2E2 Tutorial Sheet 13 Second Term[1]

5 Febuary 2006

1. (2) An equation system has solution
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it} \tag{1}$$
Sketch the phase diagram.

Solution: So our statedgy with complex solutions is to examine trajectories starting at $y_1(0) = r$ and $y_2(0) = 0$, since the complex solutions always give ellipses, circles or spirals, looking at these trajectories is enough to sketch the whole phase diagram. In this case we expect a circle node with elliptical trajectories. Putting in the initial conditions we get
$$\begin{pmatrix} r \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} \tag{2}$$
so
$$\begin{aligned} c_1 + c_2 &= r \\ 2ic_1 - 2ic_2 &= 0 \end{aligned} \tag{3}$$
giving $c_1 = c_2 = r/2$. Now, substitute this back in
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{r}{2} \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + \frac{r}{2} \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it} \tag{4}$$
and using
$$\begin{aligned} e^{2it} &= \cos 2t + i \sin 2t \\ e^{-2it} &= \cos 2t - i \sin 2t \end{aligned} \tag{5}$$
this gives the real expression
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} r \cos 2t \\ -2r \sin 2t \end{pmatrix} \tag{6}$$

This is an ellipse, it is easy to see that it satisfies the ellipse equation
$$y_1^2 + \left(\frac{y_2}{2}\right)^2 = r^2 \tag{7}$$
The factor of two on the y_2 means it goes twice as far in this direction. For very small t y_2 is negative so the trajectory starts off downwards. Hence we have a circle node with elliptical clockwise trajectories.

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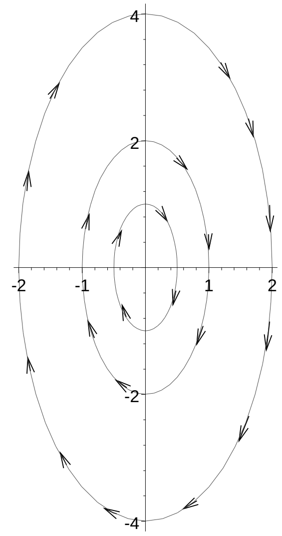

2. (3) Find the general solution for the system
$$\begin{aligned} \frac{dy_1}{dt} &= 3y_1 + y_2 \qquad (8) \\ \frac{dy_2}{dt} &= -y_1 + y_2 \qquad (9) \end{aligned}$$

Solution: This is one of those systems where there is only one eigenvalue and only one eigenvector, $\lambda = 2$ with
$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{10}$$
so the solution is of the form
$$\mathbf{y} = c_1 \mathbf{x} e^{2t} + c_2 (t\mathbf{x} + \mathbf{u}) e^{2t} \tag{11}$$
where you need to find $\mathbf{u}$ by substituting
$$\mathbf{y} = (t\mathbf{x} + \mathbf{u}) e^{2t} \tag{12}$$
back into the equation. This means the $\mathbf{u}$ vector in the extra solution is the solution to
$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Writing

$$\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$$

gives equations

$$\begin{aligned} a+b &= 1 \\ -a-b &= -1 \end{aligned}$$

These two equations are the same, as you expect, and if $b=0$ then $a=1$. Thus,

the general solution is

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{2t}$$

or

$$\mathbf{y} = \left[(c_1 + c_2 t) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] e^{2t}.$$

3. (3) Find the solution for the system

$$\begin{aligned} y_1' &= 4y_1 + y_2 \\ y_2' &= -y_1 + 2y_2. \end{aligned}$$

with initial conditions $y_1(0) = 3$ and $y_2(0) = 2$.

Solution:

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \tag{13}$$

and there is only one eigenvector,

$$\mathbf{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{14}$$

with eigenvalue $\lambda = 3$. The solution is

$$\mathbf{y} = c_1 \mathbf{x} e^{\lambda t} + c_2 (t\mathbf{x} + \mathbf{u}) e^{\lambda t} \tag{15}$$

where $\mathbf{u}$ satisfies

$$(A - \lambda \mathbf{1})\, \mathbf{u} = \mathbf{x} \tag{16}$$

and so, in this case,

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{17}$$

and a solution to this is

$$\mathbf{u} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{18}$$

and so the solution is

$$\mathbf{y} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + c_2 \left[t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t} \tag{19}$$

Now, putting $t = 0$ we get

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{20}$$

and, hence,

$$\begin{aligned} 3 &= -c_1 - c_2 \\ 2 &= c_1 \end{aligned} \tag{21}$$

aso $c_2 = 1$ and $c_2 = -5$ giving

$$\mathbf{y} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} - 5 \left[t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right] e^{3t} \tag{22}$$

or

$$\begin{aligned} y_1 &= (3+5t)e^{3t} \\ y_2 &= (2-5t)e^{3t} \end{aligned} \tag{23}$$