EMAT10001 Workshop Sheet 15.

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Work sheet

1. The general second order Runge Kutta method for $\dot{y} = f(y)$ is

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + \alpha\delta t k_1) \end{aligned} \tag{1}$$

and

$$y_{n+1} = y_n + \left[\left(1 - \frac{1}{2\alpha}\right)k_1 + \frac{1}{2\alpha}k_2\right]\delta t \tag{2}$$

Show that this gives the Taylor series up to second order. Note that $\alpha = 1/2$ gives the midpoint method. This was the last question on last week's worksheet, so lots of people didn't get a chance to think about it and it worth doing since it does give a good idea of how Runge Kutta works.

Solution: (cut and pasted from work sheet 14 solutions). Same craic as in the lectures, expand out k_2 using the Taylor series

$$k_2 = f(y_n + \alpha\delta t k_1) = f(y_n) + \left.\frac{df}{dy}\right|_{y=y_n} \alpha\delta t k_1 + \dots \tag{3}$$

then use the chain rule

$$\frac{df}{dy}k_1 = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dt} = \frac{d^2f}{dt^2} \tag{4}$$

so

$$k_2 = \dot{y}(t_n) + \alpha\delta t\ddot{y}(t_n) + \dots \tag{5}$$

so

$$y_{n+1} = y_n + \left[\left(1 - \frac{1}{2\alpha}\right)k_1 + \frac{1}{2\alpha}k_2\right]\delta t = y(t_n) + \dot{y}(t_n) + \delta t\ddot{y}(t_n) + \dots \tag{6}$$

which is the first three terms of the Taylor expansion, as required.

2. Here is a second order differential equation

$$\frac{d^2f}{dt^2} + \frac{df}{dt} - 6f = 0 \tag{7}$$

This can also be solved using an ansatz of the form $A\exp(\lambda t)$; the difference is that there will be two different λs, lets say λ_1 and λ_2. Since the equation is linear you can add them to give a solution with two arbitrary constants:

$$f(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} \tag{8}$$

This is what you expect for a second order differential equation, you need two initial conditions, here use $f(0) = 0$ and $\dot{f}(0) = -1$.

Solution: Substitute in the ansatze, so $\dot{f}(t) = A\lambda \exp(\lambda t)$ and $\ddot{f}(t) = A\lambda^2 \exp(\lambda t)$, then cancel the A and the exponential part to get

$$\lambda^2 + \lambda - 6 = 0 \tag{9}$$

which should factorize nicely if the question has been set nicely, which this has:

$$(\lambda - 2)(\lambda + 3) = 0 \tag{10}$$

that is $\lambda = 2$ or $\lambda = -3$ giving

$$f(t) = A_1 e^{2t} + A_2 e^{-3t} \tag{11}$$

Now this means $f(0) = A_1 + A_2$ and, since

$$\dot{f}(t) = 2A_1 e^{2t} - 3A_2 e^{-3t} \tag{12}$$

we have $\dot{f}(0) = 2A_1 - 3A_2$. Thus, the initial condition says $f(0) = 0$ so $A_1 = -A_2$ and $\dot{f}(0) = 2A_1 - 3A_2 = -1$, hence $5A_1 = -1$ and

$$f(t) = -\frac{1}{5}e^{2t} + \frac{1}{5}e^{-3t} \tag{13}$$

3. Here is another second order differential equation

$$\frac{d^2 f}{dt^2} - 4f = 0 \tag{14}$$

Solve this with $f(0) = 0$ and $\dot{f}(0) = 1$.

Solution: The ansatz gives

$$\lambda^2 = 4 \tag{15}$$

or

$$f(t) = A_1 e^{2t} + A_2 e^{-2t} \tag{16}$$

and then the initial condition is $A_1 + A_2 = 0$ and $2A_1 - 2A_2 = 1$ so $A_1 = 1/4$ and $A_2 = -1/4$

$$f(t) = \frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} \tag{17}$$

4. This differential equation

$$\frac{d^2 f}{dt^2} + f = 0 \tag{18}$$

with $f(0) = 0$ and $\dot{f}(0) = 1$ doesn't work so well, you end up with complex λs, however, if you keep your nerve and use the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{19}$$

it will work out; just bundle $A_1 + A_2$ into one arbitrary constant $C_1 = A_1 + A_2$ and $i(A_1 - A_2)$ into another $C_2 = i(A_1 - A_2)$; C_1 and C_2 should turn out to be real, the detour through complex numbers is just that, a detour.

Solution: The ansatz gives

$$\lambda^2 = -1 \tag{20}$$

or

$$f(t) = A_1 e^{it} + A_2 e^{-it} \tag{21}$$

Now using the Euler formula

$$f(t) = A_1(\cos t + i \sin t) + A_2(\cos t - i \sin t) \tag{22}$$

or

$$f(t) = C_1 \cos t + C_2 \sin t \tag{23}$$

Of course while C_1 and C_2 could be complex, they aren't going to be for a problem with real initial conditions. $f(0) = 0$ says $C_1 = 0$, $\dot{f}(0) = 1$ says $C_2 = 1$.

5. Some integration examples; integrate

   (a) $\int (2x + 2)e^{x^2+2x+3} dx$

   (b) $\int (x^2 + 1)/(x^3 + 3x) dx$

   (c) $\int \sqrt{x} dx$

   (d) $\int \sqrt{7x + 1} dx$

   Solution: So for the first one let $s = x^2 + 2x + 3$ and it become a simple exponential integral since $(2x + 2)dx = ds$, the second, let $s = x^3 + 3x$ and it become $(\int 1/s ds)/3$ because $(3x^2 + 3)dx = ds$. The next one is the integration formula directly with $n = 1/2$

   $$\int \sqrt{x} dx = \int x^{1/2} dx = \frac{2x^{3/2}}{3} \tag{24}$$

   and the last one is the same thing but with $s = 7x + 1$ so $ds = 7dx$.

6. Integrating the square pulse. Say f is square pulse

   $$f(t) = \begin{cases} 1 & -\pi/2 < t < \pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

   Thus $f(t)$ is one between $-\pi/2$ and $\pi/2$ but zero everywhere else. What is

   $$I = \int_{-\pi}^{\pi} f(t) \cos t dt \tag{26}$$

   what about

   $$I = \int_{-\pi}^{\pi} f(t) \cos nt dt \tag{27}$$

   for n an integer.

Solution: So the thing to realize is that $f(t)$ is zero for $t > \pi/2$ and $t < -\pi/2$, so those parts of the interval don't contribute

$$I = \int_{-\pi}^{\pi} f(t) \cos t dt = \int_{-\pi/2}^{\pi/2} f(t) \cos t dt \tag{28}$$

but now t is inside the $(-\pi/2, \pi/2)$ interval where $f(t) = 1$ so

$$I = \int_{-\pi/2}^{\pi/2} \cos t dt = \sin \pi/2 - \sin(-\pi/2) = 2 \tag{29}$$

For the version with n dealing with the $f(t)$ gets us to

$$I = \int_{-\pi/2}^{\pi/2} \cos nt dt \tag{30}$$

then let $s = nt$ so $dt = ds/n$ and $t = \pi/2$ means $s = n\pi/2$ and so on, giving us

$$I = \frac{1}{n} \int_{-n\pi/2}^{n\pi/2} \cos s ds = \frac{2}{n} \sin n\pi/2 \tag{31}$$

and this is equal to $2/n$ is n is odd, but zero is n is even, since $\sin n\pi = 0$ for integer n.