

## 2E2 A spirial example<sup>1</sup>

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Find the solution of

$$\frac{dy_1}{dt} = y_1 - 3y_2 \quad (1)$$

$$\frac{dy_2}{dt} = 3y_1 + y_2 \quad (2)$$

for initial conditions  $y_1(0) = r$  and  $y_2(0) = 0$  write this in real form. Sktech the phase diagram.

*Solution:* This time the matrix is

$$A = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$$

and so the spectrum is complex,  $\lambda_1 = 1 + 3i$  with eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

and  $\lambda_2 = 1 - 3i$  with eigenvector

$$\mathbf{x}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

The solution is then

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} e^{(1+3i)t} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} e^{(1-3i)t}.$$

Now, this means

$$\begin{pmatrix} r \\ 0 \end{pmatrix} = \mathbf{y}(0) = c_1 \begin{pmatrix} i \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

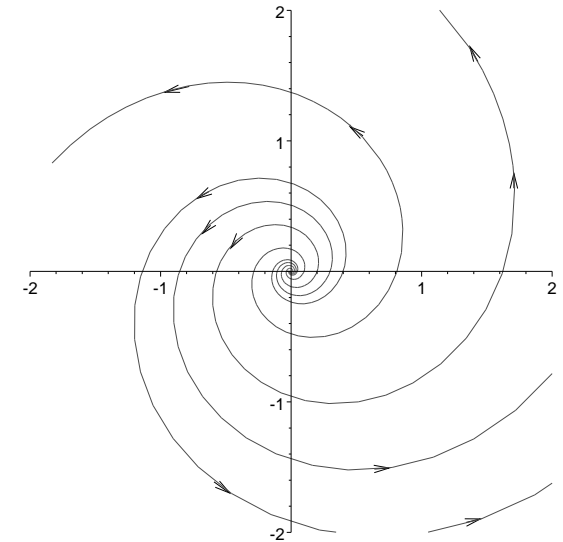
and hence  $c_1 = -ir/2$  and  $c_2 = ir/2$ . Now using  $\exp(a + ib) = \exp a \exp ib$  we have solution

$$\mathbf{y} = \frac{r}{2} \left[ \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{3it} + \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-3it} \right] e^t.$$

and so

$$\begin{aligned} \mathbf{y} &= \frac{r}{2} \left[ \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos 3t + i \sin 3t) + \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos 3t - i \sin 3t) \right] e^t \\ &= \begin{pmatrix} r \cos 3t \\ r \sin 3t \end{pmatrix} e^t \end{aligned}$$

So, this gives the outwardward spiral. Notice how fast the spiral goes in. The radius increases exponentially.



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