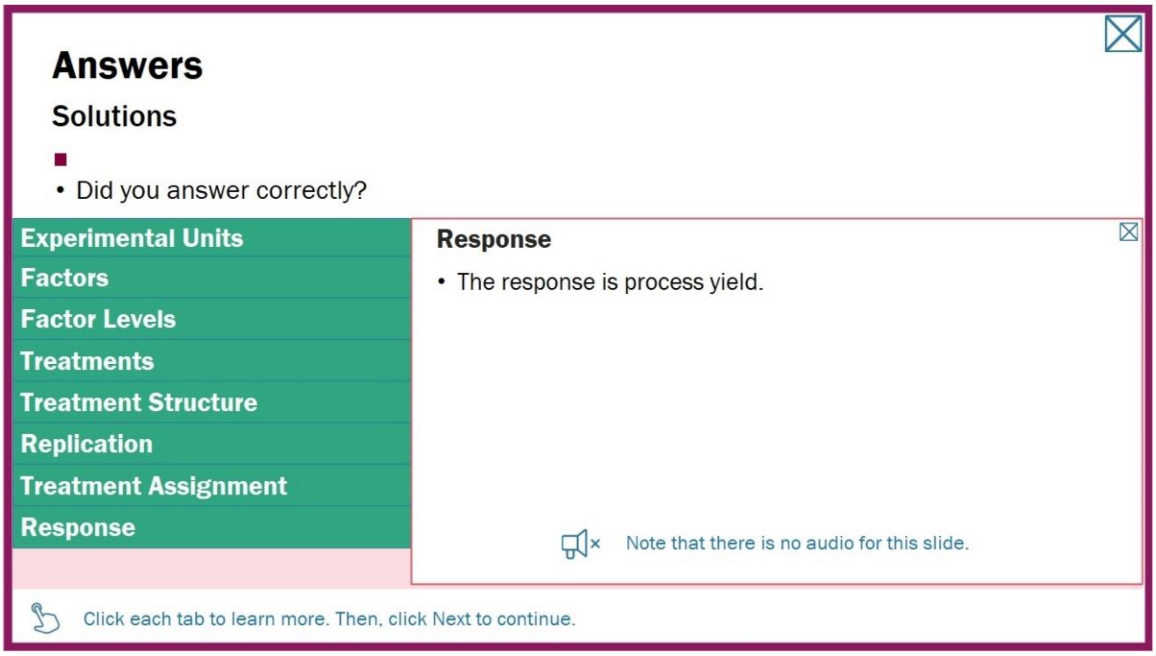
There are 2 replications.

Tab 1.7: Treatment Assignment



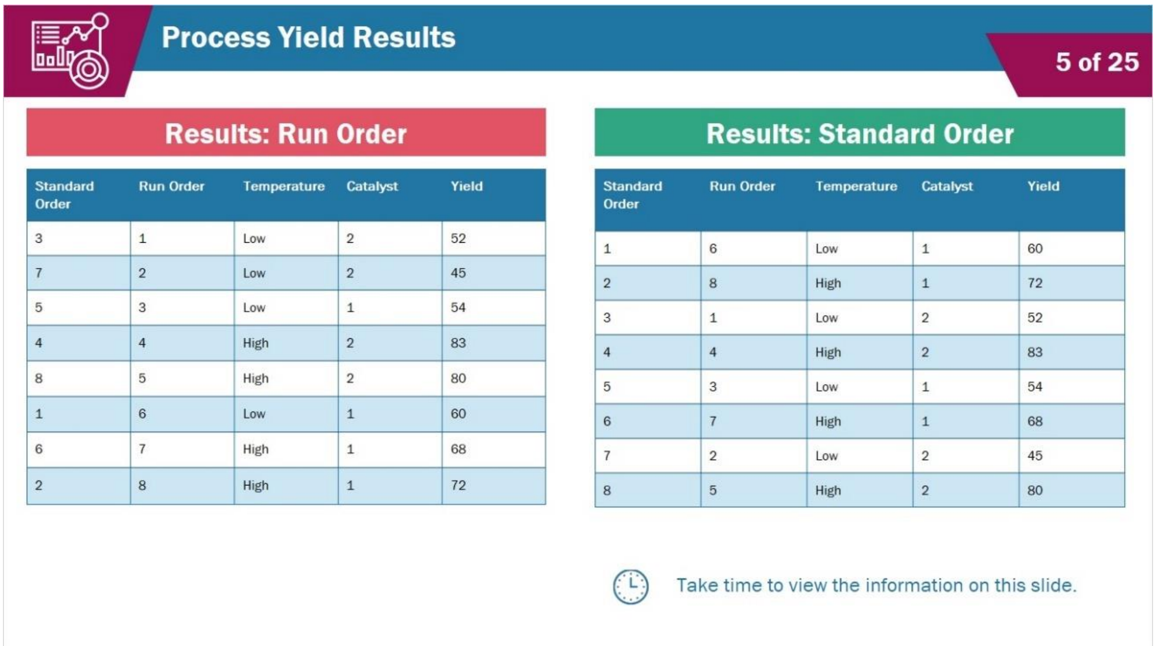
Treatment assignment is completely random.

Tab 1.8: Response



The response is process yield.

Slide 5: Process Yield Results



Results: Run Order

Standard Order	Run Order	Temperature	Catalyst	Yield
3	1	Low	2	52
7	2	Low	2	45
5	3	Low	1	54
4	4	High	2	83
8	5	High	2	80
1	6	Low	1	60
6	7	High	1	68
2	8	High	1	72

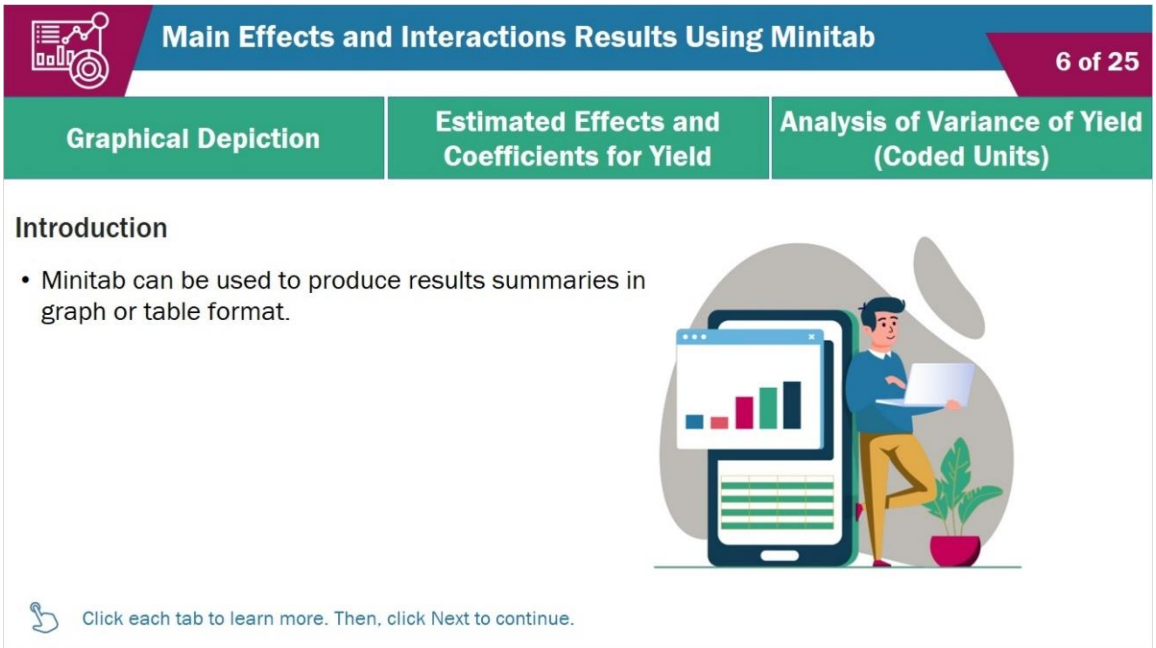
Results: Standard Order

Standard Order	Run Order	Temperature	Catalyst	Yield
1	6	Low	1	60
2	8	High	1	72
3	1	Low	2	52
4	4	High	2	83
5	3	Low	1	54
6	7	High	1	68
7	2	Low	2	45
8	5	High	2	80

The two tables here show the resulting process yields, listed in run order, and presented in standard order, to facilitate "eyeballing" the data.

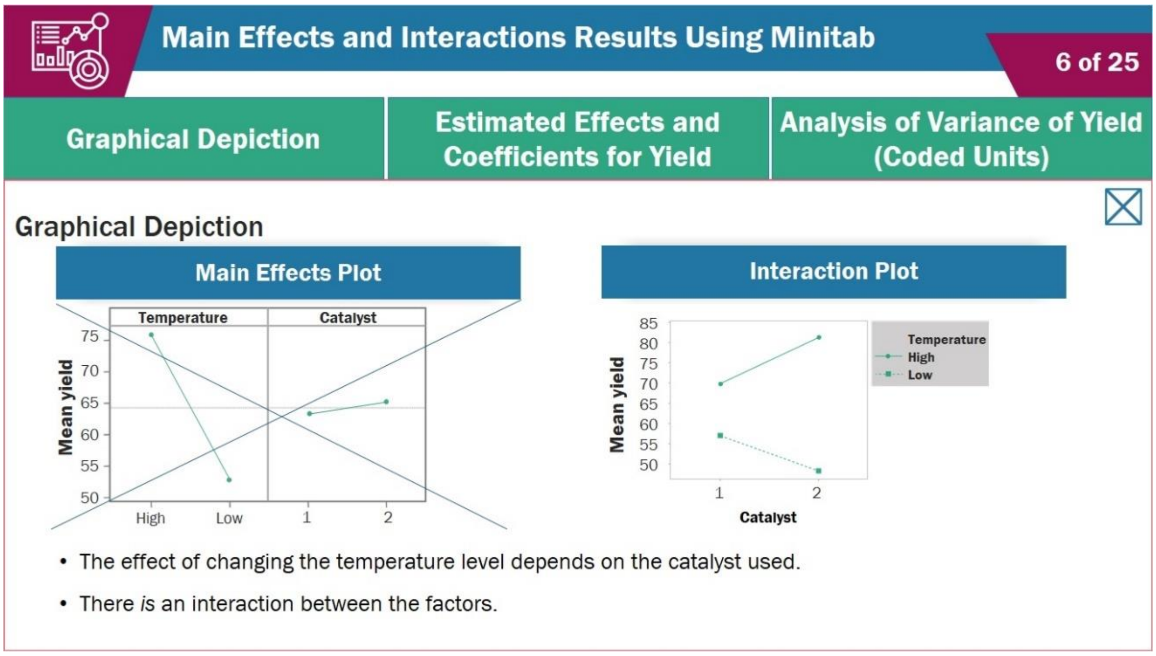
Take time to view these two tables. When you are ready, click next to continue.

Slide 6: Main Effects and Interactions Results Using Minitab



Using Minitab, key results summaries can be illustrated in graph or table format. Click each tab to learn more. When you are ready, click next to continue.

Tab 1: Graphical Depiction



As in all statistical analyses, a helpful first step is to produce simple graphs, main effects plots and an interaction plot, that show the following results.

The main effects plot suggests that there is a strong temperature effect, with yield increasing substantially when temperature is changed from Low to High, and a relatively negligible effect of changing catalyst. However, the interaction plot tells a different story. It appears that there is a modest Temperature effect when using Catalyst 1 but a much stronger Temperature effect when using Catalyst 2. Thus, the effect of changing the

Temperature level depends on the Catalyst used, that is, there is an *interaction* between the factors. Thus, the main effects plot is misleading in this case.

