EMAT10001 Workshop Sheet 14.

Conor Houghton 2014-02-02

Introduction

This worksheet is about differentiation, the Taylor series and the Runge Kutta numerical algorithm. There is the usual bounty for errors and typos, 20p to £2 depending on how serious it is.

Useful facts

- Differentiating the trigonometric functions:

$$\begin{aligned}\frac{d}{dx}\cos x &= -\sin x\\ \frac{d}{dx}\sin x &= \cos x\end{aligned} \tag{1}$$

- The Taylor expansion

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left.\frac{d^n f}{dt^n}\right|_{t=0} t^n \tag{2}$$

  This is the Taylor expansion around $t = 0$, you can expand around any point

$$f(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left.\frac{d^n f}{dt^n}\right|_{t=t_0} (t-t_0)^n \tag{3}$$

  The Taylor expansion around $t = 0$ is sometimes called the Maclaurin series.

- The Euler method for $\dot{y} = f(t, y)$ with $y(0) = y_0$

$$y_{n+1} = y_n + f(t_n, y_n)\delta t \tag{4}$$

- The midpoint method for $\dot{y} = f(t)$ with $y(0) = y_0$

$$\begin{aligned}k_1 &= f(y_n)\\ k_2 &= f(y_n + k_1\delta t/2)\end{aligned} \tag{5}$$

  then

$$y_{n+1} = y_n + k_2\delta t \tag{6}$$

  In the lecture notes this was referred to as the second order Runge Kutta method. In fact, it is a second order Runge Kutta method, we will see in the problems below that it is part of a larger family of such methods.

- The fourth order Runge Kutta method.

$$\frac{dy}{dt} = f(t, y) \tag{7}$$

Now

$$\begin{aligned} k_1 &= f(t_n, y_n) \\ k_2 &= f\left(t_n + \frac{1}{2}\delta, y_n + \frac{1}{2}\delta t k_1\right) \\ k_3 &= f\left(t_n + \frac{1}{2}\delta, y_n + \frac{1}{2}\delta t k_2\right) \\ k_4 &= f\left(t_n + \delta, y_n + \delta t k_2\right) \end{aligned} \tag{8}$$

and

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{9}$$

- Notational note:

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) \tag{10}$$

- There is some variation in how people write 'the derivative of $f(x)$ at x_0'. If you are using the prime notation it is easy, this is $f'(x_0)$, using the d/dx notation it is trickier, many people write

$$f'(x_0) = \frac{df(x_0)}{dx} \tag{11}$$

This is a poor notation in the sense that $f(x_0)$ is a constant, the variable x has been replaced by a constant x_0, so the correct notation is

$$f'(x_0) = \left.\frac{df(x)}{dx}\right|_{x=x_0} \tag{12}$$

which shows that you differentiate first and then fix x. However, this notation is cumbersome, so although I use it the other one is acceptable; it is still a bit better to use

$$f'(x_0) = \frac{df}{dx}(x_0) \tag{13}$$

to show that the differentiating is done first.

- Another notational note: there are two common notations for the binomial coefficient n choose r

$${}_nC_r = \left(\begin{array}{c} n \\ r \end{array}\right) = \frac{n!}{(n-r)!r!} \tag{14}$$

Work sheet

1. Revise differentiating again!

   (a) Find df/dx of $f(x) = \tan x$ using $\tan x = \sin x/\cos x$.

   (b) We know $\sin^2 x + \cos^2 x = 1$, differentiate both sides of this equation.

2. What is the Taylor expansion of $\tan t$ up to and include the t^3 term?

3. If $f(t) = \arctan t$ then

$$\frac{df}{dt} = \frac{1}{1+t^2} \tag{15}$$

   This is derived using the chain rule and a trick called implicit differentiation. Basically let $y = \arctan x$ so $x = \tan y$, differentiate both sides with respect to x

$$1 = \frac{dx}{dx} = \frac{d}{dx}\tan y = \frac{dy}{dx}\frac{d}{dy}\tan y \tag{16}$$

   and then do some messing to get dy/dx in terms of x. Do that!

