MAT 1

TD 1 : Nombres complexes

Ex1 : $z_1 = (2+i)^4 = (2+i)^2 \times (2+i)^2$

$= (4+4i-1)(4+4i-1)$

$= (3+4i)^2$

$= 9 + 24i - 16$

$= -7 + 24i$

$z_2' = \frac{1-3i}{1-i} = \frac{(1-3i)(1+i)}{(1-i)(1+i)}$

$= \frac{1+i-3i+3}{2} = \frac{4-2i}{2} = \frac{2(2-i)}{2}$

$z_2 = 2-i$

$z_2 = \frac{1-3i}{1-i} - \frac{5-5i}{1+2i}$

$= z_2' - \frac{(5-5i)(1-2i)}{(1+2i)(1-2i)} = \frac{5-10i-5i-10}{5} = \frac{-5-15i}{5} = -1-3i$

$= 2-i-(-1-3i)$

$z_2 = 2-i+1+3i$

$z_2 = 3+2i$

Ex2 a) $|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$

Un argument de $1+i$ est tel que :

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} = 45°$

b) On a $\forall n \in \mathbb{N}$ $|z^n| = |z|^n$

Calculons $|z_1|$ avec $z_1 = 1+i$. D'après la question précédente $|z_1| = \sqrt{2}$ donc $|(1+i)^5| = \sqrt{2}^5 = 4\sqrt{2}$

$\cos\frac{5\pi}{4} = \cancel{\cos(\ldots)} = -\frac{\sqrt{2}}{2}$

$\sin\frac{5\pi}{4} = \cancel{\sin(\ldots)} = -\frac{\sqrt{2}}{2}$

c) $(1+i)^5 = |(1+i)^5| \times (\cos\theta + i\sin\theta)$

θ étant l'argument de $(1+i)^5$

$\arg(1+i) = \frac{\pi}{4}$ et $\arg(z^n) = n\arg(z)$

Donc $\theta = 5\times\frac{\pi}{4} = \frac{5\pi}{4} = 225^0$

Donc $(1+i)^5 = \sqrt{2}^5\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ } pas la forme algébrique.

$(1+i)^5 = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -4 - 4i$

d) $|1-i| = \sqrt{2}$ $\arg(1-i) = -\frac{\pi}{4}$

Donc $(1-i)^5 = \sqrt{2}^5\left(\cos\left(-\frac{5\pi}{4}\right) + i\sin\left(-\frac{5\pi}{4}\right)\right)$

$(1-i)^5 = \sqrt{2}^5\left(\cos\frac{5\pi}{4} - i\sin\left(\frac{5\pi}{4}\right)\right)$

$= 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = -4 + 4i$

Exercice 3 forme exponentielle : $z = |z| . e^{i\theta}$

a) $|z| = \sqrt{1^2 + \sqrt{3}^3} = 2$

$\cos\theta = \frac{-\sqrt{3}}{2}$

donc $\theta = \frac{7\pi}{6} = \frac{-5\pi}{6}$

$\sin\theta = \frac{1}{2}$

donc $z = 2e^{-i\frac{5\pi}{6}}$

b) $|z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\cos\theta = \frac{\sqrt{3}}{2}$

donc $\theta = \frac{2\pi}{3}$

$\sin\theta = -\frac{1}{2}$

donc $z = e^{i\frac{2\pi}{3}}$

MAT 1 c) $|z| = \sqrt{(-\sqrt{3})^2 + 3^2} = \sqrt{3+9} = \sqrt{12}$
$= 2\sqrt{3}$

$\cos\theta = \frac{3}{2\sqrt{3}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$

donc $-\frac{\pi}{6} = \theta$

$\sin\theta = \frac{-\sqrt{3}}{2\sqrt{3}} = -\frac{1}{2}$

donc $z = 2\sqrt{3}\, e^{-i\frac{\pi}{6}}$

Exercice 4

a) $|4+4i| = \sqrt{4^2+4^2} = \sqrt{32} = \sqrt{2\times 16}$
$= 4\sqrt{2}$

donc $|z_1| = |(4+4i)^2|$
$= (4\sqrt{2})^2 = \cancel{32}\ \cancel{4}\ 32$.

