2E2 Tutorial Sheet 17 Solutions[1]

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Questions:

1. (2) Assuming the solution of

$$(1-t)y' + y = 0 \tag{1}$$

has a series expansion about $t = 0$ work out the recursion relation. Write out the first few terms and show that the series $a_2 = 0$ so the series actually terminates to give $y = A(1-t)$ for arbitrary A.

Solution: So we begin by writing

$$y = \sum_{n=0}^{\infty} a_n t^n \tag{2}$$

and so by differentiation we get

$$y' = \sum_{n=0}^{\infty} a_n n t^{n-1} \tag{3}$$

and hence

$$ty' = \sum_{n=0}^{\infty} a_n n t^n. \tag{4}$$

Thus, substituting the differential equation we get

$$\sum_{n=0}^{\infty} a_n n t^{n-1} - \sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0 \tag{5}$$

In order to make progress we need to rewrite the first of these three series so that it is in the form

$$\sum_{n=0}^{\infty} \text{stuff}_n t^n \tag{6}$$

so that all three bits in the equation match. Well, let $m = n - 1$ in the expression for y', (3), to get

$$y' = \sum_{m=0}^{\infty} a_{m+1}(m+1)t^m. \tag{7}$$

[1] Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/ 2E2.html`

In fact, this looks at first like it gives

$$y' = \sum_{m=-1}^{\infty} a_{m+1}(m+1)t^m \tag{8}$$

but the $m = -1$ term is zero, so that's fine. Now m is just an index so we can rename it n, don't get confused, this isn't the original n, we just want all parts of the equation to look the same.

In fact, we now have

$$\sum_{n=0}^{\infty} a_{n+1}(n+1)t^n - \sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0 \tag{9}$$

and we can group this all together to give

$$\sum_{n=0}^{\infty} [a_{n+1}(n+1) + (1-n)a_n]t^n = 0. \tag{10}$$

The recursion relation is

$$a_{n+1} = -\left(\frac{1-n}{1+n}\right) a_n \tag{11}$$

and this applies to n from zero upwards since that is what appears in the sum sign.

Starting at $n = 0$ we have

$$a_1 = -a_0. \tag{12}$$

For $n = 1$ we get

$$a_2 = 0 \tag{13}$$

and the series terminates here because every term is something multiplied by the one before and so if a_2 is zero the rest of the series is zero. Thus $y = a_0(1-t)$ for arbitrary a_0.

2. (2) Assuming the solution of

$$(1-t^2)y' - 2ty = 0 \tag{14}$$

has a series expansion about $t = 0$, work out the recursion relation.

Solution: Once again let

$$y = \sum_{n=0}^{\infty} a_n t^n \tag{15}$$

and, as before,

$$y' = \sum_{n=0}^{\infty} a_n n t^{n-1} \tag{16}$$

so

$$t^2 y' = \sum_{n=0}^{\infty} a_n n t^{n+1} \tag{17}$$

and finally

$$ty = \sum_{n=0}^{\infty} a_n t^{n+1}. \tag{18}$$

The equation then reads

$$\sum_{n=0}^{\infty} a_n n t^{n-1} - \sum_{n=0}^{\infty} a_n n t^{n+1} - 2\sum_{n=0}^{\infty} a_n t^{n+1}. \tag{19}$$

Once again, the first term is a problem because it doesn't have the same form as the other two. So, take

$$\sum_{n=0}^{\infty} a_n n t^{n-1} \tag{20}$$

and put $n - 1 = m + 1$ and, hence, $n = m + 2$. When $n = 0$, $m = -2$ and when $n = 1$, $m = -1$. Thus

$$\sum_{n=0}^{\infty} a_n n t^{n-1} = \sum_{m=-2}^{\infty} a_{m+2}(m+2)t^{m+1} \tag{21}$$

and, once again renaming m as n we get

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}t^{n+1} - \sum_{n=0}^{\infty} n a_n t^{n+1} - 2\sum_{n=0}^{\infty} a_n t^{n+1} = 0. \tag{22}$$

The problem now is with the range that the first sum runs over. The $n = -2$ term is no problem, it is zero, but the $n = -1$ term is a_1. Thus, we write

$$\sum_{n=-2}^{\infty} (n+2)a_{n+2}t^{n+1} = a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}t^{n+1} \tag{23}$$

and the equation becomes

$$a_1 + \sum_{n=0}^{\infty}(n+2)a_{n+2}t^{n+1} - \sum_{n=0}^{\infty} a_n n t^{n+1} - 2\sum_{n=0}^{\infty} a_n t^{n+1} = 0. \tag{24}$$

Thus

$$a_1 + \sum_{n=0}^{\infty} [(n+2)a_{n+2} - na_n - 2a_n]t^{n+1} = 0. \tag{25}$$

Notice that the summand starts with the t term. The recursion relation is therefore

$$a_{n+2} = a_n \tag{26}$$

with the additional conditions $a_1 = 0$. Hence, $a_6 = a_4 = a_2 = a_0$, $a_5 = a_3 = a_1 = 0$ and so on. The first four nonzero terms of the expansion gives

$$y = a_0(1 + t^2 + t^4 + t^6 + \dots). \tag{27}$$

3. (2) Assuming the solution of

$$y'' - 3y' + 2y = 0 \tag{28}$$

has a series expansion about $t = 0$, by substitution, work out the recursion relation. If $y(0) = 1$ and $y'(0) = 0$ what are the first three non-zero terms

