2E2 Tutorial Sheet 12 Second Term, Solutions[1]

27 January 2006

1. (3) Find the general solutions for the system

$$\begin{aligned} \frac{dy_1}{dt} &= -2y_1 + y_2 && (1)\\ \frac{dy_2}{dt} &= y_1 - 2y_2 && (2) \end{aligned}$$

Sketch the phase diagram and and describe the node.

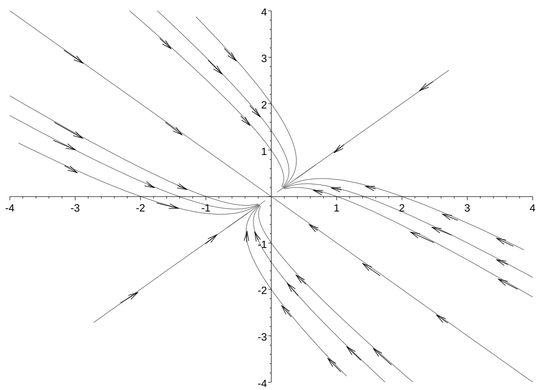
Solution: This time the eigenvalues are -3 with eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and -1 with eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus,

$$\mathbf{y} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \tag{3}$$

or

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -c_1 e^{-3t} + c_2 e^{-t} \\ c_1 e^{-3t} + c_2 e^{-t} \end{pmatrix}. \tag{4}$$

The phase diagram is



with all the arrows pointing inwards. The node is an improper node.

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2. (3) Find the solution of

$$\begin{aligned} \frac{dy_1}{dt} &= -9y_2 && (5)\\ \frac{dy_2}{dt} &= y_1 && (6) \end{aligned}$$

by considering $y_1(0) = r$ and $y_2(0) = 0$, draw the phase diagram.

Solution: Working out the eigenvalues

$$\begin{vmatrix} -\lambda & -9 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 9 = 0 \tag{7}$$

we find $\lambda = \pm 3i$ and the $3i$ eigenvector is

$$\begin{pmatrix} 0 & -9 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3i \begin{pmatrix} a \\ b \end{pmatrix} \tag{8}$$

so $a = 3ib$ and the eigenvector is, for example,

$$\mathbf{x} = \begin{pmatrix} 3i \\ 1 \end{pmatrix} \tag{9}$$

The other eigenvector is the complex conjugate of this one and so the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 3i \\ 1 \end{pmatrix} e^{3it} + c_2 \begin{pmatrix} -3i \\ 1 \end{pmatrix} e^{-3it}. \tag{10}$$

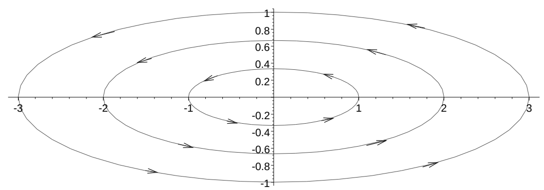
Now, if we assume that $y_1(0) = r$ and $y_2(0) = 0$ we find $c_1 = -c_2 = -ir/6$. Putting this back in and using the sine and cosine formulas we get

$$\mathbf{y} = \begin{pmatrix} r\cos 3t \\ \frac{r}{3}\sin 3t \end{pmatrix} \tag{11}$$

so we have the eillipse

$$y_1^2 + (3y_2)^2 = r \tag{12}$$

with phase diagram



3. (3) Find the solution of

$$\frac{dy_1}{dt} = -y_1 - 2y_2 \tag{13}$$
$$\frac{dy_2}{dt} = 2y_1 - y_2 \tag{14}$$

for initial conditions $y_1(0) = r$ and $y_2(0) = 0$ write this in real form. Sktech the phase diagram.

Solution: This time the matrix is

$$A = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

and so the spectrum is complex, $\lambda_1 = -1 + 2i$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and $\lambda_2 = -1 - 2i$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

The solution is then

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(-1+2i)t} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(-1-2i)t}.$$

Now, this means

$$\begin{pmatrix} r \\ 0 \end{pmatrix} = \mathbf{y}(0) = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

and hence $c_1 = -ir/2$ and $c_2 = ir/2$. Now using $\exp(a + ib) = \exp a \exp ib$ we have solution

$$\mathbf{y} = \frac{r}{2}\left[\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{2it} + \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-2it}\right] e^{-t}.$$

and so

$$\begin{aligned} \mathbf{y} &= \frac{r}{2}\left[\begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos 2t + i \sin 2t) + \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos 2t - i \sin 2t)\right] e^{-t} \\ &= \begin{pmatrix} r\cos 2t \\ r \sin 2t \end{pmatrix} e^{-t} \end{aligned}$$

So, this gives the inward spiral. Notice how fast the spiral goes in. The radius decreases exponentially.