$(4+4i)^2 = 16 + 32i - 16$
$= 32i$

Donc $\cos\theta = \frac{32}{32} = 1$

donc $\theta = 0$

$\sin\theta = 0$

D'où : $z_1 = 32e^{i0} = 32$

b) $z_2 = (4+4i)(1-i\sqrt{3}) = 4 - 4i\sqrt{3} + 4i + 4\sqrt{3}$
$z_2 = 4(\sqrt{3}+1) + 4i(1-\sqrt{3})$

$|z_2| = \sqrt{(4(\sqrt{3}+1))^2 + (4(1-\sqrt{3}))^2}$
$|z_2| = \sqrt{16(\sqrt{3}+1)^2 + 16(1-\sqrt{3})^2}$
$|z_2| = 4\sqrt{(\sqrt{3}+1)^2 + (1-\sqrt{3})^2}$
$|z_2| = 4\ \ 4 + 2\sqrt{3} + 4 - 2\sqrt{3}$
$|z_2| = 4\sqrt{8} = 4\times 2\sqrt{2} = 8\sqrt{2}$

Faire la forme exponentielle de chaque facteur :

$4+4i = 4\sqrt{2}\, e^{i\frac{\pi}{4}}$

$1 - i\sqrt{3} = 2e^{i\frac{5\pi}{6}}$

Donc : $(4+4i)(1-i\sqrt{3})$
$= 4\sqrt{2}\, e^{i\frac{\pi}{4}} \times 2e^{i\frac{5\pi}{6}}$
$= 8e^{i\left(\frac{\pi}{4}+\frac{5\pi}{6}\right)}$
$= 8e^{i\frac{13\pi}{12}}$

$$\cos\theta = \frac{4(1-\sqrt{3})}{8\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$$

$\theta = ?$

$$\sin\theta = \frac{4(\sqrt{3}+1)}{8\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

voir au dos

c) $\frac{2}{1-i} = \frac{2(1+i)}{2} = 1+i = z_3$

$|z_3| = \sqrt{2}$ $\quad \cos\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sin\theta = \frac{\sqrt{2}}{2}$ $\quad$ donc $\theta = \frac{\pi}{4}$

$z_3 = \sqrt{2}\, e^{\frac{i\pi}{4}}$

d) soit $z_4 = \frac{z^{19}}{z'^{11}}$ avec $z = 1+i$

$z' = -1+i$

$|z| = \sqrt{2}$ $\quad \arg(z) = \frac{\pi}{4}$

$|z'| = \sqrt{2}$ $\quad \cos\theta = \frac{\sqrt{2}}{2}$ $\quad \sin\theta = -\frac{\sqrt{2}}{2}$

$|z \cdot z'| = |z| \times |z'|$

$\theta = -\frac{\pi}{4}$

Donc $|z_4| = \frac{(\sqrt{2})^{19}}{(\sqrt{2})^{11}} = \sqrt{2}^{8}$

$\arg(z \cdot z') = \arg(z) + \arg(z')$

et $\arg(z_4) = 19\arg(z) - 11\arg(z')$

$= 19 \times \frac{\pi}{4} + 11 \times \frac{\pi}{4}$

$= \frac{31\pi}{4}$ $\quad$ donc $z_4 = \sqrt{2}^{8} e^{i\frac{31\pi}{4}} = 16\, e^{\frac{i31\pi}{4}}$

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MAT 1 Exercice 5

a) Par propriété de la fonction exponentielle :

$$e^{i\frac{\pi}{4}} \times e^{i\frac{\pi}{6}} = e^{i\left(\frac{\pi}{4} - \frac{\pi}{6}\right)}$$

et $\frac{\pi}{4} + \frac{\pi}{6} = \frac{3\pi}{12} - \frac{2\pi}{12} = \frac{\pi}{12}$

Donc la forme exponentielle est $e^{i\frac{\pi}{12}}$

b) $e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$e^{i-\frac{\pi}{6}} = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Donc, la forme algébrique de $e^{i\frac{\pi}{4}} \cdot e^{i\frac{\pi}{6}}$ est :

$$\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}i + \frac{\sqrt{6}}{4}i + \frac{\sqrt{2}}{4}$$

