

2E2 Tutorial Sheet 17 Solutions¹

5 March 2006

Questions:

1. (2) Assuming the solution of

$$(1 - t)y' + y = 0 \quad (1)$$

has a series expansion about $t = 0$ work out the recursion relation. Write out the first few terms and show that the series $a_2 = 0$ so the series actually terminates to give $y = A(1 - t)$ for arbitrary A .

Solution: So we begin by writing

$$y = \sum_{n=0}^{\infty} a_n t^n \quad (2)$$

and so by differentiation we get

$$y' = \sum_{n=0}^{\infty} a_n n t^{n-1} \quad (3)$$

and hence

$$ty' = \sum_{n=0}^{\infty} a_n n t^n. \quad (4)$$

Thus, substituting the differential equation we get

$$\sum_{n=0}^{\infty} a_n n t^{n-1} - \sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0 \quad (5)$$

In order to make progress we need to rewrite the first of these three series so that it is in the form

$$\sum_{n=0}^{\infty} \text{stuff}_n t^n \quad (6)$$

so that all three bits in the equation match. Well, let $m = n - 1$ in the expression for y' , (3), to get

$$y' = \sum_{m=0}^{\infty} a_{m+1} (m+1) t^m. \quad (7)$$

In fact, this looks at first like it gives

$$y' = \sum_{m=-1}^{\infty} a_{m+1} (m+1) t^m \quad (8)$$

but the $m = -1$ term is zero, so that's fine. Now m is just an index so we can rename it n , don't get confused, this isn't the original n , we just want all parts of the equation to look the same.

In fact, we now have

$$\sum_{n=0}^{\infty} a_{n+1} (n+1) t^n - \sum_{n=0}^{\infty} a_n n t^n + \sum_{n=0}^{\infty} a_n t^n = 0 \quad (9)$$

and we can group this all together to give

$$\sum_{n=0}^{\infty} [a_{n+1} (n+1) + (1-n)a_n] t^n = 0. \quad (10)$$

The recursion relation is

$$a_{n+1} = -\left(\frac{1-n}{1+n}\right) a_n \quad (11)$$

and this applies to n from zero upwards since that is what appears in the sum sign.

Starting at $n = 0$ we have

$$a_1 = -a_0. \quad (12)$$

For $n = 1$ we get

$$a_2 = 0 \quad (13)$$

and the series terminates here because every term is something multiplied by the one before and so if a_2 is zero the rest of the series is zero. Thus $y = a_0(1 - t)$ for arbitrary a_0 .

2. (2) Assuming the solution of

$$(1 - t^2)y' - 2ty = 0 \quad (14)$$

has a series expansion about $t = 0$, work out the recursion relation.

Solution: Once again let

$$y = \sum_{n=0}^{\infty} a_n t^n \quad (15)$$

¹Conor Houghton, houghton@maths.tcd.ie and <http://www.maths.tcd.ie/~houghton/2E2.html>

and, as before,

$$y' = \sum_{n=0}^{\infty} a_n n t^{n-1} \quad (16)$$

so

$$t^2 y' = \sum_{n=0}^{\infty} a_n n t^{n+1} \quad (17)$$

and finally

$$t y = \sum_{n=0}^{\infty} a_n t^{n+1}. \quad (18)$$

The equation then reads

$$\sum_{n=0}^{\infty} a_n n t^{n-1} - \sum_{n=0}^{\infty} a_n n t^{n+1} - 2 \sum_{n=0}^{\infty} a_n t^{n+1}. \quad (19)$$

