2E2 Tutorial Sheet 5 Solutions[1]

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Questions

1. (2) Solve, using Laplace transforms,

$$f'' + 4f = 1 \tag{1}$$

with $f(0) = f'(0) = 0$.

Solution: First take the Laplace transfrom of each side:

$$s^2F + 4F = \frac{1}{s} \tag{2}$$

and so

$$F = \frac{1}{s(s+2i)(s-2i)} \tag{3}$$

Now lets do partial fractions

$$\frac{1}{s(s+2i)(s-2i)} = \frac{A}{s} + \frac{B}{s+2i} + \frac{C}{s-2i} \tag{4}$$

giving

$$1 = A(s-2i)(s+2i) + Bs(s-2i) + Cs(s+2i) \tag{5}$$

hence, choosing $s = 0$ gives $A = 1/4$, $s = -2i$ gives

$$1 = B(-2i)(-4i) = -8B \tag{6}$$

hence $B = -1/8$. $s = 2i$ gives $C = -1/8$ also. Now

$$F = \frac{1}{4}\frac{1}{s} - \frac{1}{8}\frac{1}{s+2i} - \frac{1}{8}\frac{1}{s-2i} \tag{7}$$

so

$$f = \frac{1}{4} - \frac{1}{8}\left(e^{-2it} + e^{2it}\right) \tag{8}$$

then, using

$$\cos 2t = \frac{e^{2it} + e^{-2it}}{2} \tag{9}$$

we conclude

$$f = \frac{1}{4} - \frac{1}{4}\cos 2t \tag{10}$$

[1] Conor Houghton, `houghton@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~houghton/2E2.html`

2. (3) Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = e^t \tag{11}$$

with boundary conditions $f(0) = 0$ and $f'(0) = 0$.

*Solution:*Taking the Laplace transform of the equation gives

$$s^2F + 6sF + 13F = \frac{1}{s-1} \tag{12}$$

so that

$$F = \frac{1}{(s-1)(s+3+2i)(s+3-2i)}. \tag{13}$$

We write

$$\frac{1}{(s-1)(s+3+2i)(s+3-2i)} = \frac{A}{s+3-2i} + \frac{B}{s+3+2i} + \frac{C}{s-1} \tag{14}$$

giving

$$1 = A(s-1)(s+3+2i) + B(s-1)(s+3-2i) + C(s+3-2i)(s+3+2i). \tag{15}$$

$s = -3+2i$ gives

$$1 = A(-4+2i)(4i) = A(-8-16i) \tag{16}$$

so

$$A = -\frac{1}{8+16i} = -\frac{1}{8+16i}\frac{8-16i}{8-16i} = -\frac{1-2i}{40} \tag{17}$$

In the same way, $s = -3-2i$ leads to

$$B = -\frac{1+2i}{40} \tag{18}$$

and, finally, $s = 1$ gives

$$C = \frac{1}{20}. \tag{19}$$

Putting all this together we get

$$F = -\frac{1-2i}{40}\frac{1}{s+3-2i} - \frac{1+2i}{40}\frac{1}{s+3+2i} + \frac{1}{20}\frac{1}{s-1} \tag{20}$$

and so

$$\begin{aligned} f &= -\frac{1-2i}{40}e^{-(3-2i)t} - \frac{1+2i}{40}e^{-(3+2i)t} + \frac{1}{20}e^t \\ &= -\frac{1}{40}e^{-3t}\left[(1-2i)e^{2it} + (1+2i)e^{-2it}\right] + \frac{1}{20}e^t \end{aligned} \tag{21}$$

We then substitute in

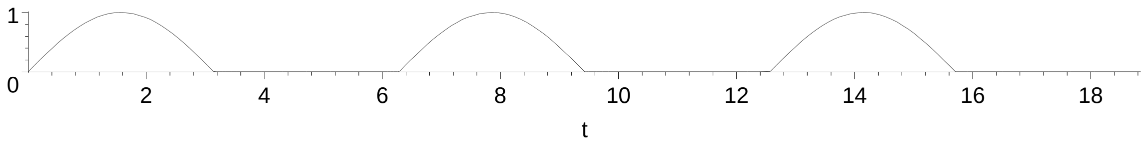
$$\begin{aligned} e^{2it} &= \cos 2t + i \sin 2t \\ e^{-2it} &= \cos 2t - i \sin 2t \end{aligned} \tag{22}$$

to end up with

$$f = -\frac{1}{20} e^{-3t} [2 \sin 2t + \cos 2t] + \frac{1}{20} e^t \tag{23}$$

3. (3) Use the formula for the Laplace transform of a periodic function to find the Laplace transform of a half-rectified wave

$$f(t) = \begin{cases} \sin t & \sin t > 0 \\ 0 & \sin t \le 0 \end{cases} \tag{24}$$



This is the form a AC current has after going through a diode and is a periodic function with period 2π. *Solution:*So we substitute this into the formula

$$\mathcal{L}(f) = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} f(t) e^{-st} dt = \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} \sin t e^{-st} dt \tag{25}$$

We need to do the integral. There are two obvious ways, the first is to split the sine into exponentials

$$\begin{aligned} \int_0^{\pi} \sin t e^{-st} dt &= \frac{1}{2i} \left(\int_0^{\pi} e^{(i-s)t} dt - \int_0^{\pi} e^{-(i+s)t} dt \right) \\ &= \frac{1}{2i} \left[\frac{1}{i - s} \left(e^{(i-s)\pi} - 1 \right) + \frac{1}{i + s} \left(e^{-(i+s)\pi} - 1 \right) \right] \end{aligned} \tag{26}$$

Now, we use

$$e^{i\pi} = e^{-i\pi} = -1 \tag{27}$$

and

$$\begin{aligned} \frac{1}{i - s} &= \frac{1}{i - s} \frac{-i - s}{-i - s} = -\frac{s + i}{s^2 + 1} \\ \frac{1}{i + s} &= \frac{1}{i + s} \frac{-i + s}{-i + s} = \frac{s - i}{s^2 + 1} \end{aligned} \tag{28}$$

to get

$$\int_0^{\pi} \sin t e^{-st} dt = \frac{1 + e^{-s\pi}}{1 + s^2} \tag{29}$$

or

$$\mathcal{L}(f) = \frac{1}{s^2+1}\frac{1+e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{s^2+1}\frac{1}{1-e^{-s\pi}} \tag{30}$$

where the final equality uses

$$1-e^{-2s\pi} = \left(1-e^{-s\pi}\right)\left(1+e^{-s\pi}\right) \tag{31}$$

The other way to do the integral is to integrate by parts. Briefly, write

$$\begin{aligned} I = \int_0^\pi \sin t e^{-st} dt &= -\frac{1}{s}\int_0^\pi \cos t e^{-st} dt \\ &= -\frac{1}{s}\left[-\frac{1}{s}\left(e^{-\pi s}+1\right)+\frac{1}{s}I\right] \end{aligned} \tag{32}$$

and solve for I to get the answer given at (**??**) above.