The Elusive Jupiter Trojan Horseshoes

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Submitted: 10th March 2021, Date of Experiment: October 2020 – February 2021

This paper presents an investigation into the Trojan orbits of the Sun-Jupiter system – specifically the dynamics of horseshoe-type asteroids which move between the system L4 and L5 points, and why today no such orbits exist. A restricted three body method with the Sun and Jupiter was used to simulate the dynamics of asteroids around the system's L4 and L5 points. This method was found to be effective for short-term simulations. A Monte Carlo simulation classed asteroid types (Horseshoe, Tadpole, unstable) based on initial conditions. Of 3300 asteroids, 16% entered tadpole orbits, and 14% entered horseshoe orbits. The distribution of these asteroids was described by zero-velocity contours, dependent on the selected boundary conditions. The narrowness of the regions in which Horseshoes were found was attributed to the Jupiter/Sun mass ratio exceeding the critical system mass ratio for stable Horseshoes - $\mu_{J/S} \approx 9.5 \cdot 10^{-4} < 8.3 \cdot 10^{-4} \approx \mu_c$. This suggests that no Horseshoes exist due to a lack of long-term stability. A $5 \cdot 10^4$yr simulation found that the most stable Horseshoes had initial conditions along $R = R_J$ with angles $\theta = -\pi/2$ and $\theta = 1.15$. We speculate that this point of stability in the proximity of the Tadpole zone could allow for the perturbation of a Tadpole into a Horseshoe through solar or gravitational effects.

I. Introduction

The Jupiter Trojans are two groups of asteroids which flank Jupiter's orbit of the Sun, at approximately 60° in front and behind. The clusters of asteroids orbit roughly at the same radius as Jupiter, and with the same period. These orbits are only possible at those positions relative to Jupiter, due to the stable Lagrange points they inhabit. This provides the asteroids with long-term stability and allows them to persist to this day. Currently, only asteroids which inhabit either one of the two flanking Lagrange points, are known to exist[1]. This paper aims to investigate another possible type of quasi-stable orbit around both of the Lagrange points these Trojans could enter, or which has possibly existed in the past – known as a Horseshoe orbit.

The restricted three body problem can be used to determine the evolution of a system of three bodies – two massive, and one of negligible mass. The system considered in this paper resembles that of the Sun and Jupiter, in circular orbits, in order to analyse the evolution of a third, massless, object representative of an asteroid.

Such a system results in a complex time and space varying gravitational potential from the superposition of the gravitational fields of the Sun and Jupiter as they orbit about a common barycentre – there is no full analytic solution to this problem, and thus numerical approximations must be used.

The following assumptions were applied to the system:

1. The Sun and Jupiter have constant fixed circular orbits with a fixed angular velocity – they orbit "on rails" and cannot be perturbed.
2. All initial positions and velocities are co-planar for both the massive objects, and the massless asteroids – the system can be represented as two-dimensional.

Jacobi's integral can be used in this system to determine the regions in the Sun-Jupiter system that an asteroid with given initial position and velocity parameters can move within. $\boldsymbol{v}$ is the asteroid velocity; $\boldsymbol{U}$ is the specific potential energy – including both the gravitational and centrifugal potentials – and $\boldsymbol{C}$ is a constant relating to the initial conditions of the asteroid.

$$\boldsymbol{v^2} = \boldsymbol{2U} - \boldsymbol{C} \ (1)$$

The asteroid must always satisfy $\boldsymbol{v^2} \geq \boldsymbol{0}$, which constrains it in regions of space, dependent on the initial conditions $\boldsymbol{C}$. This property can be used to construct equipotential zero-velocity curves which satisfy $\boldsymbol{U} = \boldsymbol{C/2}$, and constrain the particle to move where $\boldsymbol{U} \geq \boldsymbol{C/2}$.

Equilibrium points in the system exist where the net force (sum of gravitational and centrifugal forces) is zero in a rotating reference frame of angular velocity ω. In a non-rotating frame, this is equivalent to the gravitational force being equal to the centrifugal force required to rotate at angular velocity ω at a radius $\boldsymbol{r}$.

$$\boldsymbol{F} = -G\left(\frac{M_1 m}{{r_1}^2}\widehat{\boldsymbol{r_1}} + \frac{M_2 m}{{r_2}^2}\widehat{\boldsymbol{r_2}}\right) + m\boldsymbol{r}\omega^2 = 0 \ (2)$$

Points of zero net force in the rotating frame can be related to the potential via by taking a derivative.

$$\boldsymbol{F} = -m\boldsymbol{\nabla} U = 0 \ (3)$$

Therefore, these equilibrium points can be identified where the potential reaches a stationary point in both dimensions. From a potential contour plot, five of these points can be identified (for $M_1/M_2 < 0.0385$), known as Lagrange points. There are three unstable Lagrange points (L1, L2, and L3), which correlate to the three saddle stationary points along the Jupiter-Sun axis, and two stable Lagrange points (L4, and L5), which correlate to the two maxima 60° either side of Jupiter.

