1. Solve
$$\frac{df}{dt} = 5(1-f) \tag{1}$$
with $f(0) = 0$.

Answer: First, as always, make it more like a problem you already know, let
$$g(x) = 1 - f(x) \tag{2}$$
so $f(0) = 0$ means $g(0) = 1$ and $dg/dt = -df/dt$ so
$$\frac{dg}{dt} = -5g \tag{3}$$
Now substitutte $g = Aert$ and find $r = -5$ so
$$g(t) = Ae^{-5t} \tag{4}$$
and $g(0) = 1$ gives $A = 1$ and so the solution is
$$f(t) = 1 - e^{-5t} \tag{5}$$

2. Calculate a Taylor series for $\cos x$.

Answer: So
$$\frac{d\cos x}{dt} = -\sin t \tag{6}$$
and
$$\frac{d^2 \cos x}{dt^2} = -\cos t \tag{7}$$
and then it goes around again with the extra minus and you are back where you started. Now $\cos 0 = 1$ and $\sin 0 = 0$ so
$$\cos t = \sum_{n \text{ even}} (-1)^{n/2} \frac{1}{n!} \tag{8}$$

3. For $f(x, y) = xy$ work out the directional derivative in the $(1, 3)$ direction at $(1, 1)$; don't forget to normalize the direction vector.

Answer: So let $\mathbf{u} = (1, 3)$, it has length $\sqrt{10}$ so the corresponding normalized vector is $\hat{\mathbf{u}} = (1/\sqrt{10}, 3/\sqrt{10})$. Now
$$\operatorname{div} f = (y, x) \tag{9}$$
and hence $\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = (x + 3y)/\sqrt{10}$ and at $(1, 1)$ this gives $D_{\hat{\mathbf{u}}} f|_{(1,1)} = 4/\sqrt{10}$.

4. The third differential operator is curl; it acts on vector fields to give another vector field. It is only defined in three dimensions and has quite a complicated form

$$\mathrm{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \tag{10}$$

Show $\mathrm{grad}\,(\mathrm{curl}\mathbf{v}) = 0$.

Answer:This is answered through sweat, toil and tears:

$$\mathrm{grad}\,(\mathrm{curl}\mathbf{v}) = \frac{\partial^2 v_3}{\partial y \partial x} - \frac{\partial^2 v_2}{\partial z \partial x} + \frac{\partial^2 v_1}{\partial z \partial y} - \frac{\partial^2 v_3}{\partial x \partial y} + \frac{\partial^2 v_2}{\partial x \partial z} - \frac{\partial^2 v_1}{\partial y \partial z} \tag{11}$$

and since the order of the partial differentiations doesn't matter, this all cancels away to nothing.

5. Find the Fourier series for $\sin^3 t$; a quick way to do this is to regard it as a trigonometry problem, rather than a Fourier series problem, that is use the trigonometric identities to express it in terms of sines and cosines, rather than doing all the integrals: so start by writing $\sin^3 t = \sin^2 t \sin t$ and then write $\sin^2 t$ in terms of $\cos 2t$.

Answer: So this could be done with the integrals like any other Fourier series, but the funny thing is that it can also be done with trignometry.

$$\sin^3 t = \sin^2 t \sin t \tag{12}$$

Then using $\cos 2t = \cos^2 t - \sin^2 t$ and $1 = \sin^2 t + \cos^2 t$ we have

$$\sin^2 t = (1 - \cos 2t)/2 \tag{13}$$

and so

$$\sin^3 t = \frac{1}{2} \sin t - \frac{1}{2} \cos 2t \sin t \tag{14}$$

and now using your table of trignometric identities

$$\cos 2t \sin t = (-\sin t + \sin 3t)/2 \tag{15}$$

or

$$\sin^3 t = \frac{3}{4} \sin t - \frac{1}{4} \sin 3t \tag{16}$$

which is in the form of a Fourier series.