The Markov Property[1]

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The random variables X, Y and Z are said to *form a Markov chain in that order*

$$X \to Y \to Z \tag{1}$$

if and only if $p(x|y,z) = p(x|y)$ for all x, y and z. This is equivalent to requiring X, Z are conditional independent on Y:

$$p(x,z|y) = p(x|y)p(z|y) \tag{2}$$

Here is an example: let X be a random variable taking the values $\mathcal{X} = \{-1, 0, 1\}$ with probabilities

X	-1	0	1
p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(3)

Now let $Y = |X| + W_1$ where W_1 takes the values $\{0, 1\}$ with equal probabilities. Now the joint distribution is

$Y \setminus X$	-1	0	1
0	0	$\frac{1}{4}$	0
1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	$\frac{1}{8}$	0	$\frac{1}{8}$

(4)

This gives a marginal distribution

Y	0	1	2
p	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(5)

Finally let $Z = Y + W_2$ where W_2 is identical to W_1. Now the joint distribution for X, Y and Z is hard to write down since there are three variables, here is an attempt, basically there are three copies of the table, one for each value of X

	$X = -1$			$X = 0$			$X = 1$		
$Z \setminus Y$	0	1	2	0	1	2	0	1	2
0	0	0	0	$\frac{1}{8}$	0	0	0	0	0
1	0	$\frac{1}{16}$	0	$\frac{1}{8}$	$\frac{1}{8}$	0	0	$\frac{1}{16}$	0
2	0	$\frac{1}{16}$	$\frac{1}{16}$	0	$\frac{1}{8}$	0	0	$\frac{1}{16}$	$\frac{1}{16}$
3	0	0	$\frac{1}{16}$	0	0	0	0	0	$\frac{1}{16}$

(6)

Finally, the marginal distribution for Z is

Z	0	1	2	3
p	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(7)

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Now, consider, as an example,

$$\begin{aligned} p(X=0, Z=1) &= p(0,0,1) + p(0,1,1) + p(0,2,1) \\ &= \frac{1}{4} \\ &\neq p(X=0)p(Z=1) = \frac{3}{16} \end{aligned} \tag{8}$$

Now consider the conditional case, for example $Y = 1$. Since

$$p(x, z|y) = \frac{p(x, y, z)}{p(y)} \tag{9}$$

we have

$$p_{XZ|Y}(0, 1|1) = \frac{p(0, 1, 1)}{p(1)} = \frac{1}{4} \tag{10}$$

with

$$p_{X|Y}(0|1) = \frac{1}{2} \tag{11}$$

and

$$p_{Z|Y}(1|1) = 2\left(\frac{1}{16} + \frac{1}{8} + \frac{1}{16}\right) = \frac{1}{2} \tag{12}$$

So, in this case $p(x|y)p(z|y) = p(x, z|y)$, as it will be for all (x, y, z) since this is a Markov chain.