Consequently this means that an asteroid placed at any of these points with an initial velocity perpendicular to the radius drawn from the asteroid to the Sun-Jupiter barycentre with magnitude $v = \omega r = 2\pi r/P_J$ – where P_J is the orbital period of the system, will remain in that position relative to both Jupiter and the Sun forever. This results in the asteroid remaining stationary in the comoving frame, and the distance between the asteroid and both of the Sun and Jupiter remaining constant.

However, in reality, small perturbations will cause an asteroid to drift from the Lagrange point. For L1, L2, and

L3 – the saddle points – a small movement from equilibrium will cause the asteroid to be ejected from the orbit. For the maxima L4 and L5, however; a perturbed asteroid will be able to follow a closed equipotential line orbiting the Lagrange point and remain stable. This causes the asteroid to move in the comoving frame (drawing a tadpole shaped orbit, similar to the equipotential lines surround L4/L5) and for the asteroid-Jupiter/Sun distance to oscillate over time. These are known as "Tadpole" orbits. A further perturbation can push the asteroid out towards the closed equipotential contours encapsulating both L4 and L5, shaped like a horseshoe. These are known as "Horseshoe" orbits.
In the Sun-Jupiter system, all known Trojan asteroids have a variation of the Tadpole orbit – none are of the Horseshoe variety. This paper looks to investigate aspects of the Horseshoe orbit in the Sun-Jupiter system, including theoretical lifetime and regions where they could exist, to explore reasons for their absence and ways they could form in the future.

II. Method

The dynamics of a restricted 3-body system were simulated over a period of time, T, using N steps of interval a. A two body "on rails" circular orbit system, identical to the Sun-Jupiter system, was developed. Values for the masses, M_S and M_J, and the barycentre distances, R_S and R_J, were taken from literature, and subsequently the system period P and angular velocity ω were calculated using Kepler's Third Law. At a given time t_i, the Sun and Jupiter are at cartesian positions $(x_{S,i}, y_{S,i})$ and $(x_{J,i}, y_{J,i})$. The asteroid is at cartesian position $(x_{a,i}, y_{a,i})$ with velocity $(\dot{x}_{a,i}, \dot{y}_{a,i})$. To calculate velocity and position at $t_{i+1} = t_i + a$, the net gravitational force due to the Sun and Jupiter is calculated, and the RK4 method is used to accurately approximate the solution to the differential equations in 4 steps, in order to find $(x_{a,i+1}, y_{a,i+1})$ and $(\dot{x}_{a,i+1}, \dot{y}_{a,i+1})$ with greater accuracy. The positions of the Sun and Jupiter are then moved along their orbit – from polar positions $(R_J, \theta_{J,i})$ and $(R_S, \theta_{S,i})$ to $(R_J, \theta_{J,i+1})$ and $(R_S, \theta_{S,i+1})$ through an angle change of $\Delta\theta = \pm\omega a$. This process was then repeated over N iterations to obtain the final state of the system at time T, and a history of the positions each body had been at over the simulation period. From this list of positions, several key indicators of how the system evolved can be extracted. Firstly, the positions can be used to track the orbits and orbit paths in the "stationary" frame – i.e. the absolute values of object position relative to the system barycentre can be seen.Secondly, by calculating the aggregate of $\Delta\theta$ over time, and removing its value at each timestep, the "stationary" positions can be transformed to show the positions relative to the rotation of the Sun-Jupiter system. This "comoving frame" results in the positions of Jupiter, the Sun, and all Lagrange points to be stationary over time, and shows how the position of the asteroid evolves relative to the system.
Thirdly, by finding the difference between the positions of the asteroid and Jupiter at each time step, one can see how the asteroid's proximity to Jupiter evolves over time. This can be compared to the constant distance of L4 and L5 from Jupiter to see how the asteroid orbit compared to a stable Lagrange orbit over time.
Asteroids were injected into the system using an initial conditions generator. The number of asteroids to be injected, and the region of R and θ in which the asteroid should be placed were taken as arguments, and the asteroids were randomly placed within the given parameters. Velocities were based off the position of each asteroid, being selected to initially give the asteroid the angular velocity to match Jupiter at a given radius.

