2E2 Tutorial Sheet 9[1]

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Useful facts:

- Working out the eigenvalues of a 2×2 matrix: say your matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1}$$

then the characteristic equation is

$$\det(A - \lambda \mathbf{1}) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0 \tag{2}$$

The eigenvalues λ_1 and λ_2 are the two solutions of this equation.

- Working out the eigenvectors: say λ_1 is an eigenvalue of the 2×2 matrix A then to work out the eigenvalue $\mathbf{x}_1$ corresponding to λ_1 let

$$\mathbf{x}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \tag{3}$$

and solve the equation

$$A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \tag{4}$$

Doing out the matrix multiplication will give you two equations, but, all being well these will usually be the same equation so all you can do is solve a in terms of b or visa versa. To get a particular eigenvector you can choose a or b to be one.

- Solving a linear differential equation: for A a 2×2 matrix with eigenvalues λ_1 and λ_1 and corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ then if

$$\mathbf{y}' = A\mathbf{y} \tag{5}$$

the solution is

$$\mathbf{y} = C_1 \mathbf{x}_1 e^{\lambda_1 t} + C_2 \mathbf{x}_2 e^{\lambda_2 t} \tag{6}$$

where C_1 and C_2 are arbitrary constants.

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Questions

1. (3) Find the eigenvectors and eigenvalues of the following matrices

$$(i)\ \begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix} \qquad (ii)\ \begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix} \qquad (iii)\ \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \tag{7}$$

2. (2) Find the eigenvectors and eigenvalues of the following matrices

$$(i)\ \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \qquad (ii)\ \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \tag{8}$$

3. (3) Find the solution for the system

$$\frac{dy_1}{dt} = -3y_1 + 2y_2 \tag{9}$$

$$\frac{dy_2}{dt} = -2y_1 + 2y_2 \tag{10}$$