Sample paper 2008: two hour exam, do three questions.

1. (a) (9 marks) For discrete random variables X and Y define
   - The entropy $H(X)$.
   - The mutual information $I(X;Y)$.
   - The conditional entropy $H(X|Y)$.

   (b) (11 marks) Given the conditional distribution

	a	b	c
1	1/3	1/12	1/12
2	1/12	0	1/24
3	1/24	1/3	0

   for $X \in \mathcal{X} = \{1,2,3\}$ and $Y \in \mathcal{Y} = \{a,b,c\}$, find $H(X)$, $H(Y)$, $H(X|Y)$, $H(Y|X)$ and $I(X;Y)$.

2. For discrete random variables X, Y and Z

   (a) (5 marks) prove

$$H(X,Y) = H(X) + H(Y|X)$$

   (b) (5 marks) prove

$$I(X;Y) = H(Y) - H(Y|X)$$

   (c) (5 marks) prove

$$H(X,Y|Z) \geq H(X|Z)$$

   (d) (5 marks) prove

$$I(X;Z|Y) = I(Z;Y|X) - I(Z;Y) + I(X;Z)$$

3. (a) (4 marks) What is meant by a Markov chain $X \rightarrow Y \rightarrow Z$ and

   (b) (3 marks) Show that $X \rightarrow Y \rightarrow Z$ implies $Z \rightarrow Y \rightarrow X$.

   (c) (8 marks) State and prove the data processing inequality.

(d) (5 marks) Suppose that a Markov chain starts in one of n states, necks down to $k < n$ states and then fans back out to $m > k$ states. Show that the dependence of the first and last variables, X and Z is limited by the bottleneck by showing $I(X, Z) \leq \log k$.

4. (a) (4 marks) Define a source code and an instantaneous code.

   (b) (5 marks) State the Kraft inequality.

   (c) (4 marks) Define the expected length $L(C)$ of a source code $C(x)$.

   (d) (8 marks) Prove that the expected length L of any instantaneous D-ary code for a random variable X is greater than or equal to the entropy $H_D(X)$.