$$\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r} \quad (\mathbf{4})$$

Where the magnitude gives the asteroid the same angular speed of Jupiter, $\omega = 2\pi/P_J$, and the vector gives the direction of the asteroid velocity. This gives the asteroid an initially stable velocity to evolve with.
The Monte Carlo method was used to test regions in which horseshoe orbits could exist within the Sun-Jupiter system. The asteroid generator was used to place a number of asteroids, $N \approx 2300$, with initial positions between $-\pi/2 \leq 0 \leq \pi/2$ and $0.95R_J \leq R \leq 1.05R_J$. Each asteroid was simulated for a period of time ($\sim 320yrs$), and had its evolution tested to determine if its trajectory was one of a Tadpole, Horseshoe, or unstable asteroid. The asteroids were analysed in the comoving frame – with Tadpoles remaining in the $x > 0$ half of the system; Horseshoes reaching the $x < 0$, $y > 0$ quadrant of the system; and unstable asteroids exceeding a range for R, or reaching the $x < 0$, $y > 0$ quadrant before a specified time. These critical parameters in t and R used to define the separation between Horseshoes and unstable orbits were calculated by comparing fringe cases and adjusting each value until each case was consistently displayed as the correct orbit type. This method results in a series of initial positions for classified asteroid orbits, which can be plotted to demonstrate the regions in which each type of orbit can exist.
A simulation was also used to test the long-term lifetime of Horseshoe orbits in general. A small deviation in initial conditions can lead to a large reduction in lifetime, and thus a precise determination of initial conditions which led to the highest stability orbit was required. This was accomplished by running a parameter search looking to maximise the length of time an orbit remained within the previously defined horseshoe orbit zone. A sufficiently stable point was found by initially finding a somewhat stable position and searching within a close proximity locus for points of greater stability and repeating the process until a local maximum was found.

III. Results

An investigation into the Sun-Jupiter system confirmed that, as expected, the mass ratio of Jupiter to the Sun, $\mu \approx 0.0095$, is within the mass ratio for the existence for Lagrange points L4 and L5, $\mu < 0.0385$.

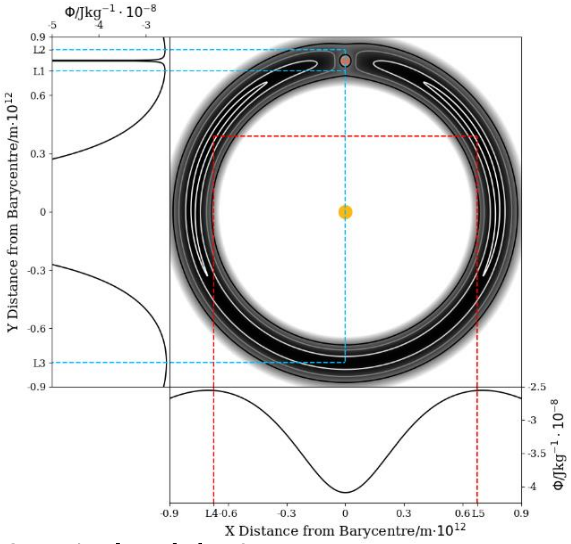

FIG 1. A plot of the Sun-Jupiter comoving gravitational potential. The top right plot shows the Sun-Jupiter system overlayed on a contour and heatmap plot of the comoving potential. The bottom plot shows the comoving potential along the $y = 0.75R_J$ slice of the system. The left plot shows the comoving potential along the $x = 0$ slice of the system. The two local maxima seen in the $y = 0.75R_J$ slice correspond to the positions of L4 and L5.

As a consequence, the zero-velocity contours in **FIG. 1** indicate two maxima flanking Jupiter at the positions of L4 and L5, approximately 60° either side of Jupiter. This agrees with the values suggested by literature[2] and mathematical derivations and lends validity to the modelling methods of this investigation.

Horseshoe and Tadpole asteroids were generated and simulated successfully using the restricted 3 body approximation.

