2E2 Tutorial Sheet 14 Second Term[1]

10 February 2006

1. (2) Find the general solution to

$$y' - 2y = -t \tag{1}$$

Solution: This follows from the general solution to

$$y' = ry + f(t) \tag{2}$$

which is

$$y = Ce^{rt} + e^{rt}\int_0^t e^{-r\tau} f(\tau) d\tau \tag{3}$$

so here $r = 2$ and $f(t) = -t$ so, using integration by parts

$$\begin{aligned} y &= Ce^{2t} - e^{2t}\int_0^t \tau e^{-2\tau} d\tau \\ &= Ce^{2t} - e^{2t}\left\{-\frac{1}{2}te^{-2t} + \frac{1}{2}\int e^{-2\tau} d\tau\right\} \\ &= Ce^{2t} - e^{2t}\left\{-\frac{1}{2}te^{-2t} - \frac{1}{4}(e^{-2t})\right\} \\ &= Ce^{2t} + \frac{t}{2} + \frac{1}{4} \end{aligned} \tag{4}$$

where $\exp 2t$ terms have been absorbed in the $C \exp 2t$.

2. (3) Find the general solution to

$$\begin{aligned} y_1' &= 5y_2 - 23 \\ y_2' &= 5y_1 + 15. \end{aligned} \tag{5}$$

with $y_1(0) = -3$ and $y_2(0) = 5$.

Solution: First of all rewrite the equation in matrix form

$$\mathbf{y}' = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix} \mathbf{y} + \begin{pmatrix} -23 \\ 15 \end{pmatrix}. \tag{6}$$

Now, the matrix

$$A = \begin{pmatrix} 0 & 5 \\ 5 & 0 \end{pmatrix} \tag{7}$$

[1] Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`

has eigenvalue $\lambda_1 = 5$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{8}$$

and eigenvalue $\lambda_1 = -5$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{9}$$

so if we write

$$\mathbf{y} = f_1\mathbf{x}_1 + f_2\mathbf{x}_2 \tag{10}$$

and subsituting this into the differential equation gives

$$(f_1' - 5f_1)\mathbf{x}_1 + (f_2' + 5f_2)\mathbf{x}_2 = \begin{pmatrix} -23 \\ 15 \end{pmatrix}. \tag{11}$$

Now to seperate the equation lets decompose the inhomogeneous part, sometimes called the forcing term, over the two eigenvectors:

$$\begin{pmatrix} -23 \\ 15 \end{pmatrix} = h_1\mathbf{x}_1 + h_2\mathbf{x}_2 \tag{12}$$

or, writing it out,

$$\begin{pmatrix} -23 \\ 15 \end{pmatrix} = \begin{pmatrix} h_1 + h_2 \\ h_1 - h_2 \end{pmatrix} \tag{13}$$

and, hence, $h_1 = -4$ and $h_2 = -19$. Putting this back into the equation leads to

$$(f_1' - 5f_1)\mathbf{x}_1 + (f_2' + 5f_2)\mathbf{x}_2 = -4\mathbf{x}_1 - 19\mathbf{x}_2 \tag{14}$$

Hence

$$f_1' - 5f_1 = -4. \tag{15}$$

Thus, this is of the form $y' = ry + f$ with $r = 5$, $f(t) = -4$ and so

$$f_1 = C_1e^{5t} - 4e^{5t}\int^t e^{-5t}dt \tag{16}$$

and so

$$f_1 = C_1e^{5t} + \frac{4}{5} \tag{17}$$

where I have committed the common notational laziness of using t inside the integration sign as well as outside, people do this a lot, because you can think of the integral as being closed off from the rest of the equation, but if it confuses you, keep using τ inside the integral. Similarly,

$$f_2' + 5f_2 = -19 \tag{18}$$

Thus, $r = -5$, $f(t) = 19$ and using integrating gives above

$$f_2 = C_2 e^{-5t} - \frac{19}{5}. \tag{19}$$

The general solution is therefore

$$\mathbf{y} = \left(C_1 e^{5t} + \frac{4}{5}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(C_2 e^{-5t} - \frac{19}{5}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{20}$$

If $y_1(0) = -3$ and $y_2(0) = 5$ then we get

$$\begin{pmatrix} -3 \\ 5 \end{pmatrix} = \left(C_1 + \frac{4}{5}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(C_2 - \frac{19}{5}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{21}$$

and hence

$$\begin{aligned} -3 &= C_1 + C_2 - 3 \\ 5 &= C_1 - C_2 + \frac{23}{5} \end{aligned} \tag{22}$$

so $C_1 = -C_2 = 1/5$ and

$$\mathbf{y} = \left(\frac{1}{5}e^{5t} + \frac{4}{5}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(-\frac{1}{5}e^{-5t} - \frac{19}{5}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{23}$$

3. (2) Find the solution to

$$\begin{aligned} y_1' &= y_1 + 3y_2 + e^t \\ y_2' &= 3y_1 + y_2 \end{aligned} \tag{24}$$

Solution: Here we have

$$\mathbf{y} = A\mathbf{y} + \begin{pmatrix} e^t \\ 0 \end{pmatrix} \tag{25}$$

where

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \tag{26}$$

this has eigenvalue $\lambda_1 = 4$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{27}$$

and eigenvalue $\lambda_1 = -2$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{28}$$

Once again, we split the forcing term over the two eigenvectors:

$$\begin{pmatrix} e^t \\ 0 \end{pmatrix} = \frac{e^t}{2}\mathbf{x}_1 + \frac{e^t}{2}\mathbf{x}_2 \tag{29}$$

We get

$$f_1 - 4f_1 = \frac{1}{2}e^t \tag{30}$$

so

$$f_1 = C_1 e^{4t} + \frac{1}{2}e^{4t}\int^t e^{-3t}dt. \tag{31}$$

and so,

$$f_1 = C_1 e^{4t} - \frac{1}{6}e^t \tag{32}$$

In the same way

$$f_2 + 2f_2 = \frac{1}{2}e^t \tag{33}$$

and so

$$f_2 = C_2 e^{-2t} + \frac{1}{2}e^{-2t}\int^t e^{3t}dt. \tag{34}$$

Integrating gives

$$f_2 = C_2 e^{-t} + \frac{1}{6}e^t \tag{35}$$

This means

$$\mathbf{y} = \left(C_1 e^{4t} - \frac{1}{6}e^t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(C_2 e^{-t} + \frac{1}{6}e^t\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{36}$$

4. (1) Rewrite $y'' + 4y' - 3y = 0$ as a system of two first order differential equations.

Solution: So $y_1 = y$, $y_2 = y_1'$ hence $y_2' = y'' = 3y - 4y' = 3y_1 - 4y_2$ giving

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= 3y_1 - 4y_2 \end{aligned} \tag{37}$$