EMAT10001 Workshop Sheet 13.

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Introduction

This worksheet is about differentiation, the exponential function and the growth equation. There is the usual bounty for errors and typos, 20p to £2 depending on how serious it is.

Useful facts

- Recall the rules of differentiation
  - Chain rule:
  $$\frac{d}{dx}f(g(x)) = \frac{d}{dx}g(x)\frac{d}{dg}f(g) \tag{1}$$
  - Product rule:
  $$\frac{d}{dx}u(x)v(x) = u(x)\frac{d}{dx}v(x) + v(x)\frac{d}{dx}u(x) \tag{2}$$
  - Powers:
  $$\frac{d}{dx}x^n = nx^{n-1} \tag{3}$$
- The exponential function as the limit of continual compounding
$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \tag{4}$$
- The series for the exponential
$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \dots \tag{5}$$
- Notational note: $df(t)/dt$ is often written $\dot{f}(t)$ and $df(x)/dx$ is often written $f'(x)$. The first notation is due to Leibnitz and is useful for remembering the chain rule and similar, the second is useful for writing the derivative at a particular point, say we want the derivative at $x = x_0$, where x_0 is a constant, we can write
$$f'(x_0) = \left.\frac{df}{dx}\right|_{x=x_0} \tag{6}$$

Work sheet

1. Revise differentiating; find df/dx for the following.

   (a) $f(x) = 1/x$

   (b) $f(x) = (1+x)^3$, do this one both by using the chain rule and by multiplying it out.

   (c) $f(x) = A\exp(\lambda x)$ where A and λ are constants.

   (d) $f(x) = x\exp(x)$

   (e) $f(x) = \exp(x^2)$

   (f) $f(x) = \exp(1/x)$

   (g) $f(x) = x\exp(1+x)$

2. Using the power series, work out the value of e to five decimal places.

3. By substituting in $f(t) = A\exp(\lambda t)$ solve

$$\frac{d}{dt}f(t) = -2f(t) \tag{7}$$

   where $f(0) = 5$. Substituting the $f(t)$ into the equation should give λ, fixing the initial condition, that is, the value of $f(0)$, should give A.

4. A breeding pair of rabbits are released on an island of endless grass. Each year each rabbit has five kits on average, these, in this approximation, grow up instantly. This rate is the average per rabbit, don't worry about the gender, the idea is that the doe have ten kits and the bucks none so the average is five. Thus, in this model if P is the population of rabbits, the rate of increase is $dP/dt = 5P$, ten kits for each member of the population and $P(0) = 2$. What is the population after seven years.

5. Caesium-137 has a half-life of about 30 years; this means that a given Caesium-137 atom has a 0.0231 chance of decaying in a given year. To check this, write down a differential equation for the decay of Caesium-137 and solve it; check that the amount has halved after 30 years.

6. The natural log is the inverse of the exponential:

$$\ln e^x = x \tag{8}$$

   Check this using your calculator for a couple of values of x.

7. Find a formula relating the decay constant and the half life; the decay constant is the chance of an individual atom decaying in a given period of time, the 0.0231 in the example above.

8. I discover a new craze, everyday, on average, I manage to persuade one new person to start in on my craze, they do the same and so on and so on. How long before everyone in the world shares my craze assuming the growth rate stays the same. Why is this assumption absurd?

9. What is the df/dx at $x = 0$ for $f = \exp(-1/x)$, what about the second derivative

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\frac{df}{dx} \tag{9}$$

at $x = 0$. Can you guess what happens to higher derivatives at $x = 0$? To do this question you need to know that the exponential $\exp(-x)$ goes to zero as x goes to infinity and, in fact, does so faster than any polynomial! This means that there is a certain amount of guessing involved since we haven't done limits, which makes this a hard question.

Exercise sheet

The difference between the work sheet and the exercise sheet is that the solutions to the exercise sheet won't be given and the problems are designed to be more suited to working on on your own, though you are free to discuss them in the work shop if you finish the work sheet problems. Selected problems from the exercise sheet will be requested as part of the continual assessment portfolio.

1. Solve

$$\frac{df}{dt} = 5f \tag{10}$$

with $f(0) = 12$.

2. Differentiate

$$f(x) = x^3 e^{x^3} \tag{11}$$

3. Solve

$$\frac{df}{dt} = 5(1 - f) \tag{12}$$

with $f(0) = 0$.

4. The growth equation is not a realistic model of growth if there is a finite resource the population requires, this might be food, or space, or available uninfected individuals. The Verhulst-Pearl equation is an alternative that includes a *carrying capacity* for the environment, growth depends not only on the population but the residual carrying capacity. A simple Velhulst-Pearl equation is

$$\frac{dP}{dt} = P(1 - P) \tag{13}$$

More complicated versions include constants which have been set to one here. Solving this equation is tricky, it involves direct integration and a partial fractions expansion. However, it is easier to check the solution is indeed a solution, the solution with $P(0) = 1/2$ is

$$P = \frac{1}{1 + \exp(-t)} \tag{14}$$

Check this.

Challenge

First three to get onto level five, that is complete four levels, of `http://www.pythonchallenge.com/` gets chocolate. Send a screenshot.