Tab 2: Estimated Effects and Coefficients for Yield

Main Effects and Interactions Results Using Minitab

6 of 25

Graphical Depiction	Estimated Effects and Coefficients for Yield	Analysis of Variance of Yield (Coded Units)

Estimated Effects and Coefficients for Yield

- Effect = Coef x 2
- SE(Effect) = SE(Coef) x 2

Term	Effect	Coef	SE Coef	T	P
Constant		64.25	1.31	49.01	0.000
Temperature	23.0	11.50	1.31	8.77	0.001
Catalyst	1.5	0.75	1.31	0.57	0.598
Temperature*Catalyst	10.0	5.00	1.31	3.81	0.019
s=3.70810					

The estimated effects for the main factors and interactions using Minitab are shown. Using Minitab terminology, the estimated effect is twice the coefficient, and the standard error of the estimated effect is twice the standard error of the coefficient. The last two columns are the t-statistics computed from the coefficient and standard error of coefficient and the associated p-values.

Tab 3: Analysis of Variance for Yield (Coded Units)

Main Effects and Interactions Results Using Minitab

6 of 25

Graphical Depiction	Estimated Effects and Coefficients for Yield	Analysis of Variance of Yield (Coded Units)

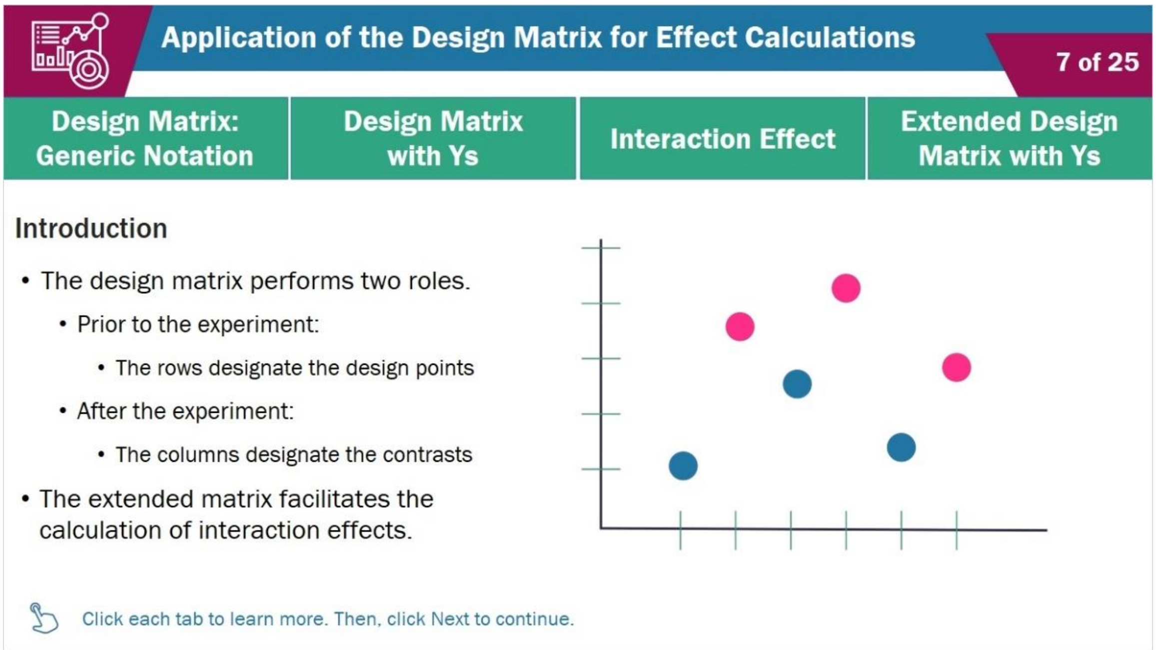
Analysis of Variance for Yield (Coded Units)

- Minitab output confirms:
  - Temperature main effect and Temperature/Catalyst interaction are statistically significant
  - Catalyst main effect is not statistically significant

Source	DF	Seq SS	Adj MS	F	P
Main Effects	2	1062.50	531.25	38.64	0.002
2-way Interactions	1	200.00	200.00	14.55	0.019
Residual Error	4	55.00	13.75		
Pure Error	4	55.00	13.75		
Total	7	1317.50			

The analysis of variance table for this 2 x 2 experiment is obtained using Minitab. The Minitab output confirms the Temperature main effect, and the Temperature / Catalyst interaction are statistically significant (p-values 0.001 and 0.019, respectively) while the Catalyst main effect is not statistically significant (p-value 0.598). Here in the table, the two main effects are combined to give a p-value of 0.002.

Slide 7: Application of the Design Matrix for Effect Calculations



The design matrix plays two roles.

Prior to the experiment, the *rows* designate the *design points, that is* the sets of conditions under which the process is to be run.

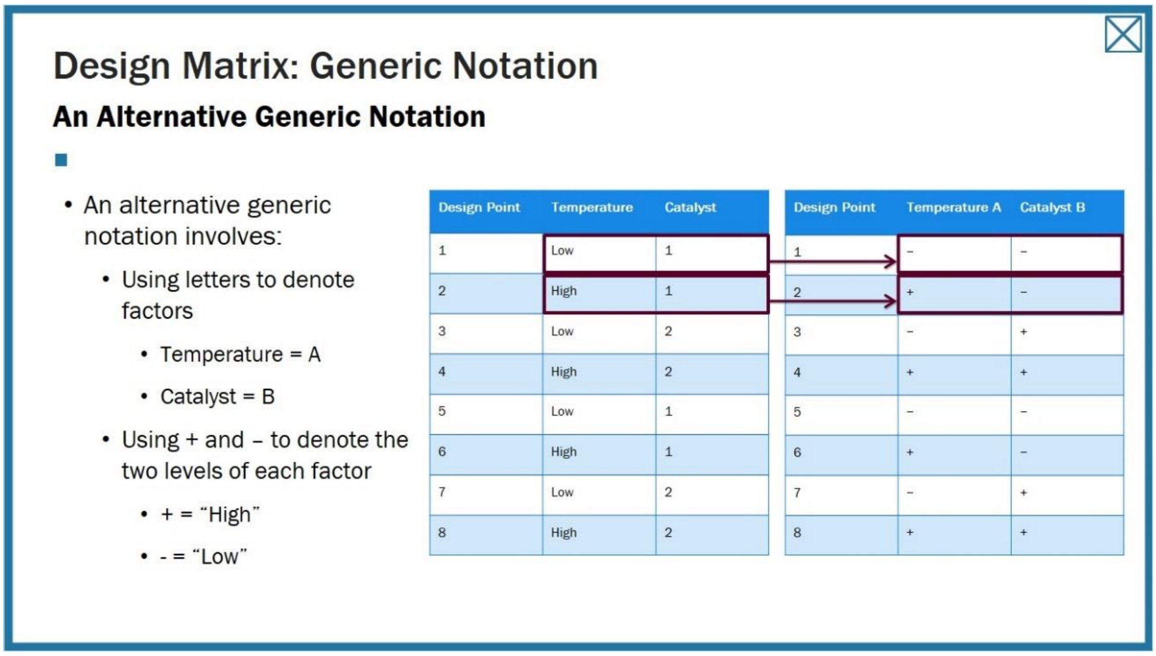
After the experiment, the *columns* designate the *contrasts,* the combinations of design point means, which measure the main effects of the factors.

The *extended* design matrix facilitates the calculation of interaction effects.

We now look at the details of the design matrix and the application of the design matrix for effect calculations.

Click each tab to learn more. When you are ready, click next to continue.

Tab 1: Design Matrix: Generic Notation



It is convenient to introduce at this stage an alternative generic notation for factor levels. Although not needed here, it will be found useful when there are several factors involved. In this design matrix, we use "High" and "Low" to denote the two levels of temperature, and the numbers 1 and 2 to denote the two levels of Catalyst.

The notation involves using the initial letters of the alphabet to denote the factors and using + and – signs to denote the two levels of each factor, with the further convention that + is referred to as "High" and – as "Low", even when the actual factor levels may not have such a direct interpretation. Thus, the High / Low designation makes sensed for a quantitative factor, such as Temperature, but not for a non-quantitative factor such as Catalyst where the numbers 1 and 2 are merely nominal designations.

With this notation, the design matrix is shown on the right. Here, "A" stands for Temperature with "Low" and "High" as they are, while "B" stands for Catalyst with "Low" and "High" standing for 1 and 2, respectively.

Tab 2: Design Matrix with Ys

Design Matrix with Ys

Facilitating Calculations of Main Effects

- The design matrix facilitates calculations of main effects.

$$\hat{A} = \tfrac{1}{4}(Y_2+Y_4+Y_6+Y_8) - \tfrac{1}{4}(Y_1+Y_3+Y_5+Y_7)$$

$$\hat{B} = \tfrac{1}{4}(Y_3+Y_4+Y_7+Y_8) - \tfrac{1}{4}(Y_1+Y_2+Y_5+Y_6)$$

Main Effects Estimates

Design Point	Temperature A	Catalyst B	Yield
1	−	−	Y_1
2	+	−	Y_2
3	−	+	Y_3
4	+	+	Y_4
5	−	−	Y_5
6	+	−	Y_6
7	−	+	Y_7
8	+	+	Y_8

The design matrix facilitates the calculations of main effects. To compute the effect of A, we take the average of Ys with a "+" sign in the "A" column of the matrix and compute the average of Ys with a "-" sign in the "A" column. The difference between these two averages is the effect of A. The calculation of effect of B is similar. The two formulas are shown here.

Tab 3: The Interaction Effect

Interaction Effect

Computing Interaction Effect of A and B

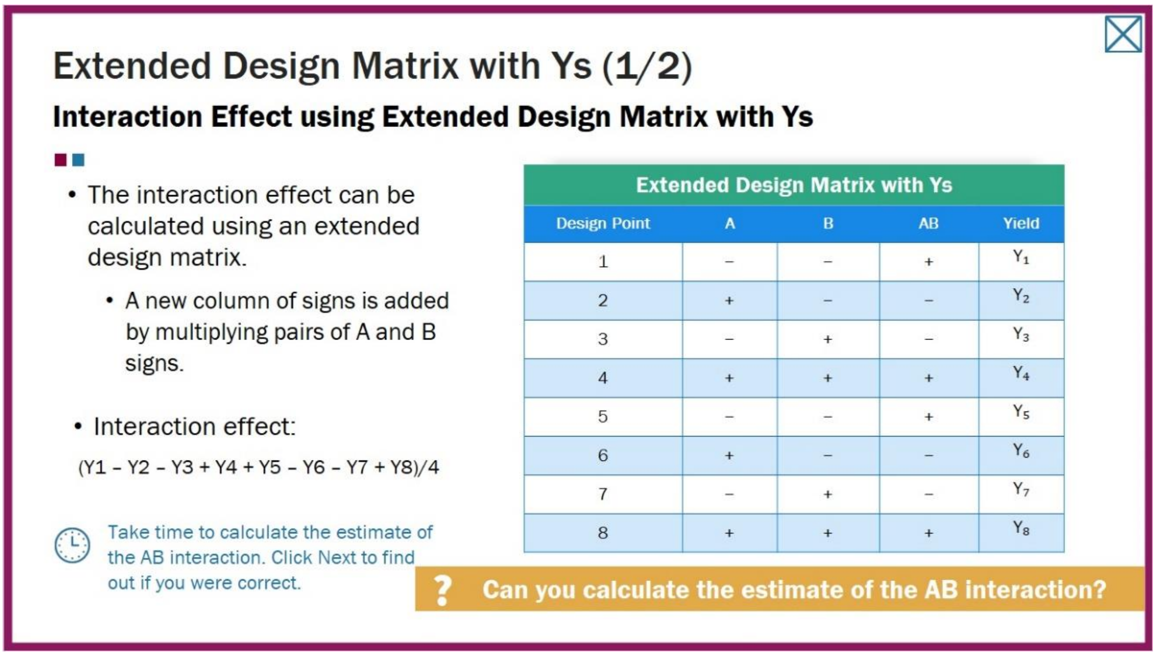
- To compute the interaction effect of A and B:
  - Compute A effect at high B: $\tfrac{1}{2}(Y_4 + Y_8) - \tfrac{1}{2}(Y_3 + Y_7)$
  - Compute A effect at low B: $\tfrac{1}{2}(Y_2 + Y_6) - \tfrac{1}{2}(Y_1 + Y_5)$
  - Take the difference between A effect at high B and A effect at low B and multiply by ½:
    $\tfrac{1}{4}(Y_4 + Y_8) - \tfrac{1}{4}(Y_3 + Y_7) - \tfrac{1}{4}(Y_2 + Y_6) + \tfrac{1}{4}(Y_1 + Y_5)$

$$= \tfrac{1}{4}(Y_4 + Y_8 + Y_1 + Y_5) - \tfrac{1}{4}(Y_3 + Y_7 + Y_2 + Y_6)$$

To calculate the interaction effect of A and B, we first compute the effect of A when B is at high level and the effect of A when B is at low level. We then take the difference between these two and multiply by a half.

