EMAT10001 Workshop Sheet 16 outline solutions.

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Work sheet

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ for

   (a) $f(x,y) = xy\sin xy$
   (b) $f(x,y) = e^{x^2+y^2}$
   (c) $f(x,y) = xe^{xy}$
   (d) $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$

   Solutions: So you just treat the other variables as constants, for brevity $\partial_x f = \partial f/\partial x$ and the same for y. This is a bad notation in every respect except being shorter than all the good notations.

   (a) $\partial_x f = y\sin xy + xy^2\cos xy$ and $\partial_y f = x\sin xy + x^2y\cos xy$
   (b) $\partial_x f = 2xe^{x^2+y^2}$ and $\partial_y f = 2ye^{x^2+y^2}$.
   (c) $\partial_x f = e^{xy} + xye^{xy}$ and $\partial_y f = x^2e^{xy}$.
   (d) $\partial_x f = 3x^2 + 6xy + 3y^2$ and $\partial_y f = 3x^2 + 6xy + 3y^2$

2. Find $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ for

   (a) $f(x,y,z) = xy\ln z$
   (b) $f(x,y,z) = x^2 + y^2 + z^2$
   (c) $f(x,y,z) = x\sin xyz$

   Solutions:

   (a) $\partial_x f = y\ln z$, $\partial_y f = x\ln z$ and $\partial_z f = xy/z$
   (b) $\partial_x f = 2x$ and so on.
   (c) $f(x,y) = x\sin xyz$

3. For $f(x,y) = x^3 + 3x^2y + 3xy^2 + y^3$ work out the directional derivative in the $(2,1)$ direction at $(1,0)$; don't forget to normalize the direction vector.

4. Find the gradient of $f(x,y) = x + y^2$ and $f(x,y) = \sqrt{x^2+y^2}$.

5. Going to three-dimensions in the obvious way, what is the gradient of

$$f(x,y,z) = \sin x + \cos y + \sin x \tag{1}$$

   at $(\pi, 0, \pi)$.

6. The diverence is a differential operator acting on a vector field to give a scalar, that's the other way around to the gradient which acts on a scalar to give a vector field. It is defined by

$$\operatorname{div}\mathbf{v}(x,y) = \nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \tag{2}$$

What is the divergence of (x, y)? What about $(y, -x)$?

7. The Laplacian operator $\Box f = \operatorname{div}(\operatorname{grad} f)$. Write down the formula for $\Box f$ in terms of partial derivatives.

8. If a surface is given by $f(x, y, z) = c$ where c is a constant, then $\operatorname{grad} f$ is perpendicular to the surface. Examine the two-dimensional version by considering $x^2 + y^2 = 1$. What is the gradient? On $x^2 + y^2 = 1$ we can write $x = \cos\theta$ and $y = \sin\theta$ since these satisfy $x^2 + y^2 = 1$. What does the gradient look like on the surface? Can you find a vector perpendicular to it, and therefore parallel to the surface.

Exercise sheet

The difference between the work sheet and the exercise sheet is that the solutions to the exercise sheet won't be given and the problems are designed to be more suited to working on on your own, though you are free to discuss them in the work shop if you finish the work sheet problems. Selected problems from the exercise sheet will be requested as part of the continual assessment portfolio.

1. Find $\partial f/\partial x$ and $\partial f/\partial y$ for
   (a) $f(x, y) = xy \ln xy$
   (b) $f(x, y) = 12x^4y + y^2$
   (c) $f(x, y) = xe^{y^2}$

2. Find $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ for
   (a) $f(x, y, z) = xyz$
   (b) $f(x, y, z) = (x - y)^2 + (y - z)^2 + (z - x)^2$

3. For $f(x, y) = xy$ work out the directional derivative in the $(1, 3)$ direction at $(1, 1)$; don't forget to normalize the direction vector.

4. The third differential operator is curl; it acts on vector fields to give another vector field. It is only defined in three dimensions and has quite a complicated form

$$\operatorname{curl}\mathbf{v} = \left(\frac{\partial v_2}{\partial z} - \frac{\partial v_3}{\partial y}, \frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}, \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x}\right) \tag{3}$$

   Show $\operatorname{grad}(\operatorname{curl}\mathbf{v}) = 0$.

Challenge

First four to email or tell me the answer:

$$
\begin{aligned}
&> + + + + +[< ++ > -] < + + + > + + + + [> + + + < -] > [< + + ++ > -]\\
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&+ + + + + > + < [> [>> + > + <<< -] >>> [<<< + >>> -] <<<> [< + > -]\\
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&[<< + >>>>> + <<< -] << [>> + << -] >>>>> .[-]\\
&<<<<<<< . >>>>>> [< + > -] <<< -]
\end{aligned}
\tag{4}
$$

The start of what?