

2E2 Tutorial Sheet 15 Second Term, Solutions¹

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Consider the non-linear differential equation

$$y'' = y - y^2 \quad (1)$$

- (1) By defining $y_1 = y$ and $y_2 = y_1'$ convert this into two first order equations.
- (1) The stationary points are the points where $y_1' = y_2' = 0$, find the two stationary points for this equation.
- (2) Consider the $y_1 = 0$ stationary point, linearize the equations near this point by assuming $y_1 \ll 1$. Solve the corresponding linear equations. What sort of stationary point is this?
- (2) Consider the $y_1 = 1$ stationary point, linearize the equations near this point by assuming $y_1 = 1 + \eta$ where $\eta \ll 1$. Solve the corresponding linear equations. What sort of stationary point is this?
- (2) Try and draw the whole phase diagram, first draw in the two stationary points and then try and join the lines, remember the lines don't cross.

Solution: First we change the system into a pair of first order equations, $y_1 = y$ and

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1(1 - y_1). \end{aligned} \quad (2)$$

Setting $y_1' = y_2' = 0$ gives $y_2 = 0$ and $y_1(y_1 - 1) = 0$ so this has two critical points, one at $(y_1, y_2) = (0, 0)$ and the second at $(y_1, y_2) = (1, 0)$.

Near $(0, 0)$ the system linearizes to the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1. \end{aligned} \quad (3)$$

which has eigenvalue $\lambda_1 = 1$ corresponding to eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (4)$$

and eigenvalue $\lambda_1 = -1$ corresponding to eigenvector

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (5)$$

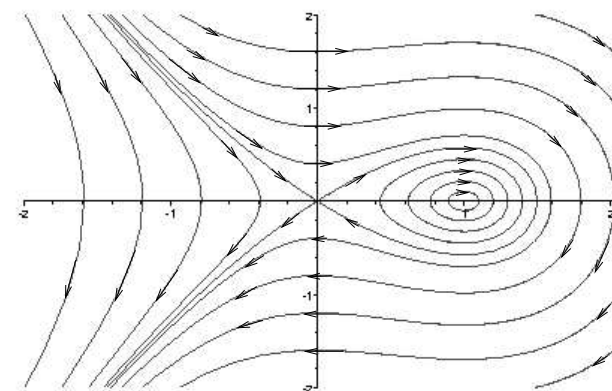
It is a saddlepoint.

Near $(1, 0)$ write $y_1 = 1 + \eta$ to get

$$\begin{aligned} \eta' &= y_2 \\ y_2' &= -\eta \end{aligned} \quad (6)$$

so the eigenvalues are $\lambda = \pm i$ and the critical point is a center.

To draw the phase plane, draw the saddlepoint and the circle and try to join them up. The answer is



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