The Markov Property[1]

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The random variables X, Y and Z are said to *form a Markov chain in that order*

$$X \to Y \to Z \tag{1}$$

if and only if $p(x|y,z) = p(x|y)$ for all x, y and z. This is equivalent to requiring X, Z are conditional independent on Y:

$$p(x,z|y) = p(x|y)p(z|y) \tag{2}$$

As an extremely simple example consider snakes and ladders: imagine a game of snakes and ladders with no snakes and no ladders, for each turn you flip a coin and move one or two squares depending on whether you get a head or a tail. This is a Markov chain: the probability distributions of positions at the nth throw depends only on your current position and not on how you got their. To make this more definite let X_1 be your position after one throw, X_2 after two and X_3 after three. Thus

$$p_{X_1}(1) = p_{X_1}(2) = 1/2 \tag{3}$$

Now X_3 is not independent of X_1. If $X_1 = 1$ then two heads gives $X_3 = 3$, a head and a tails or visa versa, gives $X_3 = 4$ and two tails puts $X_3 = 5$.

$$\begin{aligned} p_{X_3|X_1}(3|1) = p_{X_3|X_1}(5|1) &= 1/4 \\ p_{X_3|X_1}(4|1) &= 1/2 \end{aligned} \tag{4}$$

whereas, if $X_1 = 2$ you are starting one further along and

$$\begin{aligned} p_{X_3|X_1}(4|2) = p_{X_3|X_1}(6|2) &= 1/4 \\ p_{X_3|X_1}(5|2) &= 1/2 \end{aligned} \tag{5}$$

we can add to get

$$\begin{aligned} p_{X_3}(3) &= 1/8 \\ p_{X_3}(4) &= 3/8 \\ p_{X_3}(5) &= 3/8 \\ p_{X_3}(6) &= 1/8 \end{aligned} \tag{6}$$

but, more importantly, clearly, as stated above and as we would guess, the conditional distributions are different for different values of X_1.

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Now, X_2 and X_3 are also dependent, however, if you know X_2 knowing X_1 will not tell you any more about X_3, all the dependence of X_3 on X_1 comes through X_2 and $X_1 \rightarrow X_2 \rightarrow X_3$. For completeness, here are the conditional probabilities

$$\begin{aligned}
p_{X_3|X_2}(3|2) = p_{X_3|X_2}(4|2) &= 1/2 \\
p_{X_3|X_2}(4|3) = p_{X_3|X_2}(5|3) &= 1/2 \\
p_{X_3|X_2}(5|4) = p_{X_3|X_2}(6|4) &= 1/2
\end{aligned} \tag{7}$$

The important point is, that if, say $X_2 = 3$ we know X_3 is equally likely to be four or five. There are also two possible values of X_1, for X_2 to be three, we could have had $X_1 = 1$ or $X_1 = 2$, but knowing which it was does not affect the distribution for X_3. In fact $p(x_1, x_3|X_2 = 3) = 1/4$ for each of the possible values of x_1 and x_3, just as $p(x_1|X_2 = 3) = 1/2$ and $p(x_3|X_2 = 3) = 1/2$ for each possible value of x_1 and x_3.