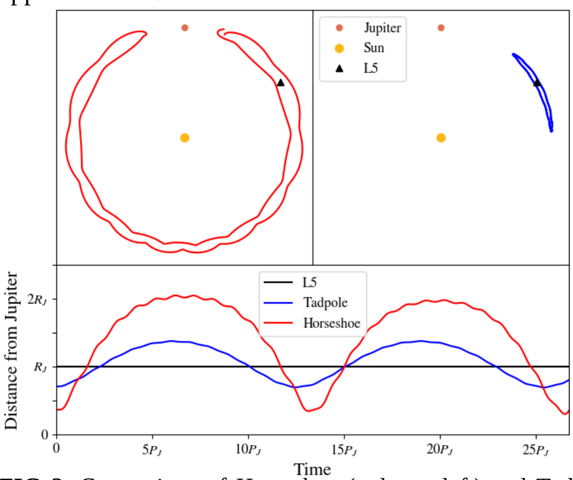

FIG 2. Comparison of Horseshoe (red, top left) and Tadpole (blue, top right) orbits in the comoving frame, and their distance from Jupiter over time. The distance plot (bottom) shows how far each orbit strays from L5. The evolution of these systems (including in the stationary frame) can be seen in these animations: https://imgur.com/a/C83i8QA (Horseshoe); https://imgur.com/a/53z5HOe (Tadpole).

Tadpoles remain close to Jupiter over long periods of time – orbiting around one of L4 or L5, as can be seen from **FIG. 2**. This shows that the asteroid closely flanks Jupiter, as seen from the existing Trojans, and slowly oscillates back and forth, remaining extremely close to a constant orbit radius of R_J. Horseshoes, on the other hand, move from one side of Jupiter to the other – orbiting both L4 and L5, as seen in **FIG. 2**. The asteroid is in a pursuit orbit with Jupiter. Initially, the asteroid starts in a higher orbit with respect to the Sun than Jupiter, and thus moves slower relative to the planet. Over time, Jupiter will catch up to the asteroid, and the asteroid will begin to enter Jupiter's zone of influence and be pulled towards the planet. This acceleration acts to pull the asteroid down into a lower orbit with respect to the Sun, at which point it is moving quicker relative to Jupiter. Over time, the asteroid catches up with Jupiter, and is again pulled towards the planet. This time, the acceleration acts to pull the asteroid up into a higher orbit with respect to the Sun (similar to the original orbit), and the cycle repeats – leading to the orbit drawing a horseshoe shape in the comoving frame.

The Tadpole follows a tight, closed, path around the L5 point. The low amplitude and predictable oscillations suggest that this orbit is stable. Long-term simulations of order 10^7 years of an asteroid in a tadpole orbit showed the asteroid to remain stable, even with a relatively large timestep which is prone to introducing aggregating inaccuracies. This finding is supported by current observations of Jupiter's Trojans[3] – which all follow a variety of the Tadpole orbit[1] – as well as numerical integrations in literature[4] which find them stable up to 10^7 years in accurate simulations, including gravitational effects from the outer planets.

The Horseshoe follows a much wider, fluctuating path around both L4 and L5. As seen from **FIG. 2**, the orbit does not appear to be closed – the final position of the orbit deviates slightly from the initial conditions. This suggests that the orbit is not stable, and the asteroid's closest approach to Jupiter changes over time. One explanation for this is that with every approach to Jupiter, the acceleration from the planet results in a larger change in velocity than the previous approach, leading to increasingly closer approaches to Jupiter. The result of this would be the "horns" of the orbit in the comoving frame creeping closer to Jupiter after each cycle, until the asteroid eventually enters Jupiter's sphere of gravitational influence and is ejected from the Horseshoe orbit. This suggests that the stability of the Horseshoe orbit is far lower than the Tadpole orbit. This is supported by the lack of Horseshoe Trojans today – any that have existed in the past have been destabilised over time.

The approximations used in this investigation lend a combination of additional unphysical stability and instability to the simulated asteroids – namely a lack of external gravitational perturbations (such as from other planets[3]), the simulation being restricted to two dimensions (asteroids having an inclination of zero results in higher stability[3]) Jupiter having a zero eccentricity orbit (increasing the stability of its co-orbitals[4]), a lack of inter-asteroid dynamics (on one hand preventing increased stability through mutual orbits, but on the other preventing collisions and velocity fluctuations), and a lack of solar effects/perturbations (such as the Yarkovsky effect[5], or the effects of solar radiation over a long time period).

Over a long timescale this may cause some significant deviation between the model and the physical system;

however, we believe that the reproduction of a stable tadpole orbit, and a quasi-stable orbit shows that the model is sufficiently accurate for short to mid-length timescale simulations of the system.