Once again, the first term is a problem because it doesn't have the same form as the other two. So, take

$$\sum_{n=0}^{\infty} a_n n t^{n-1} \quad (20)$$

and put $n-1 = m+1$ and, hence, $n = m+2$. When $n = 0$, $m = -2$ and when $n = 1$, $m = -1$. Thus

$$\sum_{n=0}^{\infty} a_n n t^{n-1} = \sum_{m=-2}^{\infty} a_{m+2} (m+2) t^{m+1} \quad (21)$$

and, once again renaming m as n we get

$$\sum_{n=-2}^{\infty} (n+2) a_{n+2} t^{n+1} - \sum_{n=0}^{\infty} n a_n t^{n+1} - 2 \sum_{n=0}^{\infty} a_n t^{n+1} = 0. \quad (22)$$

The problem now is with the range that the first sum runs over. The $n = -2$ term is no problem, it is zero, but the $n = -1$ term is a_1 . Thus, we write

$$\sum_{n=-2}^{\infty} (n+2) a_{n+2} t^{n+1} = a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} t^{n+1} \quad (23)$$

and the equation becomes

$$a_1 + \sum_{n=0}^{\infty} (n+2) a_{n+2} t^{n+1} - \sum_{n=0}^{\infty} n a_n t^{n+1} - 2 \sum_{n=0}^{\infty} a_n t^{n+1} = 0. \quad (24)$$

Thus

$$a_1 + \sum_{n=0}^{\infty} [(n+2) a_{n+2} - n a_n - 2 a_n] t^{n+1} = 0. \quad (25)$$

Notice that the summand starts with the t term. The recursion relation is therefore

$$a_{n+2} = a_n \quad (26)$$

with the additional conditions $a_1 = 0$. Hence, $a_6 = a_4 = a_2 = a_0$, $a_5 = a_3 = a_1 = 0$ and so on. The first four nonzero terms of the expansion gives

$$y = a_0(1 + t^2 + t^4 + t^6 + \dots). \quad (27)$$

3. (2) Assuming the solution of

$$y'' - 3y' + 2y = 0 \quad (28)$$

has a series expansion about $t = 0$, by substitution, work out the recursion relation. If $y(0) = 1$ and $y'(0) = 0$ what are the first three non-zero terms

Solution: Again

$$y = \sum_{n=0}^{\infty} a_n t^n \quad (29)$$

so

$$y' = \sum_{n=0}^{\infty} n a_n t^{n-1} \quad (30)$$

and

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} \quad (31)$$

Thus,

$$\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} - 3 \sum_{n=0}^{\infty} n a_n t^{n-1} + 2 \sum_{n=0}^{\infty} a_n t^n = 0 \quad (32)$$

Again, we want to make each part look the same. As before, changing the index gives

$$y' = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} t^n. \quad (33)$$

The same thing can be done with the y'' : let $m = n-2$ to get

$$\sum_{n=0}^{\infty} n(n-1) a_n t^{n-2} = \sum_{m=-2}^{\infty} (m+1)(m+2) a_{m+2} t^m \quad (34)$$

and the $m = -2$ and $m = -1$ terms are both zero, so, renaming the m as n we get

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}t^n - 3 \sum_{n=0}^{\infty} (n+1)a_{n+1}t^n + 2 \sum_{n=0}^{\infty} a_n t^n = 0 \quad (35)$$

and this gives

$$\sum_{n=0}^{\infty} [(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n]t^n = 0. \quad (36)$$

The recursion relation is

$$(n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + 2a_n = 0. \quad (37)$$

Now apply the initial conditions, $y(0) = 1$ implies that $a_0 = 1$, $y'(0) = 0$ implies $a_1 = 0$. For $n = 0$ the recursion relation gives

$$2a_2 - 3a_1 + 2a_0 = 0 \quad (38)$$

and so $a_2 = -a_0 = -1$. Next $n = 1$ gives

$$6a_3 - 6a_2 + 2a_1 = 0 \quad (39)$$

and so $a_3 = a_2 = -a_0 = -1$. Therefore the first three nonzero terms are

$$y = 1 - t^2 - t^3 + \dots \quad (40)$$

4. (2) Assuming the solution of

$$y'' - 3t^2y = 0 \quad (41)$$

has a series expansion about $t = 0$ work out the recursion relation and write out the first four non-zero terms if $y(0) = 1$ and $y'(0) = 1$.

