2E2 Tutorial Sheet 24, Solutions[1]

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This problem sheet relates to solving the Dirichlet problem for the heat equation.. Dirichlet boundary conditions fix the values at the end. Consider an iron bar on which heat obeys the heat equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

where $u(x,t)$ and the boundary conditions are

$$u(0,t) = u(\pi,t) = 0 \tag{2}$$

1. (2) Writing $u(x,t) = T(t)X(x)$ show this equation is equivalent to the equations

$$\begin{aligned} \frac{d^2X}{dx^2} &= EX \\ \frac{dT}{dt} &= ET \end{aligned} \tag{3}$$

   where E is some constant.

   Solution: The main trick for multi-variable problems like this is to try and seperate the equation for u into a function of x and t into two equations, one for a function of x and the other for a function of t. This way, we are able to use the ordinary differential equation methods we have already learned to solve the more complicated example where there is more than one variables. There are a number of trick for doing this depending on the equation, but the one that works for the heat equation, along with a number of other important examples, is the seperation of variables method. According to this method, you assume the function u can be written as the multiple of two function, one, which we will call T is just a function of t and the other, which we call X is just a function of x. Obviously, not every function of x and t can be split in this way, we are just imagining that u can be. Although we won't make much of it here, in fact most functions can be written as the sum of functions that are of this form and, for a linear equation, that's good enough.

   So, we make the assumption and substitute $u = TX$ into the equation. The x differenciation only acts on the X and, in turn, the X only depends on x. In the same way, the t differentiation only acts on T and T in turn only depends on t. Thus, the equation becomes

$$T\frac{d^2X}{dx^2} = \frac{dT}{dt}X \tag{4}$$

[1]Conor Houghton, `houghton@maths.tcd.ie` and `http://www.maths.tcd.ie/~houghton/2E2.html`

Now divide across by TX

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{T}\frac{dT}{dt} \tag{5}$$

We still have only one equation, but it has a very suprising form, all the x stuff is on one side and all the t stuff is on the other. x and t are supposed to be independent variables: x doesn't depend on t and t doesn't depend on x. Hence, we should be able to hold x fixed and change t and still have the equation stay true. But holding x fixed and changing t can only change the right hand side of the equation, not the left, and that would be a contradition, the equation wouldn't hold, unless it was a fact that the right hand side didn't change when t was changed, in other words, unless the right hand side of the equation is a constant. Basically, by the same arguement the left hand side must be a constant too, so the heat equation, using seperation of variables comes down to an equation which only makes sense if

$$\frac{1}{T}\frac{dT}{dt} \tag{6}$$

is a constant

$$\frac{1}{T}\frac{dT}{dt} \tag{7}$$

is a constant. Usually, this constant is given the name E and hence, by this suprising trick, the seperation of variables assumption $u = X(x)T(t)$ has turned one equation with two variables into two equations with one variable:

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= E \\ \frac{1}{T}\frac{dT}{dt} &= E \end{aligned} \tag{8}$$

The only thing is we don't know the value of this new constant E, in fact, if you solve these two equations to get X and T and multiply them together to get u, you have a solution to the heat equation, no matter what value E has, but we will see that only special values of E give solutions that satisfy the boundary conditions.

Multiply the first equation by X and the second by T to get the answer.

$$\begin{aligned} \frac{d^2X}{dx^2} &= EX \\ \frac{dT}{dt} &= ET \end{aligned} \tag{9}$$

2. (2) Solve these equations and argue from the boundary conditions that E must be negative. Writing $E = -k^2$ calculate what values of k satisfy the boundary conditions.

Solution: Now the T equation is easy to solve, either by guessing the answer and substituting it in, or by integration or indeed by using the Laplace transform, we get

$$u = Ae^{Et} \tag{10}$$

For the second X there are two possibile types of solution depending on the sign of E, basically, we will see, if E is positive, then the X equation is solved by real exponentials, but if it is negative, it is solved by trignometric functions. What we find is that the corresponding u can only solve the boundary condition if we have the trignometric solution, if E is negative.

