2E2 Tutorial Sheet 9 Solutions[1]

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Questions

1. (3) Find the eigenvectors and eigenvalues of the following matrices

$$(i)\quad \begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix} \qquad (ii)\quad \begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix} \qquad (iii)\quad \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \tag{1}$$

Solution: In (i) the characteristic equation is

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & -6-\lambda \end{vmatrix} = 0 \tag{2}$$

so

$$(4-\lambda)(-6-\lambda) = 0 \tag{3}$$

So $\lambda = 4$ or $\lambda = -6$. Taking the $\lambda = 4$ first

$$\begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 4\begin{pmatrix} a \\ b \end{pmatrix} \tag{4}$$

so $4a = 4a$ and $-6b = 4b$, hence $b = 0$ and a is arbitrary, taking $a = 1$ an eigenvalue 4 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5}$$

Taking the $\lambda = -6$

$$\begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -6\begin{pmatrix} a \\ b \end{pmatrix} \tag{6}$$

so $4a = -6a$ and $-6b = -6b$, hence $a = 0$ and b is arbitrary, taking $b = 1$ a eigenvalue -6 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

In (ii) the characteristic equation is

$$\begin{vmatrix} 10-\lambda & -4 \\ 18 & -12-\lambda \end{vmatrix} = 0 \tag{8}$$

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so

$$(10-\lambda)(-12-\lambda)-(-4)18=0 \tag{9}$$

or

$$\lambda^2+2\lambda-48=0 \tag{10}$$

Solve this gives us $\lambda=6$ or $\lambda=-8$. Taking the $\lambda=6$ first

$$\begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=6\begin{pmatrix} a \\ b \end{pmatrix} \tag{11}$$

so the first equation is $10a-4b=6a$ or $a=b$, the other equation is $18a-12b=6b$ which is also $a=b$. Taking $a=1$ an eigenvalue 6 eigenvector is,

$$\mathbf{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{12}$$

Taking the $\lambda=-8$ next

$$\begin{pmatrix} 10 & -4 \\ 18 & -12 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=-8\begin{pmatrix} a \\ b \end{pmatrix} \tag{13}$$

so the first equation is $10a-4b=-8a$ or $9a=2b$, the other equation is $18a-12b=-8b$ which is also $9a=2b$. Taking $a=2$, $b=9$ an eigenvalue -8 eigenvector is,

$$\mathbf{x}=\begin{pmatrix} 2 \\ 9 \end{pmatrix} \tag{14}$$

In (iii) the characteristic equation is

$$\begin{vmatrix} -\lambda & r \\ r & -\lambda \end{vmatrix}=0 \tag{15}$$

so

$$\lambda^2-r^2=0 \tag{16}$$

or

$$\lambda=\pm r \tag{17}$$

Taking the $\lambda=r$ first

$$\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=r\begin{pmatrix} a \\ b \end{pmatrix} \tag{18}$$

so the equation is $rb=ra$ or $a=b$. Taking $a=1$ an eigenvalue r eigenvector is,

$$\mathbf{x}=\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{19}$$

Taking the $\lambda = -r$

$$\begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -r \begin{pmatrix} a \\ b \end{pmatrix} \tag{20}$$

so the equation is $rb = -ra$ or $a = -b$. Taking $a = 1$ an eigenvalue $-r$ eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{21}$$

2. (2) Find the eigenvectors and eigenvalues of the following matrices

$$(i) \quad \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \qquad (ii) \quad \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \tag{22}$$

Solution: In (i) the characteristic equation is

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \tag{23}$$

so

$$(3-\lambda)(-3-\lambda) - 16 = 0 \tag{24}$$

or

$$\lambda^2 + 2\lambda - 48 - 25 = 0 \tag{25}$$

Solve this gives us $\lambda = \pm 5$. Taking the $\lambda = 5$ first

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 5 \begin{pmatrix} a \\ b \end{pmatrix} \tag{26}$$

so the first equation is $3a + 4b = 5a$ or $a = 2b$, the other equation is $4a - 3b = 5b$ which is also $a = 2b$. Taking $a = 2$ an eigenvalue 5 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{27}$$

Taking $\lambda = -5$ next

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -5 \begin{pmatrix} a \\ b \end{pmatrix} \tag{28}$$

so the first equation is $3a+4b = -5a$ or $2a = -b$, the other equation is $4a-3b = -5b$ which is also $2a = -b$. Taking $a = 1$ an eigenvalue -5 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{29}$$

In (ii) the characteristic equation is

$$\begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = 0 \tag{30}$$

so

$$\lambda^2 + 9 = 0 \tag{31}$$

or

$$\lambda = \pm 3i \tag{32}$$

Taking the $\lambda = 3i$ first

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3i \begin{pmatrix} a \\ b \end{pmatrix} \tag{33}$$

so the equation is $3b = 3ia$ or $a = -ib$. Taking $b = 1$ an eigenvalue $3i$ eigenvector is,

$$\mathbf{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \tag{34}$$

Taking the $\lambda = -3i$

$$\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -3i \begin{pmatrix} a \\ b \end{pmatrix} \tag{35}$$

so the equation is $3b = -3ia$ or $a = ib$. Taking $a = 1$ an eigenvalue $-3i$ eigenvector is,

$$\mathbf{x} = \begin{pmatrix} i \\ 1 \end{pmatrix} \tag{36}$$

3. (3) Find the solution for the system

$$\frac{dy_1}{dt} = -3y_1 + 2y_2 \tag{37}$$

$$\frac{dy_2}{dt} = -2y_1 + 2y_2 \tag{38}$$

Solution: This equation is $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix}$$

We can find the eigenvalues, the characteristic equation is

$$\begin{vmatrix} -3-\lambda & 2 \\ -2 & 2-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2) + 4 = \lambda^2 + \lambda - 2 = 0$$

so that $\lambda_1 = 1$ and $\lambda_2 = -2$.

Next, we need the eigenvectors. First, λ_1:

$$\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{39}$$

so $-3a + 2b = a$ or $b = 2a$, hence, choosing $a = 1$ we get

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{40}$$

For λ_2:

$$\begin{pmatrix} -3 & 2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -2 \begin{pmatrix} a \\ b \end{pmatrix} \tag{41}$$

so $-3a + 2b = -2a$ giving $a = 2b$, choosing $b = 1$ gives

$$\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{42}$$

Now, in general the solution is

$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t} \tag{43}$$

so, here,

$$\mathbf{y} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} \tag{44}$$