The Monte Carlo method was used to search for the zones within the comoving frame horseshoe orbits could exist. As there are no exact definitions for any of the three types of orbits (horseshoe, tadpole, unstable), boundary conditions had to be approximated and then fine-tuned by manually classifying fringe cases and altering parameters until they are correctly classified.

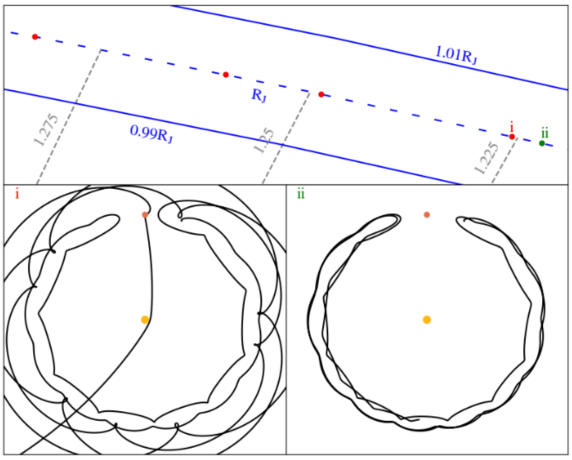

FIG 3. Asteroids distributed near an unstable-horseshoe boundary in a section of the Sun-Jupiter system (top, blue indicating the polar radius, grey indicating the polar angle, where $y = 0, x > 0$ is taken as zero). The boundary exists between asteroid ii (horseshoe) asteroid i (unstable). The bottom left and right plots show the comoving evolution of the unstable and horseshoe asteroids, respectively. Based on these fringe cases, a horseshoe was defined as remaining (in the comoving frame) within $0.85R_J < R < 1.15R_J$ and reaching the $x < 0, y > 0$ quadrant after $t > T/6$. A tadpole was defined as remaining within the $x > 0$ half of the system. An unstable asteroid was defined as exceeding any of the horseshoe conditions.

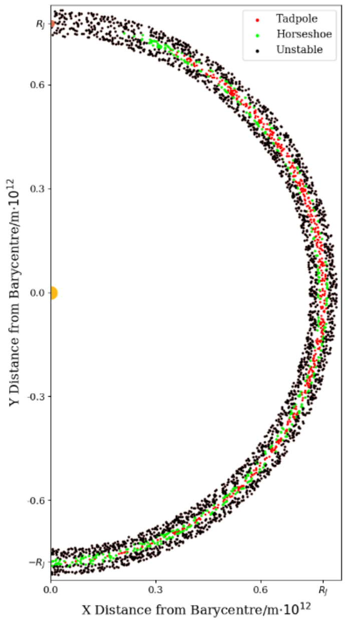

FIG 4. The Monte Carlo simulation to show which initial positions resulted in which type of asteroid over a 340yr simulation. The general distribution of tadpoles (red) and horseshoes (green) roughly follow their orbit contours – closed around L5 and encapsulating both L4/L5, respectively. The simulation used $N = 3231$ asteroids with initial conditions of $0.95R_J \leq R \leq 1.05R_J$ and $\pi/2 \leq \theta \leq -\pi/2$, and simulated them for $T \approx 340$yrs. Of the 3231 asteroids, 2282 (71%) were unstable and escaped, 504 (16%) were Tadpoles, and 445 (14%) were Horseshoes. The $x < 0$ half of the plot is identical, due to symmetry.

A horseshoe orbit was defined as an orbit which creates at least one and a half, full, clearly stable horseshoes in the comoving frame – equivalent to surviving 3 closest approaches to Jupiter. This definition was decided upon, as several fringe asteroids, including asteroid i, completed one full horseshoe orbit; however, did not return to a stable configuration and were subsequently ejected at the next closest approach. Anything less than this number of closest approaches could not be confidently identified as a Horseshoe. It could be argued that simply entering any portion of a horseshoe orbit classifies the asteroid as a Horseshoe – in which case, the regions in which horseshoe orbits can exist would be wider than presented.

The distribution of asteroid type across the system – as seen in **FIG 4.** – appears to generally follow the equipotential contours in **FIG. 1**, as previously hypothesised. Tadpoles are generally confined to areas which are contained by a closed contour centred on L5 – such contours describe a zone of influence of L5, within which all orbits are Tadpoles orbiting about the point in the comoving frame. Horseshoes exist over a much wider region of the system – with two clear blocks either side of the L5 influence zone. Again, this loosely follows the shape of a potential contour encapsulating both L4 and L5 – approaching close to Jupiter, and connecting at the opposite side of the Sun, reflected by the abundance of Horseshoe orbits at $\theta \leq -\pi/3$ and $\theta \geq 1.1$. Unstable asteroids tend to be on a near-horseshoe type orbit – following horseshoe contours which approach too close to Jupiter, leading to them to being destabilised and being ejected from their orbit.