So, to go through that, if E is positive, we make that clear by writing it as the square of something, the square of a real number is positive, so we write $E = k^2$ where k denotes a real number, the square root of E. Now, it is easy to check, for example, by substituting back in, or solving by converting to first order and using matrices or even by using Laplace that the solution is

$$X = C_1 e^{kx} + C_2 e^{-kx} \tag{11}$$

In other words, there are lots of ways to solve the equation, the simplest is probably to kind of remember the solution and check you guess by substitution:

$$\begin{array}{rcl} X' &=& C_1 k e^{kx} - kC_2 e^{-kx} \\ X'' &=& C_1 k^2 e^{kx} + k^2 C_2 e^{-kx} = EX \\ & (12) & \end{array}$$

On the other hand, if E is negative, we can make it clear by writting it is minus the square of something, so we write $E = -k^2$, in which case the solution is

$$X = C_1 \cos kx + C_2 \sin kx \tag{13}$$

Again, there are a number of ways of seeing these are the solutions, the first is to know they are the solutions and quickly check that differenciating X twive gives the right thing, another is to split the equation into two first order equations and the write in matrix form, as we did after Christmas and the third is to use the Laplace transform.

Now, the u that comes from the positive case can never satisfy the boundary conditions,

$$u(0, t) = u(\pi, t) = 0 \tag{14}$$

because we have

$$u = TX = \left(C_1 e^{kx} + C_2 e^{-kx}\right) e^{k^2 t} \tag{15}$$

where the A has been absorbed into the other two arbitrary constants. and so, if

$$u(0, t) = 0 \tag{16}$$

then

$$(C_1 + C_2)\, e^{k^2 t} \tag{17}$$

so that $C_1 = -C_2$. Now

$$u(\pi, t) = C_1 \left(e^{k\pi} - e^{-k\pi}\right) e^{k^2 t} \tag{18}$$

which is only zero if

$$e^{k\pi} - e^{-k\pi} = 0 \tag{19}$$

which never happens for non-zero π and k.

Choosing the negative solutions, this gives

$$u = TX = [C_1 \cos(kx) + C_2 \sin(kx)]\, e^{-k^2 t} \tag{20}$$

so

$$u(t, 0) = C_1 e^{-k^2 t} \tag{21}$$

and only gives zero when $C_1 = 0$, hence

$$u(\pi, t) = C_2 \sin(k\pi) e^{-k^2 t} \tag{22}$$

and this is zero if $k = n$ where n is any integer. Hence, if $C_1 = 0$ and E is fixed to be

$$E = -n^2 \tag{23}$$

where n is an integer we get a u that solves the heat equation and satisfies the boundary conditions. Hence, solutions are of the form

$$u = A_n \sin nx e^{-n^2 t} \tag{24}$$

where, for convenience, we are using the name A_n for the arbitrary constant when $E = -n^2$.

In fact, this is a linear equation so the sum of solutions is a solution and so

$$u = \sum_{n=0}^{\infty} A_n \sin nx e^{-n^2 t} \tag{25}$$

is a solution. This is the clever thing, we start off looking for one solution, but end up with a whole family of solutions, one for each n and, since the equation is linear we can just add them all together. There is still the initial condition to deal with, in general, matching the initial condition comes down to Fourier series, but we have an easier example in the next question.

3. (2) Say the initial condition is

$$u(x,0) = f(x) = \sin x - \sin 4x \tag{26}$$

what is $u(x,t)$.

Solution: So the general solution is

$$u = \sum_{n=0}^{\infty} A_n \sin nx e^{-n^2 t} \tag{27}$$

and at $t = 0$ this gives

$$u(x,0) = \sum_{n=0}^{\infty} A_n \sin nx \tag{28}$$

which matches up with the initial condition if $A_1 = 1$, $A_4 = -1$ and all the others are zero. Hence putting these As back in

$$u(x,t) = \sin xe^{-t} - \sin 4xe^{-16t} \tag{29}$$