

## 2E2 Tutorial Sheet 4 Solutions<sup>1</sup>

6 November 2005

### Questions

1. (4) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \delta(t - 1) \quad (1)$$

subject to the initial conditions  $f(0) = 0$ ,  $f'(0) = 1$ .

*Solution:* soln The only thing that is unusual is that there is a delta function. We take the Laplace transform using

$$\mathcal{L}(\delta(t - a)) = e^{-as} \quad (2)$$

hence

$$(s^2 + 2s - 3)F - 1 = e^{-s} \quad (3)$$

Now, if we do partial fractions on  $1/(s^2 + 2s - 3)$  we get

$$\frac{1}{s^2 + 2s - 3} = -\frac{1}{4(s + 3)} + \frac{1}{4(s - 1)} \quad (4)$$

Hence

$$F = \left( -\frac{1}{4(s + 3)} + \frac{1}{4(s - 1)} \right) (1 + e^{-s}) \quad (5)$$

Since

$$\mathcal{L} \left( -\frac{1}{4}e^{-3t} + \frac{1}{4}e^t \right) = -\frac{1}{4(s + 3)} + \frac{1}{4(s - 1)} \quad (6)$$

then, by the third shift theorem we have

$$f = \left( -\frac{1}{4}e^{-3t} + \frac{1}{4}e^t \right) + H_1(t) \left( -\frac{1}{4}e^{-3t+3} + \frac{1}{4}e^{t-1} \right) \quad (7)$$

2. (4) Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = 0 \quad (8)$$

with boundary conditions  $f(0) = 0$  and  $f'(0) = 1$  and get your answer into a real form.

*Solution:* So, taking the Laplace transform of the equation we get,

$$s^2F - 1 + 6sF + 13F = 0 \quad (9)$$

and, hence,

$$F = \frac{1}{s^2 + 6s + 13}. \quad (10)$$

Now, using minus b plus or minus the square root of b squared minus four a c all over two a, we get

$$s^2 + 6s + 13 = 0 \quad (11)$$

if

$$s = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm 2i \quad (12)$$

which means

$$s^2 + 6s + 13 = (s + 3 - 2i)(s + 3 + 2i) \quad (13)$$

Next, we do the partial fraction expansion,

$$\frac{1}{s^2 + 6s + 13} = \frac{A}{s + 3 - 2i} + \frac{B}{s + 3 + 2i} \quad (14)$$

and multiplying across we get

$$1 = A(s + 3 + 2i) + B(s + 3 - 2i) \quad (15)$$

therefore we choose  $s = -3 + 2i$  to get

$$A = \frac{1}{4i} = -\frac{i}{4} \quad (16)$$

and  $s = -3 - 2i$  to get

$$B = -\frac{1}{4i} = \frac{i}{4} \quad (17)$$

and so

$$F = -\frac{i}{4} \frac{1}{s + 3 - 2i} + \frac{i}{4} \frac{1}{s + 3 + 2i}. \quad (18)$$

If we take the inverse transform

$$\begin{aligned} f &= -\frac{i}{4}e^{-(3-2i)t} + \frac{i}{4}e^{-(3+2i)t} \\ &= \frac{i}{4}e^{-3t}(e^{-2it} - e^{2it}) \\ &= \frac{1}{2}e^{-3t} \sin 2t \end{aligned} \quad (19)$$

<sup>1</sup>Conor Houghton, houghton@maths.tcd.ie, see also <http://www.maths.tcd.ie/~houghton/2E2.html>