

2E2 Tutorial Sheet 11 Second Term, Solutions¹

22 January 2006

1. (2) For the system

$$\begin{aligned}\frac{dy_1}{dt} &= -3y_1 + 2y_2 \\ \frac{dy_2}{dt} &= -2y_1 + 2y_2\end{aligned}$$

The solution is

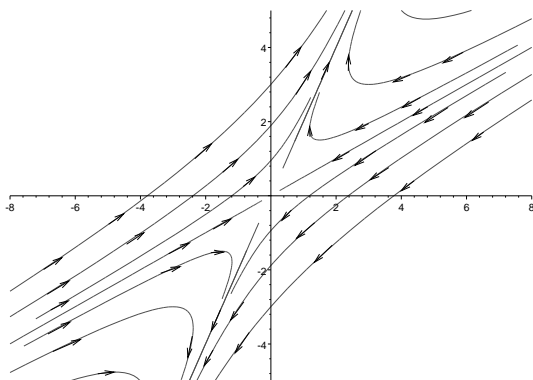
$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^t + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} \quad (1)$$

Sketch the phase diagram and describe the stationary point.

Solution: So, any point that starts on the

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad (2)$$

eigenvector will move inwards, since $c_1 = 0$ and $c_2 \exp -2t$ gets small as t increases, anywhere on the other eigenvectors will move straight outwards. If you aren't on either eigenvector, the amount along the negative eigenvalue eigenvector decreases and the amount along the positive eigenvalue eigenvector increases and so you move outwards getting closer and closer to the positive eigenvalue line. The phase diagram is



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where the arrows go outwards except on the line defined by \mathbf{x}_2 . The stationary point is a saddle point.

2. (2) For the system

$$\frac{dy_1}{dt} = 3y_1 + y_2 \quad (3)$$

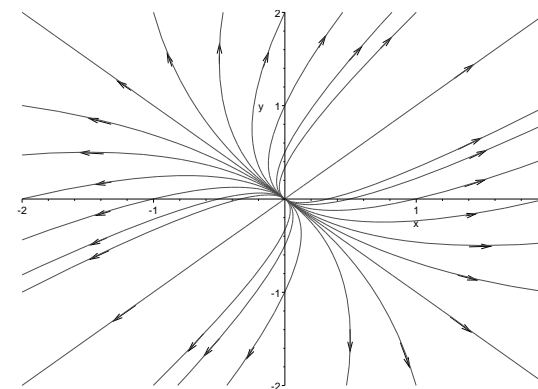
$$\frac{dy_2}{dt} = y_1 + 3y_2 \quad (4)$$

The solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}. \quad (5)$$

Sketch the phase diagram and describe the stationary point.

Solution: The phase-diagram is



with all the lines going outward. Notice they are all tending towards the same direction as \mathbf{x}_1 .

3. (4) Find the general solutions for the system

$$\frac{dy_1}{dt} = 2y_1 - y_2 \quad (6)$$

$$\frac{dy_2}{dt} = -4y_2 \quad (7)$$

Sketch the phase diagram and describe the stationary point.

Solution: So here

$$A = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix}$$

and the spectrum² is $\lambda_1 = 2$ corresponding to

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

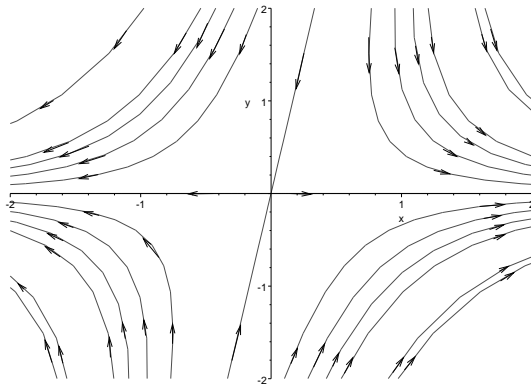
and $\lambda_2 = -4$ corresponding to

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

so the general solution is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 6 \end{pmatrix} e^{-4t}.$$

The phase diagram is



²The set of eigenvalues of a matrix is sometimes called its spectrum