EMAT10001 Workshop Sheet 14 outline solutions.

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Work sheet

1. Revise differentiating again!

   (a) Find df/dx of $f(x) = \tan x$ using $\tan x = \sin x/\cos x$.

   (b) We know $\sin^2 x + \cos^2 x = 1$, differentiate both sides of this equation.

   Solutions: I guess you could use the quotient rule for the first one, but I can never remember it so I'll use the product rule. First let $y = 1/\cos x$ so, using the chain rule

$$\frac{dy}{dx} = \sin x/\cos^2 x \tag{1}$$

Thus

$$\frac{d}{dx}\tan x = \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \tag{2}$$

As for the second,

$$\frac{d}{dx}\sin^2 x = 2\sin x\cos x \tag{3}$$

whereas

$$\frac{d}{dx}\cos^2 x = -2\sin x\cos x \tag{4}$$

which add to zero, which is good since $d/dx1$ is obviously zero.

2. What is the Taylor expansion of $\tan t$ up to and include the t^3 term?

   Solutions: Well we already saw that

$$\frac{d}{dt}\tan t = \cos^{-2} t \tag{5}$$

Differentiating that gives

$$\frac{d^2}{dt^2}\tan t = 2\sin t\cos^{-3} t = 2\tan t\sec^2 t \tag{6}$$

where I've used the notation $\sec t = \tan t$. Next

$$\frac{d^3}{dt^3}\tan t = 2\sec^2 t\sec^2 t + 4\tan^2 t\sec^4 t \tag{7}$$

Hence, using $\sec 0 = 1$ and $\tan 0 = 1$

$$\tan t = t + \frac{1}{3}t^3 + O(t^4) \tag{8}$$

3. If $f(t) = \arctan t$ then
$$\frac{df}{dt} = \frac{1}{1+t^2} \tag{9}$$
This is derived using the chain rule and a trick called implicit differentiation. Basically let $y = \arctan x$ so $x = \tan y$, differentiate both sides with respect to x
$$1 = \frac{dx}{dx} = \frac{d}{dx}\tan y = \frac{dy}{dx}\frac{d}{dy}\tan y \tag{10}$$
and then do some messing to get dy/dx in terms of x. Do that!

Solutions: Well it's mostly done
$$1 = \frac{dy}{dx}\sec^2 y \tag{11}$$
and we just need to get back to something involving $\tan y$, in fact
$$\sin^2 y + \cos^2 y = 1 \tag{12}$$
when you divide by $\cos^2 y$ gives
$$\tan^2 y + 1 = \sec^2 y \tag{13}$$
so our equation is
$$1 = \frac{dy}{dx}(1+\tan^2 y) \tag{14}$$
which gives the answer since $\tan y = x$.

4. The Taylor expansion of $\arctan x$ is
$$\arctan x = \sum_{n\,\mathrm{odd}} (-1)^{(n-1)/2}\frac{x^n}{n} \tag{15}$$
Check this as far as the x^3 term. [There is a very elegant derivation of this Taylor series, basically $1/(1+x)$ has a known Taylor expansion, up to a sign this is asked in the exercise sheet; this allows you to calculate the expansion of $1/(1+x^2)$ and from this you can read off all the derivative you need for the expansion of $\arctan x$, if you are feeling ambitious you can try to do this.]

Solutions: So
$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \tag{16}$$
and using the chain rule
$$\frac{d^2}{dx^2}\arctan x = -\frac{2x}{(1+x^2)^2} \tag{17}$$
and
$$\frac{d^3}{dx^3}\arctan x = -\frac{2(1+x^2)^2 - 8x^2(1+x^2)}{(1+x^2)^4} \tag{18}$$
and substituting in $x = 0$ gives one, zero and -2, as required.

