2E2 A sprial example[1]

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Find the solution of

$$\frac{dy_1}{dt} = y_1 - 3y_2 \quad (1)$$
$$\frac{dy_2}{dt} = 3y_1 + y_2 \quad (2)$$

for initial conditions $y_1(0) = r$ and $y_2(0) = 0$ write this in real form. Sktech the phase diagram.

Solution: This time the matrix is

$$A = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

and so the spectrum is complex, $\lambda_1 = 1 + 3i$ with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and $\lambda_2 = 1 - 3i$ with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

The solution is then

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1+3i)t} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(1+3i)t}.$$

Now, this means

$$\begin{pmatrix} r \\ 0 \end{pmatrix} = \mathbf{y}(0) = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

and hence $c_1 = -ir/2$ and $c_2 = ir/2$. Now using $\exp(a + ib) = \exp a \exp ib$ we have solution

$$\mathbf{y} = \frac{r}{2}\left[\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{3it} + \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-3it}\right] e^t.$$

and so

$$\begin{aligned} \mathbf{y} &= \frac{r}{2}\left[\begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos 3t + i \sin 3t) + \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos 3t - i \sin 3t)\right] e^t \\ &= \begin{pmatrix} r\cos 3t \\ r \sin 3t \end{pmatrix} e^t \end{aligned}$$

So, this gives the outwardward spiral. Notice how fast the spiral goes in. The radius increases exponentially.

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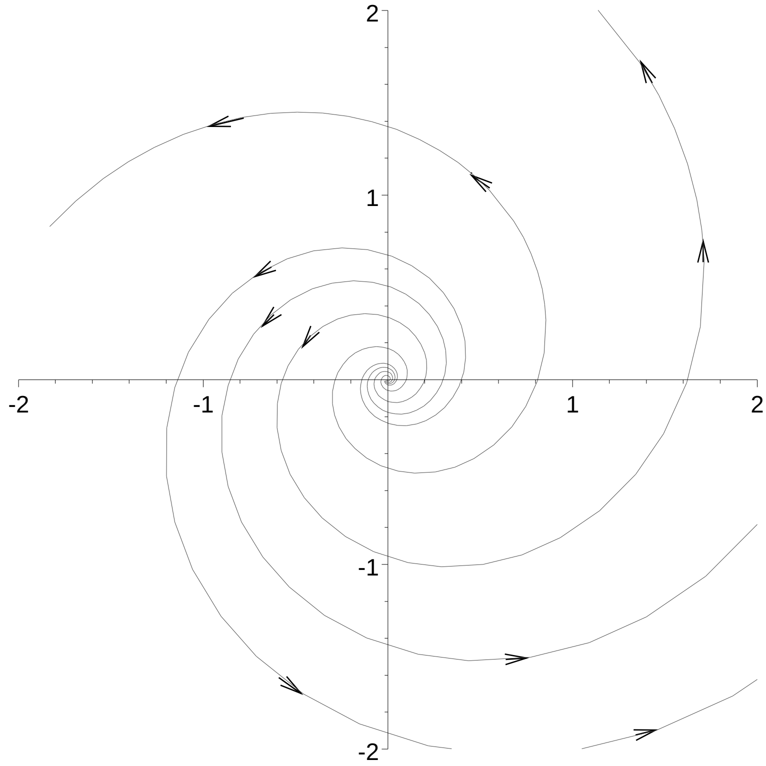
[1] Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`



2
1
-2
-1
1
2
-1
-2