2E2 Tutorial Sheet 2, Solutions[1]

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Questions In the answers here I use the expression *subsidiary equation*, this is a name sometimes given for the Laplace transform of the differential equations, hence, the subsidiary equation is the equation you solve to get F.

1. (2) Find the Laplace transform of both sides of the differential equation

$$2\frac{df}{dt} = 1$$

with initial conditions $f(0) = 4$. By solving the resulting equations find $F(s)$. Based on the Laplace trasforms you know, decide what $f(t)$ is.

Solution: Using linearity of $\mathcal{L}$, plus the property of Laplace transforms of derivatives, we get

$$\begin{aligned} \mathcal{L}\left(2\frac{dx}{dt}\right) &= \mathcal{L}(1) \\ 2\mathcal{L}\left(\frac{df}{dt}\right) &= \frac{1}{s} \\ 2sF(s) - 8 &= \frac{1}{s} && (1) \\ & && (2) \end{aligned}$$

This means that

$$F(s) = \frac{4}{s} + \frac{1}{2s^2} \quad (3)$$

and, since, $\mathcal{L}(t^n) = n!/s^{n+1}$

$$f = 4 + \frac{1}{2}t \quad (4)$$

To verify that this solves the equation note that $f(0) = 4$ as required and $f' = 1/2$.

2. (2) Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 1 \quad (5)$$

with boundary conditions $f(0) = f'(0) = 0$. You will need to do partial fractions.

Solution: First, take the Laplace transform of the equation. Since $f'(0) = f(0) = 0$, if $\mathcal{L}(f) = F(s)$ then $\mathcal{L}(f') = sF(s)$ and $\mathcal{L}(f'') = s^2F(s)$. Thus, the subsidiary equation is

$$s^2F - 4sF + 3F = \frac{1}{s} \quad (6)$$

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and so

$$\begin{aligned} (s^2 - 4s + 3)F &= \frac{1}{s} \\ F &= \frac{1}{s}\frac{1}{s^2 - 4s + 3} && (7) \end{aligned}$$

and, since $s^2 - 4s + 3 = (s-3)(s-1)$, this gives

$$F = \frac{1}{s(s-3)(s-1)} \quad (8)$$

Before we can invert this, we need to do a partial fraction expansion.

$$\begin{aligned} \frac{1}{s(s-3)(s-1)} &= \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1} \\ 1 &= A(s-3)(s-1) + Bs(s-1) + Cs(s-3) && (9) \end{aligned}$$

So substituting in $s = 0$ we get $A = 1/3$, $s = 3$ gives $B = 1/6$ and $s = 1$ gives $C = -1/2$. Hence

$$F = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)} \quad (10)$$

and so

$$f(t) = \frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^t \quad (11)$$

3. (2) Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 0 \quad (12)$$

with boundary conditions $f(0) = 1$ and $f'(0) = 1$.

Solution: In this example there are non-zero boundary conditions. Since

$$\begin{aligned} \mathcal{L}(f') &= sF - f(0) && (13) \\ \mathcal{L}(f'') &= s^2F - sf(0) - f'(0) && (14) \end{aligned}$$

the subsidiary equation in this case is

$$s^2F - s - 1 - 4sF + 4 + 3F = 0 \quad (15)$$

so

$$(s^2 - 4s + 3)F = s - 3. \quad (16)$$

Hence

$$F = \frac{1}{s-1} \quad (17)$$

and

$$f(t) = e^t \quad (18)$$

4. (2) Using the Laplace transform solve the differential equation

$$f'' - 4f' + 3f = 0 \tag{19}$$

with boundary conditions $f(0) = 1$ and $f'(0) = 0$.

Solution: In this example there are also non-zero boundary conditions. Since

$$\mathcal{L}(f') = sF - f(0) \tag{20}$$
$$\mathcal{L}(f'') = s^2F - sf(0) - f'(0) \tag{21}$$

the subsidiary equation in this case is

$$s^2F - s - 4sF + 4 + 3F = 0 \tag{22}$$

so

$$(s^2 - 4s + 3)F = s - 4. \tag{23}$$

or

$$F = \frac{s-4}{(s-3)(s-1)} \tag{24}$$

We do partial fractions

$$\frac{s-4}{(s-3)(s-1)} = \frac{A}{s-3} + \frac{B}{s-1} \tag{25}$$

implying

$$s - 4 = A(s-1) + B(s-3) \tag{26}$$

Choosing $s = 3$ give $A = -1/2$ and chosing $s = 1$ gives $B = 3/2$ so

$$F = -\frac{1}{2}\frac{1}{s-3} + \frac{3}{2}\frac{1}{s-1} \tag{27}$$

and

$$f = -\frac{1}{2}e^{3t} + \frac{3}{2}e^{t} \tag{28}$$