2E2 Tutorial Sheet 6 Solutions[1]

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Questions:

1. (2) Verify the formula $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$ in the case where $f = \exp(2t)$ and $g = \exp(2t)$.

   Solution: So,

$$\begin{aligned} e^{2t} * e^{-2t} &= \int_0^t e^{2\tau} * e^{2(t-\tau)} d\tau = \int_0^t e^{2t} d\tau \\ &= e^{2t} \int_0^t d\tau = te^{2t} \end{aligned} \tag{1}$$

   Now, this means

$$\mathcal{L}\left(e^{2t} * e^{2t}\right) = \mathcal{L}\left(te^{2t}\right) = \frac{1}{(s-2)^2} \tag{2}$$

   by the shift theorem. Doing it from the formula for the Laplace transform gives

$$\mathcal{L}\left(e^{2t} * e^{2t}\right) = \left[\mathcal{L}\left(e^{2t}\right)\right]^2 = \frac{1}{(s-2)^2} \tag{3}$$

2. (3) Find the convolution $(f * g)(t)$ when $f(t) = t$, $g(t) = e^{2t}$ $(t \geq 0)$.

   Solution: From the definition of convolutions

$$\begin{aligned} (f * g)(t) &= \int_0^t f(\tau) g(t-\tau)\, d\tau = \int_0^t \tau e^{2(t-\tau)}\, d\tau \\ &= \int_0^t \tau e^{2t} e^{-2\tau}\, d\tau = e^{2t} \int_0^t \tau e^{-2\tau}\, d\tau \\ \text{Use integration by parts with} \quad & u = \tau, \quad dv = e^{-2\tau}\, d\tau \\ & du = d\tau, \quad v = -\frac{1}{2} e^{-2\tau} \\ &= e^{2t} \int_0^t u\, dv = e^{2t} \left([uv]_0^t - \int_0^t v\, du \right) \\ &= e^{2t} \left(\left[-\frac{\tau}{2} e^{-2\tau} \right]_0^t - \int_0^t -\frac{1}{2} e^{-2\tau}\, d\tau \right) \\ &= e^{2t} \left(-\frac{t}{2} e^{-2t} + 0 + \frac{1}{2} \int_0^t e^{-2\tau}\, d\tau \right) \end{aligned}$$

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$$
\begin{aligned}
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left[-\frac{1}{2}e^{-2\tau}\right]_0^t \\
&= -\frac{t}{2} + \frac{e^{2t}}{2}\left(-\frac{1}{2}e^{-2t} + \frac{1}{2}\right) \\
&= -\frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{2t}
\end{aligned}
$$

3. (3) Use the convolution theorem to find the function $f(t)$ with

$$\mathcal{L}(f) = \frac{1}{s^2(s-4)}. \tag{4}$$

Solution: We know $\mathcal{L}(t)) = \frac{1}{s^2}$ and $\mathcal{L}(e^{4t}) = \frac{1}{s-4}$. From the convolution theorem, we see

$$\mathcal{L}(f) = \frac{1}{s^2(s-4)} = \mathcal{L}(t)\mathcal{L}(e^{4t}) = \mathcal{L}(t * e^{4t})$$

so that $f(t)$ is the convolution $t * e^{4t}$.

$$
\begin{aligned}
f(t) &= \int_0^t \tau e^{4(t-\tau)}\,d\tau \\
&= \int_0^t \tau e^{4t}e^{-4\tau}\,d\tau = e^{4t}\int_0^t \tau e^{-4\tau}\,d\tau \\
\text{Use integration by parts with}\quad & U = \tau, \quad dV = e^{-4\tau}\,d\tau \\
& dU = d\tau, \quad V = -\frac{1}{4}e^{-4\tau} \\
&= e^{4t}\int_0^t U\,dV = e^{4t}\left([UV]_0^t - \int_0^t V\,dU\right) \\
&= e^{4t}\left(\left[-\frac{\tau}{4}e^{-4\tau}\right]_0^t - \int_0^t -\frac{1}{4}e^{-4\tau}\,d\tau\right) \\
&= e^{4t}\left(-\frac{t}{4}e^{-4t} + 0 + \frac{1}{4}\int_0^t e^{-4\tau}\,d\tau\right) \\
&= -\frac{t}{4} + \frac{e^{4t}}{4}\left[-\frac{1}{2}e^{-4\tau}\right]_0^t \\
&= -\frac{t}{4} + \frac{e^{4t}}{4}\left(-\frac{1}{4}e^{-4t} + \frac{1}{4}\right) \\
&= -\frac{t}{4} - \frac{1}{16} + \frac{1}{16}e^{4t}
\end{aligned}
$$