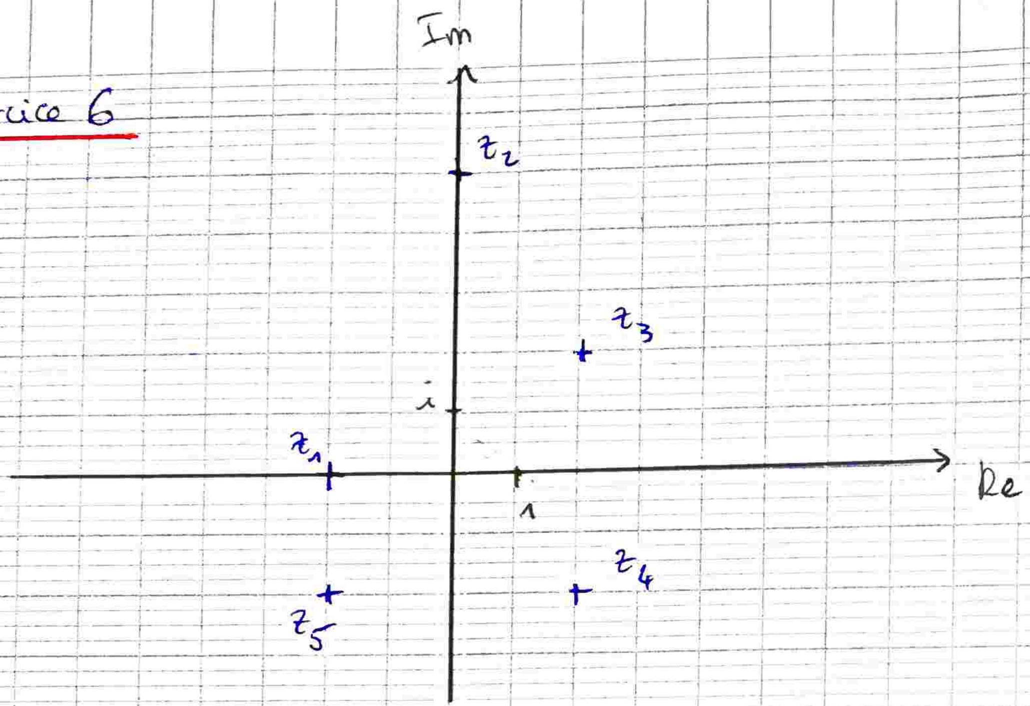
$$= \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4}i = \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}}i$$

c) Donc par identification :

$$e^{i\frac{\pi}{12}} = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}-1}{2\sqrt{2}}i$$

Donc $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}$ et $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}$

Exercice 6



a) $z_1 = -2 \quad |z_1| = 2 \quad \theta_1 = \pi$

$z_1 = 2e^{i\pi}$

b) $z_2 = 5i \quad |z_2| = 5 \quad \theta_2 = \frac{\pi}{2}$

$z_2 = 5e^{i\frac{\pi}{2}}$

c) $z_3 = 2 + 2i \quad |z_3| = 2 \quad \theta_3 = \frac{\pi}{4}$

$z_3 = 2e^{i\frac{\pi}{4}}$

d) $z_4 = 2 - 2i \quad |z_4| = 2 \quad \theta_4 = -\frac{\pi}{4}$

$z_4 = 2e^{-i\frac{\pi}{4}}$

e) $z_5 = -2 - 2i \quad |z_5| = 2 \quad \theta_5 = -\frac{3\pi}{4}$

$z_5 = 2e^{-\frac{3\pi i}{4}}$

Exercice 7

a) $\bar{z} = 2e^{-i\frac{\pi}{4}} \qquad -z = -2e^{i\frac{\pi}{4}}$

$z = \sqrt{2} + i\sqrt{2}$ donc $iz = -\sqrt{2} + i\sqrt{2}$

donc $iz = 2e^{i\frac{3\pi}{4}}$

$\frac{1}{z} = \frac{1}{2e^{i\frac{\pi}{4}}} = \frac{1}{2} \times e^{-i\frac{\pi}{4}}$