2E2 Tutorial Sheet 3 Solutions[1]

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Questions

1. (2) Using the Laplace transform solve the differential equation
$$f'' - 2f' + f = 0 \tag{1}$$
with boundary conditions $f'(0) = 1$ and $f(0) = 0$.
Solution: Taking the Laplace transform we get
$$s^2F - 1 - 2F + a^2F = 0 \tag{2}$$
and hence
$$F = \frac{1}{(s-1)^2} \tag{3}$$
which means that
$$f = te^{at} \tag{4}$$

2. (2) Using the Laplace transform solve the differential equation
$$f'' + f' - 6f = e^{-3t} \tag{5}$$
with boundary conditions $f(0) = f'(0) = 0$.
Solution: So, as before, the subsidiary equation is
$$s^2F + sF - 6F = \frac{1}{s+3} \tag{6}$$
or
$$F = \frac{1}{(s+3)^2(s-2)} \tag{7}$$
As before, we do partial fractions
$$\begin{aligned} \frac{1}{(s+3)^2(s-2)} &= \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2} \\ 1 &= A(s+3)(s-2) + B(s-2) + C(s+3)^2 \end{aligned} \tag{8}$$
$s = -3$ gives $B = -1/5$ and $s = 2$ gives $C = 1/25$. Putting in $s = 1$ we find
$$1 = -4A + \frac{1}{5} + \frac{16}{25} \tag{9}$$
and so $A = -1/25$. Putting all this together says that
$$f = -\frac{1}{25}e^{-3t} - \frac{t}{5}e^{-3t} + \frac{1}{25}e^{2t} \tag{10}$$

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3. (2) Use Laplace transform methods to solve the differential equation
$$f'' + 2f' - 3f = \begin{cases} 1, & 0 \le t < c \\ 0, & t \ge c \end{cases} \tag{11}$$
subject to the initial conditions $f(0) = f'(0) = 0$. Notice that the right hand side is $1 - H_1(t)$.

Solution: Taking Laplace transforms of both sides and using the tables for the Laplace transform of the right hand side function, leads to
$$\begin{aligned} (s^2+2s-3)F &= \frac{1-e^{-cs}}{s} \\ F &= \frac{1-e^{-cs}}{s(s^2+2s-3)} \\ &= (1-e^{-cs})\frac{1}{s(s-1)(s+3)} \\ &= (1-e^{-cs})\left(\frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3}\right) \end{aligned} \tag{12}$$
Concentrating on the partial fractions part, we have
$$\begin{aligned} \frac{1}{s(s-1)(s+3)} &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+3} \\ 1 &= A(s-1)(s+3) + Bs(s+3) + Cs(s-1) \\ \underline{s=0:} & \\ 1 &= -3A \\ A &= -\frac{1}{3} \\ \underline{s=1:} & \\ 1 &= 0 + 4B + 0 \\ B &= \frac{1}{4} \\ \underline{s=-3:} & \\ 1 &= 0 + 012C \\ C &= \frac{1}{12} \end{aligned}$$
Hence we have
$$F = (1-e^{-cs})\left(-\frac{1}{3}\frac{1}{s} + \frac{1}{4}\frac{1}{s-1} + \frac{1}{12}\frac{1}{s+3}\right) \tag{13}$$

4. (2) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \tag{14}$$

subject to the initial conditions $f(0) = f'(0) = 0$. You should begin by rewriting the right-hand side in terms of the Heaviside function:

$$H_1(t) - H_2(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \tag{15}$$

*Solution:*So the thing here is to rewrite the right hand side of the equations in terms of Heaviside functions. Remember the definition of the Heaviside function:

$$H_a(t) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases} \tag{16}$$

so the Heaviside function is zero until a and then it is one. The right hand side is zero until $t = 1$ and then it is one until $t = 2$ and then it is zero again. Consider $H_1(t) - H_2(t)$, this is zero until you reach $t = 1$, then the first Heaviside function switches on, the other one remains zero. Things stay like this until you reach $t = 2$, then the second Heaviside function switches on aswell and you get $1 - 1 = 0$. Thus

$$H_1(t) - H_2(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \tag{17}$$

Now, using

$$\mathcal{L}(H_a(t)) = \frac{e^{-as}}{s} \tag{18}$$

we take the Laplace transform of the differential equation:

$$s^2F + 2sF - 3F = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} \tag{19}$$

This gives

$$\begin{aligned} (s^2 + 2s - 3)F &= \frac{1}{s}\left(e^{-s} - e^{-2s}\right) \\ F = \frac{1}{s(s-1)(s+3)}\left(e^{-s} - e^{-2s}\right) \end{aligned} \tag{20}$$

Now,

$$\frac{1}{s(s-1)(s+3)} = -\frac{1}{3s} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)} \tag{21}$$

and we know that

$$\mathcal{L}\left(-\frac{1}{3} + \frac{1}{4}e^t + \frac{1}{12}e^{-3t}\right) = -\frac{1}{3} + \frac{1}{4(s-1)} + \frac{1}{12(s+3)} \tag{22}$$

In other word, if it wasn't for the expontentials we'd know the little f. However, we know from the third shift thereom that the affect of the exponential e^{-as} is to change t to $t - a$ and to introduce an overall factor of $H_a(t)$. Thus

$$f = H_1(t)\left(-\frac{1}{3} + \frac{1}{4}e^{t-1} + \frac{1}{12}e^{-3t+3}\right) - H_2(t)\left(-\frac{1}{3} + \frac{1}{4}e^{t-2} + \frac{1}{12}e^{-3t+6}\right) \tag{23}$$