$$= \tfrac{1}{4}\,(Y_4 + Y_8 + Y_1 + Y_5) \qquad - \tfrac{1}{4}\,(Y_3 + Y_7 + Y_2 + Y_6)$$

Tab 4: Extended Design Matrix with Ys



Extended Design Matrix with Ys

Design Point	A	B	AB	Yield
1	–	–	+	Y_1
2	+	–	–	Y_2
3	–	+	–	Y_3
4	+	+	+	Y_4
5	–	–	+	Y_5
6	+	–	–	Y_6
7	–	+	–	Y_7
8	+	+	+	Y_8

A big bonus of introducing the design matrix is that the interaction effect can be calculated in a similar way using a simple augmentation of the basic design matrix. This involves adding a new column of signs found by multiplying pairs of A and B signs, using the standard convention that multiplying like signs results in + and multiplying opposite signs results in –. More explicitly,

+ x + = +,

– x – = +,

+ x – = –,

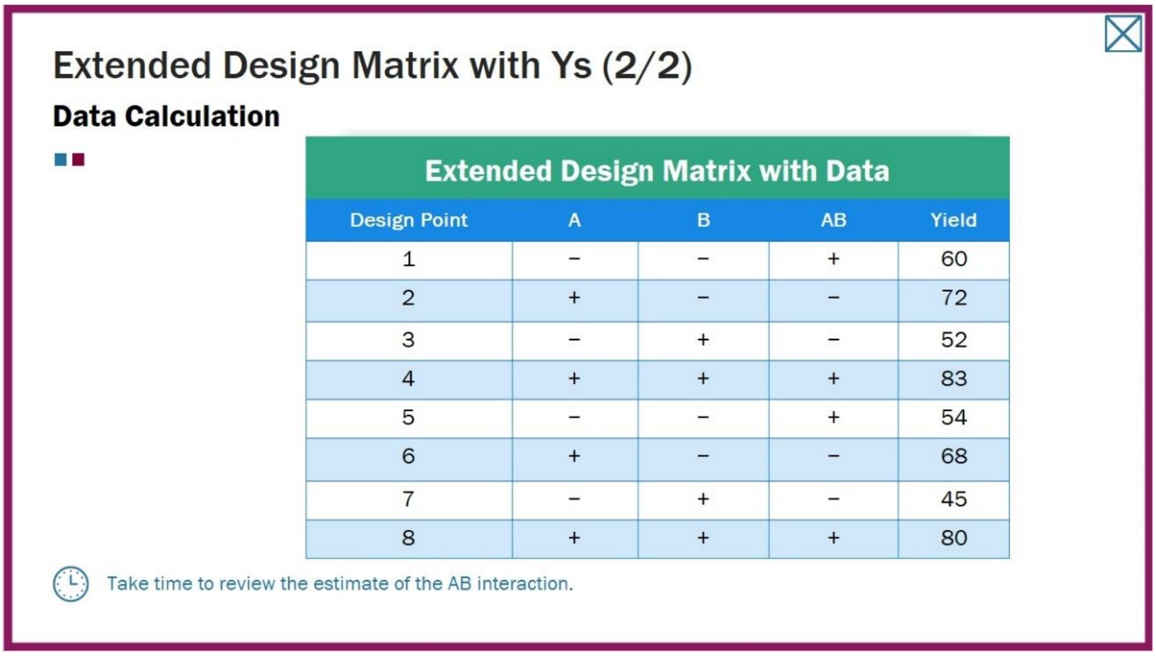
– x + = –.

Applying this to the basic design matrix, including results, with the new "interaction" column headed AB leads to the extended design matrix. Using the extended design matrix, the interaction effect can be calculated easily.

As an exercise, take time to calculate the estimate of the AB interaction.

When you are ready, click next to learn if you calculated correctly.

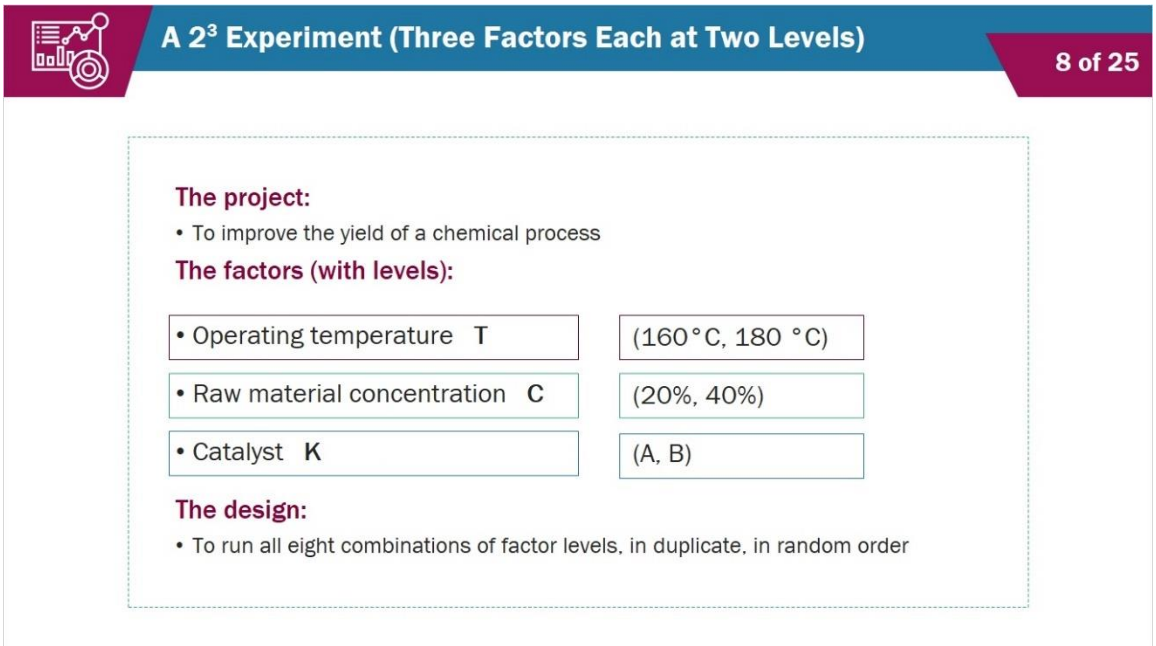
Tab 4.1: Data Calculation



Design Point	A	B	AB	Yield
1	−	−	+	60
2	+	−	−	72
3	−	+	−	52
4	+	+	+	83
5	−	−	+	54
6	+	−	−	68
7	−	+	−	45
8	+	+	+	80

Take time to review the estimate of the AB interaction. When you are ready, click next to continue.

Slide 8: A 2^3 Experiment (Three Factors Each at Two Levels)



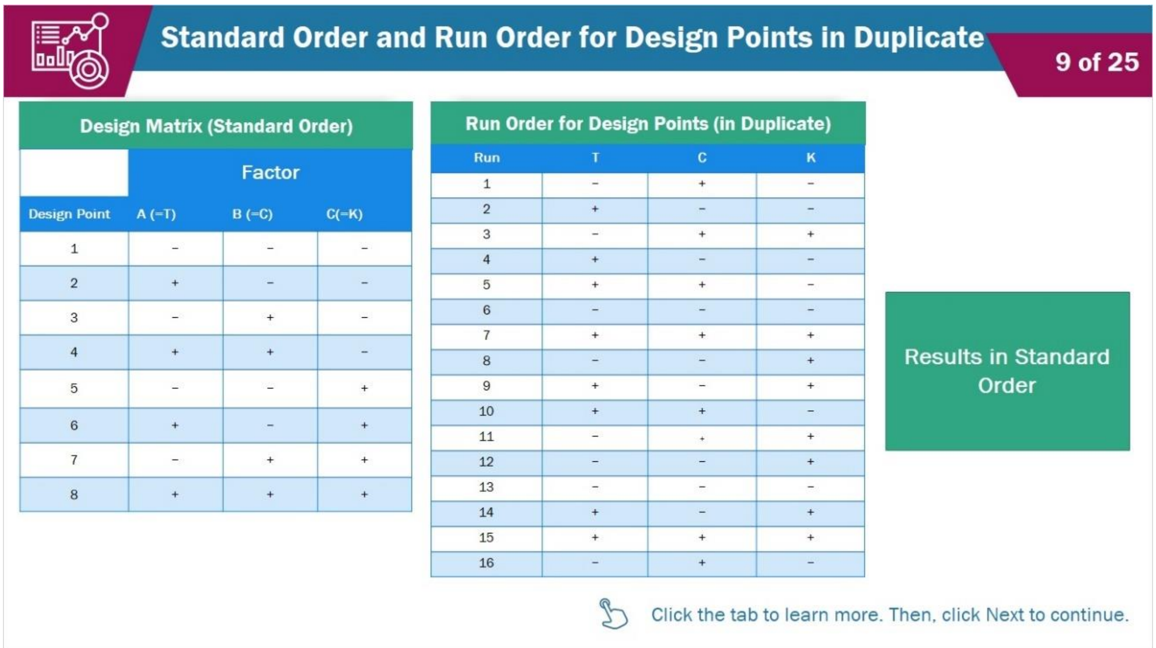
Let's take another example. In a chemical plant, a project was undertaken with a view to improving the yield of a chemical process. Three factors were chosen and, following a brainstorming session, an alternative level to the current operating level was chosen for each of the three factors. The three factors, with their current and alternative levels, were

operating Temperature T,	160 °C and 180 °C,
raw material Concentration, C,	20% and 40%, and

catalyst K, A and B.

The experimental design involved running the process at all eight possible combinations of factor levels, in duplicate, in random time order.

