**Intro**

The Jupiter Trojans are two groups of asteroids which flank Jupiter’s orbit of the Sun, at approximately in front and behind. The clusters of asteroids appear to be on an identical orbit around the Sun as Jupiter – orbiting roughly at the same radius as Jupiter, and with the same period. These orbits are only possible at those positions relative to Jupiter, due to the stable Lagrange points they inhabit. This provides the asteroids which make up the Jupiter Trojans with long-term stability which allows them to persist to this day. Currently, only asteroids which inhabit either one of the two flanking Lagrange points, having Tadpole orbits, are known to exist. This paper aims to investigate another possible type of quasi-stable orbit around both of the Lagrange points these Trojans could enter, or which has possibly existed in the past – known as a Horseshoe orbit.

The restricted three body problem can be used to determine the evolution of a system of three bodies – two massive, and one of negligible mass. The system considered in this paper takes initial parameters resembling the Sun and Jupiter, in circular orbits, in order to analyse the evolution of a third, massless, object representative of an asteroid.

Such a system results in a complex time and space varying gravitational field resulting from the superposition of the gravitational fields of the Sun and Jupiter as they orbit about a common barycentre – there is no full analytic solution to this problem, and thus numerical approximations must be used.

The following assumptions were applied to the system:

1. The Sun and Jupiter have constant fixed circular orbits around one another which are invariant over time – they orbit “on rails” and cannot be perturbed.
2. All initial positions and velocities are co-planar for both the massive objects, and the massless asteroids – the system can be represented as two-dimensional.

Jacobi’s integral can be used in this system to determine the regions in the Sun-Jupiter system that an asteroid with given initial position and velocity parameters can move within. is the asteroid velocity; is the specific potential energy – including both the gravitational and centrifugal potentials – and is a constant relating to the initial conditions of the asteroid.

The asteroid must always satisfy , which constrains it in regions of space, dependent on the initial conditions . This property can be used to construct equipotential zero-velocity curves which satisfy , and constrain the particle to move where .

Equilibrium points in the system exist where the net force (sum of gravitational and centrifugal forces) is zero in a rotating reference frame of angular velocity . In a non-rotating frame, this is equivalent to the gravitational force being equal to the centrifugal force required to rotate at angular velocity at a radius .

Points of zero net force in the rotating frame can be related to the potential via by taking a derivative.

Therefore, these equilibrium points can be identified where the potential reaches a stationary point in both dimensions. From a potential contour plot, five of these points can be identified (for ), known as Lagrange points. There are three unstable Lagrange points (L1, L2, and L3), which correlate to the three saddle stationary points along the Jupiter-Sun axis, and two stable Lagrange points (L4, and L5), which correlate to the two maxima either side of Jupiter.

The consequence of this, is that an asteroid placed at any of these points with an initial velocity perpendicular to the radius drawn from the asteroid to the Sun-Jupiter barycentre with magnitude – where is the orbital period of the system, will remain in that position relative to both Jupiter and the Sun forever. This results in the asteroid remaining stationary in the comoving frame, and the distance between the asteroid and both of the Sun and Jupiter remaining constant.

However, in reality, small perturbations will cause an asteroid to drift from the Lagrange point. For L1, L2, and L3 – the saddle points – a small movement from equilibrium will cause the asteroid to be ejected from the orbit. For the maxima L4 and L5, however; a perturbed asteroid will be able to follow a closed equipotential line orbiting the Lagrange point and remain stable. This causes the asteroid to move in the comoving frame (drawing a tadpole shaped orbit, similar to the equipotential lines surround L4/L5) and for the asteroid-Jupiter/Sun distance to oscillate over time. These are known as “Tadpole” orbits. A further perturbation can push the asteroid out towards the closed equipotential contours encapsulating both L4 and L5, shaped like a horseshoe. These are known as “Horseshoe” orbits.

In the Sun-Jupiter system, all known Trojan asteroids have a variation of the Tadpole orbit; and none being of the Horseshoe variety. This paper looks to investigate aspects of the Horseshoe orbit in the Sun-Jupiter system, including theoretical lifetime and regions where they could exist, to explore reasons for their absence and ways they could form in the future.

**Method**

The dynamics of a restricted 3-body system were simulated over a period of time, , using steps of interval . A two body “on rails” circular orbit system, identical to the Sun-Jupiter system, was developed. Values for the masses, and , and the barycentre distances, and , were taken from literature, and subsequently the system period and angular velocity were calculated using Kepler’s Third Law. At a given time , the Sun and Jupiter are at cartesian positions and . The asteroid is at cartesian position with velocity . To calculate velocity and position at , the net gravitational force due to the Sun and Jupiter is calculated, and the RK4 method is used to accurately approximate the solution to the differential equations in 4 steps, in order to find and . The positions of the Sun and Jupiter are then moved along their orbit – from polar positions and to and through an angle change of . This process was then repeated over iterations to obtain the final state of the system at time , and a history of the positions each body had been at over the simulation period. From this list of positions, several key indicators of how the system evolved can be extracted.

Firstly, the raw positions can be used to track the orbits and orbit paths in the “stationary” frame – i.e. the absolute values of object position relative to the system barycentre can be seen.

Secondly, by calculating the aggregate of over time, and removing its value at each timestep, the raw positions can be transformed to show the positions relative to the rotation of the Sun-Jupiter system. This “comoving frame” results in the positions of Jupiter, the Sun, and all Lagrange points to be stationary over time, and shows how the position of the asteroid evolves relative to the system.

Thirdly, by finding the difference between the positions of the asteroid and Jupiter at each time step, one can see how the asteroid’s proximity to Jupiter evolves over time. This can be compared to the constant distance of L4 and L5 from Jupiter to see how the asteroid orbit compared to a stable Lagrange orbit over time.

Asteroids were injected into the system using an initial conditions generator. The number of asteroids to be injected, and the region of and in which the asteroid should be placed were taken as arguments, and the asteroids were randomly placed within the given parameters. Velocities were based off the position of each asteroid, being selected to initially give the asteroid the angular velocity to match Jupiter at a given radius.

Where the magnitude gives the asteroid the same angular speed of Jupiter, , andthe vector gives the direction of the asteroid velocity. This gives the asteroid an initially stable velocity to evolve with.

The Monte Carlo method was used to test regions in which horseshoe orbits could exist within the Sun-Jupiter system. The asteroid generator was used to place a number of asteroids, , with initial positions between and . Each asteroid was simulated for a period of time (, and had its evolution tested to determine if its trajectory was one of a Tadpole, Horseshoe, or unstable asteroid. The asteroids were analysed in the comoving frame – with Tadpoles remaining in the half of the system; Horseshoes reaching the , quadrant of the system; and unstable asteroids exceeding a range for , or reaching the , quadrant before a specified time. These critical parameters in and used to define the separation between Horseshoes and unstable orbits were calculated by comparing fringe cases and adjusting each value until each case was consistently displayed as the correct orbit type. This method results in a series of initial positions for classified asteroid orbits, which can be plotted to demonstrate the regions in which each type of orbit can exist.

A simulation was also used to test the long-term lifetime of Horseshoe orbits in general. A small deviation in initial conditions can lead to up to a tenfold reduction in lifetime, and thus a precise determination of initial conditions which led to the highest stability orbit was required. This was accomplished by running a parameter search looking to maximise the length of time an orbit remained within the previously defined horseshoe orbit zone. A sufficiently stable point was found by initially finding a somewhat stable position, and searching within a close proximity locus for points of greater stability, and repeating the process until a local maximum was found.