2007/8 Annual Exam Q1

1. a) define the KL divergence.

p(x) & q(x) distributions for some set $\mathcal{X}$ of outcomes

$$D(p\|q) = \sum_{x\in\mathcal{X}} p(x)\log \frac{p(x)}{q(x)}$$

b) state Jensen's inequality

if f is a cap like$^{(*)}$ / convex fn & X a random variable

$$\mathbb{E} f(X) \geq f(\mathbb{E}X) \quad \text{notation in bk}$$

or

$$\langle f(X)\rangle \geq f(\langle X\rangle) \quad \text{notation in lectures.}$$

moreover if f is strictly cap-like equality in (*) implies $\Pr\{X = \langle X\rangle\} = 1$.

c) Prove the KL divergence is ~~non zero~~ non-negative, when is it zero?

Pf let $\mathcal{A} = \{x\in\mathcal{X} \mid p(x) > 0\}$ support set

$$-D(p\|q) = -\sum_{x\in\mathcal{X}} p(x)\log\frac{p(x)}{q(x)} = \sum_{x\in\mathcal{A}} p(x)\log\frac{q(x)}{p(x)}$$

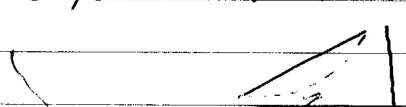
apply the Jensen inequality with ~~X~~ random variable $\frac{q(X)}{p(X)}$

log is _cap-like_!

(*) ~~$E(x_0 +$~~

cap-like $f((1-t)x_0 + tx_1) \leq (1-t)f(x_0) + tf(x_1)$

"convex"



$$-D(p\|q) \leq \log \sum_{x\in\mathcal{A}} p(x) \frac{q(x)}{p(x)} \quad (1)$$
$$= \log \sum_{x\in\mathcal{A}} q(x)$$
$$\leq \log \sum_{x\in\mathcal{X}} q(x) \quad (2) \text{ log monotonic}$$
$$= \log 1 = 0.$$

$\therefore \quad D(p\|q) \geq 0.$

for equality we need equality at (1) & (2)

(1) $\Rightarrow$ $\frac{q(x)}{p(x)} = c$ c a const.
$q(x) = c\,p(x)$

(2) $\Rightarrow \sum_{x\in\mathcal{A}} q(x) = 1 \Rightarrow c = 1 \therefore p(x) = q(x) \;\forall x \in \mathcal{X}.$

d) find examples where KL divergence is not symmetric in p & q. Is it ever symmetric when $p \neq q$.

easy to find an example where it is not symmetric, it is generically the case.

p	0	1
	1/3	2/3

q	0	1
	1/2	1/2

$$D(p\|q) = \tfrac{1}{3}\log\tfrac{2}{3} + \tfrac{2}{3}\log\tfrac{4}{3} = \log\tfrac{2}{3} + \tfrac{2}{3}\log 2 = \tfrac{5}{3} - \log 3$$
$$D(q\|p) = \tfrac{1}{2}\log\tfrac{3}{2} + \tfrac{1}{2}\log\tfrac{3}{4} = \log 3 - \tfrac{3}{2}$$
& these aren't equal.

for symmetry we want p & q not equal but

$$p\log\frac{p}{q} + p'\log\frac{p'}{q'} = q\log\frac{q}{p} + q'\log\frac{q'}{p'}$$

$p' = 1-p$ ie. $p \quad p'$

$q' = 1-q$. $\quad q' \quad q'$

easy to see $p = 1-q$ works

eg. $\frac{1}{3} \quad \frac{2}{3}$ & $\frac{2}{3} \quad \frac{1}{3}$

so p & q aren't equal but they are symmetric