Slide 9: Standard Order and Run Order for Design Points in Duplicate



Design Matrix (Standard Order)

	Factor		
Design Point	A (−T)	B (−C)	C(−K)
1	−	−	−
2	+	−	−
3	−	+	−
4	+	+	−
5	−	−	+
6	+	−	+
7	−	+	+
8	+	+	+

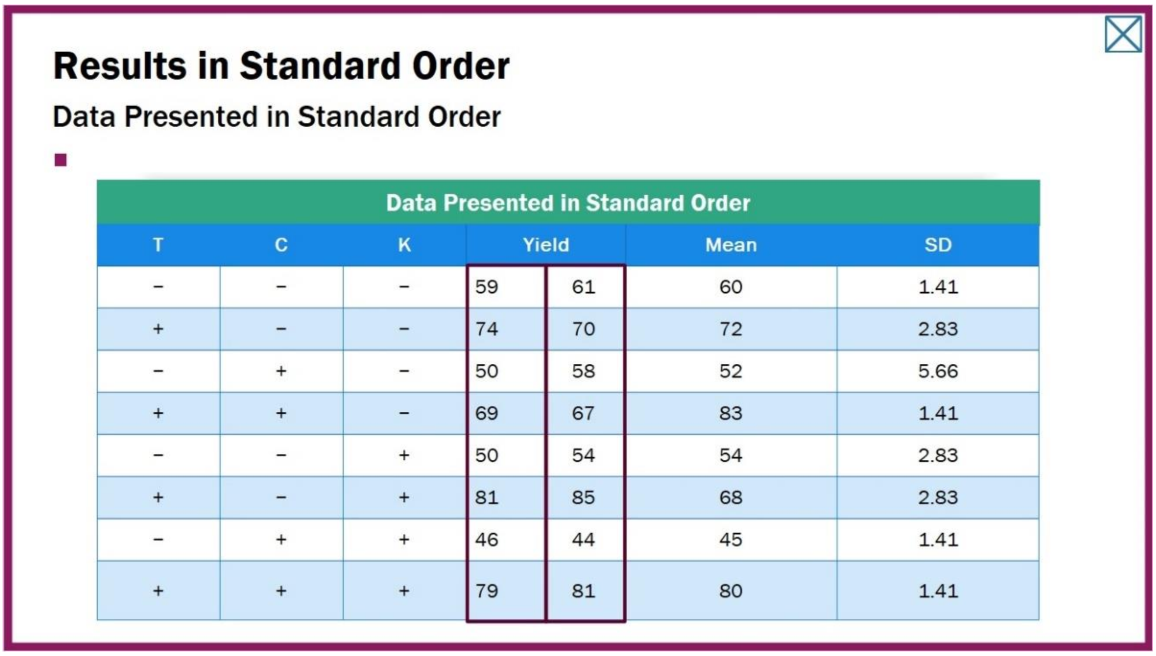
Run Order for Design Points (in Duplicate)

Run	T	C	K
1	−	+	−
2	+	−	−
3	−	+	+
4	+	−	−
5	+	+	−
6	−	−	−
7	+	+	+
8	−	−	+
9	+	−	+
10	+	+	−
11	−	+	+
12	−	−	+
13	−	−	−
14	+	−	+
15	+	+	+
16	−	+	−

The design matrix in standard order is shown on the left. Each design point corresponds to a factor level combination. The run order for design points in duplicate is shown on the right.

Click the tab to view the results in standard order. When you are ready, click next to continue.

Tab 1: Results in Standard Order



Results in Standard Order

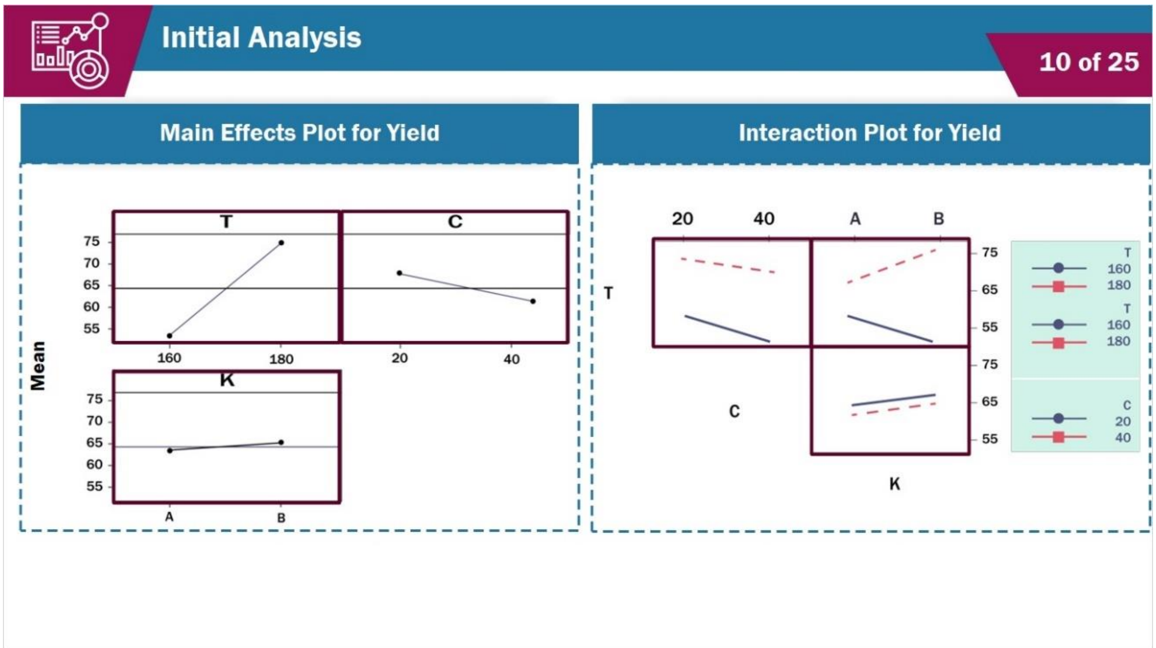
Data Presented in Standard Order

Data Presented in Standard Order

T	C	K	Yield		Mean	SD
−	−	−	59	61	60	1.41
+	−	−	74	70	72	2.83
−	+	−	50	58	52	5.66
+	+	−	69	67	83	1.41
−	−	+	50	54	54	2.83
+	−	+	81	85	68	2.83
−	+	+	46	44	45	1.41
+	+	+	79	81	80	1.41

When the experiment was completed, the data were presented in standard order, in duplicate pairs. The mean and standard deviation for each pair are calculated.

Slide 10: Initial Analysis



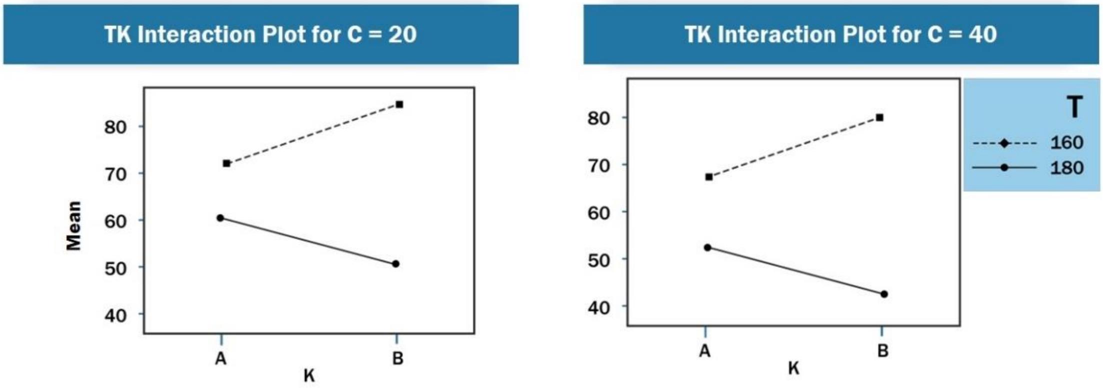
The Main Effects plots show what appears to be a substantial effect on process yield of raising temperature from 160 degrees to 180, a smaller and negative effect of increasing raw material concentration from 20% to 40% and a relatively negligible effect of changing catalyst.

The Interactions plots shows similar large Temperature and small Catalyst effects for both levels of Concentration, (as seen in the top left and bottom right plots), but different and opposite Catalyst effects at high and low Temperature, as seen in the top right plot.

Slide 11: ## Three-factor Interaction

Three-factor Interaction

- A 3-factor interaction may manifest itself as different 2-factor interactions at the different levels of the third factor.



There remains the possibility of a 3-factor interaction which would manifest itself as different 2-factor interactions at the different levels of the third factor. This can be seen by producing an appropriate pair of 2-factor interaction plots from subsets of the data created by splitting on the two levels of the third factor. Given the evident 2-factor interaction of Catalyst and Temperature, it may be of interest to split the data into high and low levels of concentration and check whether this interpretation remains the same at both levels or whether different forms of interaction emerge.

The 2-factor interaction patterns are very similar suggesting no significant 3-factor interaction. Formal analysis will be conducted to confirm this finding.

Slide 12: ## Calculating the Main Effects

Calculating the Main Effects

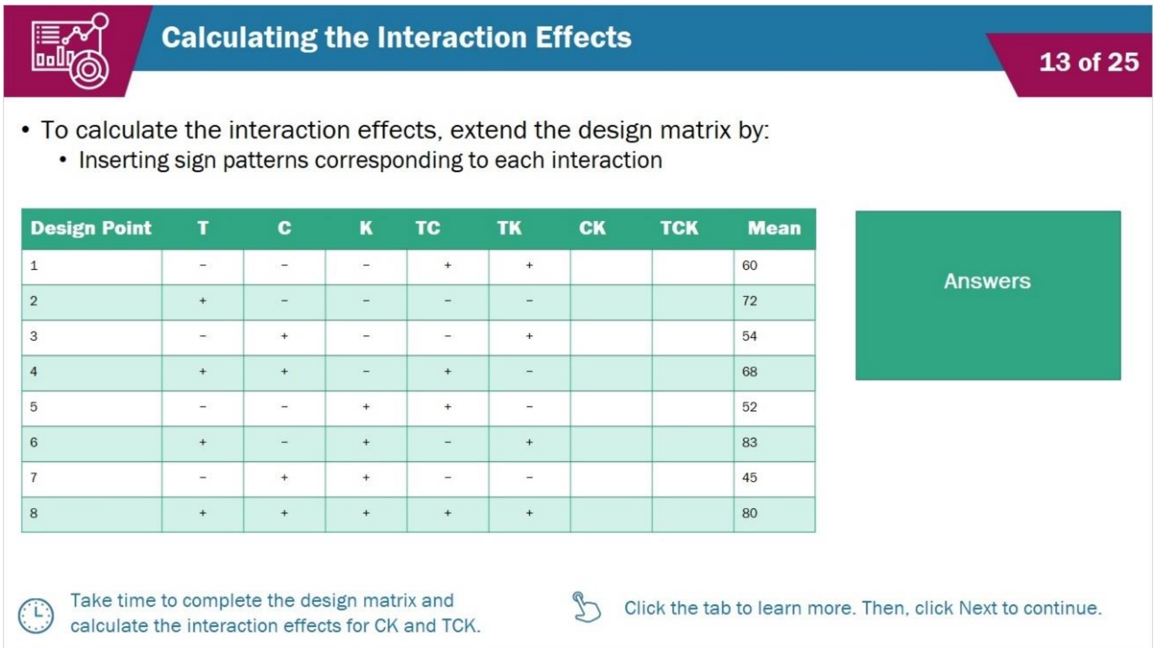
- To calculate the main effects, append the mean yield at each design point to each column of signs, sum, and divide by four.

T	C	K	Mean
–	–	–	60
+	–	–	72
–	+	–	54
+	+	–	68
–	–	+	52
+	–	+	83
–	+	+	45
+	+	+	80

$$\hat{K} = (-60 - 72 - 54 - 68 + 52 + 83 + 45 + 80)/4 = 1.5$$

To calculate the main effects, simply append the mean yield at each design point to each column of signs, sum and divide by 4. The main effect for catalyst is 1.5.