Solution: Again

$$y = \sum_{n=0}^{\infty} a_n t^n \tag{29}$$

so

$$y' = \sum_{n=0}^{\infty} n a_n t^{n-1} \tag{30}$$

and

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} \tag{31}$$

Thus,

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 3\sum_{n=0}^{\infty} na_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0 \tag{32}$$

Again, we want to make each part look the same. As before, changing the index gi ves

$$y' = \sum_{n=0}^{\infty} na_n t^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} t^n. \tag{33}$$

The same thing can be done with the y'': let $m = n - 2$ to get

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{m=-2}^{\infty} (m+1)(m+2)a_{m+2} t^m \tag{34}$$

and the $m = -2$ and $m = -1$ terms are both zero, so, renaming the m as n we get

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}t^n - 3\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + 2\sum_{n=0}^{\infty}a_n t^n = 0 \tag{35}$$

and this gives

$$\sum_{n=0}^{\infty}[(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n]t^n = 0. \tag{36}$$

The recursion relation is

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0. \tag{37}$$

Now apply the initial conditions, $y(0) = 1$ implies that $a_0 = 1$, $y'(0) = 0$ implies $a_1 = 0$. For $n = 0$ the recursion relation gives

$$2a_2 - 3a_1 + 2a_0 = 0 \tag{38}$$

and so $a_2 = -a_0 = -1$. Next $n = 1$ gives

$$6a_3 - 6a_2 + 2a_1 = 0 \tag{39}$$

and so $a_3 = a_2 = -a_0 = -1$. Therefore the first three nonzero terms are

$$y = 1 - t^2 - t^3 + \dots. \tag{40}$$

4. (2) Assuming the solution of

$$y'' - 3t^2 y = 0 \tag{41}$$

has a series expansion about $t = 0$ work out the recursion relation and write out the first four non-zero terms if $y(0) = 1$ and $y'(0) = 1$.

Solution: We substitute

$$y = \sum_{n=0}^{\infty} a_n t^n \tag{42}$$

into the equation. This gives

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} 3a_n t^{n+2} = 0 \tag{43}$$

The problem here is with the powers of t. The easiest thing is to change everything to the highest power, in this case $n + 2$. Hence, put $m + 2 = n - 2$ in the first sum

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{m=-4}^{\infty}(m+4)(m+3)a_{m+4}t^{m+2}. \tag{44}$$

and substitute that back into the equation, writing m as n:

$$\sum_{n=-4}^{\infty}(n+4)(n+3)a_{n+4}t^{n+2}-\sum_{n=0}^{\infty}3a_nt^{n+2}=0 \tag{45}$$

and so the problem now is that the ranges are different. We need to take out the first few term of the first sum, well, the $n=-4$ and $n=-3$ terms are zero and so

$$\sum_{n=-4}^{\infty}(n+4)(n+3)a_{n+4}t^{n+2}=2a_2+6a_3t+\sum_{n=0}^{\infty}(n+4)(n+3)a_{n+4}t^{n+2}. \tag{46}$$

Now the equation reads

$$2a_2+6a_3t+\sum_{n=0}^{\infty}(n+4)(n+3)a_{n+4}t^{n+2}-\sum_{n=0}^{\infty}3a_nt^{n+2}=0 \tag{47}$$

or

$$2a_2+6a_3t+\sum_{n=0}^{\infty}\left[(n+4)(n+3)a_{n+4}-3a_n\right]t^{n+2}=0. \tag{48}$$

Thus

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_{n+4} &= \frac{3}{(n+4)(n+3)}a_n \end{aligned} \tag{49}$$

where the recursion relation applies for $n=0,1,\ldots$. Now, $y(0)=1$ implies $a_0=1$ and $y'(0)=1$ implies $a_1=1$, next, with $n=0$, t he recursion gives

$$a_4=\frac{1}{4}a_0=\frac{1}{4} \tag{50}$$

and with $n=1$

$$a_5=\frac{3}{20}a_1=\frac{3}{20}. \tag{51}$$

Now since $a_2=a_3=0$ the $n=2$ recusion gives $a_6=0$ and the $n=3$ recursion g ives $a_7=0$. However, $n=4$ gives

$$a_8=\frac{3}{32}a_4=\frac{3}{128} \tag{52}$$

and so

$$y=1+t+\frac{1}{4}t^4+\frac{3}{20}t^5+\frac{3}{128}t^8+\ldots. \tag{53}$$

Aside. In the above we made all the powers the same as the highest power, this is usually the easiest thing, but it is just a matter of convenience. If we had decided to make them equal the smallest power i nstead, we would have substituted $n+2 = m-2$ in the second sum to get

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=4}^{\infty} 3a_{n-4}t^{n-2} = 0 \tag{54}$$

and we would then remove the first four term from the first sum to get

$$2a_2 + 6a_3 t + \sum_{n=4}^{\infty} \left[n(n-1)a_n t^{n-2} - 3a_{n-4}\right] t^{n-2} = 0 \tag{55}$$

and so

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_n &= \frac{3}{n(n-1)} a_{n-4} \end{aligned} \tag{56}$$

where now the recursion relation applies to $n = 4, 5, \ldots$ because that is what is in the sum. Another way of proceeding is to define $a_{-4} = a_{-3} = a_{-2} = a_{-1} = 0$ and then rewrite the equation as

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} 3a_{n-4}t^{n-2} = 0 \tag{57}$$

and carry on from there.