There are some notable discrepancies between the contour lines and the asteroid distribution, however. Tadpoles look to extend far past the zone of influence of L5 along $R \approx R_J$, with a few even approaching the $\theta \approx -\pi/2$ region. The width of the Horseshoe region is also incredibly thin, even compared to the Tadpole region. The L5 zone of influence is approximately as wide as would be suggested by **FIG 1.**, however, the Horseshoe region past this is significantly smaller than would be expected before reaching the unstable asteroid region – leaving just a thin strip for Horseshoes to occupy. This mismatch between the potential contours and the asteroid positions suggests that the boundary conditions of the Monte Carlo method were too strict. The width agreement with Tadpoles – unaffected by the boundary conditions – and the disagreement in Horseshoe width – directly affected by the boundary conditions – shows that the requirement for three stable closest approaches to be too conservative. Relaxing this would increase the width without necessarily compromising the accuracy of the simulation – such a change would allow increasingly unstable horseshoe orbits to be classed as horseshoes, however it is difficult to judge where the line should be drawn. The presence of Tadpoles past the L5 zone of influence, along $R \approx R_J$ to $\theta \approx -\pi/2$ also suggest another error is present. We believe that these "Tadpoles" are actually initially Horseshoes on the boundary of the L5 zone which are trapped near L3 ($R = R_J, \theta = -\pi/2$). It is possible for these asteroids to be slow moving compared to other, more obvious Horseshoes due to their proximity to the L5 zone. Extreme cases may not have crossed to the $x \leq 0$ half of the system at all, while some may have approached $x = 0$ along the $R = R_J$ line but were ensnared by L3 and never crossed over to the other side of the system. While the latter is incredibly unlikely due to the unstable nature of L3 further amplified by the simulation timestep, the former suggests that the length of the simulation was insufficient. Relaxing the definition of a Horseshoe does not change the fact that they have an inherent instability in the Sun-Jupiter system. Other systems are known to have stable horseshoe orbits, most notably the Saturn-Janus-Epimetheus system. There are two factors which differentiate this system from the Sun-Jupiter system. Firstly, Epimetheus is approximately a tenth of the mass of Janus, which affects orbits significantly compared to the asteroid having negligible mass. Secondly, the mass ratio of the two main bodies in the system, Janus and Saturn – $\mu \approx 3.3 \cdot 10^{-9}$, is significantly lower than that of Jupiter and the Sun – $\mu \approx 9.5 \cdot 10^{-4}$. Previous investigations into Horseshoe orbits[6] found that the critical mass ratio for their stability is approximately:

$$\mu_c \leq 8.3 \cdot 10^{-4} \quad (5)$$

The Sun-Jupiter system lies a small margin beyond this stability requirement, whereas Saturn-Janus lies well within it. This proximity to the critical ratio could explain why Sun-Jupiter Horseshoe orbits can exist in a quasi-stable state, but never settle into a fully stable state. Such a dependency on the mass ratio is logical, as Horseshoe orbits are a consequence of the stable Lagrange points L4 and L5, which require a mass ratio of $\mu_L < 0.0385$. For $\mu_c < \mu < \mu_L$, the L5 zone of influence takes up a large portion of the stability region along the orbiter radius. As such, the potential contours which correspond to Horseshoe orbits are too narrow for an asteroid to survive and thus are not stable. For $\mu \leq \mu_c$, the stability zones sufficiently increase in width for Horseshoe orbits to be fully contained in their related contours, and thus will be stable over long periods of time.

Due to their instability, there exists a maximum possible lifetime for Horseshoe orbits in the Sun-Jupiter system. An estimate for the upper limit of the lifetime of a Horseshoe orbit, Γ, within a system of orbital period P and mass ratio μ has been found to be[1]

$$\Gamma \leq P/\mu^{5/3} \quad (6)$$

For the Sun-Jupiter system, this results in an approximate value of $\Gamma \approx 10^6$yrs.

