EMAT10001 Workshop Sheet 15.

Conor Houghton 2014-02-12

Introduction

Since there was no lecture this week this worksheet partly revizes worksheets 13 and 14, it also includes some integration. There is the usual bounty for errors and typos, 20p to £2 depending on how serious it is. There is no exercize sheet for this week.

Useful facts

- Differentiating the trigonometric functions:

$$\begin{aligned} \frac{d}{dx}\cos x &= -\sin x \\ \frac{d}{dx}\sin x &= \cos x \end{aligned} \tag{1}$$

- The Taylor expansion around $t=0$

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n f}{dt^n} \right|_{t=0} t^n \tag{2}$$

Integration

Integration is the inverse of differentiation:

$$\int_a^b \frac{df}{dt} dt = f(b) - f(a) \tag{3}$$

and

$$\frac{d}{dt}\int_a^t f dt = f(t) \tag{4}$$

The fundamental theorem of calculus tells us that the integral is also the area under the curve, thus

$$A = \int_a^b f(t) dt \tag{5}$$

is the area enclosed by $f(t)$ and the t axis as t goes from $t=a$ to $t=b$.

You can work out some standard integrals from knowing that integration is backwards diffentiation; these are often written without the limits, so

$$\int t^n dt = \frac{t^{n+1}}{n+1} + C \tag{6}$$

means

$$\frac{d}{dt}\left(\frac{t^{n+1}}{n+1}+C\right)=t^n \tag{7}$$

no matter what constant value is given to C. This means

$$\int_a^b t^n dt = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \tag{8}$$

and this is often written

$$\int_a^b t^n dt = \left.\frac{t^{n+1}}{n+1}\right|_a^b \tag{9}$$

So

$$\begin{aligned}
\int t^n dt &= \frac{t^{n+1}}{n+1}+C\\
\int e^t dt &= e^t + C\\
\int \sin t dt &= -\cos t + C\\
\int \cos t dt &= -\sin t + C\\
\int \frac{1}{t} dt &= \ln t + C
\end{aligned} \tag{10}$$

Other integrals can often be done by a change of variables, for example, if

$$I=\int e^{2t}dt \tag{11}$$

let $s = 2t$ and $ds/dt = 2$ means $ds = 2dt$, this manner of treating the differential like a fraction and multiplying across by the dt shouldn't make you think you can always get away with this sort of thing, but there is a theorem that says you can here, so

$$I=\frac{1}{2}\int e^s ds = \frac{1}{2}e^s + C = \frac{1}{2}e^{2t}+C \tag{12}$$

Another, harder version would be

$$I=\int te^{t^2}dt \tag{13}$$

Use $s = t^2$ so $ds/dt = 2t$ or $ds = 2tdt$ and

$$I=\int te^{t^2}dt = \frac{1}{2}\int e^s dt = \frac{1}{2}e^{t^2}+C \tag{14}$$

If there are limits you need to change them too, so

$$I=\int_0^2 \exp t/2dt \tag{15}$$

If you let $s = t/2$ then $2ds = dt$ and when $t = 0$ then $s = 0$ and when $t = 2$ then $s = 1$ so the integral becomes

$$I = 2\int_0^1 \exp s ds = 2e - 2 \tag{16}$$

Work sheet

1. The general second order Runge Kutta method for $\dot{y} = f(y)$ is

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + \alpha\delta t k_1) \end{aligned} \tag{17}$$

   and

$$y_{n+1} = y_n + \left[\left(1 - \frac{1}{2\alpha}\right)k_1 + \frac{1}{2\alpha}k_2\right]\delta t \tag{18}$$

   Show that this gives the Taylor series up to second order. Note that $\alpha = 1/2$ gives the midpoint method. This was the last question on last week's worksheet, so lots of people didn't get a chance to think about it and it worth doing since it does give a good idea of how Runge Kutta works.

2. Here is a second order differential equation

$$\frac{d^2f}{dt^2} + \frac{df}{dt} - 6f = 0 \tag{19}$$

   This can also be solved using an ansatz of the form $A\exp(\lambda t)$; the difference is that there will be two different λs, lets say λ_1 and λ_2. Since the equation is linear you can add them to give a solution with two arbitrary constants:

$$f(t) = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t} \tag{20}$$

   This is what you expect for a second order differential equation, you need two initial conditions, here use $f(0) = 0$ and $\dot{f}(0) = -1$.

3. Here is another second order differential equation

$$\frac{d^2f}{dt^2} - f = 0 \tag{21}$$

   Solve this with $f(0) = 0$ and $\dot{f}(0) = 1$.

4. This differential equation

$$\frac{d^2f}{dt^2} + f = 0 \tag{22}$$

   with $f(0) = 0$ and $\dot{f}(0) = 1$ doesn't work so well, you end up with complex λs, however, if you keep your nerve and use the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{23}$$

it will work out; just bundle $A_1 + A_2$ into one arbitrary constant $C_1 = A_1 + A_2$ and $i(A_1 - A_2)$ into another $C_2 = i(A_1 - A_2)$; C_1 and C_2 should turn out to be real, the detour through complex numbers is just that, a detour.

5. Some integration examples; integrate

   (a) $\int (2x+2)e^{x^2+2x+3}dx$

   (b) $\int (x^2+1)/(x^3+3x)dx$

   (c) $\int \sqrt{x}dx$

   (d) $\int \sqrt{7x+1}dx$

6. Integrating the square pulse. Say f is square pulse

$$f(t) = \begin{cases} 1 & -\pi/2 < t < \pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

   Thus $f(t)$ is one between $-\pi/2$ and $\pi/2$ but zero everywhere else. What is

$$I = \int_{-\pi}^{\pi} f(t) \cos t dt \tag{25}$$

   what about

$$I = \int_{-\pi}^{\pi} f(t) \cos nt dt \tag{26}$$

   for n an integer.