4. The Taylor expansion of $\arctan x$ is

$$\arctan x = \sum_{n\,\mathrm{odd}} (-1)^{(n-1)/2}\frac{x^n}{n} \tag{17}$$

   Check this as far as the x^3 term. [There is a very elegant derivation of this Taylor series, basically $1/(1+x)$ has a known Taylor expansion, up to a sign this is asked in the exercise sheet; this allows you to calculate the expansion of $1/(1+x^2)$ and from this you can read off all the derivative you need for the expansion of $\arctan x$, if you are feeling ambitious you can try to do this.]

5. What is $\arctan 1$?

6. Use the Taylor expansion of $\arctan 1$ to write down a formula for calculating π and do the first six terms.

7. The series for $\arctan 1$ is disastrously slow if you want to use it to calculate π; in fact getting ten decimal places correct requires 5,000,000,000 terms. However, there are other approaches to using the $\arctan x$ series. According to Wikipedia, in 1699, English mathematician Abraham Sharp used this series with $x = \sqrt{1/3}$ to compute π to 71 digits, breaking the previous record of 39 digits, his record only stood for seven years when a new and faster converging series for π was found; this was

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239} \tag{18}$$

   You can prove this with clever use of trigonometric identities, you shouldn't do that here, but you should use the formula to find an approximation of π, take three terms

for example. By the way, there is a graph at
`http://en.wikipedia.org/wiki/File:Record_pi_approximations.svg`
showing how the number of known digits of π has changed over history.

8. The binomial theorem tells us that

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r \tag{19}$$

Prove this using the Taylor expansion.

9. Find the first three terms of the Taylor expansion of $\sqrt{1+x}$. Don't bother with this one if time is short since we've already done lots.

10. The general second order Runge Kutta method for $\dot{y} = f(y)$ is

$$\begin{aligned} k_1 &= f(y_n) \\ k_2 &= f(y_n + \alpha\delta t k_1) \end{aligned} \tag{20}$$

and

$$y_{n+1} = y_n + \left[\left(1 - \frac{1}{2\alpha}\right) k_1 + \frac{1}{2\alpha} k_2\right] \delta t \tag{21}$$

Show that this gives the Taylor series up to second order. Note that $\alpha = 1/2$ gives the midpoint method.

Exercise sheet

The difference between the work sheet and the exercise sheet is that the solutions to the exercise sheet won't be given and the problems are designed to be more suited to working on on your own, though you are free to discuss them in the work shop if you finish the work sheet problems. Selected problems from the exercise sheet will be requested as part of the continual assessment portfolio.

1. Revise differentiating; find df/dx of
   (a) $f(x) = 1 - \cos^2 x$.
   (b) $f(x) = 1/(1-x)$.

2. Calculate a Taylor series for $\cos x$.

3. Plot the behavior of different truncated Taylor series for $\cos x$ and compare them to $\cos x$ itself, truncate after one, two, four, eight and ten terms and plot over -2π to 2π.

4. We saw the natural log in last week's problem sheet, $\ln x = y$ means $x = e^y$. now prove using the chain rule that
$$\frac{d}{dx} \ln x = \frac{1}{x} \tag{22}$$

5. Calculate the Taylor series for $\ln(1+t)$ around $t = 0$.

6. Calculate a Taylor series for $1/(1-x)$. Test the accuracy of the series after three terms for $x = 0$, $x = 0.5$, $x = 0.75$ and $x = 0.9$.

7. Write a program to implement the Euler method and the second and fourth order Runge-Kutta methods for solving differential equations for $\dot{y} = f(y)$; use these to solve the equation $\dot{P} = P(1-P)$ mentioned last week and check the error of the various methods.

Challenge

First four to get onto level ten, that is have completed nine levels, of
`http://www.pythonchallenge.com/`
gets chocolate. Send a screenshot.