The greatest stability was found to lie along the line $r = R_J$. This was due to the incredibly narrow horseshoe regions at the fringe of the L5 influence zone. Any asteroids placed at positions slightly beyond R_J are likely to stray past the zone in which Horseshoes are contained and fall into an unstable orbit after a much shorter period of time. A variety of values of θ were found to be very stable - $\theta \approx 1.15$ which roughly corresponds to the top of the L5 influence zone, and $\theta \approx -\pi/2$ which roughly corresponds to the position of L3.

$\boldsymbol{\theta}$/rad	$\boldsymbol{-\pi/2}$	$\mathbf{-1.4}$	$\mathbf{-1.3}$	$\mathbf{-1.1}$
$\boldsymbol{\Gamma}$/yrs	$5 \cdot 10^4$	$3 \cdot 10^4$	$2.4 \cdot 10^4$	$3.1 \cdot 10^3$

TAB 1. The lifetimes of Horseshoes with initial conditions $R = R_J$ and $-\pi/2 \leq \theta \leq -1.1$. Lifetimes decrease as θ tends to zero. The value of $5 \cdot 10^4$ for $\theta = -\pi/2$ represents a lower limit on the lifetime, as it was infeasible to run several simulations for longer than this time without sacrificing accuracy.

The bottom of the L5 influence zone is not well defined as shown in **FIG. 4**. The asteroids displayed in **TAB. 1** all followed Horseshoe orbits, which supports the hypothesis that the Tadpole asteroids extending far past L5 in **FIG. 4** are in fact slow moving Horseshoes, and the Monte Carlo simulation was run for an insufficient length of time.

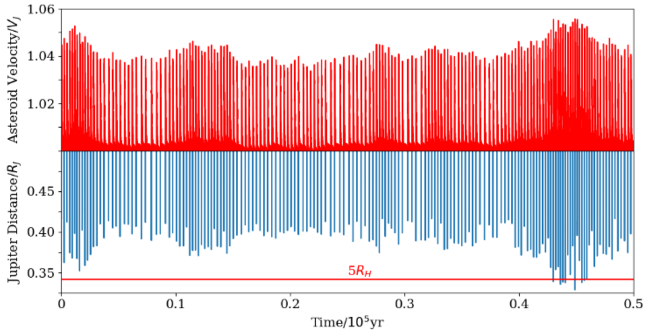

FIG 5. The velocity and distance evolution of a Horseshoe with initial conditions $R = R_J$, $\theta = 1.15$. This position approximately coincides with the top of the L5 influence zone. The simulation had a timestep of 5000s and ran for 50000yrs. This is $0.05\Gamma_{HS}$. As expected, the two plots are closely correlated – a close approach to Jupiter results in the asteroid having a higher velocity. The comoving oscillatory period varies throughout the simulation, shown by a reduction in frequency at $T \approx 0.24 \cdot 10^5$yrs and an increase at $T \approx 0.44 \cdot 10^5$yrs. The increase in frequency coincides with an increase in speed/decrease in closest approach, and vice versa. In general, the asteroid remains at distance $d \geq 5R_H$ from Jupiter and only briefly crosses this threshold during the high frequency, high velocity phase at $T \approx 0.44 \cdot 10^5$. The asteroid then proceeds to return to the modal closet approach of $d \approx 0.4R_J$, with a similar decrease in velocity.

FIG. 5 shows the evolution of a Horseshoe's velocity and distance from Jupiter over time. In contrast to the previously hypothesised continual increase in closest approach over time, the closest approach appears to oscillate about a stable value of $d \approx 0.4R_J$ and occasionally exceeds this significantly. It is unclear what causes this sudden increase in closest approach. Periods of lower velocity appear to correlate to periods of lower orbital frequency and are in general shortly followed by periods of higher velocity with higher orbital frequency. This may indicate some form of orbital resonance between the asteroid and Jupiter, the dynamics of which are unclear, or may indicate an aggregating error caused by the timestep used. The asteroid also crosses the proposed[6] $d \approx 5R_H$ limit for closest approach to Jupiter and remains in a Horseshoe orbit. The asteroid only crosses this threshold for 7 non-consecutive closest approaches, and only exceeds it by a marginal amount. Although the limit is a rough empirical guideline based on different simulations, this could suggest that the timestep used was insufficient to maintain a high level of accuracy over the course of the simulation. The initial positions of one of the most stable Horseshoes ($R = R_J$, $\theta = 1.15$) being on the fringe of the L5 influence zone allows for the possibility of a Tadpole crossing the boundary into a Horseshoe orbit via some form of perturbation. The Yarkovsky effect results from asymmetric re-emission of solar radiation from a spinning body and has been found to significantly alter small ($R \leq 1km$) Trojan objects over a timescale of hundreds of millions of years[5]. Over a long time period, the Yarkovsky effect could cause a fringe Tadpole to fall into a Horseshoe orbit due to a change in velocity. However, it is likely that the orbit would be incredibly unstable and would quickly be ejected. Another candidate for Tadpole perturbation is an interaction with another local body. Mutual orbits and differences in velocity could cause two Tadpole asteroids to collide[7] and perturb the smaller body into a Horseshoe orbit. Long-period comets, such as those from the Oort cloud, could collide with or gravitationally perturb a small Trojan body[8] into a Horseshoe region. While an unlikely event to occur, such an interaction has the potential of producing longer lifetime Horseshoes, similar to that shown in **FIG. 5**.

