2E2 Tutorial Sheet 10 Solutions[1]

15 January 2006

Questions

1. (2) Find the general solution for the system

$$\frac{dy_1}{dt} = 3y_1 + y_2 \tag{1}$$
$$\frac{dy_2}{dt} = y_1 + 3y_2 \tag{2}$$

Solution: The eigenvectors and eigenvalues of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \tag{3}$$

are $\lambda_1 = 4$ with

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4}$$

and $\lambda_2 = 2$ with

$$\mathbf{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{5}$$

so the general soln is

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{2t}. \tag{6}$$

2. (3) Find the solution of the system

$$\frac{dy_1}{dt} = 3y_1 + 4y_2 \tag{7}$$
$$\frac{dy_2}{dt} = 4y_1 - 3y_2 \tag{8}$$

with $y_1(0) = 2$ and $y_2(0) = -1$.

he characteristic equation is

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \tag{9}$$

[1] Conor Houghton, houghton@maths.tcd.ie, see also http://www.maths.tcd.ie/~houghton/2E2.html

so

$$(3-\lambda)(-3-\lambda)-16=0 \tag{10}$$

or

$$\lambda^2+2\lambda-48-25=0 \tag{11}$$

Solve this gives us $\lambda = \pm 5$. Taking the $\lambda = 5$ first

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 5\begin{pmatrix} a \\ b \end{pmatrix} \tag{12}$$

so the first equation is $3a+4b=5a$ or $a=2b$, the other equation is $4a-3b=5b$ which is also $a=2b$. Taking $a=2$ an eigenvalue 5 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{13}$$

Taking $\lambda = -5$ next

$$\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = -5\begin{pmatrix} a \\ b \end{pmatrix} \tag{14}$$

so the first equation is $3a+4b=-5a$ or $2a=-b$, the other equation is $4a-3b=-5b$ which is also $2a=-b$. Taking $a=1$ an eigenvalue -5 eigenvector is,

$$\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{15}$$

Hence, the solution is

$$\mathbf{y} = C_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-5t} \tag{16}$$

or

$$\begin{aligned} y_1 &= 2C_1e^{5t}+C_2e^{-5t} \qquad (17) \\ y_2 &= C_1e^{5t}-2C_2e^{-5t} \qquad (18) \end{aligned}$$

So, for $t=0$ we have

$$\begin{aligned} y_1(0)=2 &= 2C_1+C_2 \qquad (19) \\ y_2(0)=-1 &= C_1-2C_2 \qquad (20) \end{aligned}$$

giving $C_1 = 3/5$ and $C_2 = 4/5$ so

$$\begin{aligned} y_1 &= \frac{6}{5}e^{5t}+\frac{4}{5}e^{-5t} \qquad (21) \\ y_2 &= \frac{3}{5}e^{5t}-\frac{8}{5}e^{-5t} \qquad (22) \end{aligned}$$

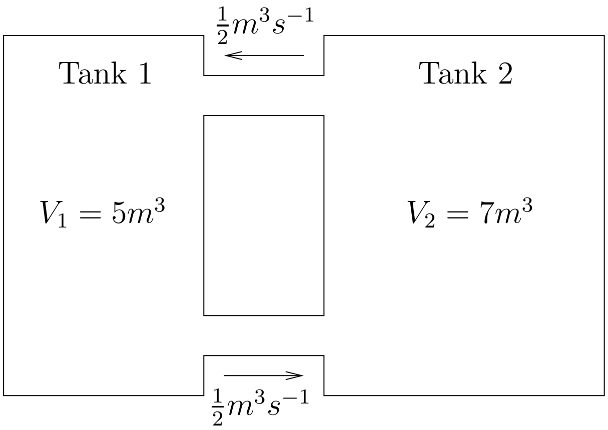

Figure 1: Two containers with flow between them.

3. (3) As illustrated in Fig. 1, two large containers are connected and American style sandwich spead is pumped between them at a rate of $1/2m^3s^{-1}$. One container has volume $5m^3$, the other $7m^3$. Both are full of spread. Initially the smaller container contains pure jam, the second container has $5m^3$ of jam and $2m^3$ of peanut butter. Assume perfect mixing and so on.
(i) Write down the differential equation for $y_1(t)$ and $y_2(t)$, the amount of peanut butter in the first and second container.
(ii) Solve it to find $y_1(t)$ and $y_2(t)$ explicitly.
(iii) Use the initial data to find the values of the constants in the solution.

Solution: Well if there is y_1 peanut butter in the small container then the concentration of the spead in the small container is $y_1/5$ and so $y_1/10$ is flowing out per second. In the same way $y_2/7$ is the concentration of peanut butter in the second tank and so $y_2/14$ per second is going from the large tank to the small one. This means the equations are

$$y_1' = -\frac{1}{10}y_1 + \frac{1}{14}y_2 \tag{23}$$
$$y_2' = \frac{1}{10}y_1 - \frac{1}{14}y_2 \tag{24}$$

This equation can be rewritten

$$\mathbf{y}' = \begin{pmatrix} -\frac{1}{10} & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} \end{pmatrix} \mathbf{y} \tag{25}$$

We work out the eigenvalues

$$\begin{vmatrix} -\frac{1}{10} - \lambda & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} - \lambda \end{vmatrix} = \left(\frac{1}{10} + \lambda\right)\left(\frac{1}{14} + \lambda\right) - \frac{1}{140} \tag{26}$$
$$= \lambda^2 + \frac{6}{35}\lambda = 0 \tag{27}$$

This means that there are two eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = -6/35$. The corresponding eigenvectors are given by

$$\begin{pmatrix} -\frac{1}{10} & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \tag{28}$$

which has solutions of the form $a = 10$ and $b = 14$ for λ_1 and

$$\begin{pmatrix} -\frac{1}{10} & \frac{1}{14} \\ \frac{1}{10} & -\frac{1}{14} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -\frac{6}{35} \begin{pmatrix} a \\ b \end{pmatrix} \tag{29}$$

for λ_2. This has solution $a = -1$ and $b = 1$. Thus, the general solution is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 \begin{pmatrix} 10 \\ 14 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{6}{35}t} \tag{30}$$

For part (iii), matching with $y_1(0) = 0$ and $y_2(0) = 2$, this gives $c_1 = 1/12$ and $c_2 = 5/6$ and hence

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 10 \\ 14 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-\frac{6}{35}t} \tag{31}$$

By the way, clearly the exponetially decaying part goes away with time so that

$$\lim_{t\to\infty} \mathbf{y} = \begin{pmatrix} \frac{5}{6} \\ \frac{7}{6} \end{pmatrix} \tag{32}$$