5. What is $\arctan 1$?

   Solutions:
$$\arctan 1 = \pi/4 \tag{19}$$

6. Use the Taylor expansion of $\arctan 1$ to write down a formula for calculating π and do the first six terms.

   Solutions: Well we have
$$\pi = 4\sum_{n\,\mathrm{odd}}(-1)^{(n-1)/2}\frac{1}{n} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots \tag{20}$$
   and so this give 3.3396 whici isn't very close, in fact, this series approximation of π is very slow to converge, there are other, similar, series that converge much faster.

7. The series for $\arctan 1$ is disasterously slow if you want to use it to calculate π; in fact getting ten decimal places correct requires 5,000,000,000 terms. However, there are other approaches to using the $\arctan x$ series. According to Wikipedia, in 1699, English mathematician Abraham Sharp used this series with $x = \sqrt{1/3}$ to compute π to 71 digits, breaking the previous record of 39 digits, his record only stood for seven years when a new and faster converging series for π was found; this was
$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \tag{21}$$
   You can prove this with clever use of trignometric identities, you shouldn't do that here, but you should use the formula to find an approximation of π, take three terms for example. By the way, there is a graph at
   `http://en.wikipedia.org/wiki/File:Record_pi_approximations.svg`
   showing how the number of known digits of π has changed over history.

   Solutions: Well
$$4\arctan\frac{1}{5} = \frac{4}{5} - \frac{4}{3}\frac{1}{125} + \frac{4}{5}\frac{1}{3125} + \ldots \approx 0.789589333 \tag{22}$$
   and
$$\arctan\frac{1}{239} = \frac{1}{239} - \frac{1}{3}\frac{1}{57121} + \ldots \approx 0.00417826 \tag{23}$$

8. The binomial theorem tells us that
$$(1+x)^n = \sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) x^r \tag{24}$$
   Prove this using the Taylor expansion.

   Solutions: Just a matter of differentiating:
$$f(x) = (1+x)^n \tag{25}$$

so

$$\frac{df}{dx} = n(1+x)^{n-1} \tag{26}$$

and so on, so

$$\frac{d^r f}{dx^r} = n(n-1)(n-2)\dots(n-r+1)(1+x)^{n-r} \tag{27}$$

and so on until $n = r$, in which case this is a constant and the next term is zero, hence the Taylor expansion is

$$(1+x)^n = \sum_{r=0}^{n} \frac{n(n-1)\dots(n-r+1)}{r!} x^r \tag{28}$$

and this is the binomial expansion since

$$\frac{n(n-1)\dots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!} \tag{29}$$

9. Find the first three terms of the Taylor expansion of $\sqrt{1+x}$. Don't bother with this one if time is short since we've already done lots.

   Solutions: Ok, so

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots \tag{30}$$

10. The general second order Runge Kutta method for $\dot{y} = f(y)$ is

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + \alpha\delta t k_1) \end{aligned} \tag{31}$$

   and

$$y_{n+1} = y_n + \left[\left(1 - \frac{1}{2\alpha}\right)k_1 + \frac{1}{2\alpha}k_2\right]\delta t \tag{32}$$

   Show that this gives the Taylor series up to second order. Note that $\alpha = 1/2$ gives the midpoint method.

   Solutions: Same craic as in the lectures, expand out k_2 using the Taylor series

$$k_2 = f(y_n + \alpha\delta t k_1) = f(y_n) + \left.\frac{df}{dy}\right|_{y=y_n} \alpha\delta t k_1 + \dots \tag{33}$$

   then use the chain rule

$$\frac{df}{dy}k_1 = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dt} = \frac{d^2 f}{dt^2} \tag{34}$$

   so

$$k_2 = \dot{y}(t_n) + \alpha\delta t\ddot{y}(t_n) + \dots \tag{35}$$

   so

$$y_{n+1} = y_n\left[\left(1 - \frac{1}{2\alpha}\right)k_1 + \frac{1}{2\alpha}k_2\right]\delta t = y(t_n) + \dot{y}(t_n) + \delta t\ddot{y}(t_n) + \dots \tag{36}$$

   which is the first three terms of the Taylor expansion, as required.