472 2007/8 q2 b/c.

(a) $X \to Y \to Z$ Markov iff $p(z|y,x) = p(z|y)$

(b) $X \to Y \to Z$

[redacted]

$$p(x,z|y) = \frac{p(x,y,z)}{p(y)} = \frac{p(x,y)p(z|x,y)}{p(y)} = \frac{p(x,y)p(z|y)}{p(y)}$$

$$= \underbrace{p(x|y)p(z|y)}_{\text{symmetric in } x \& z}$$

ie. $X \to Y \to Z$ iff $X \& Z$ are condit. indep. which is a symmetric condit. ind.

c)

p_X	0	2
	$\frac{1}{2}$	$\frac{1}{2}$

$p_{Y\|X=0}$	0	1
	$\frac{1}{2}$	$\frac{1}{2}$

$p_{Y\|X=1}$	1	2
	$\frac{1}{2}$	$\frac{1}{2}$

X \ Y	0	1	p_X
0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
2	0	$\frac{1}{4}$	$\frac{1}{4}$
p_Y	$\frac{1}{2}$	$\frac{1}{2}$	

$$I(X,Y) = \sum p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

$$= \frac{1}{4}\log\frac{1/4}{1/8} + \frac{1}{4}\log\frac{1/4}{1/4} + \frac{1}{4}\log\frac{1/4}{1/4} + \frac{1}{4}\log\frac{1/4}{1/8}$$

$$= \frac{1}{2}\log 2 = \frac{1}{2}$$

Z \ X	0	1	
0	1/8	0	1/8
1	1/4	1/8	3/8
2	1/8	1/4	3/8
3	0	1/8	1/8
	1/2	1/2	

eg if Z is 1 & X is 0

$1/8 + 1/8$ $\quad$ $1/8$

$\uparrow$ $\leftarrow$ $y=0$ $z=1$.

$y=1$ $z=0$

$$I(X,Z) = \sum p(x,y) \log \frac{p(x,y)}{p(x)p(y)}$$

$$= \frac{1}{8}\log 2 + \frac{1}{4}\log \frac{1/4}{3/16} + \frac{1}{8}\log \frac{1/8}{3/16} + \frac{1}{4}\log \frac{1/4}{3/16} + \frac{1}{8}\log \frac{1/8}{3/16} + \frac{1}{8}\log 2$$

$$= \frac{1}{4} + \frac{1}{2}\log \frac{4}{3} + \frac{1}{4}\log \frac{2}{3}$$

$$\frac{1}{4} + \frac{3}{4}\log 4 - \frac{3}{4}\log 3 = \frac{1}{4} + \frac{3}{2} - \frac{3}{4}\log 3$$

$$= \frac{7}{4} - \frac{3}{4}\log 3 \leq \frac{1}{2}$$

ie

$$\frac{1}{4}\log \frac{1/4 \cdot 4}{3/2 \cdot 3} \qquad \frac{1}{8}\log \frac{1/8}{3/8 \cdot 1/2}$$

$$I(X,Z) = \frac{1}{8}\log 2 + \frac{1}{4}\log \frac{6}{3} + \frac{1}{8}\log \frac{2}{3} + \frac{1}{8}\log \frac{2}{3} + \frac{1}{4}\log \frac{4}{3} + \frac{1}{8}\log 2$$

$$= \frac{1}{4} + \frac{1}{2}\log \frac{6}{3} + \frac{1}{4}\log \frac{2}{3}$$

$$= \frac{1}{4} + \frac{1}{2}\log 4 - \frac{1}{2}\log 3 + \frac{1}{4}\log 2 - \frac{1}{4}\log 3$$

$$= \frac{3}{2} - \frac{3}{4}\log 3 \approx .311. \leq \frac{1}{2}.$$