2E2 Tutorial Sheet 4 Solutions[1]

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Questions

1. (4) Use Laplace transform methods to solve the differential equation

$$f'' + 2f' - 3f = \delta(t-1) \tag{1}$$

subject to the initial conditions $f(0) = 0$, $f'(0) = 1$.

*Solution:*soln The only thing that is unusual is that there is a delta function. We take the Laplace transform using

$$\mathcal{L}(\delta(t-a)) = e^{-as} \tag{2}$$

hence

$$(s^2 + 2s - 3)F - 1 = e^{-s} \tag{3}$$

Now, if we do partial fractions on $1/(s^2 + 2s - 3)$ we get

$$\frac{1}{s^2 + 2s - 3} = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \tag{4}$$

Hence

$$F = \left(-\frac{1}{4(s+3)} + \frac{1}{4(s-1)}\right)\left(1 + e^{-s}\right) \tag{5}$$

Since

$$\mathcal{L}\left(-\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}\right) = -\frac{1}{4(s+3)} + \frac{1}{4(s-1)} \tag{6}$$

then, by the third shift theorem we have

$$f = \left(-\frac{1}{4}e^{-3t} + \frac{1}{4}e^{t}\right) + H_1(t)\left(-\frac{1}{4}e^{-3t+3} + \frac{1}{4}e^{t-1}\right) \tag{7}$$

2. (4) Using the Laplace transform solve the differential equation

$$f'' + 6f' + 13f = 0 \tag{8}$$

with boundary conditions $f(0) = 0$ and $f'(0) = 1$ and get your answer into a real form.

*Solution:*So, taking the Laplace transform of the equation we get,

$$s^2F - 1 + 6sF + 13F = 0 \tag{9}$$

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and, hence,

$$F = \frac{1}{s^2 + 6s + 13}. \tag{10}$$

Now, using minus b plus or minus the square root of b squared minus four a c all over two a, we get

$$s^2 + 6s + 13 = 0 \tag{11}$$

if

$$s = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm 2i \tag{12}$$

which means

$$s^2 + 6s + 13 = (s + 3 - 2i)(s + 3 + 2i) \tag{13}$$

Next, we do the partial fraction expansion,

$$\frac{1}{s^2 + 6s + 13} = \frac{A}{s + 3 - 2i} + \frac{B}{s + 3 + 2i} \tag{14}$$

and multiplying across we get

$$1 = A(s + 3 + 2i) + B(s + 3 - 2i) \tag{15}$$

therefore we choose $s = -3 + 2i$ to get

$$A = \frac{1}{4i} = -\frac{i}{4} \tag{16}$$

and $s = -3 - 2i$ to get

$$B = -\frac{1}{4i} = \frac{i}{4} \tag{17}$$

and so

$$F = -\frac{i}{4}\frac{1}{s + 3 - 2i} + \frac{i}{4}\frac{1}{s + 3 + 2i}. \tag{18}$$

If we take the inverse transform

$$\begin{aligned} f &= -\frac{i}{4}e^{-(3-2i)t} + \frac{i}{4}e^{-(3+2i)t} \\ &= \frac{i}{4}e^{-3t}(e^{-2it} - e^{2it}) \\ &= \frac{1}{2}e^{-3t}\sin 2t \end{aligned} \tag{19}$$