Slide 13: Calculating the Interaction Effects



As with 2-factor designs, the estimated interaction effects may be calculated using similar calculations involving an extension of the design matrix where the sign pattern corresponding to each interaction is the product of the sign patterns of the constituent factors. As an exercise, take time to complete the design matrix and calculate the interaction effects for CK and TCK.

When you are ready, click the tab to learn if you calculated the interaction effects correctly.

Tab 1: Answers

Answers

Completed Design Matrix and Interactions for CK and TCK

Did you answer correctly?

Take time to view the information on this slide.

Design Point	T	C	K	TC	TK	CK	TCK	Mean
1	-	-	-	+	+	+	-	60
2	+	-	-	-	-	+	+	72
3	-	+	-	-	+	-	+	54
4	+	+	-	+	-	-	-	68
5	-	-	+	+	-	-	+	52
6	+	-	+	-	+	-	-	83
7	-	+	+	-	-	+	-	45
8	+	+	+	+	+	+	+	80

TK: (+ 60 – 72 + 54 – 68 – 52 + 83 – 45 + 80)/4 = 10
TCK: (– 60 + 72 + 54 – 68 + 52 – 83 – 45 + 80)/4 = 0.5

Did you complete the design matrix and calculate the correct interaction effects for CK and TCK?

Take time to review the answers. When you are ready, click next to continue.

Slide 14: Calculating S

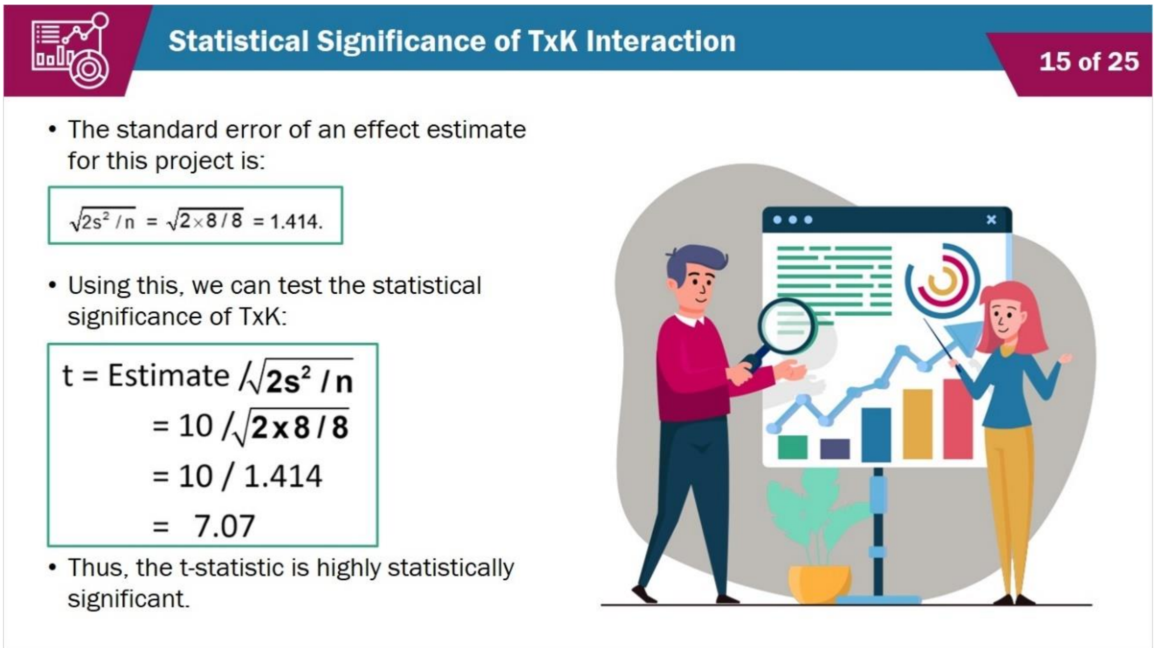
Calculating S

- If there are replicate measurements, calculating a suitable estimate of σ, the standard deviation of experimental "error" is straightforward.

T	C	K	Yield		Variance = ½(diff)²
-	-	-	59	61	2
+	-	-	74	70	8
-	+	-	50	58	32
+	+	-	69	67	2
-	-	+	50	54	8
+	-	+	81	85	8
-	+	+	46	44	2
+	+	+	79	81	
				Total	64
				s^2	8
				s	2.828

The calculation of a suitable estimate of ☐, the standard deviation of experimental "error", is straightforward in the presence of replicate measurements. The estimated variance is 8, standard deviation 2.83, based on 8 degrees of freedom.

Slide 15: ## Statistical Significance of TxK



The standard error of an effect estimate, that is, of the difference between the means of two sets of 8 individual measurements, is

$$\sqrt{2s^2/n} = \sqrt{2\times 8/8} = 1.414.$$

Equipped with this, we can proceed to test the statistical significance of estimated effects and calculate confidence intervals for effects. Thus, the t-statistic for testing the statistical significance of the TK interaction effect is

$$t = 10 / 1.414 = 7.07,$$

highly statistically significant by the usual standards.

Slide 16: Minitab Analysis

Minitab Analysis

- Minitab uses a computing algorithm design for multiple regression to produce these results.

Estimated Effects for Yield

Term	Effect	SE	T	P
T	23.0	1.414	16.26	0.000
C	-5.00	1.414	-3.54	0.008
K	1.5	1.414	1.06	0.320
T*C	1.5	1.414	1.06	0.320
T*K	10.0	1.414	7.07	0.000
C*k	0.0	1.414	0.00	1.000
T*C*k	0.5	1.414	0.35	0.733

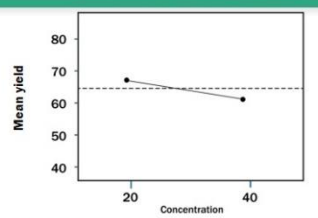
S = 2.82843

Minitab's Stat / DOE command can be set up to produce effect estimates and standard errors. For the current example, the relevant output follows. Unfortunately, the format of this output is driven by computing considerations. Minitab uses a computing algorithm designed for multiple regression to produce these results.

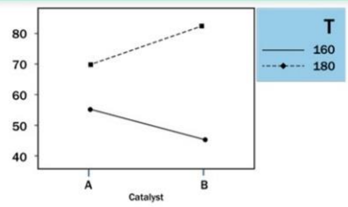
Slide 17: Optimum Operating Conditions

Optimum Operating Conditions

Concentration Effect



Catalyst by Temperature Interaction Plot



! The SE of a mean of two measurements is $\sqrt{s^2/n} = \sqrt{8/2} = 2$

Mean Yields and SE

Temperature	Concentration	Catalyst	Mean	SE
160	20	A	60.00	2.00
180	20	A	72.00	2.00
160	40	A	54.00	2.00
180	40	A	68.00	2.00
160	20	B	52.00	2.00
180	20	B	83.00	2.00
160	40	B	45.00	2.00
180	40	B	80.00	2.00

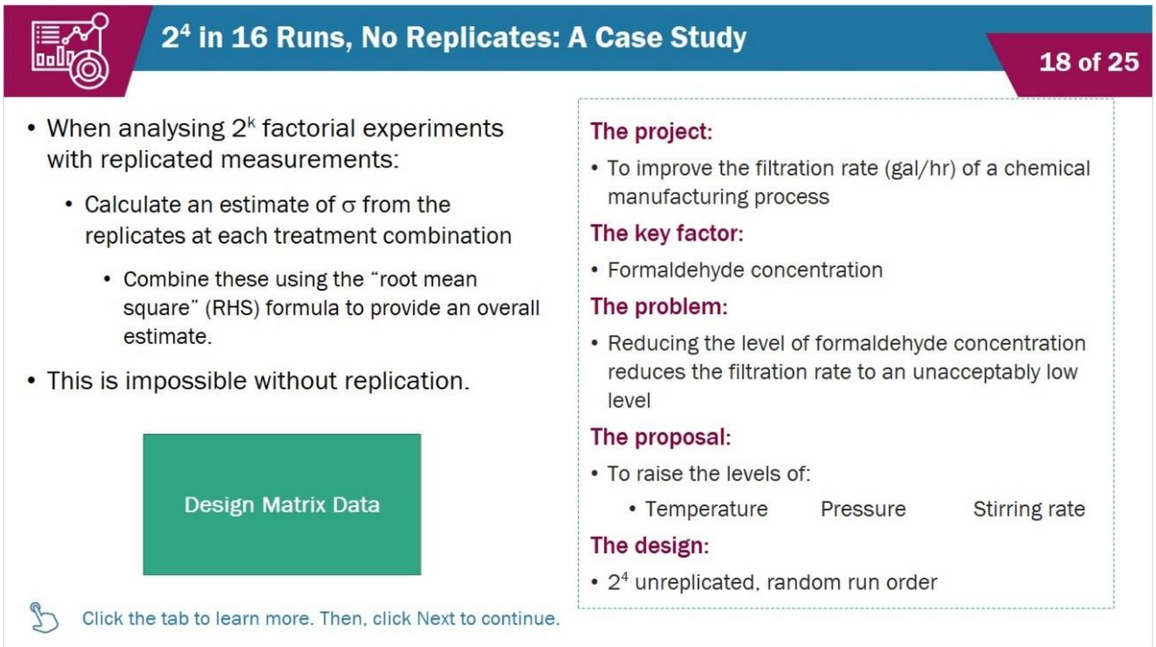
Best operating conditions

The optimum operating conditions are readily identifiable from the graphs. Using raw material concentration of 20% gives higher yield irrespective of the temperature and catalyst settings; the combination of Temperature at 180°C and Catalyst B gives higher yield irrespective of Concentration.

For a more precise numerical answer to this question, selecting the appropriate Means option in the relevant Minitab command produces the following table of mean yields for individual treatments (factor level combinations), with standard errors. (Note that the standard error of a mean of 2 (duplicate) measurements is $\sqrt{s^2/n} = \sqrt{8/2} = 2$).

This confirms that the best operating conditions are temperature at 180 °C, raw material concentration at 20% and using catalyst B.

Slide 18: 2^4 in 16 Runs, No Replicates: A Case Study



When analysing 2^k factorial experiments with replicated measurements at each treatment, that is, each factor level combination, an estimate of ☐ may be calculated from the replicates at each treatment combination and these may be combined using the "root mean square" (RMS) formula to provide an overall estimate. When there is no replication, that is, there is just one measurement of the response at each treatment combination, this is not possible.

Let's look at a case study on Improving the filtration rate (gal/hr) of a chemical manufacturing process. The key factor for the improvement of the filtration rate is Formaldehyde concentration where reducing the level of formaldehyde concentration reduces the filtration rate to an unacceptably low level. A proposal is to raise levels of Temperature, Pressure and Stirring rate. The design of the experiment is 2^4 unreplicated, random run order.

Click the tab to view the design matrix data for this project. Then when you are ready, click next to continue.

Tab 1: Design Matrix Data

Design Matrix Data

Data

- Each row corresponds to one condition.
- A filtration rate is measured at each of the 16 reaction conditions.

Temperature	Pressure	Formaldehyde Concentration	Stirring Rate	Filtration Rate
Low	Low	Low	Low	45
High	Low	Low	Low	71
Low	High	Low	Low	48
High	High	Low	Low	65
Low	Low	High	Low	68
High	Low	High	Low	60
Low	High	High	Low	80
High	High	Hgh	Low	65
Low	Low	Low	High	43
High	Low	Low	High	100
Low	High	Low	High	45
High	High	Low	Hgh	104
Low	Low	High	High	75
High	Low	High	High	86
Low	High	High	High	70
High	High	High	Hgh	96

The design matrix is shown in this slide where each row corresponds to one condition. A response (filtration rate) was measured at each of the 16 reaction conditions.