IV. Conclusions

The restricted 3-body RK4 method was used to successfully recreate asteroid orbits similar to those seen in the Jupiter Trojans and was then used to explore properties of the Sun-Jupiter system, specifically to investigate the lack of horseshoe-type orbits. The Horseshoe orbits that were simulated were far less stable compared to their tadpole counterparts – migrating from one Lagrange point to the other in the comoving frame. This was found to be due to their pursuit-like orbit around the Sun with Jupiter, resulting in significant changes in velocity and closest approaches to Jupiter, ultimately leading to their ejection from a stable orbit. Due to their instability, any Horseshoes which may have existed have long since been ejected from the system, leaving only the stable Tadpoles. A Monte Carlo simulation of asteroid initial conditions supported the hypothesis that asteroid types were bound to regions of equal comoving potential, with Tadpole regions encapsulating one of L4 or L5, and Horseshoe regions encapsulating both. The simulation results were not in complete agreement with the suggested contour shapes. This is likely due to stricter-than-required boundaries placed on the definition of a Horseshoe, based on horseshoe-unstable fringe cases, and an insufficient simulation time period. With additional resources and time, a future investigation could explore this disagreement further by altering the boundary conditions and simulation length. From the results, the conclusion that the horseshoe zones were too narrow to contain fully stable Horseshoes were drawn. The mass ratio of Jupiter to the Sun was found to be marginally above the critical mass ratio for the existence of stable Horseshoes – $\mu_{J-S} \approx 9.5 \cdot 10^{-4} < 8.3 \cdot 10^{-4} \approx \mu_c$. This critical ratio explains why stable horseshoe-type orbits are known to exist in other systems, such as Epimetheus in the Saturn-Janus system ($\mu \approx 3.3 \cdot 10^{-9}$), and why Sun-Jupiter Horseshoes are quasi-stable. A future investigation using similar methods could test how the size of these Horseshoe zones change as the system mass ratio is changed and determine what allows Horseshoe orbits to remain stable in systems within the critical mass ratio.

Simulated Horseshoes were found to remain stable for approximately $5 \cdot 10^4 yrs$ – well within the theorised maximum lifetime of $\Gamma \leq P/\mu^{5/3} \approx 10^6$yrs for the Sun-Jupiter system. The initial conditions of the most stable Horseshoes were found to lie on the Jupiter orbit radius ($R = R_J$) at angles correlating to the position of L3 ($\theta = -\pi/2$) and the fringe of the L5 influence zone ($\theta \approx 1.15$). Differences in the closest approach limit between the approximate literature value ($d \approx 5R_H$) and simulation results were slight and could be attributed to a lack of accuracy caused by a large simulation timestep. Unexplained oscillations in closest approach frequency correlated to closest approach distance were also observed. Future works could look to simulate horseshoes over a range of timesteps to fully test the Horseshoe lifetime equation, the closest approach limit to Jupiter, as well as whether the orbital frequency oscillations are an artefact of timestep errors. The increased stability of Horseshoes which reside on the fringes of the L5 zone of influence indicate that a perturbation leading to a Tadpole-to-Horseshoe movement is possible. Slow moving solar effects (such as the Yarkovsky effect) could lead to the destabilisation of Tadpoles over period of hundreds of millions of years, while inter-Tadpole dynamics or long-period comets (perhaps from the Oort cloud) could dislodge smaller asteroids from the L5 zone of influence and into a horseshoe orbit. The former could be tested in the future by implementing the physical sizes of asteroids and their rotations into the simulation, and the latter could be tested by evolving the system using an N-body method to account for inter-Trojan dynamics, as well as simulating comets which can interact with the asteroids.

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