Algorithms Worksheet 1

For each part of a question write the answer and include workings. Each question is worth two marks, except Q3 which is worth four, there is also two marks for attendance, but the number of marks are capped at ten.

1. This question is about estimating the algorithmic complexity of evaluating a polynomial. Here, consider fixed sized variables, so multiplication and addition take roughly one step, irrespective of how many digits the number has. Once again, powers are calculated by multiplication.

   a) What is the big-oh complexity of evaluating, that is finding the value of $p(x)$, of an order n polynomial

   $$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

   using straight-forward substitution?

   b) Horner's method is a quicker method for evaluating a polynomial. If x_o is the value that the polynomial needs to be evaluated on, let $b_n = a_n$ and then

   $$b_{n-1} = a_{n-1} + x_o b_n$$

   and

   $$b_{n-2} = a_{n-2} + x_0 b_{n-1}$$

   right down to

   $$b_0 = a_0 + x_0 b_1$$

   and $b_0 = p(x_0)$ is the answer. What is the big-oh complexity?

   Solution: So calculating x^i is $i-1$ multiplications, there are faster ways to work out powers, but you are told that they are calculated by multiplication; multiplying by a_i is one more, so that is i calculations, thus evaluating the polynomial is $1+2+3+\ldots n$ multiplications, along with $n-1$ additions; thus this is $\Theta(n^2)$. However, using Horner's method each b_i is a few calculations and there are n b_is, so that means it is $\Theta(n)$.

2. This question is about the asymptotic behavior of different functions, in each case give big-Theta for $T(n)$; if $T(n)$ was the worst case run-time this would give big-Oh. There is no need to give any working for this problem.

   a) $T(n) = n^5 + \frac{1}{n} + n(n-1)(n+2)^4$
   b) $T(n) = n^2 \log n + n^3$
   c) $T(n) = 2^n + n!$
   d) $T(n) = \sum_{i=0}^{n} i$
   e) $T(n) = \sqrt{n}n + n$
   f) $T(n) = n^2/\log n + n$
   g) $T(n) = (n^5 + 345n^4 + 36n)/(n^2 + 2n + 1)$
   h) $T(n) = 1/(n^2 + 2n + 1)$

i) $T(n) = [(n+1)(n+2)(n+3)]/[(n+4)(n+5)]$

j) $T(n) = n!/(n-1)!$

Solution: So just take the leading term in n each time

a) $\Theta(n^6)$

b) $\Theta(n^3)$

c) $\Theta(n!)$

d) $\Theta(n^2)$

e) $\Theta(\sqrt{n}n)$

f) $\Theta(n^2/\log n)$

g) $\Theta(n^3)$

h) $\Theta(1)$

i) $\Theta(n)$

j) $\Theta(n)$

3. This question is about solving recursion relations using telescoping. In each case find the value of $T(n)$ by telescoping. Check your answer by substitution, it is permissible to combine these two steps by using telescoping to come up with an ansatz and then substituting it to fix values in the ansatz. Write down the big-Theta for the solution.

   a) $T(n) = T(n-1) + 3$ with $T(0) = 1$
   b) $T(n) = T(n-1) + 3$ with $T(1) = 1$
   c) $T(n) = 2T(n-1) + 3$ with $T(0) = 1$
   d) $T(n) = 3T(n-1) + 2$ with $T(0) = 1$

   Solution: So

$$T(n) = T(n-1) + 3 = T(n-2) + 3 + 3 = T(n-3) + 3 \cdot 3 = \dots$$

It would be easy to solve this directly by telescoping, but lets use an ansatz, since there is clearly a 3 for each iteration we'll try $T(n) = 3n + A$, substituting in

$$3n + A = 3n - 3 + A + 3$$

so the equation holds for all A, alternatively substituting a more general ansatz of the form $T(n) = Bn + A$ would give you

$$Bn + A = Bn - B + A + 3$$

which holds for all A and $B = 3$. Either way $T(n) = 3n + A$, now the initial condition is $T(0) = 1$ but setting $n = 0$ gives $T(0) = A$ so $A = 1$ and the solution is $T(n) = 3n+1$. For the next part the question is exactly the same except $T(1) = 1$ so taking $T(n) = 3n + A$ again $T(1) = 3 + A = 1$ and hence $A = -2$. For the next part

$$T(n) = 2T(n-1) + 3 = 4T(n-2) + 2 \cdot 3 = 2^3T(n-3) + (4 + 2 + 1) \cdots 3 = \cdots$$

Doing this directly by telescoping requires skill because you need to know that $2^{n-1} + 2^{n-2} + \ldots + 1 = 2^n - 1$, but you could guess

$$T(n) = 2^nA + B$$

and hope for the best, substituting in gives

$$2^nA + B = 2^nA + 2B + 3$$

so this is a solution when $B = -3$, hence

$$T(n) = A2^n - 3$$

and the initial condition gives $T(0) = A - 3 = 1$ so $A = 4$ and

$$T(n) = 2^{n+2} - 3$$

Finally, for the last $T(n) = A3^n + B$ is obviously a good ansatz based on the previous example, now

$$A3^n + B = A3^n + 3B + 2$$

so $B = -1$ and the initial condition gives $A = 2$, hence

$$T(n) = 2 \cdot 3^n - 1$$

4. Use telescoping to guess an ansatz and then solve $T(n) = T(n-1) + 3n$ with $T(1) = 1$. What is the corresponding big-Theta?

   Solution: So telescoping gives a bit that looks like $3n + 3(n-1) + 3(n-2) + \ldots$, hence a good ansatz is $T(n) = An^2 + Bn + C$, substituting that in gives

$$An^2 + Bn + C = A(n-1)^2 + B(n-1) + C + 3n = An^2 - 2An + A + Bn - B + C + 3n$$

   or, after cancelling

$$-2An + A - B + 3n = 0$$

   so $A = 3/2$ and $B = A$. Thus

$$T(n) = \frac{3}{2}n^2 - \frac{3}{2}n + C$$

   and the initial condition means $C = 1$.