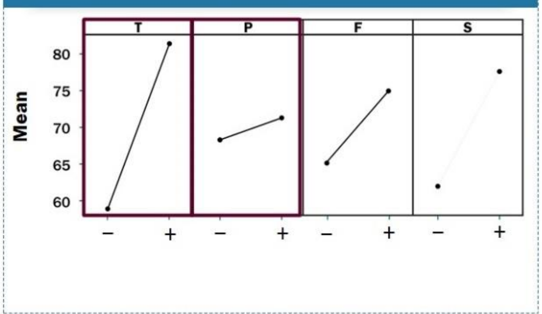
Slide 19: Initial Analysis

Initial Analysis

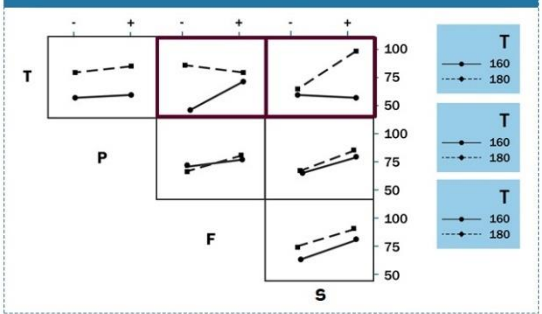
19 of 25

- The plots provide an initial view of the experimental variation.

Main Effects Plot



Interaction Plots



Main effect plots and 2-factor interaction plots are shown, provide an initial view of the experimental variation. From the main effects plot, it appears that increasing Temperature results in a substantial increase in filtration rate. Changing Pressure appears to have negligible effect. There may also be some interactions between Temperature and Concentration and between Temperature and Stirring rate.

Slide 20: Minitab Design of Experiment

Minitab Design of Experiment

- The Minitab command DOE calculates estimates of all 15 main effects and interactions.

Estimated Effects

Term	Effect
T	21.625
P	3.125
F	9.875
S	14.625
T*P	0.125
T*F	-18.125
T*S	16.625
P*F	2.375
P*S	-0.375
F*S	-1.125
T*P*F	1.875
T*P*S	4.125
T*F*S	-1.625
P*F*S	-2.625
T*P*F*S	1.375

The effects may be calculated by hand using the extended design matrix. Alternatively, the Minitab command DOE may be used to calculate estimates of all 15 main effects and interactions.

Slide 21: Alternative Statistical Analyses in the Absence of Replicates

Alternative Statistical Analyses in the Absence of Replicates

Normal Plot of Effects	Pareto Chart	Lenth Method

Introduction

- Without replicates, we cannot compute a standard estimate for σ.
  - Alternative approaches to statistical analysis are required.
- The best approach is to use a combination of the Normal plot of effects, the Pareto chart and the Lenth method.

Click each tab to learn more. Then, click Next to continue.

The lack of replicates means that we are unable to compute a standard estimate for s. Thus, alternative approaches to statistical analysis are required in circumstances such as these.

There are three methods that can be used: a Normal plot, a pareto analysis or the Lenth method.

The best approach is to combine all these three methods.

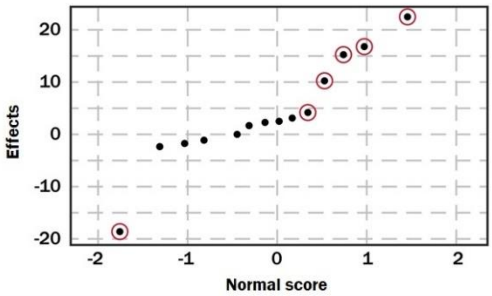
Click each tab to learn more. When you are ready, click next to continue.

Tab 1: **Normal Plot of Effects**

Normal Plot of Effects

Normal Plot

- This approach distinguishes between "significant" and chance effects.
- A few real effects will appear as exceptions in a normal plot.



Estimated Effects	
Term	**Effect**
T	21.625
P	3.125
F	9.875
S	14.625
T*P	0.125
T*F	-18.125
T*S	16.625
P*F	2.375
P*S	-0.375
F*S	-1.125
T*P*F	1.875
T*P*S	4.125
T*F*S	-1.625
P*F*S	-2.625
T*P*F*S	1.375

A Normal plot of effects provides an approach to distinguishing between "significant" effects and effects that may be ascribed to chance. If all effects were null, the 15 effect estimates shown above would constitute a simple random sample from a Normal distribution with constant standard deviation (being the standard error of an effect).

A Normal plot may be drawn, as shown in the Figure. This suggests one highly exceptional estimated effect and a few others that also appear to deviate substantially from the rest. Matching readings from the plot with the estimated effects, these may be identified as the T main effect, F main effect, S main effect, T by F interaction, T by S interaction and T by P by S three-way interaction.

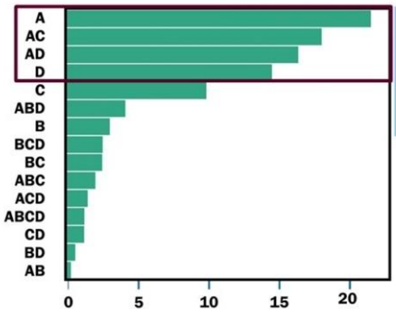
The remaining 11 estimated effects lie on a more or less straight line, indicating that their values are consistent with chance variation in a Normal distribution.

Tab 2: Pareto Chart

Pareto Chart

Pareto Analysis

- Pareto analysis distinguishes between substantial and negligible effects without referring to statistical models incorporating chance variation.



Factor	Name
A	T
B	P
C	F
D	S

The Pareto bar chart illustrates:

- The magnitudes of the effects, from largest to smallest
- The separation of the vital effects from the trivial effects

alternative approach to distinguishing between substantial and negligible effects that does not refer to statistical models incorporating chance variation is Pareto analysis.

A Pareto chart is a bar chart where the bars represent the magnitudes (irrespective of sign, + or –) of the effects, arranged in order from largest to smallest. This is designed to show the separation of the "vital few" from the "trivial many".

There appears to be a distinction between the four biggest (in magnitude) effects and the rest.

Tab 3: Lenth Method

Lenth Method (1/4)

An Alternative Method for Estimating s

- Given several normal values with mean 0 and their absolute values (magnitudes, or values without signs), it may be shown that:
  - SD(Normal values) ≈ 1.5 × median(Absolute values)
- Apply this formula to effects to produce an initial PSE_0 (Pseudo Standard Error)
  - Given the small number of effects with mean ≠ 0, this PSE_0 is inflated.
- Add a refinement by using the formula: $\boxed{PSE \approx 1.5 \times \text{median}(\text{Abs values} < 2.5 \times PSE_0)}$

The Lenth method provides an alternative method for estimating s.

The basis for Lenth's analysis is that, given several values sampled from a Normal distribution with mean 0 and common standard deviation, and given their absolute values (magnitudes, or values without signs), then it may be shown that

$$\text{SD(Normal values)} \approx 1.5 \times \text{median(Absolute values)}.$$

We first apply this formula to the effects to get an initial pseudo standard error PSE_0. Given that there are a small number of effects with non-zero means, this initial pseudo standard error PSE_0 is inflated. We then apply a refinement step using the formula

$$PSE \approx 1.5 \times \text{median(Abs values} < 2.5 \times PSE_0\text{)}.$$

Tab 3.1: Calculating the PSE

Lenth Method (2/4)

Take time to view the information on this slide.

Calculating the PSE

- Ignore +/- signs.
- Order the effects from smallest to largest.

Effects

Term	Effect
T	21.625
P	3.125
F	9.875
S	14.625
T*P	0.125
T*F	-18.125
T*S	16.625
P*F	2.375
P*S	-0.375
F*S	-1.125
T*P*F	1.875
T*P*S	4.125
T*F*S	-1.625
P*F*S	-2.625
T*P*F*S	1.375

Absolute Value of Effects

Term	Effect
T	21.625
P	3.125
F	9.875
S	14.625
T*P	0.125
T*F	-18.125
T*S	16.625
P*F	2.375
P*S	0.375
F*S	1.125
T*P*F	1.875
T*P*S	4.125
T*F*S	1.625
P*F*S	2.625
T*P*F*S	1.375

Ordered Effects

No.	Term	Effect
1	T*P	0.125
2	P*S	0.375
3	F*S	1.125
4	T*P*F*S	1.375
5	T*F*S	1.625
6	T*P*F	1.875
7	P*F	2.375
8	P*F*S	2.625
9	P	3.125
10	T*F*S	4.125
11	F	9.875
12	S	14.625
13	T*S	16.625
14	T*F	18.125
15	T	21.625

Let's now demonstrate the calculation of PSE.

The list of effects is shown on the left. We first ignore the signs of the effect, equivalently we take the absolute values of the effects.

The next step is to order the effects from smallest to largest.

Take time to review this information. Then click next to continue.

Tab 3.2: **Finding the Median and Calculating the PSE0**

Lenth Method (3/4)

Finding the Median and Calculating the PSE_0

- The median is 2.625.
  - Lenth's PSE_0 of SE is:

    $\boxed{1.5 \times 2.625 = 3.94}$
- Since 2.5 x 3.94 = 9.84, this entails excluding the last five effects.

Ordered Effects List

No.	Term	Effect
1	T*P	0.125
2	P*S	0.375
3	F*S	1.125
4	T*P*F*S	1.375
5	T*F*S	1.625
6	T*P*F	1.875
7	P*F	2.375
8	P*F*S	2.625
9	P	3.125
10	T*F*S	4.125
11	F	9.875
12	S	14.625
13	T*S	16.625
14	T*F	18.125
15	T	21.625

In a sorted list of 15, the 8th is the median, here 2.625, so that Lenth's estimate of effect standard error would be

$$1.5 \times 2.625 = 3.94.$$

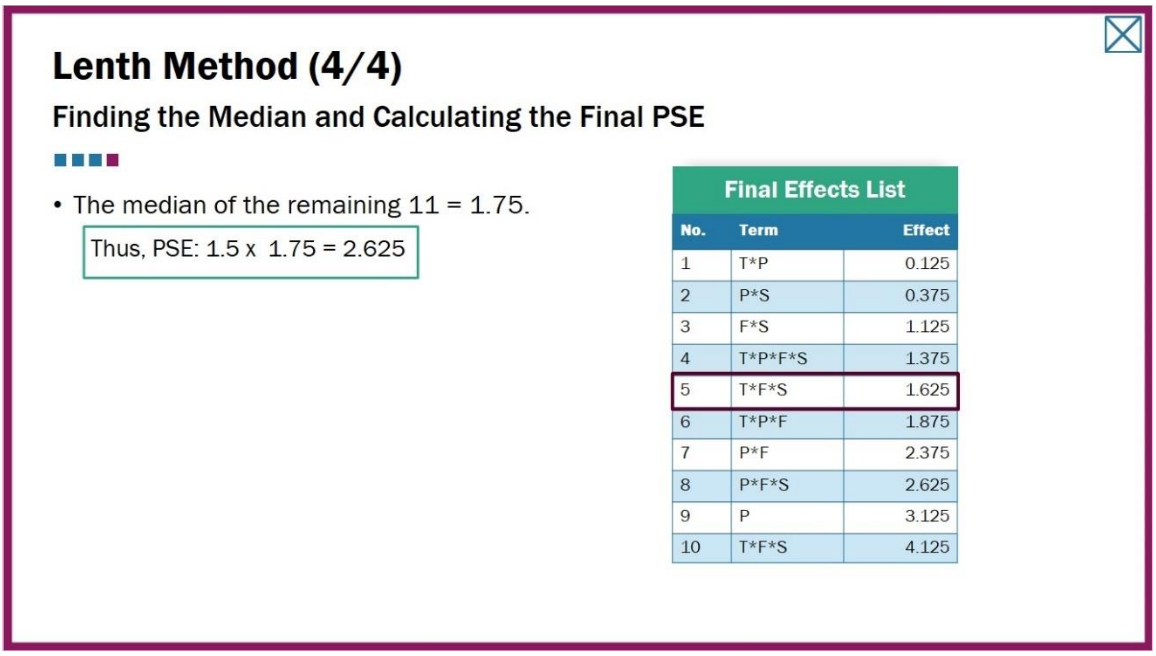
Since it is unlikely that all effects are null, Lenth takes this estimate as a preliminary estimate and refines it by excluding all effects that exceed 2.5 times this estimate (on the basis that the chances of a null effect estimate exceeding this value are negligible) and repeating the exercise on the remaining effects.

