

Bayes Rule¹

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Consider X and Y , two random variables with X taking values in \mathcal{X} and Y in \mathcal{Y} . The *joint probability* is

$$p_{X,Y}(x,y) = \text{probability that } X=x \text{ and } Y=y \quad (1)$$

Here is an example with $\mathcal{X} = \{0, 1\}$ and $\mathcal{Y} = \{0, 1, 2\}$

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1/4 & 1/8 \\ 1 & 1/16 & 3/8 \\ 2 & 1/16 & 1/8 \end{array} \quad (2)$$

So the probability of $(X, Y) = (0, 1)$ is $1/16$.

From the joint probability we can define the *marginal distributions*

$$\begin{aligned} p_X(x) &= \text{probability that } X = x \text{ irrespective of what } Y \text{ is} \\ p_Y(y) &= \text{probability that } Y = y \text{ irrespective of what } X \text{ is} \end{aligned} \quad (3)$$

and, it follows that

$$\begin{aligned} p_X(x) &= \sum_{y \in \mathcal{Y}} p_{X,Y}(x,y) \\ p_Y(y) &= \sum_{x \in \mathcal{X}} p_{X,Y}(x,y) \end{aligned} \quad (4)$$

For the example above

$$\begin{array}{c|cc} X & 0 & 1 \\ \hline & 3/8 & 5/8 \end{array} \quad (5)$$

and

$$\begin{array}{c|ccc} Y & 0 & 1 & 2 \\ \hline & 3/8 & 7/16 & 3/16 \end{array} \quad (6)$$

We can also define the *conditional probabilities*

$$\begin{aligned} p_{X|Y}(x|y) &= \text{probability that } X = x \text{ if } Y = y \\ p_{Y|X}(y|x) &= \text{probability that } Y = y \text{ if } X = x \end{aligned} \quad (7)$$

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These are calculated using Bayes rule, basically this says that the probability of $X = x$ and $Y = y$ is the probability of $X = x$ multiplied by the probability of $Y = y$ given that $X = x$:

$$p_{X,Y}(x, y) = p_X(x)p_{Y|X}(y|x) \quad (8)$$

and, similarly

$$p_{X,Y}(x, y) = p_Y(y)p_{X|Y}(x|y) \quad (9)$$

and, of course, this means

$$\begin{aligned} p_{Y|X}(y|x) &= \frac{p_{X,Y}(x, y)}{p_X(x)} \\ p_{X|Y}(x|y) &= \frac{p_{X,Y}(x, y)}{p_Y(y)} \end{aligned} \quad (10)$$

so in the example above For the example above

$$\begin{array}{c|cc} X|Y=0 & 0 & 1 \\ \hline & 2/3 & 1/3 \end{array} \quad (11)$$

$$\begin{array}{c|cc} X|Y=1 & 0 & 1 \\ \hline & 1/7 & 6/7 \end{array} \quad (12)$$

$$\begin{array}{c|cc} X|Y=2 & 0 & 1 \\ \hline & 1/3 & 2/3 \end{array} \quad (13)$$

and

$$\begin{array}{c|ccc} Y|X=0 & 0 & 1 & 2 \\ \hline & 4/6 & 1/6 & 1/6 \end{array} \quad (14)$$

$$\begin{array}{c|ccc} Y|X=1 & 0 & 1 & 2 \\ \hline & 1/5 & 3/5 & 1/5 \end{array} \quad (15)$$

Finally, since $p_{X,Y}(x, y) = p_{Y,X}(y, x)$ we can write Bayes rule in the more familiar form

$$p_{Y|X}(y|x) = \frac{p_Y(y)p_{X|Y}(x|y)}{p_X(x)} \quad (16)$$