Solution: We substitute

$$y = \sum_{n=0}^{\infty} a_n t^n \quad (42)$$

into the equation. This gives

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} 3a_n t^{n+2} = 0 \quad (43)$$

The problem here is with the powers of t . The easiest thing is to change everything to the highest power, in this case $n + 2$. Hence, put $m + 2 = n - 2$ in the first sum

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{m=-4}^{\infty} (m+4)(m+3)a_{m+4} t^{m+2}. \quad (44)$$

and substitute that back into the equation, writing m as n :

$$\sum_{n=-4}^{\infty} (n+4)(n+3)a_{n+4} t^{n+2} - \sum_{n=0}^{\infty} 3a_n t^{n+2} = 0 \quad (45)$$

and so the problem now is that the ranges are different. We need to take out the first few term of the first sum, well, the $n = -4$ and $n = -3$ terms are zero and so

$$\sum_{n=-4}^{\infty} (n+4)(n+3)a_{n+4} t^{n+2} = 2a_2 + 6a_3 t + \sum_{n=0}^{\infty} (n+4)(n+3)a_{n+4} t^{n+2}. \quad (46)$$

Now the equation reads

$$2a_2 + 6a_3 t + \sum_{n=0}^{\infty} (n+4)(n+3)a_{n+4} t^{n+2} - \sum_{n=0}^{\infty} 3a_n t^{n+2} = 0 \quad (47)$$

or

$$2a_2 + 6a_3 t + \sum_{n=0}^{\infty} [(n+4)(n+3)a_{n+4} - 3a_n] t^{n+2} = 0. \quad (48)$$

Thus

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_{n+4} &= \frac{3}{(n+4)(n+3)} a_n \end{aligned} \quad (49)$$

where the recursion relation applies for $n = 0, 1, \dots$. Now, $y(0) = 1$ implies $a_0 = 1$ and $y'(0) = 1$ implies $a_1 = 1$, next, with $n = 0$, the recursion gives

$$a_4 = \frac{1}{4} a_0 = \frac{1}{4} \quad (50)$$

and with $n = 1$

$$a_5 = \frac{3}{20} a_1 = \frac{3}{20}. \quad (51)$$

Now since $a_2 = a_3 = 0$ the $n = 2$ recursion gives $a_6 = 0$ and the $n = 3$ recursion gives $a_7 = 0$. However, $n = 4$ gives

$$a_8 = \frac{3}{32} a_4 = \frac{3}{128} \quad (52)$$

and so

$$y = 1 + t + \frac{1}{4} t^4 + \frac{3}{20} t^5 + \frac{3}{128} t^8 + \dots \quad (53)$$

Aside. In the above we made all the powers the same as the highest power, this is usually the easiest thing, but it is just a matter of convenience. If we had decided to make them equal the smallest power instead, we would have substituted $n+2 = m-2$ in the second sum to get

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=4}^{\infty} 3a_{n-4} t^{n-2} = 0 \quad (54)$$

and we would then remove the first four term from the first sum to get

$$2a_2 + 6a_3 t + \sum_{n=4}^{\infty} [n(n-1)a_n t^{n-2} - 3a_{n-4}] t^{n-2} = 0 \quad (55)$$

and so

$$\begin{aligned} a_2 &= 0 \\ a_3 &= 0 \\ a_n &= \frac{3}{n(n-1)} a_{n-4} \end{aligned} \quad (56)$$

where now the recursion relation applies to $n = 4, 5, \dots$ because that is what is in the sum. Another way of proceeding is to define $a_{-4} = a_{-3} = a_{-2} = a_{-1} = 0$ and then rewrite the equation as

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - \sum_{n=0}^{\infty} 3a_{n-4} t^{n-2} = 0 \quad (57)$$

and carry on from there.