Since

$$2.5 \times 3.94 = 9.84,$$

this entails excluding the last five effects in the ordered list.

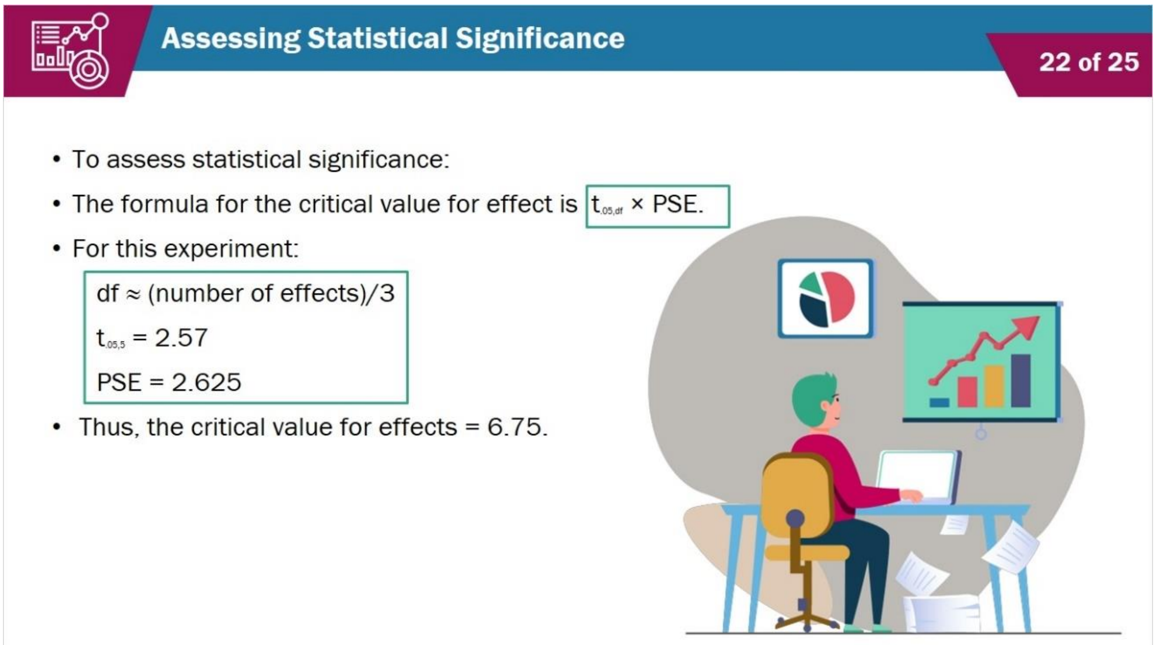
Tab 4: Finding the Median and Calculating the Final PSE



We finally determine the median of the remaining 11, 1.75, and calculating

$$PSE = 1.5 \times 1.75 = 2.625.$$

Slide 22: Assessing Statistical Significance



To convert this into a critical value for determining statistically significant effects, Lenth multiplies by a critical t-value with degrees of freedom equal to total number of effects divided by 3. This formula was arrived at by a combination of probabilistic simulation, trial and error and judgement.

In this case, it leads to

$$df = 15 / 3 = 5.$$

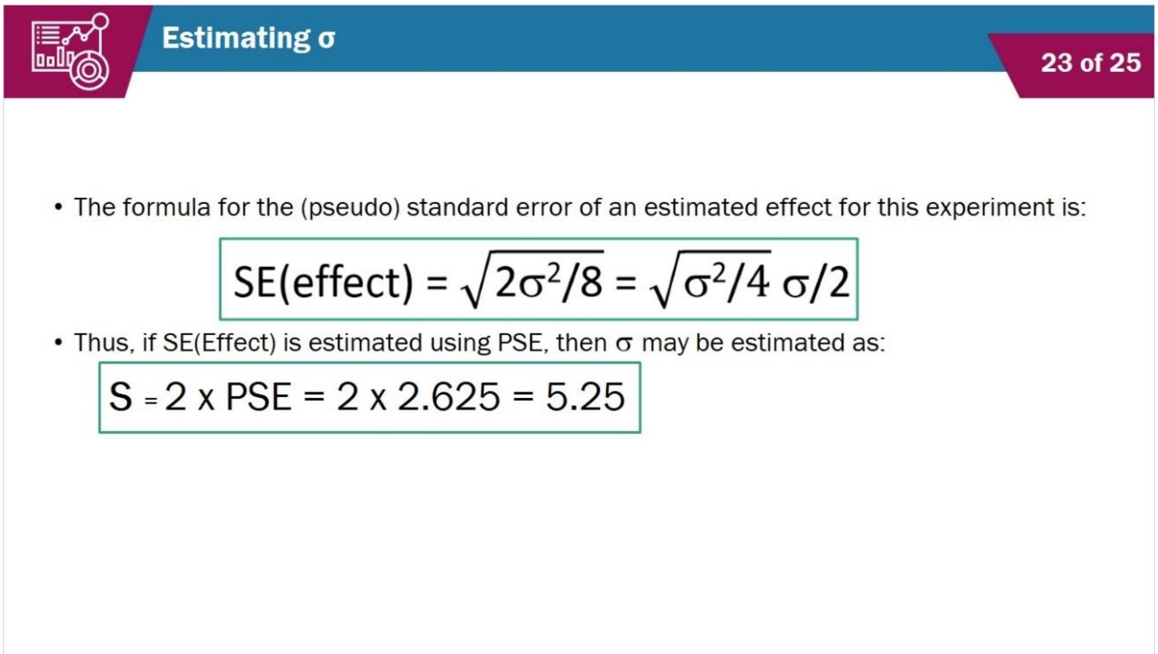
The corresponding 5% critical value for t is

$$t_{0.05,\,5} = 2.57,$$

so that the critical value for effects is

$2.57 \times 2.625 = 6.75$

Slide 23: **Estimating б**



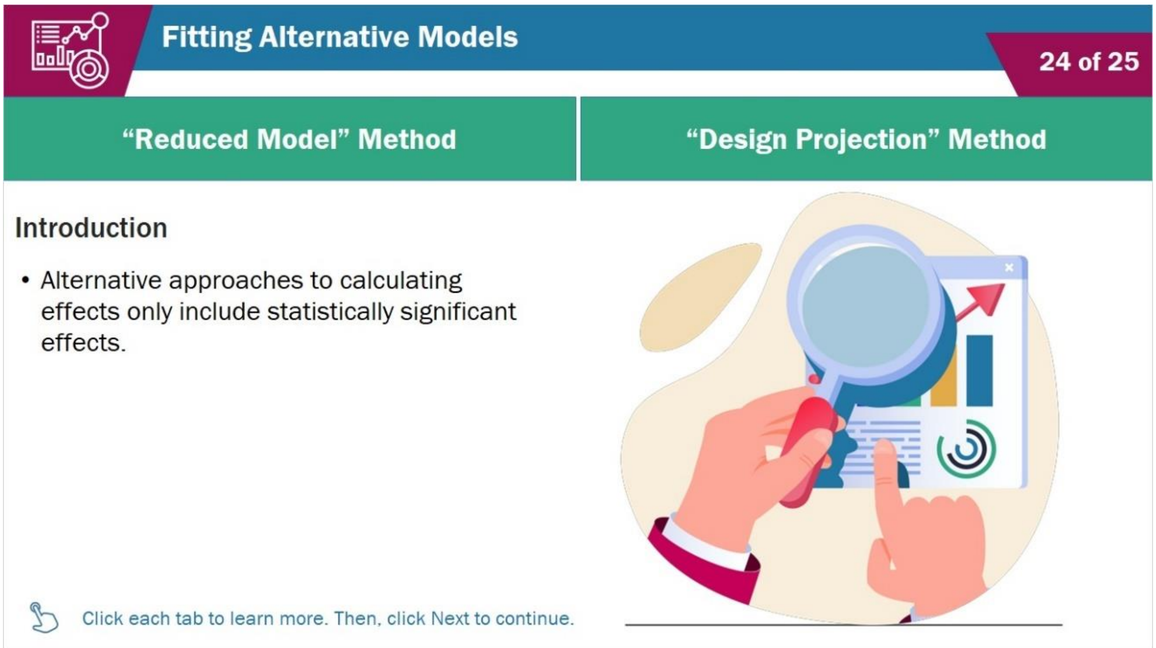
The formula for the estimated standard error of an effect, in this case, a difference of two means based on 8 measurements each, is

$$SE(Effect) = \sqrt{2\sigma^2 / 8} = \sqrt{\sigma^2 / 4} = \sigma / 2;$$

Thus, if SE(Effect) is estimated using PSE, then ☐ may be estimated as

$$s = 2 \times PSE = 2 \times 2.625 = 5.25$$

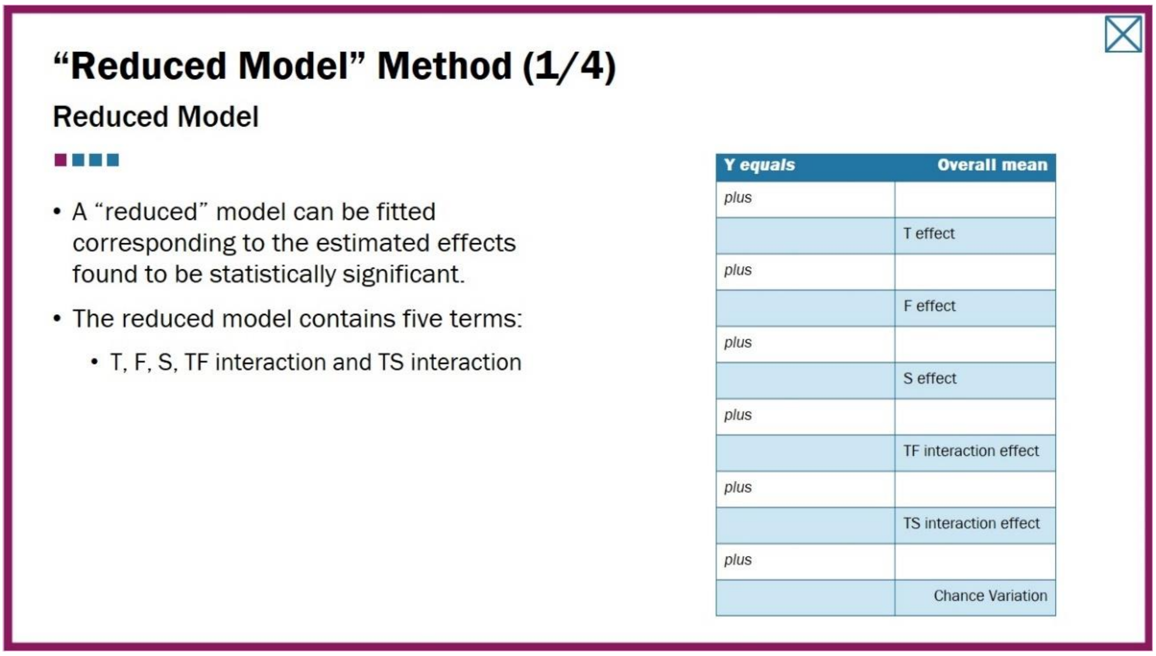
Slide 24: **Fitting Alternative Models**



Implicit in the calculation of the effects as shown previously was an underlying statistical model according to which every factor level and combination of factor levels contributes to the variation in filtration rate in the previous case study. We are now going to look at an alternative approach which include only statistically significant effects.

Click each tab to learn about fitting different models. When you are ready, click next to continue.

Tab 1: **“Reduced Model” Method**



However, since we have seen that most of the estimated experimental effects reflect chance variation, this leads to the suggestion of fitting a "reduced" model corresponding to the estimated effects that have been found to be statistically significant.

The reduced model contains five terms, namely T, F, S, TF interaction, TS interaction. More specifically, the filtration rate Y equals to the overall mean plus T effect, plus F effect, plus S effect, plus TF interaction effect, plus TS interaction effect, and plus chance variation.

Tab 1.1: **Estimated Effects (Minitab)**

"Reduced Model" Method (2/4)

Estimated Effects (Minitab)

- Use Minitab to fit the reduced model and obtain the effect estimates.
- T-values are highly significant.

Estimated Effects and Coefficients for FR (Coded Units)

Term	Effect	Coef	SE Coef	T	P
Constant		70.063	1.104	63.44	0.000
T	21.625	10.812	1.104	9.79	0.000
F	9.875	4.938	1.104	4.47	0.001
S	14.625	7.312	1.104	6.62	0.000
T*F	-18.125	-9.062	1.104	-8.21	0.000
T*S	16.625	8.313	1.104	7.53	0.000

s = 4.41730

Using Minitab to fit the reduced model, we obtain the effect estimates. This corresponds with the results found earlier, with t-values that are very highly significant.

Tab 1.2: **Analysis of Variance (Basis for s)**

"Reduced Model" Method (3/4)

Analysis of Variance (Basis for s)

Analysis of Variance

Source	DF	SS	MS	F-Value	P-Value
T	1	1870.6	1870.56	95.86	0.000
F	1	390.1	390.06	19.99	0.001
S	1	855.6	855.56	43.85	0.000
T*F	1	1314.1	1314.06	67.34	0.000
T*S	1	1105.6	1105.56	56.66	0.000
Error	10	195.1	19.51		
Total	15				

S = 4.42

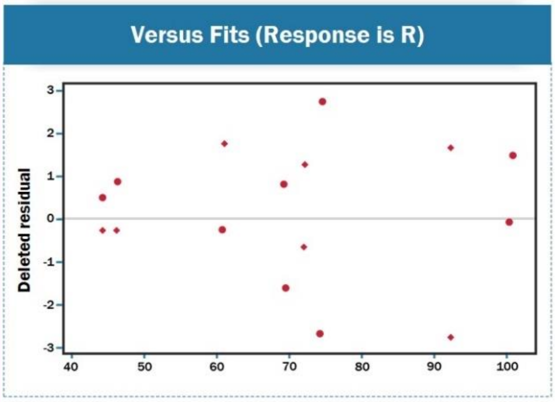
Minitab also produces an Analysis of Variance table.

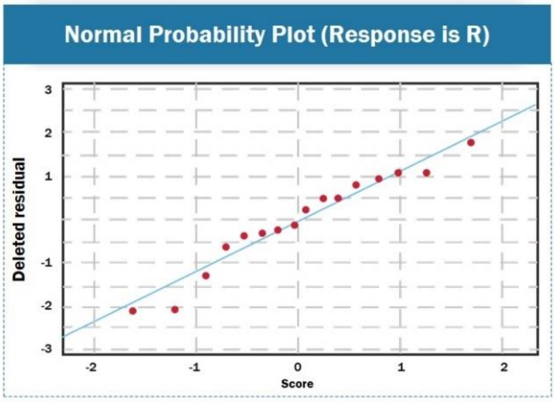
From this, an estimate of s and the corresponding degrees of freedom may be deduced, which is 4.42.

Tab 1.3: Diagnostics: Unremarkable, with no Disturbing Pattern

“Reduced Model” Method (4/4)

Diagnostics: Unremarkable, with no Disturbing Pattern





The diagnostic plots are produced, which are unremarkable since we do not observe any outlier or disturbing pattern.

Tab 2: “Design Projection” Method

“Design Projection” Method (1/2)

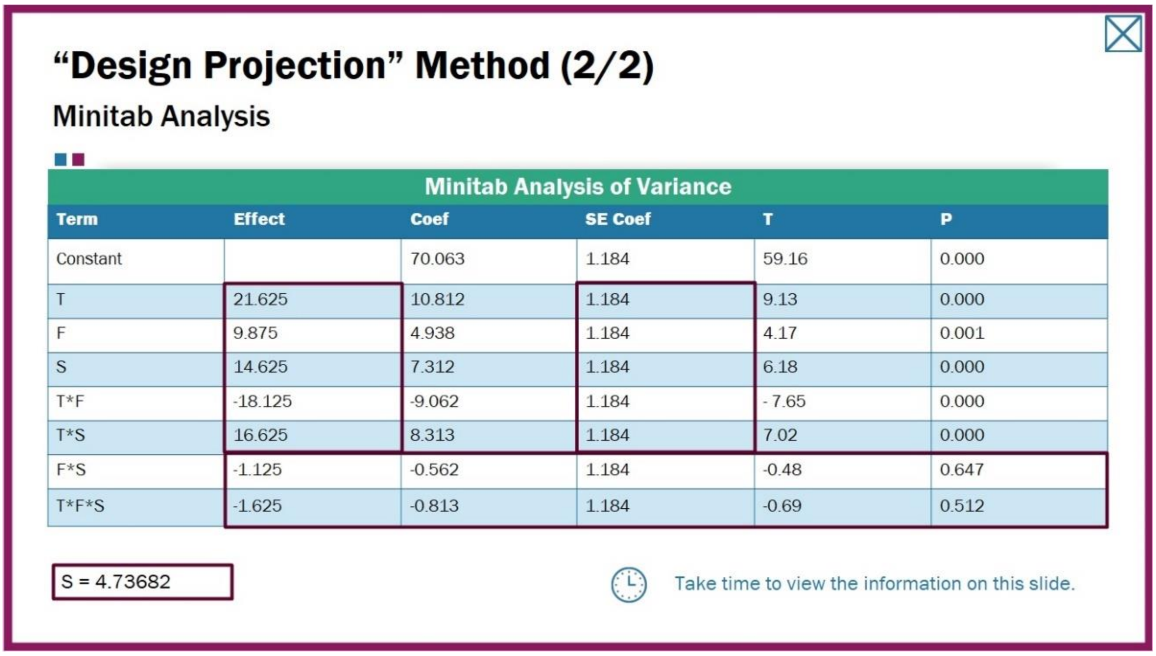
Excluding Pressure Factor

- Pressure is not statistically significant and can be excluded.
- Leave 2^3, duplicated.

T	F	S	FR
–	–	–	45
–	–	–	48
+	–	–	71
+	–	–	65
–	+	–	68
–	+	–	80
+	+	–	60
+	+	–	65
–	–	+	43
–	–	+	45
+	–	+	100
+	–	+	104
–	+	+	75
–	+	+	70
+	+	+	86
+	+	+	96

An alternative approach to reducing the model is the so-called "design projection" method. This involves noting that the Pressure factor is not involved in any statistically significant estimated effects, either on its own through its main effect or in interaction effects involving the other factors. Thus, Pressure may be excluded, effectively reducing the 2^4 design with no replication to a 2^3 with duplicates at each design point.

Tab 2.1: Minitab Analysis



“Design Projection” Method (2/2)

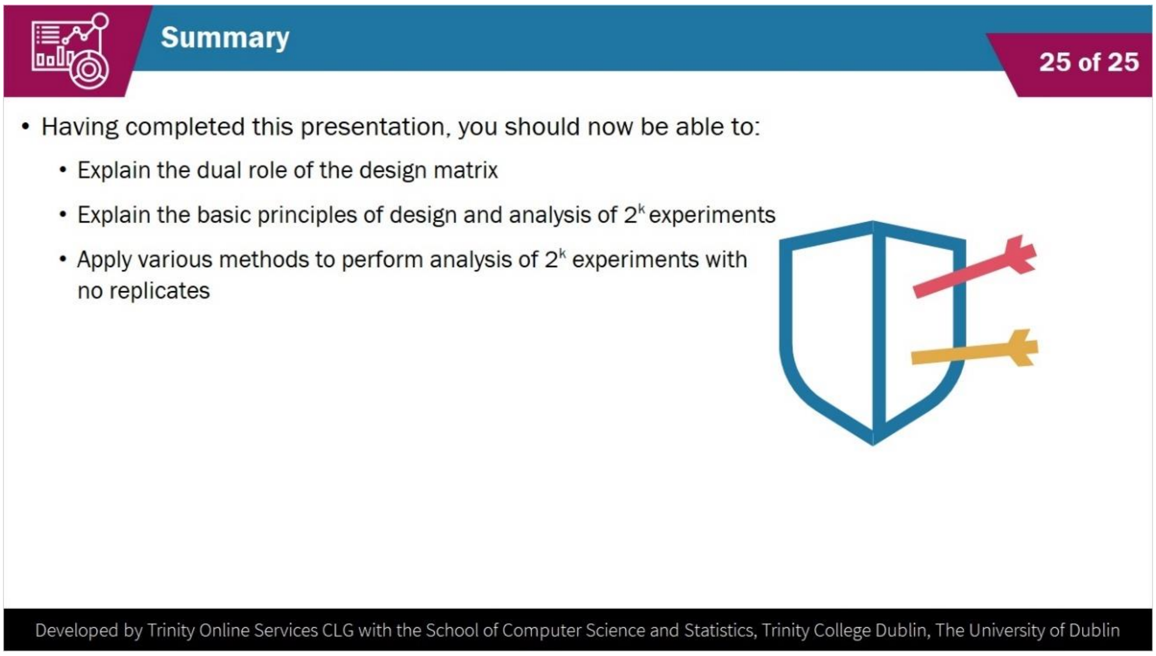
Minitab Analysis

Minitab Analysis of Variance					
Term	**Effect**	**Coef**	**SE Coef**	**T**	**P**
Constant		70.063	1.184	59.16	0.000
T	21.625	10.812	1.184	9.13	0.000
F	9.875	4.938	1.184	4.17	0.001
S	14.625	7.312	1.184	6.18	0.000
T*F	-18.125	-9.062	1.184	- 7.65	0.000
T*S	16.625	8.313	1.184	7.02	0.000
F*S	-1.125	-0.562	1.184	-0.48	0.647
T*F*S	-1.625	-0.813	1.184	-0.69	0.512

The Analysis of Variance table obtained from Minitab is shown with s = 4.73682.

Take time to view the information on this slide. Then click next to continue.

Slide 25: Summary



Having completed this presentation, you should now be able to:

- Explain the dual role of the design matrix,
- Explain the basic principles of design and analysis of 2^k experiments, and
- Apply various methods to perform analysis of 2^k experiments with no replicates